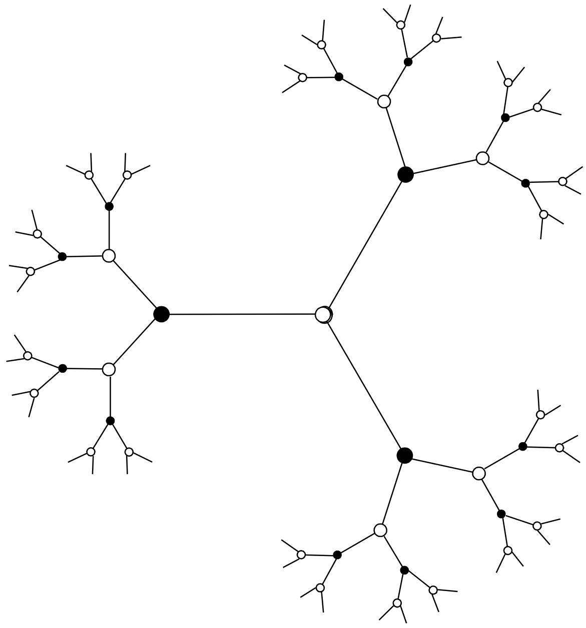

Figure 1: Serre-Tree of $\mathrm{SL}_2(\mathbb{Q}_2)$

This figure shows the Serre-tree associated with the p-adic Lie group $\mathrm{SL}_2(\mathbb{Q}_2)$. This tree is a geometric object which plays a role analogous to that of the upper half plane in the theory of $\mathrm{SL}_2(\mathbb{R})$. In particular, the tree is a metric space (with a $\mathbb{Z}$-valued metric), and the group acts transitively by isometries. We sketch how to construct the tree.

Construction

In general, if K is a field with discrete valuation ν, the tree for $\mathrm{SL}_2(K)$ is constructed as follows. Let $V = K^2$ and $\mathcal{O} = \{x \in K \mid \nu(x) \geq 0\}$ the associated discrete valuation ring. We consider free $\mathcal{O}$-submodules $L \subset V$ of rank 2, and call them $\mathcal{O}$-lattices. Two lattices L and L' are homothetic if $L = cL'$ for some $c \in K^\times$. The set of homothety classes $[L]$ is the vertex set $\mathcal{V}$ of our tree, and it remains to define the set $\mathcal{E}$ of edges. To this end one defines a $\mathbb{Z}$-valued metric on $\mathcal{V}$. We let $\pi \in \mathcal{O}$ be a uniformizing element, that is, an element with $\nu(\pi) = 1$. Then one can show that if L, L' are two lattices, and L has $\mathcal{O}$-basis $\{v_1, v_2\}$, then one can find $a, b \in \mathbb{Z}$ such that $\{\pi^a v_1, \pi^b v_2\}$ is a basis for L'. One defines

$$d([L], [L']) = |b - a|,$$

and then has to check that it does not depend on the chosen representatives. Finally we say that there is an edge between $[L]$ and $[L']$ if the distance is $d([L],[L']) = 1$. It is then a theorem that the graph $(\mathcal{E}, \mathcal{V})$ is a tree, see [1].

If the residue field $\mathcal{O}/\mathfrak{m}$ is finite and has p elements, one can furthermore show that the tree one obtains is complete of order $p+1$, and this explains the above picture: It shows a tree each of whose nodes has degree $2+1=3$.

Generalizations

A similar construction to the one outlined above works for higher rank groups $\mathrm{SL}_n(K)$ and leads to so called *affine Bruhat-Tits buildings*. These objects are simplicial complexes which are pasted together from copies of the complex $\Sigma = \Sigma(W)$ associated to an affine Weyl group. In the above rank-1-example, the group is $\tilde{A}_1 = \mathbb{Z}$, which is generated by two reflections of $\mathbb{R}$. The two generators give rise to the two edge colors (black and white) which can be seen in the above picture. In the general case, if W has $k+1$ generators (that is, if the group has rank k,) then the building is a $k+1$-colorable simplicial complex. See [3] for an introduction to the theory of buildings.

Another generalization allows metrics with values in arbitrary ordered abelian groups Λ and leads to the theory of Λ-buildings and Λ-trees ([5], [4]) and Λ-buildings ([2]).

References

[1] Jean-Pierre Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell; Corrected 2nd printing of the 1980 English translation. MR **1954121 (2003m:**20032)

[2] Curtis D. Bennett, *Affine Λ-buildings. I*, Proc. London Math. Soc. (3) **68** (1994), no. 3, 541–576, DOI 10.1112/plms/s3-68.3.541. MR **1262308 (94m:**20068)

[3] Kenneth S. Brown, *Buildings*, Springer Monographs in Mathematics, Springer-Verlag, New York, 1998. Reprint of the 1989 original. MR **1644630 (99d:**20042)

[4] Ian Chiswell, *Introduction to Λ-trees*, World Scientific Publishing Co. Inc., River Edge, NJ, 2001. MR **1851337 (2003e:**20029)

[5] John W. Morgan, *Λ-trees and their applications*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), no. 1, 87–112, DOI 10.1090/S0273-0979-1992-00237-9. MR **1100579 (92e:**20017)