GAUSS-MANIN CONNECTIONS FOR ARRANGEMENTS

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Abstract. We construct a universal Gauss-Manin connection for the moduli space of an arrangement of hyperplanes in the cohomology of a complex rank one local system. We prove that the eigenvalues of this connection are integral linear combinations of the weights which define the local system.

1. Introduction

Let \( A = \{H_1, \ldots, H_n\} \) be a hyperplane arrangement in \( \mathbb{C}^\ell \), with complement \( M = M(A) = \mathbb{C}^\ell \setminus \bigcup_{j=1}^n H_j \). Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \) be a collection of weights. Associated to \( \lambda \), we have a rank one representation \( \rho : \pi_1(M) \to \mathbb{C}^* \) given by \( \gamma_j \mapsto t_j = \exp(-2\pi i \lambda_j) \) for any meridian loop \( \gamma_j \) about the hyperplane \( H_j \) of \( A \), and a corresponding rank one local system \( \mathcal{L} \) on \( M \). The need to calculate the local system cohomology \( H^*(M; \mathcal{L}) \) arises in various contexts. In topology, local systems may be used to study the Milnor fiber of the non-isolated hypersurface singularity at the origin obtained by coning the arrangement, see [6, 5]. In mathematical physics, local systems on complements of arrangements arise in the Aomoto-Gelfand theory of multivariable hypergeometric integrals [1, 10, 13] and the representation theory of Lie algebras and quantum groups. These considerations lead to solutions of the Knizhnik-Zamolodchikov differential equation in conformal field theory [16, 17]. Here, a central problem is the determination of the Gauss-Manin connection on \( H^*(M(A); \mathcal{L}) \) for certain arrangements, and local systems arising from certain weights. In this paper, we study the Gauss-Manin connection on \( H^*(M(A); \mathcal{L}) \) for all arrangements, and local systems arising from arbitrary weights.

The arrangements which arise in the context of the K-Z equations are the discriminantal arrangements of Schechtman-Varchenko [16]. The monodromy corresponding to the Gauss-Manin connection is a representation of the fundamental group of the moduli space of all these arrangements [17]. This moduli space is a configuration space. Moduli spaces of arbitrary arrangements with a fixed combinatorial type were defined and investigated by Terao [15]. He identified the moduli spaces of certain arrangements, and determined the Gauss-Manin connection for certain weights. A priori, the eigenvalues of this connection are rational functions of the weights. Terao found that the eigenvalues are, in fact, integral linear combinations of the weights. He asked if this is always the case. In Theorem 4.3, we prove that the eigenvalues of the Gauss-Manin connection are indeed integral linear combinations of the weights for all arrangements and all weights.

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Fix the combinatorial type of an arrangement $\mathcal{A}$, and let $\mathcal{B}$ be a component of the moduli space of arrangements of type $\mathcal{A}$. There is a fiber bundle $\pi : M \to B$ over this moduli space. The fibers of this bundle are complements, $\pi^{-1}(b) = M(\mathcal{A}_b)$, of arrangements combinatorially equivalent to $\mathcal{A}$, so are diffeomorphic to $M(\mathcal{A})$. Given weights $\lambda$, we used stratified Morse theory in [1] to construct a complex which computes $H^*(M(\mathcal{A}); L)$. In fact, we constructed a universal complex $(\mathcal{K}_\lambda^\bullet(\mathcal{A}), \Delta^\bullet(x))$ with the property that the specialization $x_j \mapsto t_j = \exp(-2\pi i \lambda_j)$ calculates $H^*(M(\mathcal{A}); L)$. Here, $x = (x_1, \ldots, x_n)$ are coordinates on the complex $n$-torus $(\mathbb{C}^*)^n$, and $\Lambda = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the coordinate ring. This construction is reviewed in Section 4.

At $b \in \mathcal{B}$, we have the corresponding universal complex $(\mathcal{K}_\lambda^\bullet(\mathcal{A}_b), \Delta^\bullet(t))$ and the cohomology of the latter. Loops in $\mathcal{B}$ based at $b$ induce automorphisms of all these objects, and hence yield representations of $\pi_1(\mathcal{B}, b)$. We call the representation $\pi_1(\mathcal{B}, b) \to \text{Aut}_\Lambda \mathcal{K}_\lambda^\bullet(\mathcal{A}_b)$ the universal Gauss-Manin representation. Let $1 = (1, \ldots, 1)$ be the identity element of $(\mathbb{C}^*)^n$. Let $y = (y_1, \ldots, y_n)$ be the coordinates of $T_1(\mathbb{C}^*)^n = \mathbb{C}^n$, the holomorphic tangent space of $(\mathbb{C}^*)^n$ at $1$. The exponential map $T_1(\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ is induced by $\mathbb{C} \to \mathbb{C}^*, y_j \mapsto e^{y_j} = x_j$. We call the formal connection associated to the universal Gauss-Manin representation the universal Gauss-Manin connection.

Since the complex $(\mathcal{K}_\lambda^\bullet(\mathcal{A}_b), \Delta^\bullet(t))$ computes the cohomology of the local system $\mathcal{L}$ on $M(\mathcal{A}_b)$ corresponding to the weights $\lambda$, the representation $\pi_1(\mathcal{B}, b) \to \text{Aut}_\mathbb{C} H^*(M(\mathcal{A}_b); L)$ is induced by the representation $\pi_1(\mathcal{B}, b) \to \text{Aut}_\mathbb{C} \mathcal{K}_\lambda^\bullet(\mathcal{A}_b)$. We realize the latter as the specialization at $t$ of the universal Gauss-Manin representation. Similarly, the specialization $y_j \mapsto \lambda_j$ of the universal Gauss-Manin connection yields the Gauss-Manin connection on the local system cohomology. In Section 3, we show that the eigenvalues of the universal Gauss-Manin representation are monomials in the $x_j$ with integer exponents. In Section 6, we show that the eigenvalues of the universal Gauss-Manin connection are linear forms in the $y_j$ with integer coefficients. It follows that the eigenvalues of the Gauss-Manin connection in local system cohomology are integral linear combinations of the weights, answering Terao’s question affirmatively for all arrangements and all weights.

The universal Gauss-Manin connection may be viewed as a connection on a combinatorial object, the Aomoto complex. We use the notation and results of [12, 13]. Let $A = A(\mathcal{A})$ be the Orlik-Solomon algebra of $\mathcal{A}$ generated by the 1-dimensional classes $a_j$, $1 \leq j \leq n$. It is the quotient of the exterior algebra generated by these classes by a homogeneous ideal, hence a finite dimensional graded $\mathbb{C}$-algebra. There is an isomorphism of graded algebras $H^*(M; C) \simeq A(\mathcal{A})$. For weights $\lambda$, the Orlik-Solomon algebra is a cochain complex with differential given by multiplication by $a_\lambda = \sum_{j=1}^n \lambda_j a_j$. The Aomoto complex $(\mathcal{A}_\lambda^\bullet(\mathcal{A}), a_\lambda \wedge)$ is a universal complex with the property that the specialization $y_j \mapsto \lambda_j$ calculates $H^*(\mathcal{A}_\lambda^\bullet, a_\lambda \wedge)$. Here, $R = \mathbb{C}[y_1, \ldots, y_n]$ is the coordinate ring of $\mathbb{C}^n$, the holomorphic tangent space of $(\mathbb{C}^*)^n$ at $1$. In [1, Thm. 2.13], we showed that the Aomoto complex $(\mathcal{A}_\lambda^\bullet(\mathcal{A}), a_\lambda \wedge)$ is chain equivalent to the linearization of the universal complex $(\mathcal{K}_\lambda^\bullet(\mathcal{A}), \Delta^\bullet(x))$.

Call a system of weights or the corresponding local system combinatorial if local system cohomology is quasi-isomorphic to Orlik-Solomon algebra cohomology,

$$H^*(M(\mathcal{A}); L) \simeq H^*(\mathcal{A}_\lambda^\bullet(\mathcal{A}), a_\lambda \wedge).$$

The set of combinatorial weights is open and dense in $\mathbb{C}^n$. See [8, 14] for sufficient conditions. For combinatorial weights, the Gauss-Manin connection in local system
cohomology coincides with that in the cohomology of the Orlik-Solomon algebra. Thus, if the eigenvalues of the former are integer linear combinations of the weights, then so are those of the latter. In Section 3, we show that, in fact, the eigenvalues of the Gauss-Manin connection in Orlik-Solomon algebra cohomology are integer linear combinations of the weights for all weights. Since the Aomoto complex $A^*_R(\mathcal{A})$ is the linearization of the universal complex $K^*_{\mathcal{A}}$, results on the universal Gauss-Manin representation on $K^*_{\mathcal{A}}$ inform on the universal Gauss-Manin connection on $A^*_R(\mathcal{A})$, and its specializations, for arbitrary weights.

Call a system of weights or the corresponding local system non-resonant if the Betti numbers of $M$ with coefficients in the associated local system $\mathcal{L}$ are minimal. The set of non-resonant weights is open and dense in $\mathbb{C}^n$, but does not coincide with the set of combinatorial weights. The cohomology of non-resonant local systems is known. A detailed account, including sufficient conditions, is found in [13]. For non-resonant weights we have

$$H^q(M; \mathcal{L}) = 0 \text{ for } q \neq \ell, \text{ and } \dim H^\ell(M; \mathcal{L}) = |e(M)|,$$

where $e(M)$ is the Euler characteristic, see [8, 13, 18]. If the weights are both combinatorial and non-resonant, the Gauss-Manin connection may be studied effectively by combinatorial means. In particular, explicit bases for the single non-vanishing cohomology group are known, see [9]. Using such a basis, Aomoto-Kita [1] computed the Gauss-Manin connection matrices in the case where $\mathcal{A}$ is a general position arrangement. Terao [15] obtained these connection matrices for a larger class of arrangements. Like the eigenvalues, the entries of these matrices were known to be rational functions of the weights. Both Aomoto-Kita and Terao found that these entries were, in fact, integer linear combinations of the weights, and Terao asked if this is the case in general. Our work here was motivated by these results and this question.

2. COHOMOLOGY COMPLEXES

For an arbitrary complex local system $\mathcal{L}$ on the complement of an arrangement $\mathcal{A}$, we used stratified Morse theory in [3] to construct a complex $(K^*(\mathcal{A}), \Delta^*)$, the cohomology of which is naturally isomorphic to $H^*(M; \mathcal{L})$, the cohomology of $M$ with coefficients in $\mathcal{L}$. We now recall this construction in the context of rank one local systems, and record several related complexes and relevant results from [3, 13].

Choose coordinates $u = (u_1, \ldots, u_\ell)$ on $\mathbb{C}^\ell$, and let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, with complement $M = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{j=1}^n H_j$. We assume throughout that $\mathcal{A}$ contains $\ell$ linearly independent hyperplanes. For each $j$, let $f_j$ be a linear polynomial with $H_j = \{u \in \mathbb{C}^\ell \mid f_j(u) = 0\}$. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ be a system of weights. Associated to $\lambda$, we have

(1) a flat connection on the trivial line bundle over $M$, with connection form $\nabla = d + \omega_\lambda \wedge : \Omega^0 \to \Omega^1$, where $d$ is the exterior differential operator with respect to the coordinates $u$, $\omega_\lambda = \sum_{j=1}^n \lambda_j d \log(f_j)$, and $\Omega^q$ is the sheaf of germs of holomorphic differential forms of degree $q$ on $M$;

(2) a rank one representation $\rho : \pi_1(M) \to \mathbb{C}^\ast$, given by $\rho(\gamma_j) = t_j$, where $t = (t_1, \ldots, t_n) \in (\mathbb{C}^\ast)^n$ is defined by $t_j = \exp(-2\pi i \lambda_j)$, and $\gamma_j$ is any meridian loop about the hyperplane $H_j$ of $\mathcal{A}$; and

(3) a rank one local system $\mathcal{L} = \mathcal{L}_t = \mathcal{L}_\lambda$ on $M$ associated to the representation $\rho$ (resp., the flat connection $\nabla$).
Note that weights $\lambda$ and $\lambda'$ yield equivalent connections, and identical representations and local systems if $\lambda - \lambda' \in \mathbb{Z}^n$.

**Remark 2.1.** The arrangement $\mathcal{A}$ determines a Whitney stratification of $\mathbb{C}^\ell$, with codimension zero stratum given by the complement $M$. To describe the strata of higher codimension, recall that an edge of $\mathcal{A}$ is a nonempty intersection of hyperplanes. Associated to each codimension $p$ edge $X$, there is a stratum $S_X = X \setminus \bigcup Y$, where the union is over all edges $Y$ of $\mathcal{A}$ which satisfy $Y \subsetneq X$. Note that $S_X = M(A^X)$ may be realized as the complement of the arrangement $A^X$ in $X$, see [12].

Let $\mathcal{F}$ be a complete flag (of affine subspaces) in $\mathbb{C}^\ell$,

\[
\mathcal{F} : \emptyset = F^{-1} \subset F^0 \subset F^1 \subset \cdots \subset F^\ell = \mathbb{C}^\ell, \tag{2.1}
\]

transverse to the stratification determined by $\mathcal{A}$, so that $\dim \mathcal{F}^q \cap S_X = q - \text{codim } S_X$ for each stratum, where a negative dimension indicates that $\mathcal{F}^q \cap S_X = \emptyset$. For an explicit construction of such a flag, see [12, §1]. Let $M^q = \mathcal{F}^q \cap M$ for each $q$. Let $K^q = H^q(M^q, M^q - 1; \mathcal{L})$, and denote by $\Delta^q$ the boundary homomorphism $H^q(M^q, M^q - 1; \mathcal{L}) \to H^{q+1}(M^{q+1}, M^q; \mathcal{L})$ of the triple $(M^{q+1}, M^q, M^q - 1)$. The following compiles several results from [3].

**Theorem 2.2.** Let $\mathcal{L}$ be a complex rank one local system on the complement $M$ of an arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$.

1. For each $q$, $0 \leq q \leq \ell$, we have $H^i(M^q, M^{q-1}; \mathcal{L}) = 0$ if $i \neq q$, and $\dim_C H^q(M^q, M^{q-1}; \mathcal{L}) = b_q(\mathcal{A})$ is equal to the $q$-th Betti number of $M$ with trivial local coefficients $\mathbb{C}$.

2. The system of complex vector spaces and linear maps $(K^*, \Delta^*)$,

\[
K^0 \xrightarrow{\Delta^0} K^1 \xrightarrow{\Delta^1} K^2 \longrightarrow \cdots \longrightarrow K^{\ell-1} \xrightarrow{\Delta^{\ell-1}} K^\ell,
\]

is a complex $(\Delta^{q+1} \circ \Delta^q = 0)$. The cohomology of this complex is naturally isomorphic to $H^*(M; \mathcal{L})$, the cohomology of $M$ with coefficients in $\mathcal{L}$.

The dimensions of the terms of the complex $(K^*, \Delta^*)$ are independent of $t$ (resp., $\lambda$, $\mathcal{L}$). Write $\Delta^* = \Delta^*(t)$ to indicate the dependence of the complex on $t$, and view these boundary maps as functions of $t$. Let $\Lambda = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ring of complex Laurent polynomials in $n$ commuting variables, and for each $q$, let $K^q_\Lambda = \Lambda \otimes_\mathbb{C} K^q$.

**Theorem 2.3** ([12, Thm. 2.9]). For an arrangement $\mathcal{A}$ of $n$ hyperplanes with complement $M$, there exists a universal complex $(K^*_\Lambda, \Delta^*(x))$ with the following properties:

1. The terms are free $\Lambda$-modules, whose ranks are given by the Betti numbers of $M$, $K^q_\Lambda \simeq \Lambda^{b_q(A)}$.

2. The boundary maps, $\Delta^q(x) : K^q_\Lambda \to K^{q+1}_\Lambda$ are $\Lambda$-linear.

3. For each $t \in (\mathbb{C}^*)^n$, the specialization $x \mapsto t$ yields the complex $(K^*_\Lambda, \Delta^*(t))$, the cohomology of which is isomorphic to $H^*(M; \mathcal{L}_t)$, the cohomology of $M$ with coefficients in the local system associated to $t$.

The entries of the boundary maps $\Delta^q(x)$ are elements of the Laurent polynomial ring $\Lambda$, the coordinate ring of the complex algebraic $n$-torus. Via the specialization $x \mapsto t \in (\mathbb{C}^*)^n$, we view them as holomorphic functions $(\mathbb{C}^*)^n \to \mathbb{C}$. Similarly, for each $q$, we view $\Delta^q(x)$ as a holomorphic map $\Delta^q : (\mathbb{C}^*)^n \to \text{Mat}(\mathbb{C})$, $t \mapsto \Delta^q(t)$.

**Remark 2.4.** If $t = 1$ is the identity element of $(\mathbb{C}^*)^n$, the associated local system $\mathcal{L}_1$ is trivial. Consequently, the specialization $x \mapsto 1$ yields a complex $(K^*_\Lambda, \Delta^*(1))$.
whose cohomology gives $H^*(M; \mathbb{C})$. Since $\dim K^q = b_q(A) = \dim H^q(M; \mathbb{C})$, the boundary maps of this complex are necessarily trivial, $\Delta^q(1) = 0$ for each $q$.

There is an analogous universal complex for the cohomology, $H^*(A^*, a_\Lambda \wedge)$, of the Orlik-Solomon algebra $A = A(A)$. Let $R = \mathbb{C}[y_1, \ldots, y_n]$ be the polynomial ring. The Aomoto complex $(A^*_R, a_\Lambda \wedge)$ has terms $A^q_R = R \otimes_\mathbb{C} A^q \simeq R^{b_q(A)}$, and boundary maps given by $p(y) \otimes \eta \mapsto \sum y_j p(y) \otimes a_j \wedge \eta$. For $\lambda \in \mathbb{C}^n$, the specialization $y \mapsto \lambda$ of the Aomoto complex $(A^*_R, a_\lambda \wedge)$ yields the Orlik-Solomon algebra complex $(A^*, a_\lambda \wedge)$.

A choice of basis for the Orlik-Solomon algebra of $A$ yields a basis for each term $A^q_R$ of the Aomoto complex. Let $\mu^q(y)$ denote the matrix of $a_\lambda \wedge : A^q_R \to A^{q+1}_R$ with respect to a fixed basis. The following results were established in [4].

**Theorem 2.5.**

1. For each $q$, the entries of $\mu^q(y)$ are integral linear forms in $y_1, \ldots, y_n$.
2. For any arrangement $A$, the Aomoto complex $(A^*_R, \mu^q(y))$ is chain equivalent to the linearization of the universal complex $(K^*_A, \Delta^*(x))$.

3. **GAUSS-MANIN REPRESENTATIONS**

Let $A$ be an arrangement in $\mathbb{C}^\ell$ as above, and let $B$ be a smooth, connected component of the moduli space of arrangements with the combinatorial type of $A$. We refer to [13, 15] for the precise definition of this moduli space, and to Section 6 for an example. In this section, we extend the constructions of the previous section to produce a number of representations of the fundamental group of $B$ related to the cohomology of the complement of $A$ with coefficients in a rank one local system.

Recall from [13, §3] that there is a fiber bundle $\pi : M \to B$. For $b \in B$, the fiber $M_b = \pi^{-1}(b)$ is the complement, $M_b = M(A_b)$, of the arrangement $A_b$ combinatorially equivalent to $A$. The closure, $\overline{M}_b$, of the fiber is homeomorphic to $\mathbb{C}^\ell$, and admits a Whitney stratification determined by the arrangement $A_b$ as in Remark 2.1. Let $\mathcal{F}_b$ be a flag in $\overline{M}_b$ that is transverse to $A_b$ as in (2.1). Evidently, these flags may be chosen to vary smoothly with $b$.

Given $t \in (\mathbb{C}^*)^n$ (or weights $\lambda \in \mathbb{C}^n$) and $b \in B$, denote the corresponding local system on $M_b$ by $\mathcal{L}(b)$. In this context, the construction of the previous section yields vector bundles $K^q$ over $B$ for $0 \leq q \leq \ell$ as follows. For $b \in B$, let $M^q_b = \mathcal{F}_b^q \cap M_b$ and

$$K^q(b) = H^q(M^q_b, M^{q-1}_b; \mathcal{L}(b)).$$

Since $\pi : M \to B$ is locally trivial, the natural projection $\pi^q : K^q \to B$, where $K^q = \bigcup_{b \in B} K^q(b)$, is a vector bundle. The transition functions of this vector bundle are locally constant.

If $\gamma : I \to B$ is a path, then the induced bundle $\gamma^*(K^q)$ is trivial. Consequently there is a canonical linear isomorphism $K^q(\gamma(0)) \to K^q(\gamma(1))$, from the fiber over the initial point of $\gamma$ to that over the terminal point, which depends only on the homotopy class of the path. Fix a basepoint $b \in B$, and write $K^* = K^*(b)$. The operation of parallel translation of fibers over curves in $B$ in the vector bundle $\pi^q : K^q \to B$ provides a complex representation of rank $b_q(A)$,

$$\Phi^q : \pi_1(B, b) \to \text{Aut}_\mathbb{C}(K^q).$$

**Lemma 3.1.** If $t = 1$ is the identity element of $(\mathbb{C}^*)^n$, then for each $q$ the corresponding representation $\Phi^q$ is trivial, $\Phi^q(1) = \text{id} : K^q \to K^q$ for all $\gamma \in \pi_1(B, b)$. 

Proof. For the trivial local system \( \mathcal{L}(b) = \mathbb{C} \) associated to \( t = 1 \), the long exact cohomology sequence of the pair \((M_b^q, M_b^{q-1})\) splits into short exact sequences

\[
0 \to H^i(M_b^q, M_b^{q-1}; \mathbb{C}) \to H^i(M_b^q; \mathbb{C}) \to H^i(M_b^{q-1}; \mathbb{C}) \to 0,
\]

see [11, III.3] and [9, Rem. 5.4]. In particular, the \( q \)-th relative cohomology group \( K^q \) is canonically isomorphic to \( H^q(M_b^q; \mathbb{C}) = H^q(M_b; \mathbb{C}) \), the \( q \)-th cohomology of \( M_b \) with constant coefficients \( \mathbb{C} \), see Remark [24]. Consequently, the representation \( \Phi^q(1) : \pi_1(B, b) \to \text{Aut}_C H^q((M_b; \mathbb{C})) \) may be realized as arising from the Gauss-Manin connection on \( B \) corresponding to the trivial weight vector \( \lambda = 0 \in \mathbb{C}^n \), cf. [10, §3]. As this connection is trivial, so is the corresponding representation. □

Denote the boundary homomorphism of the triple \((M_b^{q+1}, M_b^q, M_b^{q-1})\) in cohomology with local coefficients \( \mathcal{L}(b) \) determined by \( t \) by \( \Delta^q(t) = \Delta^q(b) : K^q \to K^{q+1} \).

**Proposition 3.2.** For each \( t \in (\mathbb{C}^*)^n \) and each \( \gamma \in \pi_1(B, b) \), the automorphisms \( \Phi^q(\gamma) \), \( 0 \leq q \leq \ell \), comprise a chain automorphism \( \Phi^q(\gamma) \) of the complex \((K^*, \Delta^*(t))\).

Proof. By Lemma [3.1], the result holds at \( t = 1 \). Therefore it holds for \( t \) close to 1. The result follows. □

To indicate the dependence of the representation \( \Phi^q \) on \( t \in (\mathbb{C}^*)^n \), write \( \Phi^q = \Phi^q(t) \). We abbreviate the above result by writing \( \Phi^q(t) : \pi_1(B, b) \to \text{Aut}_C(K^*) \) in this notation. For \( \gamma \in \pi_1(B, b) \), the automorphism \( \Phi^q(\gamma) = \Phi^q(t)(\gamma) \) may be viewed as a holomorphic function of \( t \):

\[
\Phi^q(\gamma) : (\mathbb{C}^*)^n \to \text{Aut}_C(K^q), \quad t \mapsto \Phi^q(t)(\gamma).
\]

Recall that \( \Lambda = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \). Let \((K^*_b, \Delta^*(x))\) be the universal complex of the arrangement \( A_b \) from Theorem [2.3]. By the continuity of the functions \( \Phi^q(\gamma) \), we have the following extension of this result.

**Theorem 3.3.** For each \( \gamma \in \pi_1(B, b) \), there is a chain map \( \phi^q(x)(\gamma) : K^*_b \to K^*_b \) so that the specialization \( x \mapsto t \) yields the chain automorphism \( \Phi^q(t)(\gamma) \) of the complex \((K^*, \Delta^*(t))\). In other words, there is a universal representation \( \Phi^q(x) : \pi_1(B, b) \to \text{End}_\Lambda(K^*_b) \) which specializes to the representation \( \Phi^q(t) : \pi_1(B, b) \to \text{Aut}_C(K^*) \).

Call \( \Phi^q(x) : \pi_1(B, b) \to \text{End}_\Lambda(K^*_b) \) the universal Gauss-Manin representation.

**Theorem 3.4.** For each \( q \) and each \( \gamma \in \pi_1(B, b) \), the eigenvalues of \( \Phi^q(x)(\gamma) \) are monomials of the form \( x_1^{m_1} \cdots x_n^{m_n} \), where \( m_j \in \mathbb{Z} \).

Proof. Given \( q \) and \( \gamma \in \pi_1(B, b) \), let \( r(x) \) be an eigenvalue of \( \Phi^q(x)(\gamma) \). Then \( r(t) \) is an eigenvalue of \( \Phi^q(t)(\gamma) \in \text{End}_C(K^q) \cong \text{GL}(b_q(A), \mathbb{C}) \) for every \( t \in (\mathbb{C}^*)^n \). It follows that the function \( r : (\mathbb{C}^*)^n \to \mathbb{C}, t \mapsto r(t) \) is single-valued and has no poles. Thus, \( r(x) \) is a Laurent polynomial in \( x_1, \ldots, x_n \). Write \( r(x) = x_1^{m_1} \cdots x_n^{m_n} \cdot p(x) \), where \( p(x) \) is a polynomial. Since \( \Phi^q(t)(\gamma) \) is an automorphism for every \( t \in (\mathbb{C}^*)^n \), we have \( p(t) \neq 0 \) for all \( t \). Thus, \( p(x) = c \in \mathbb{C}^* \) is a non-zero constant, and \( r(x) = c \cdot x_1^{m_1} \cdots x_n^{m_n} \) is a unit in \( \Lambda \). Using Lemma [3.1], we have \( c = 1 \). □

Thus for every \( \gamma \in \pi_1(B, b) \), the maps \( \phi^q(x)(\gamma) \) are automorphisms, so we write \( \Phi^q(x) : \pi_1(B, b) \to \text{Aut}_\Lambda(K^*_b) \).

**Corollary 3.5.** For each \( t \in (\mathbb{C}^*)^n \), the eigenvalues of the automorphism \( \Phi^q(t)(\gamma) \) are evaluations \( r(t) \) of the monomial functions \( r(x) \).
Given $t \in (\mathbb{C}^*)^n$ with associated local system $\mathcal{L}(b)$ on $M_b$, there are also vector bundles $H^q \to B$ over the moduli space, defined by $H^q = \bigcup_{b \in B} H^q(M_b; \mathcal{L}(b))$ for each $q$. As above, parallel translation of the fibers in this bundle over curves in the base gives rise to a representation

$$\Psi^q = \Psi^q(t) : \pi_1(B, b) \to \operatorname{Aut}_\mathbb{C}(H^q(M_b; \mathcal{L}(b))).$$

By Theorem 3.2, the cohomology representation $\Psi^q(t) = \Phi^q(t)^*$ is induced by $\Phi^q(t)$. In other words, for each $\gamma \in \pi_1(B, b)$, the automorphism $\Psi^q(t)(\gamma)$ in cohomology is induced by the automorphism $\Phi^q(t)(\gamma)$ of the complex $(K^*, \Delta^*(t))$.

**Theorem 3.6.** The cohomology representation $\Psi^q(t) = \Phi^q(t)^*$ is induced by $\Phi^q(t)$. In other words, for each $\gamma \in \pi_1(B, b)$, the automorphism $\Psi^q(t)(\gamma)$ in cohomology is induced by the automorphism $\Phi^q(t)(\gamma)$ of the complex $(K^*, \Delta^*(t))$.

**Corollary 3.7.** For each $t \in (\mathbb{C}^*)^n$, the eigenvalues of the automorphism $\Psi^q(t)(\gamma)$ are evaluations of monomial functions.

### 4. Gauss-Manin Connections

The vector bundles $K^q \to B$ and $H^q \to B$ over the moduli space constructed above support Gauss-Manin connections corresponding to the representations $\Phi^q(t)$ and $\Psi^q(t)$ of the fundamental group of $B$. We now study these connections in light of the results of the previous section.

There is a well known equivalence between vector bundles equipped with flat connections and local systems. If $V$ is such a bundle with connection $\nabla$, this equivalence is given by $(V, \nabla) \mapsto V^\nabla$, where $V^\nabla$ is the local system, or locally constant sheaf, of horizontal sections. Under this equivalence, the vector bundles and Gauss-Manin connections above correspond to the local systems on $B$ which, by abuse of notation, we denote by the same symbols. To translate information concerning these local systems, or the fundamental group representations which induce them, to the context of the associated Gauss-Manin connections, we proceed as follows.

Let $\gamma \in \pi_1(B, b)$, and choose a representative path $\tilde{g} : I \to B$. Then, of course, $\tilde{g}(0) = \tilde{g}(1) = b$, so $\tilde{g}$ defines a map $g : S^1 \to B$ from the circle to $B$. If $\phi$ denotes one of the representations $\Phi^q(t)$ or $\Psi^q(t)$, we obtain a corresponding representation of $\pi_1(S^1, 1) = \langle \zeta \rangle = \mathbb{Z}$, given by $\zeta \mapsto X$, where $X = \phi(\gamma)$.

Now let $V \to B$ be one of the vector bundles $K^q \to B$ or $H^q \to B$, and let $g^*(V)$ be the induced vector bundle over the circle. Pulling back the relevant Gauss-Manin connection, we have a corresponding connection on the bundle $g^*(V)$, which is necessarily a trivial vector bundle. Specifying the flat connection on this trivial bundle amounts to choosing a log, $Y$, of the matrix $X$ arising from the above representation. In summary:

**Proposition 4.1.** The connection matrix $Y$ satisfies $X = \exp(-2\pi i Y)$.

The representations $\Phi^q(t)$ and $\Psi^q(t)$ are induced by the universal Gauss-Manin representation $\Phi^q(x) : \pi_1(B, b) \to \operatorname{Aut}_\Lambda(K^*)$, see Theorems 3.3 and 3.4. We now define a corresponding universal connection. Recall from Theorem 2.7 that the Aomoto complex $(\Lambda_{\mathbb{C}}^*, \mu^*) = (\Lambda_{\mathbb{R}}^*(b), \mu^*_b)$ is chain equivalent to the linearization (at $1 \in (\mathbb{C}^*)^n$) of the universal complex $(\Lambda^*, \Delta^*(x))$. For appropriate choices of bases, we may assume without loss that the Aomoto complex is equal to this linearization, see the proof of [1, Thm. 2.13].
where \( x_t \). It follows that \( \Omega^q \) of the automorphism \( \Phi^q \).

**Proof.** Recall from Theorem 3.4 that the eigenvalues of \( \Phi^q \) are linear forms in \( y_1, \ldots, y_n \), with integer coefficients, see Theorem 2.3.4. We call the collection \( \Omega^*(y) \) the universal Gauss-Manin connection.

**Theorem 4.2.** The eigenvalues of the universal Gauss-Manin connection are integral linear forms in the variables \( y_1, \ldots, y_n \). In other words, for each \( \gamma \in \pi_1(\mathcal{B}, b) \) and each \( q \), the eigenvalues of \( \Omega^q(y)(\gamma) \) are integral linear forms in \( y_1, \ldots, y_n \).

**Proof.** Recall from Theorem 3.4 that the eigenvalues of \( \Phi^q(x)(\gamma) \) are monomials of the form \( x_1^{m_1} \cdots x_n^{m_n} \). Since \( \Omega^q(y)(\gamma) \) is the linear term in the power series expansion of \( \Phi^q(\exp(y))((\gamma)) \) in \( y \), the result follows.

This result leads to the following theorem, which is the main result of this paper.

**Theorem 4.3.** For any arrangement \( \mathcal{A} \) and any system of weights \( \lambda \), the eigenvalues of the Gauss-Manin connection in local system cohomology are evaluations of linear forms with integer coefficients, and are thus integral linear combinations of the weights.

**Proof.** Given \( \gamma \in \pi_1(\mathcal{B}, b) \), the endomorphism \( \Omega^q(y)(\gamma) \) of \( A^n_R \) is the linearization of the automorphism \( \Phi^q(x)(\gamma) \) of \( K_X^q \). Recall from Lemma 3.3 that \( \Phi^q(1)(\gamma) = \text{id} \). It follows that \( \Omega^q(y)(\gamma) \) may be realized as a logarithmic derivative of \( \Phi^q(x)(\gamma) \) at \( t = 1 \). This being the case, we have

\[
\Phi^q(x)(\gamma) = \exp(\Omega^q(y)(\gamma)),
\]

where \( x = \exp(y) \). Since \( t = \exp(-2\pi i \lambda) \), the specialization \( x \mapsto t \) yields

\[
\Phi^q(t)(\gamma) = \exp(\Omega^q(-2\pi i \lambda)(\gamma)).
\]

Thus a Gauss-Manin connection matrix \( Y(\gamma) \) satisfies \(-2\pi i Y(\gamma) = \Omega^q(-2\pi i \lambda)(\gamma) \), see Proposition 4.1. Now the entries of \( \Omega^q(y)(\gamma) \) are linear forms in the variables \( y_1, \ldots, y_n \). Consequently, we have \( \Omega^q(-2\pi i \lambda)(\gamma) = -2\pi i \Omega^q(\lambda)(\gamma) \). Therefore, the specialization \( y \mapsto \lambda \) yields the Gauss-Manin connection matrix \( Y(\gamma) = \Omega^q(\lambda)(\gamma) \).

By Theorem 4.2, the eigenvalues of the connection matrix \( \Omega^q(\lambda)(\gamma) \) are evaluations at \( \lambda \) of linear forms with integer coefficients. Passage to cohomology yields a connection matrix \( \Omega^*(\lambda)(\gamma) \) corresponding to the cohomology representation \( \Psi^q(t)(\gamma) \), whose eigenvalues satisfy the same condition.

5. Combinatorial Gauss-Manin Connections

In this section, we investigate the combinatorial implications of the universal Gauss-Manin connection defined on the Aomoto complex. Recall that the Aomoto complex, \( (A^*_R(\mathcal{A}), a_\gamma \wedge) \), is a universal complex with the property that the specialization \( y_j \mapsto \lambda_j \) calculates the Orlik-Solomon algebra cohomology \( H^*(A^*(\mathcal{A}), a_\lambda \wedge) \).

Let \( A^q \rightarrow \mathcal{B} \) be the vector bundle over the moduli space whose fiber at \( b \) is \( A^q(\mathcal{A}_b) \), the \( q \)-th graded component of the Orlik-Solomon algebra of the arrangement \( \mathcal{A}_b \). Given weights \( \lambda \), the cohomology of the complex \( (A^*(\mathcal{A}_b), a_\lambda \wedge) \) gives rise
to an additional vector bundle $H^q(A) \to B$ whose fiber at $b$ is the $q$-th cohomology group of the Orlik-Solomon algebra, $H^q(A^\bullet(A_b), a\Lambda \wedge)$. Like their topological counterparts studied in the previous sections, these combinatorial vector bundles admit combinatorial Gauss-Manin connections.

Fix a basepoint $b \in B$, and denote the Aomoto complex of $A_b$ by simply $(A^\bullet_R(A_b), a\Lambda \wedge)$. As before, let $\mu^\star(y)$ denote the boundary map with respect to a given basis. Recall from [14] that the universal Gauss-Manin connection is comprised of the representations $\Omega^\star(\gamma) : \pi_1(B, b) \to \text{End}_R(A^\bullet_R(B))$.

**Proposition 5.1.** For each $\gamma \in \pi_1(B, b)$, the endomorphisms $\Omega^\star(\gamma)(\gamma)$, $0 \leq q \leq \ell$, comprise a chain map $\Omega^\star(\gamma)(\gamma)$ of the Aomoto complex $(A^\bullet_R(b), \mu^\star(y))$.

**Proof.** For $\gamma \in \pi_1(B, b)$, we have an automorphism $\Phi^\star(x)(\gamma)$ of the universal complex $(K^\bullet, \Delta^\bullet(x))$ by Theorems [3.3 and 3.4]. Write $\Phi^\star = \Phi^\star(x)(\gamma)$ and $\Delta^q = \Delta^q(x)$, and consider these maps as matrices with entries in $\Lambda$. Then, for each $q$,

$$\Delta^q \cdot \Phi^{q+1} = \Phi^q \cdot \Delta^q.$$  

Now make the substitution $x = \exp(y)$, and denote power series expansions in $y$ by $\Delta^q = \sum_{k \geq 0} \Delta_k^q$ and $\Phi^q = \sum_{k \geq 0} \Phi_k^q$. In this notation, $\Omega^\star(y)(\gamma) = \Phi_1^q$.

Comparing terms of degree two in the above equality, we obtain

$$\Delta_0^q \cdot \Phi_2^{q+1} + \Delta_1^q \cdot \Phi_1^{q+1} + \Delta_2^q \cdot \Phi_0^{q+1} = \Phi_0^q \cdot \Delta_2^q + \Phi_1^q \cdot \Delta_1^q + \Phi_2^q \cdot \Delta_0^q.$$  

By Theorem [2.5.2], the linearization of $\Delta^q$ is equal to the boundary map of the Aomoto complex, $\Delta^q \mid \Delta_1^q = \mu^q(y)$. Also, Lemma [3.1] implies that $\Phi_0^q = \text{id}$ and $\Phi_0^{q+1} = \text{id}$, and by Remark [2.4], we have $\Delta_0^q = 0$. These facts, together with (5.1), imply that $\mu^q(y) \cdot \Phi_1^{q+1} = \Phi_1^q \cdot \mu^q(y)$. In other words, $\Phi^\star = \Omega^\star(y)(\gamma)$ is a chain map of the Aomoto complex.

We conclude by illustrating the results of the previous sections with an explicit example. Let $A$ be the arrangement in $\mathbb{C}^2$ with hyperplanes

$$H_1 = \{u_1 + u_2 = 0\}, \quad H_2 = \{2u_1 + u_2 = 0\},$$

$$H_3 = \{3u_1 + u_2 = 0\}, \quad H_4 = \{1 + 5u_1 + u_2 = 0\}.$$
6.1. **Universal Complexes.** We first record the universal complex $K^\Lambda_\bullet$ and the Aomoto complex $A^\mu_R$ of $\mathcal{A}$. The universal complex is equivalent to the cochain complex of the maximal abelian cover of the complement $M = M(\mathcal{A})$. For any $t \in (\mathbb{C}^*)^4$, the specializations at $t$ of the two complexes are quasi-isomorphic. The latter complex may be obtained by applying the Fox calculus to a presentation of the fundamental group of the complement, see for instance [16]. A presentation of this group is

(6.1) \[ \pi_1(M) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid [\gamma_3 \gamma_1, \gamma_2], [\gamma_1 \gamma_2, \gamma_3], [\gamma_i, \gamma_4] \text{ for } i = 1, 2, 3 \rangle, \]

where $[\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}$. Using this presentation, we obtain

\[
K^\Lambda_\bullet : \Lambda \xrightarrow{\Delta^0} \Lambda^4 \xrightarrow{\Delta^1} \Lambda^5,
\]

where, in matrix form, $\Delta^0 = \Delta^0(x) = \begin{bmatrix} x_3 - x_2 x_3 & 1 - x_3 & 1 - x_4 & 0 & 0 \\ x_1 x_3 - 1 & x_1 - x_3 x_4 & 0 & 1 - x_4 & 0 \\ 1 - x_2 & x_1 x_2 - 1 & 0 & 0 & 1 - x_4 \\ 0 & 0 & x_1 - 1 & x_2 - 1 & x_3 - 1 \end{bmatrix}$.

By Theorem 2.3.3, the Aomoto complex $A^\mu_R$ is the linearization of the complex $K^\Lambda_\bullet$. Fixing the nbc-basis for the Orlik-Solomon algebra of $\mathcal{A}$ yields a corresponding basis for $A^\mu_R$. With respect to this basis, the Aomoto complex is given by

\[
A^\mu_R : R \xrightarrow{\mu^0} R^4 \xrightarrow{\mu^1} R^5,
\]

where $\mu^0 = \mu^0(y) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ and

\[
\mu^1 = \mu^1(y) = \begin{bmatrix} -y_2 & -y_3 & -y_4 & 0 & 0 \\ y_1 + y_3 & -y_3 & 0 & y_4 & 0 \\ -y_2 & y_1 + y_2 & 0 & 0 & -y_4 \\ 0 & 0 & y_1 & y_2 & y_3 \end{bmatrix}.
\]

6.2. **The Moduli Space and Related Bundles.** The moduli space of the arrangement $\mathcal{A}$ was studied in detail by Terao [17], see also [13, Example 10.4.2]. This moduli space may be described as

\[
B = B(\mathcal{A}) = \left\{ \begin{pmatrix} z_0^1 & z_0^2 & z_0^3 & z_0^4 & 1 \\ z_1^1 & z_1^2 & z_1^3 & z_1^4 & 0 \\ z_2^1 & z_2^2 & z_2^3 & z_2^4 & 0 \\ z_3^1 & z_3^2 & z_3^3 & z_3^4 & 0 \end{pmatrix} \mid D_{i,j,k} = 0 \text{ if } \{i, j, k\} = \{1, 2, 3\} \right\}.
\]

Here, $(z_0^i : z_1^i : z_2^i : z_3^i) \in \mathbb{C}P^2$ for $1 \leq i \leq 4$, and $D_{i,j,k}$ denotes the determinant of the submatrix of the above matrix with columns $i$, $j$, and $k$, for $1 \leq i < j < k \leq 5$. This moduli space is smooth, see [13, Prop. 9.3.3]. Recall the fiber bundle $\pi : M \rightarrow B$ of [13, §3], with fiber $\pi^{-1}(b) = M_b = M(\mathcal{A}_b)$, the complement of the arrangement $\mathcal{A}_b$ combinatorially equivalent to $\mathcal{A}$. The total space of this bundle is given by

\[
M = \{ (b, u) \in B \times \mathbb{C}^2 \mid u \in M_b \}.
\]

For brevity, in [13, §3] below, we calculate various Gauss-Manin representation and connection matrices for a single element $\alpha \in \pi_1(B, b_0)$, where $b_0 \in B$ is the
basepoint (corresponding to \( \mathcal{A} \)) given below. View \( S^1 \) as the set of complex numbers of length one, and define \( g : S^1 \to B(\mathcal{A}) \), \( s \mapsto g(s) \), by the following formula.

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
1 & 2 & 3 & 5 & 0 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
\quad\quad\quad
\begin{pmatrix}
\frac{3 + x}{1} & \frac{3 + x}{1} & 3 & 5 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix}
\]

Note that \( g \) is a loop based at \( \mathcal{b}_0 \) about the divisor defined by \( D_{1,2,5} = 0 \) in \( B \setminus B \), so represents an element \( \alpha \) of the fundamental group \( \pi_1(\mathcal{B}, \mathcal{b}_0) \). We will determine the action of \( \alpha \) on the universal complex \( K^n_\bullet \).

For this, consider the induced bundle \( g^*(\mathcal{M}) \), with total space

\[
E = \{(s, (b, u)) \in S^1 \times (B \times \mathbb{C}^2) \mid g(s) = b \text{ and } u \in M_b \},
\]

and projection \( \pi'(s, (b, u)) = s \). A similar bundle over \( S^1 \) arises in the context of configuration spaces. We refer to [2] as a general reference on configuration spaces and braid groups. Let \( F_n(C) = \{v \in \mathbb{C}^n \mid v_i \neq v_j \text{ if } i \neq j \} \) be the configuration space of \( n \) ordered points in \( C \), the complement of the braid arrangement.

There is a well known bundle \( p : F_{n+1}(C) \to F_n(C) \), which admits a section. Writing \( F_{n+1}(C) = \{(v, w) \in F_n(C) \times C \mid w \in C \setminus \{v_j\}\} \), the projection is \( p(v, w) = v \).

The fiber of this bundle is \( p^{-1}(v) = C \setminus \{v_j\} \), the complement of \( n \) points in \( C \).

Define \( g_1 : S^1 \to F_4(C) \) by \( g_1(s) = \left(\frac{3 + x}{1}, \frac{3 + x}{1}, 3, 5, 0\right) \). This loop represents the standard generator \( A_{1,2} \) of the pure braid group \( F_4 = \pi_1(F_4(C), v_0) \), the fundamental group of the configuration space \( F_4(C) \), where \( v_0 = (1, 2, 3, 4) \). Let \( g_1^*(F_5(C)) \) be the pullback of the bundle \( p : F_5(C) \to F_4(C) \) along the map \( g_1 : S^1 \to F_4(C) \). The bundle \( g_1^*(F_5(C)) \) has total space

\[
E_1 = \{(s, (v, w)) \in S^1 \times (F_4(C) \times C) \mid g_1(s) = v \text{ and } w \in C \setminus \{v_j\}\},
\]

and projection \( \pi'(s, (v, w)) = s \). The two induced bundles \( g^*(\mathcal{M}) \) and \( g_1^*(F_5(C)) \) are related as follows. If \( v = g_1(s) \) and \( w \in C \setminus \{v_j\} \), it is readily checked that the point \( u = (-1, w) \) is in \( M_{g_1(s)} \), the fiber of \( g^*(\mathcal{M}) \) over \( s \in S^1 \). This defines a map \( h : E_1 \to E \), \( (s, (v, w)) \mapsto (s, (g_1(s), u)) \), where \( u = (-1, w) \). Checking that \( \pi' \circ h = \pi' \), we see that \( h : g_1^*(F_5(C)) \to g^*(\mathcal{M}) \) is a map of bundles.

6.3. Universal Representations and Connections. The bundles \( g_1^*(F_5(C)) \) and \( g^*(\mathcal{M}) \) admit compatible sections, induced by the section of the configuration space bundle \( p : F_5(C) \to F_4(C) \) and the bundle map \( h \) defined above. Consequently, upon passage to fundamental groups, we obtain a commutative diagram, with split short exact rows, as follows.

\[
\begin{array}{cccccc}
1 & \to & \pi_1(C \setminus \{v_j\}) & \to & \pi_1(E_1) & \to & \pi_1(S^1) & \to & 1 \\
& & \downarrow h_* & & \downarrow h_* & & \parallel & & \\
1 & \to & \pi_1(M) & \to & \pi_1(E) & \to & \pi_1(S^1) & \to & 1
\end{array}
\]

Via the bundle map \( h \), the fiber \( \pi_1(C \setminus \{v_j\}) \) of \( g_1^*(\xi_1) \) may be realized as the intersection of the line \( \{v_1 = -1\} \) with the fiber of \( g^*(\xi) \) in \( \mathbb{C}^2 \). Thus, the map \( h_* : \pi_1(C \setminus \{v_j\}) \to \pi_1(M) \) is the natural projection of the free group on four generators, \( \pi_1(C \setminus \{v_j\}) = F_4 = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle \), onto the group \( \pi_1(M) \) with presentation [1]. Let \( \zeta \) denote the standard generator of \( \pi_1(S^1) \), mapping to \( A_{1,2} \in F_4 = \pi_1(F_4(C), v_0) \) and to \( \alpha \in \pi_1(B, \mathcal{b}_0) \) under the homomorphisms induced
by the maps $g_1$ and $g$. The action of $\zeta$ on the free group $F_4$ coincides with that of $A_{1,2}$ on $F_4$, and is well known. It is given by the Artin representation:

$$\zeta(\gamma_i) = A_{1,2}(\gamma_i) = \begin{cases} \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} & \text{if } i = 1, \\ \gamma_1 \gamma_2 \gamma_1^{-1} & \text{if } i = 2, \\ \gamma_1 & \text{otherwise.} \end{cases}$$

By virtue of the commutativity of the above diagram, this action descends to an action of $\alpha \in \pi_1(B, b_0)$ on $\pi_1(M)$ defined by the same formula. The resulting action of $\alpha$ on the universal complex $K^*_\Lambda$—the universal Gauss-Manin representation—may be determined using the Fox calculus, see for instance [8] for similar computations. The action on $K^*_\Lambda$ is trivial since $\alpha$ acts on $\pi_1(M)$ by conjugation. The action on $K^*_\Lambda$ is familiar. It is obtained by applying the Gassner representation to the pure braid $A_{1,2}$. We suppress the calculation of the action of $\alpha$ on $K^*_\Lambda$, and record only the result below.

Denote the universal Gauss-Manin representation and connection matrices corresponding to $\alpha \in \pi_1(B, b_0)$ by $\Phi^q = \Phi^q(x)(\alpha)$ and $\Omega^q = \Omega^q(y)(\alpha)$ respectively. These matrices are given by $\Phi^0 = 1$, $\Omega^0 = 0$,

$$\Phi^1 = \begin{bmatrix} 1 - x_1 + x_1 x_2 & 1 - x_2 & 0 & 0 \\ x_1 - x_1^2 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Omega^1 = \begin{bmatrix} y_2 & -y_2 & 0 & 0 \\ -y_1 & y_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and provide chain maps of the universal and Aomoto complexes as indicated below.

$$\Lambda \xrightarrow{\Delta^0} \Lambda^4 \xrightarrow{\Delta^1} \Lambda^5 \quad \quad R \xrightarrow{\mu^0} R^4 \xrightarrow{\mu^1} R^5$$

6.4. Non-Resonant Local Systems. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be a system of weights in $\mathbb{C}^4$, and $t = (t_1, t_2, t_3, t_4)$ the corresponding point in $(\mathbb{C}^*)^4$. The induced local system $\mathcal{L}$ on $M$ is non-resonant, and $H^2(M; \mathcal{L}) \cong \mathbb{C}^2$, provided the rank of the matrix $\Delta^1(t)$ is equal to three. If this is the case, then $t \neq 1$ and rank $\Delta^0(t) = 1$.

Let $\Xi(x) : \Lambda^5 \to \Lambda^2$ and $\Upsilon(y) : R^5 \to R^2$ be the linear maps with matrices

$$\Xi = \Xi(x) = \begin{bmatrix} x_4 - 1 & 0 & 0 \\ 0 & x_4 - 1 & x_3 - x_2 x_3 \\ x_1 x_3 - 1 & x_1 - x_1 x_3 & 1 - x_2 \\ 1 - x_2 & x_1 x_2 - 1 \end{bmatrix} \quad \quad \Upsilon = \Upsilon(y) = \begin{bmatrix} y_4 & 0 \\ 0 & y_4 \\ y_1 + y_3 & -y_1 \\ -y_2 & y_1 + y_2 \end{bmatrix}.$$
Via $\Xi : K^*_{\Lambda} \to \Lambda^2$ and $\Upsilon : A^2_{R} \to R^2$, the chain maps $\Phi^* : K_{\Lambda}^* \to K_{\Lambda}^*$ and $\Omega^* : A^*_R \to A^*_R$ induce maps $\overline{\Phi} : \Lambda^2 \to \Lambda^2$ and $\overline{\Omega} : R^2 \to R^2$, given by

$$\overline{\Phi} = \begin{bmatrix} x_1x_2 & 0 \\ x_2 - 1 & 1 \end{bmatrix} \quad \text{and} \quad \overline{\Omega} = \begin{bmatrix} y_1 + y_2 & 0 \\ y_2 & 0 \end{bmatrix}.$$

Specializing at $t \in (\mathbb{C}^*)^4$ and $\lambda \in \mathbb{C}^4$ yields the Gauss-Manin representation matrix $\Psi^2(t)(\alpha)$ and the corresponding Gauss-Manin connection matrix $\Omega^*(\lambda)(\alpha)$ in the cohomology of the non-resonant local system $L$. These matrices are

$$\Psi^2(t)(\alpha) = \overline{\Phi}(t) = \begin{bmatrix} t_1t_2 & 0 \\ t_2 - 1 & 1 \end{bmatrix} \quad \text{and} \quad \Omega^*(\lambda)(\alpha) = \overline{\Omega}(\lambda) = \begin{bmatrix} \lambda_1 + \lambda_2 & 0 \\ \lambda_2 & 0 \end{bmatrix}.$$

Up to a transpose, the latter recovers Terao’s calculation of the connection matrix corresponding to the divisor $D_{1,2,5}$, denoted by $\Omega_4$ in [13, Example 10.4.2].

6.5. Resonant Local Systems. Now let $L$ be a non-trivial resonant local system on $M$. Such a local system corresponds to a point $t \notin (\mathbb{C}^*)^4$ satisfying $t_1t_2t_3 = 1$ and $t_4 = 1$. For each such $t$, we have $H^1(M; L) \cong \mathbb{C}$ and $H^2(M; L) \cong \mathbb{C}^3$. Gauss-Manin representation and connection matrices corresponding to the loop $\gamma \in \pi_1(B, b)$ in resonant local system cohomology may be obtained by methods analogous to those used in the non-resonant case above.

Define $\Xi : \Lambda^5 \to \Lambda^3$ and $\Upsilon : R^5 \to R^3$ by

$$\Xi = \begin{bmatrix} x_1x_2 - 1 & 0 & 0 \\ x_2 - 1 & 0 & 0 \\ 0 & x_2 - 1 & 0 \\ 0 & 1 - x_1 & x_3 - 1 \\ 0 & 0 & 1 - x_2 \end{bmatrix} \quad \text{and} \quad \Upsilon = \begin{bmatrix} y_1 + y_2 & 0 & 0 \\ y_2 & 0 & 0 \\ 0 & -y_1 & y_3 \end{bmatrix}.$$

As before, $\Xi \circ \Delta^1 = 0$, and $\Upsilon \circ \mu^1 = 0$, and $\Xi$ is the linearization of $\Xi$. For each $t \in (\mathbb{C}^*)^4$ satisfying $t_1t_2t_3 = 1$ and $t_4 = 1$, we have rank $\Xi(t) = 3$. So the projection $\mathbb{C}^5 \cong K^2 \to H^2(M; L) \cong \mathbb{C}^3$ may be realized as the specialization $\Xi(t)$.

Via $\Xi : K^*_{\Lambda} \to \Lambda^3$ and $\Upsilon : A^2_{R} \to R^3$, the chain maps $\Phi^* : K_{\Lambda}^* \to K_{\Lambda}^*$ and $\Omega^* : A^*_R \to A^*_R$ induce $\overline{\Phi} : \Lambda^3 \to \Lambda^3$ and $\overline{\Omega} : R^3 \to R^3$, given by

$$\overline{\Phi} = \begin{bmatrix} x_1x_2 & 0 & 0 \\ x_1x_2 - 1 & 0 & 0 \\ 0 & 1 - x_3 & 0 \end{bmatrix} \quad \text{and} \quad \overline{\Omega} = \begin{bmatrix} y_1 + y_2 & 0 & 0 \\ 0 & y_1 + y_2 & -y_3 \end{bmatrix}.$$

Specializing yields the Gauss-Manin representation matrix $\Psi^2(t)(\alpha)$ and the corresponding Gauss-Manin connection matrix $\Omega^*(\lambda)(\alpha)$ in the second cohomology of the resonant local system $L$.

Using the universal complex $K_{\Lambda}^*$ and the conditions satisfied by a point $t \in (\mathbb{C}^*)^4$ inducing the resonant local system $L$, one can show that the Gauss-Manin representation matrix $\Psi^1(t)(\alpha)$ in first cohomology is given by $\Psi^1(t)(\alpha) = t_1t_2$. The corresponding Gauss-Manin connection matrix is, of course, $\Omega^*(\lambda)(\alpha) = \lambda_1 + \lambda_2$.

Remark 6.6. For this arrangement, every local system $L$ is combinatorial. Given $L$, there are weights $\lambda \in \mathbb{C}^4$ for which the local system cohomology $H^*(M; L)$ is quasi-isomorphic to Orlik-Solomon algebra cohomology $H^*(A^*, a_{\Lambda})$. Thus, for such weights, the combinatorial Gauss-Manin connection matrices in the cohomology of the Orlik-Solomon algebra coincide with the Gauss-Manin connection matrices in local system cohomology computed above.
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