Milnor fibers of arrangements

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Plan

- Formulation of main result
- Projective closure and good compactification
- Local cohomology sequence
- Main result and proof
- Computations
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Milnor fiber

- $Q = \prod_{i=1}^{n} \alpha_i$, a reduced defining polynomial of arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$.
- $\alpha_i \colon \mathbb{C}^\ell \to \mathbb{C}$, a nonzero linear functional; Q is homogeneous of degree n.
- $H_i = \ker \alpha_i$, a linear hyperplane in $\mathbb{C}^\ell$; $\mathcal{A} = \{H_1, \ldots, H_n\}$; $|\mathcal{A}| = n$.
- $Q^{-1}(0) = \bigcup_{i=1}^{n} H_i$.
- $Q \colon \mathbb{C}^\ell - \bigcup_{i=1}^{n} H_i \to \mathbb{C}^*$, the Milnor fibration.
- $F = Q^{-1}(1)$, the *Milnor fiber of* $\mathcal{A}$; $\dim(F) = 2\ell - 2$; $F \simeq$ an $(\ell - 1)$-complex.

Remark: If $\ell = 2$, then Q has an isolated singularity at 0, and $F \cong$ a punctured surface $\simeq \bigvee^{n-1} S^1$. For $\ell \geq 3$, Q has a nonisolated singularity at 0.

Combinatorics

- $L = L(\mathcal{A}) = \{\bigcap_{i \in S} H_i \mid S \subseteq \mathcal{A}\}$, the intersection lattice of $\mathcal{A}$.
- $X \leq Y$ in L iff $X \supseteq Y$.

The main result

Conjecture

The betti numbers of $F = Q^{-1}(1)$ are determined by L.

Equivalently, the betti numbers of F are determined by the underlying matroid of $\mathcal{A}$.

Attribution: Randell (Orlik?) circa 1985; in published form [F-, 1986].

Theorem (F-, 2016)

The first betti number $b_1(F) = \dim H^1(F, \mathbb{Q})$ is determined by L.

Remark: By the Lefschetz Hyperplane Section Theorem, we can assume $\ell = 3$.

Projective closure of F

Let $S = \{[x : y : z : w] \in \mathbb{P}^3 \mid Q(x, y, z) = w^n\}$.

S is a (compact) projective surface in $\mathbb{P}^3$, and $F \cong S \cap \{w \neq 0\}$.

Proposition

The surface S is simply-connected.

The complement $D := S - F = S \cap \{w = 0\}$ is a copy of $\bigcup_{i=1}^n \bar{H}_i \subset \mathbb{P}^2$; here $\bar{H}_i$ is the projective image of H_i.

The singular set Σ of S is the (discrete) set of singular points of D in the plane $\{w = 0\}$.

Resolution of S

Let $\pi \colon \tilde{S} \to S$ be a resolution of singularities:

- $\tilde{S}$ is a smooth;
- π restricts to an isomorphism $\tilde{S} - \pi^{-1}(\Sigma) \to S - \Sigma$;
- $\pi^{-1}(\Sigma)$ is a divisor with normal crossings.

Let $\tilde{D} = \pi^{-1}(D)$. We can identify F with $\tilde{S} - \tilde{D}$.

$\tilde{D}$ is a normal crossing divisor.

Remark: $\tilde{S}$ is simply-connected, obtained from S by a sequence of blow-ups.

Exact sequence

Using $H^1(S) = 0$, from the exact sequence of the pair $(\tilde{S}, F)$, we have an exact sequence

$$H^1(\tilde{S}) = 0 \longrightarrow H^1(F) \longrightarrow H^2(\tilde{S}, F) \xrightarrow{\iota} H^2(\tilde{S}) \longrightarrow H^2(F) \longrightarrow \cdots .$$

Let $\tilde{U}$ be a regular neighborhood of $\tilde{D}$ in $\tilde{S}$.
By excision

$$H^2(S, F) \cong H^2(\tilde{U}, \tilde{U} - \tilde{D}),$$

so we have an exact sequence

$$H^1(\tilde{S}) = 0 \longrightarrow H^1(F) \longrightarrow H^2(\tilde{U}, \tilde{U} - \tilde{D}) \xrightarrow{\iota} H^2(\tilde{S}) \longrightarrow H^2(F) \longrightarrow \cdots .$$

Weights

Henceforth all cohomology groups have coefficients in $\mathbb{Q}$.

Proposition

$\iota\colon H^2(\tilde{S}, F) \to H^2(\tilde{S})$ *is zero.*

Proof.

The local cohomology $H^2(\tilde{U}, \tilde{U} - \tilde{D})$ carries a mixed Hodge structure. Because $\tilde{D}$ is a divisor with normal crossings in the smooth surface $\tilde{S}$, the weights of $H^2(\tilde{U}, \tilde{U} - \tilde{D})$ are 3 and 4. (The weight-three part "$d\log(z_1) \wedge dz_2$" comes from nonsingular points of $\tilde{D}$; the weight-four part "$d\log(z_1) \wedge d\log(z_2)$" comes from singular points of $\tilde{D}$.) Since $\tilde{S}$ is smooth and compact of dimension two, $H^2(\tilde{S})$ has a pure mixed Hodge structure of weight two. The map ι is a morphism of Hodge structures, and hence must vanish.

□

Corollary

$H^1(F) \cong H^2(\tilde{U}, \tilde{U} - \tilde{D})$.

A short exact sequence

We have the exact sequence

$$0 \longrightarrow H^1(\tilde{U}) \longrightarrow H^1(\tilde{U} - \tilde{D}) \longrightarrow H^2(\tilde{U}, \tilde{U} - \tilde{D}) \xrightarrow{\eta} H^2(\tilde{U}) \longrightarrow \cdots .$$

The map η factors:

$$\begin{array}{ccc} H^2(\tilde{U}, \tilde{U} - \tilde{D}) & \xrightarrow{\eta} & H^2(\tilde{U}) \\ \cong \uparrow & & \uparrow \\ H^2(S, F) & \xrightarrow{\iota} & H^2(S) \end{array}$$

hence vanishes as well.
Thus we obtain

Theorem

There is an exact sequence in rational cohomology

$$0 \longrightarrow H^1(\tilde{U}) \xrightarrow{\rho} H^1(\tilde{U} - \tilde{D}) \longrightarrow H^1(F) \longrightarrow 0.$$

Isolated singularities with $\mathbb{C}^*$-action

It remains only to show the map $\rho\colon H^1(\tilde{U}) \to H^1(\tilde{U} - \tilde{D})$ is determined by L. In [Suciu, 2014] it is shown that the target $H^1(\tilde{U} - \tilde{D}) \cong H^1(\partial \tilde{U})$ (the boundary of the Milnor fiber) is determined by L.

Let $p \in \Sigma$ correspond to a point of multiplicity r in D. Choose coordinates so that $p = [0 : 0 : 1 : 0]$. Then in a neighborhood U_p of p the surface S has local equation

$$Q_p(x, y) = w^n,$$

where Q_p is a homogeneous polynomial of degree r.

This is a weighted homogeneous singularity, with weights (r, r, n). There is a corresponding (local) $\mathbb{C}^*$-action $t \cdot (x, y, w) = (t^a x, t^a y, t^b w)$, where $ar = bn = \operatorname{lcm}(r, n)$.

Equivariant resolution

By [Orlik-Wagreich, 1971], there is a “good” resolution $\pi_p \colon \tilde{U}_p \to U_p$ which carries a $\mathbb{C}^*$-action, determined by the weights (r, r, n).

Sketch of construction: The boundary ∂U_p is a three-manifold with (fixed-point free) circle action, hence is obtained by plumbing S^1-bundles over surfaces. The resolution π_p is obtained by plumbing the associated D^2-bundles using the same labelled graph. The exceptional divisor is the union of the zero sections.

The plumbing graph of $\tilde{U}_p$

By [O-W, 1971], the plumbing graph of $\tilde{U}_p$ is star-shaped, with all genera equal zero except for the central vertex, whose genus is given by

$$g(v) = \frac{1}{2}(\frac{r}{b} - 1)(r - 2).$$

For example, $r = 3$, $n = 6$, yields $g(v) = 1$, while $g(v) = 0$ if $r = 2$ or if r and n are coprime.

The regular neighborhood $\tilde{U}$

The regular neighborhood $\tilde{U}$ is obtained by plumbing as well: the local pieces $\tilde{U}_p$ are assembled by plumbing onto Hopf bundles (with D^2 fibers) over the hyperplanes.

The plumbing graph $\tilde{\Gamma}$ is obtained from the Hasse diagram Γ of $\bar{L} := L - \{\mathbb{C}^3, 0\}$ by replacing replacing $p \in L^{[2]}$ with the plumbing graph for $\tilde{U}_p$, with p as the central vertex.

$\tilde{U} - \tilde{D} \simeq \partial\tilde{U}$ is obtained by plumbing S^1-bundles according to the same labelled graph $\tilde{\Gamma}$.

Remark: $\partial\tilde{U}$ the the boundary of the Milnor fiber defined in [Suciu, 2014]. It is a $\mathbb{Z}_n$-cover (corresponding to a character determined by L) of the boundary manifold ∂U of $\bar{\mathcal{A}}$ in $\mathbb{P}^2$ (also determined by L) studied by Jiang-Yau, Westlund, Hironaka, Cohen-Suciu, who described it as a graph manifold, obtained by plumbing Seifert manifolds according to Γ.

Computing $b_1(\tilde{U} - \tilde{D})$

From the exact sequence we have

Corollary

$b_1(F) = b_1(\partial\tilde{U}) - b_1(\tilde{D})$

$b_1(\partial\tilde{U})$ is determined by the depths of certain characters in the characteristic variety of the boundary manifold ∂U. (Difficult to calculate.) This is the relevant jumping locus.

There is a combinatorial procedure to compute $b_1(\partial\tilde{U})$, although an elegant formulation is yet to be worked out.

Computing $b_1(\tilde{D})$

Proposition

If $g(v) = 0$ for all $v \in \tilde{\Gamma}$, then

$$\begin{aligned} b_1(\tilde{D}) &= b_1(\tilde{\Gamma}) \\ &= b_1(\Gamma) \\ &= e - v + 1. \end{aligned}$$

There is a combinatorial procedure to compute $b_1(\tilde{D})$ in general, although an elegant formulation is yet to be worked out.

Multi-arrangements

If $\mathcal{A}$ is a multi-arrangement, $Q = \prod_{i=1}^{n} \alpha_i^{m_i}$. The surface S has non-isolated singularities coming from $m_i \geq 2$.

This case can be treated in a similar way: the initial normalization step in the resolution is described by the equivariant resolution of weighted homogeneous curve singularities $x^m = w^n$ (transverse to the singular curves).

The plumbing graph $\tilde{\Gamma}$ will have "whiskers" on the rank-one vertices.

Monodromy

There may be a way to use this picture to understand monodromy eigenspaces of $H^1(F)$. These won't be described by piecing together local descriptions.

Last slide

This work owes almost everything to the things I've learned from my friends and teachers over the years (starting in 1979 with Orlik), most of whom are in the room.
Thanks to everyone (old friends and new friends, and former student) for coming, for the nice words, and for the birthday wishes.

(May 30, 1955).