2009 LSU Math Contest
Team Problems

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

You have one hour and fifteen minutes to complete the entire team session.

These 11 questions require exact numerical or algebraic answers. Hand written exact answers must be written with fractions reduced, radicals simplified, and denominators rationalized. Do not make an approximation for π or other irrational numbers. Answers must be exact.

The tiebreaker for the team competition is time. *If your team reaches a point where you are satisfied or expect that you will not have more solutions in the allotted time, then you may wish to turn in your paper a little early to get a time advantage.*

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1 How many subsets with three elements can be formed from the set $\{1, 2, \ldots, 20\}$ so that 4 is a factor of the product of the three numbers in the subset?

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2 How many four digit numbers with all digits different and which contain the digit 7 are there ?

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3 Let a, b, c, and m be positive real numbers. Find for which m, independently of a, b, and c, at least one of the following three equations has real roots:

$$ax^2 + bx + cm = 0, \tag{1}$$
$$bx^2 + cx + am = 0, \tag{2}$$
$$cx^2 + ax + bm = 0. \tag{3}$$

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4 Let $f(x)$ be a decreasing function defined on $(0, +\infty)$. Describe the set of all real numbers a such that

$$f(2a^2 + a + 1) < f(3a^2 - 4a + 1).$$

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5 Find a pair of positive integers x and y, $x \leq y$, such that $\sqrt{x} + \sqrt{y} = \sqrt{2009}$.

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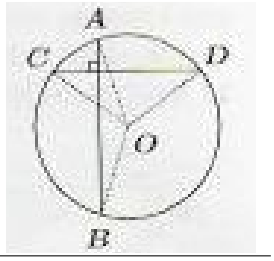
6 In a circle having center O, the chords AB and CD are perpendicular to each other and neither chord passes through the center. Find the measure (in degrees) of the angle which is the sum $\angle AOD + \angle BOC$.



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7 A circle has center O. Lines PA and PB are tangent to the circle respectively at A and B. Points E and F belong respectively to segments PA and PB and the line EF is tangent to the circle. Knowing that $\angle EPF = \angle EOF$ find the measure of $\angle EPF$ (in radians).

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8 Let p, q, r be any real numbers such that $pq \neq -1 \neq qr$. Simplify

$$\arctan \frac{p-q}{1+pq} + \arctan \frac{q-r}{1+qr}.$$

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9 A cube of cheese

$$C = \{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$$

is cut along the planes $x = y$, $y = z$, and $z = x$. How many pieces are there?

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10 Alice and Bob are in a hardware store. The store sells colored sleeves that fit over keys to distinguish them. The following conversation takes place:

A: Are you going to cover your keys?

B: I would like to, but there are only 7 colors and I have 8 keys.

A: Yes, but you could always distinguish a key by noticing that the red key next to the green key was different from the red key next to the blue key.

B: You must be careful what you mean by *next to* or *three keys over from* since you can turn the key ring over and the keys are arranged in a circle.

A: Even so, you don't need 8 colors.

What is the smallest number of colors needed to distinguish $n \geq 6$ keys if all the keys are to be covered?

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11 Find the largest prime factor of

$$(25!)^3 - (24!)^3.$$
