Generating Functions and Highest Weight Representations

Mathematics Department

Louisiana State University

Overview
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- Special Functions and Representation Theory
- (Classical) Generating Functions: some Examples
- Reproducing Kernel Hilbert Spaces: $\mathcal{H}(\mathcal{S}, V)$
- Bounded operators $\Theta : L^2(X, d\mu) \to \mathcal{H}(\mathcal{S}, V)$
- Kernels and Generating Functions
- Restriction Principle
- Highest Weight Representations
- Generating Functions (revisited)

Classical Generating Functions
Generating Functions
Laguerre
Hermite
Generalizations
Goals

Classical Generating Functions

Classical Generating Functions
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Suppose $a_n(x)$ is a sequence of real valued functions. The **generating function** associated to $\{a_k : k = 1, 2, \ldots\}$ is an analytic function, $a(z, x)$, such that

$$a(z,x) = \sum_{k=0}^{\infty} a_k(x) z^k.$$

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Laguerre Polynomials

$$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}).$$

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Laguerre Polynomials

$$L_n^{\alpha}(x) = \frac{1}{n!} e^{x} x^{-\alpha} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}).$$

Generating Function:

$$(1-z)^{-\alpha-1} e^{\frac{xz}{z-1}} = \sum_{k=0}^{\infty} L_n^{\alpha}(x) z^n \quad |z| < 1.$$

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Hermite Polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}).$$

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Hermite Polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}).$$

Generating Function:

$$e^{(-z^2+2zx)} = \sum_{k=0}^{\infty} H_k(x) \frac{z^k}{k!} \quad z \in \mathbb{C}.$$

Generalizations

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Each of these formulae are classical and easy to prove in that it is possible to compute the Maclaurin series for the generating function to produce the corresponding series.

In some generalizations one finds that the powers z^n that appear in the series

$$\sum_n a_n(x) z^n$$

are replaced by a more general class of functions.

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are replaced by a more general class of functions.

Generalize Laguerre functions: $\ell_{\mathbf{n}}^{\mu}$ are certain distinguished function defined of symmetric cones and are paired with generalized power functions $\Psi_{\mathbf{m}}$ that lie in a reproducing kernel Hilbert Space. The generating function takes the form

$$\Delta(e-z)^{-\nu} \int_K e^{-(kx|(1+z)(1-z)^{-1})}\, dk = \sum_{\mathbf{m}\geq 0} d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \ell_{\mathbf{m}}^{\nu}(x) \Phi_{\mathbf{m}}(z).$$

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Goals:

1. Place these formulas in a more general context. Specifically, we show that a bounded operator Θ defined on an L^2 space with values in a reproducing kernel Hilbert space is necessarily given by a kernel from which is derived a generating function for an appropriately chosen system of functions in L^2.
2. When the L^2 space and the reproducing kernel Hilbert space are equivalent realizations of a highest weight representation we obtain new characterizations of the generating functions and relationships amongst the given systems in terms of operations involving the representation.

Reproducing Kernel Hilbert Spaces

Reproducing Kernel Hilbert Spaces

Suppose

- $\mathcal{S}$- locally compact Hausdorff space
- V - complex Hilbert space with inner product $(\cdot \mid \cdot)_V$

Let $\mathcal{H}(\mathcal{S}, V)$ be a Hilbert space of continuous V-valued functions on $\mathcal{S}$.

We say $\mathcal{H}(\mathcal{S}, V)$ is a **reproducing kernel Hilbert Space** if for each $z \in \mathcal{S}$ the linear map $E_z : \mathcal{H}(\mathcal{S}, V) \to V$ given by $E_z(f) = f(z)$ is continuous and has dense range. This assumption implies that the adjoint

$$Q(\cdot, z) := E_z^*$$

is continuous, injective, and has the reproducing property:

$$(f(z) \mid v)_V = (f \mid Q(\cdot, z)v) \,,$$

for each $f \in \mathcal{H}(\mathcal{S}, V)$ and $v \in V$.

Bounded Operators
Vector-valued L^2 spaces
Linear Operators
Proposition
proof
Generating Function
proof

RKHS-valued operators on an L^2 space

Vector-valued L^2 spaces

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proof

Let W be a finite dimensional complex Hilbert Space and $\mathrm{Herm}^+(W)$ the convex cone of nonnegative definite operators on W; i.e. each $E \in \mathrm{Herm}^+(W)$ is a bounded operator and satisfies

$$(Ew \mid w)_W \geq 0,$$

for each $w \in W$.

Let X be a measure space with positive measure μ and ν a $\mathrm{Herm}^+(W)$-valued measurable on X. We write $d\mu_\nu(x) = \nu(x)d\mu$.

Let $L^2(X, W, d\mu_\nu)$ denote the space of measurable W-valued functions f such that

$$\|f\|^2 = \int_X (\nu(x)f(x) \mid f(x))\, d\mu(x) < \infty.$$

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A linear map Θ on $L^2(X, W, d\mu_\nu)$ with values in $\mathcal{H}(\mathcal{S}, V)$ is said to be an integral transform given by a kernel $K = K^\Theta$ if Θ can be written in the form

$$\Theta(f)(z) = \int_X K(z, x)\nu(x) f(x)\, d\mu(x),$$

for all $z \in \mathcal{S}$. Pointwise, the kernel $K(z, x)$ is a map from W to V.

Proposition

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Proposition: Suppose $\Theta : L^2(X, W, d\mu_\nu) \to \mathcal{H}(\mathcal{S}, V)$ is a bounded linear map. Then Θ is an integral transform given by a kernel K. For each $z \in \mathcal{S}$ and $v \in V$ the map $x \mapsto K(z, x)^* v$ is in $L^2(X, W, d\mu_\nu)$.

proof

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Let $z \in \mathcal{S}$ and $v \in V$. Then

$$\begin{aligned} (\Theta f(z) \mid v) &= (\Theta f \mid Q(\cdot, z)v) \\ &= (f \mid \Theta^*(Q(\cdot, z)v)) . \end{aligned}$$

Let $k_{z,v} = \Theta^*(Q(\cdot, z)v)$. Then $k_{z,v} \in L^2(X, W, d\mu_\nu)$, for all $z \in \mathcal{S}$ and $v \in V$.

For each $x \in X$ the map

$$v \mapsto k_{z,v}(x) : V \to W$$

is a linear and hence continuous.

Let $B(V, W)$ be the space of W-valued linear maps on V and let $s_z(x) \in B(V, W)$ be given by $s_z(x)v = k_{z,v}(x)$.

Let $K(z, x) \in B(W, V)$ be the adjoint of $s_z(x)$.

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We then have

$$\begin{aligned}(\Theta f(z) \mid v) &= (f \mid k_{z,v}) \\ &= \int_X (\nu(x) f(x) \mid s_z(x) v)\ d\mu(x) \\ &= \int_X (K(z,x)\nu(x) f(x) \mid v)\ d\mu(x)\end{aligned}$$

and hence

$$\Theta f(z) = \int_X K(z,x)\nu(x) f(x)\, d\mu(x),$$

for all $f \in L^2(X, W, d\mu_\nu)$ and $z \in \mathcal{S}$.

Since $K(z,\cdot)^* v = k_{z,v}$ for all $z \in \mathcal{S}$ and $v \in V$ we have $K(z,\cdot)^* v \in L^2(X, W, d\mu_\nu)$

Generating Function

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Theorem [Davidson 2004]: Suppose $\mathbb{H} \subset L^2(X, W, d\mu_\nu)$ is a separable Hilbert subspace and $\Theta : \mathbb{H} \to \mathcal{H}(\mathcal{S}, V)$ is a bounded linear map into a reproducing kernel Hilbert space with kernel K^Θ. Suppose $\{e_i : i \in I\}$ is a basis of $\mathbb{H}$ that has a dual basis $\{\check{e}_i : i \in I\}$. Set $E_i = \Theta\check{e}_i \in \mathcal{H}(\mathcal{S}, V)$. Then

$$K^\Theta(z, \cdot)^* v = \sum_{i \in I} (v | E_i(z))\, e_i,$$

where convergence is with respect to the L^2-norm.

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Let $v \in V$. For each $z \in \mathcal{S}$ we have $K^{\Theta}(z,\cdot)^* v \in \mathbb{H}$ and

$$\begin{aligned} K^{\Theta}(z,\cdot)^* v &= \sum_{i \in I} (K^{\Theta}(z,\cdot)^* v | \check{e}_i)\, e_i \\ &= \sum_{i \in I} (v | \Theta \check{e}_i(z))\, e_i \\ &= \sum_{i \in I} (v | E_i(z))\, e_i . \end{aligned}$$

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Let $v \in V$. For each $z \in \mathcal{S}$ we have $K^\Theta(z,\cdot)^* v \in \mathbb{H}$ and

$$\begin{aligned} K^\Theta(z,\cdot)^* v &= \sum_{i \in I} (K^\Theta(z,\cdot)^* v | \check{e}_i)\ e_i \\ &= \sum_{i \in I} (v | \Theta \check{e}_i(z))\ e_i \\ &= \sum_{i \in I} (v | E_i(z))\ e_i. \end{aligned}$$

We will call $K^\Theta(z,\cdot)^* v$ the **generating function** for $\{e_i : i \in \mathbb{N}\}$.

Restriction Principle
Restriction Princ.
Bargmann-Segal
Laplace Transform
Cayley Transform
Laguerre functions

Restriction Principle

Olafsson-Orsted Restriction Principle

Restriction Principle
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- M - complex manifold
- $\mathcal{H}$ - Reproducing Kernel Hilbert Space of holomorphic functions on M
- X - totally real submanifold of M
- $R : \mathcal{H} \to L^2(X, d\mu)$ defined by $RF(x) = D(x)F(x)$, where $D(x)$ is some positive multiplier.
- Suppose R is densely defined, injective, dense range. Then we can polarize R^*:

$$R^* = U\sqrt{RR^*},$$

to get a unitary map $U : L^2(X, d\mu) \to \mathcal{H}$.

Bargmann-Segal Transform: an Example

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- $M = \mathbb{C}$, $\mathcal{H}$= Fock space of $\mathbb{C}$.
- $X = \mathbb{R}$,
- $R : \mathcal{H} \to L^2(\mathbb{R}, dx)$ given by $RF(x) = e^{\frac{-x^2}{2}} F(x)$.
- Polarization: $R^* = U\sqrt{RR^*}$ where $U : L^2(\mathbb{R}, dx) \to \mathcal{H}$ is given by

$$Uf(z) = c\int_{\mathbb{R}} e^{-x^2+2xz-z^2/2} f(x)\, dx,$$

the *Bargmann-Segal transform*.

Define $h_k(x) = e^{-x^2} H_k(\sqrt{2}x)$. Then $\check{h}_k(x) = \frac{1}{2^k k!} h_k(x)$ and

$$U(\check{h}_k)(z) = \frac{c}{k!}\left(\frac{z}{\sqrt{2}}\right)^k .$$

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Applying the theorem, a change of variable, and conjugation gives the classical generating function:

$$e^{-z^2+2zx} = \sum_{k=0}^{\infty} H_k(x)\frac{z^k}{k!}.$$

The Laplace Transform: an Example

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- $M = \{z \in \mathbb{C} : RE(z) > 0\}$, $\mathcal{H}_\alpha(M)$ is the Hilbert space of holomorphic functions F on M such that

$$\|F\|^2 = \int_M |F(z)|^2 \, x^{\alpha-1} \, dxdy.$$

- $X = \mathbb{R}^+$,
- $R : \mathcal{H} \to L^2(\mathbb{R}^+, x^\alpha dx)$ given by $RF(x) = F(x)$.
- Polarization: $R^* = U\sqrt{RR^*}$ where $U = \mathcal{L} : L^2(\mathbb{R}^+, x^\alpha dx) \to \mathcal{H}_\alpha$ is given by

$$\mathcal{L}f(z) = \int_0^\infty e^{-zt} f(t) \, t^\alpha \, dt,$$

the *Laplace transform*.

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Cayley Transform: Let $\mathcal{D}$ be the unit disk. The Cayley Transform $c : \mathcal{D} \to M$ is defined by $c(z) = \frac{1+z}{1-z}$. It induces a unitary operator $C : \mathcal{H}_\alpha(M) \to \mathcal{H}_\alpha(D)$, defined by

$$CF(z) = (1-z)^{-(\alpha+1)} F\left(\frac{1+z}{1-z}\right),$$

where $\mathcal{H}_\alpha(D)$ is the space of holomorphic functions F on $\mathcal{D}$ such that

$$\|F\|^2 = \int_{\mathcal{D}} |F(z)|^2 \, (1-|z|^2)^{\alpha-2} dz.$$

Cayley Transform

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$$\|F\|^2 = \int_{\mathcal{D}} |F(z)|^2 \, (1-|z|^2)^{\alpha-2} dz.$$

Cayley-Laplace Transform The map

$$\Theta = C \circ \mathcal{L} : L^2(\mathbb{R}^+, x^\alpha dx) \to \mathcal{H}_\alpha(\mathcal{D})$$

defines a unitary operator given by

$$\Theta f(z) = (1-z)^{-(\alpha+1)} \int_{\mathbb{R}^+} e^{-\frac{1+w}{1-w}t} f(t)\, t^\alpha dt$$

Laguerre functions

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Laguerre functions: $\ell_n^\alpha(x) = e^{-x} L_n^\alpha(2x) \in L^2(R^+, x^\alpha dx)$ and forms an orthogonal system.

A calculation the gives that $\Theta(\check{\ell}_n^\alpha)(z) = z^n$.

Applying the theorem and conjugation gives:

$$(1-z)^{-(\alpha+1)} e^{-\frac{1+w}{1-w}t} = \sum_{n=0}^{\infty} \ell_n^\alpha(t) z^n.$$

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Applying the theorem and conjugation gives:

$$(1-z)^{-(\alpha+1)} e^{-\frac{1+w}{1-w}t} = \sum_{n=0}^{\infty} \ell_n^\alpha(t) z^n.$$

This is equivalent to

$$(1-z)^{-\alpha-1} e^{\frac{xz}{z-1}} = \sum_{k=0}^{\infty} L_n^\alpha(x) z^n \quad |z| < 1.$$

Highest Weight Representations
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Highest Weight Representations

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- Laguerre functions $\sim$ Highest Weight representation theory of $SL(2,\mathbb{R}) \backsimeq SU(1,1)$.
- Hermite functions $\sim$ Highest Weight representation theory of the metaplectic group.

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- G - Hermitian Symmetric Group
- K - maximal compact subgroup
- $\mathcal{D} = G/K$ - Complex manifold and G acts of $\mathcal{D}$ by biholomorphic diffeomorphisms.
- Let $\mathfrak{g}_\circ$ and $\mathfrak{k}_\circ$ be the Lie algebra of G and K, resp.
- Let $\mathfrak{g}$ and $\mathfrak{k}$ be the complexification of the $\mathfrak{g}$ and $\mathfrak{k}$.

- Our assumptions imply

$$\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_- .$$

- $\overline{\mathfrak{p}_\pm} = \mathfrak{p}_\mp$
- $\mathcal{D} \subset \mathfrak{p}_+$

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- Suppose π is an irreducible representation of G on $\mathbb{H}$.
  - π extends to $\mathfrak{g}_\circ$; the derived representation.
  - π extends to $\mathfrak{g}$ by complex linearity.
- Suppose there is a nonzero vector $v \in \mathbb{H}$ such that $\pi(x)v = 0$ for all $x \in \mathfrak{p}_+$. (Say v is annihilated by $\mathfrak{p}_+$.) Then π is called a **highest weight representation**.

Let V be the set of all vectors in $\mathbb{H}$ annihilated by $\mathfrak{p}_+$. Then K acts on V irreducibly by $\lambda(k) = \pi(k)|_V$.

The association

$$\pi \longrightarrow \lambda$$

is a correspondence:

irred h.w.reps of G $\leftrightarrow$ irred rep of K

irred unitary h.w.reps of G $\leftrightarrow$ $\Lambda \subset$ irred rep of K [EHW,J]

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For each $\lambda \in \Lambda$ we can associate a RKHS, $\mathcal{H}(\mathcal{D}, V_\lambda)$ on which G acts by a multiplier representation $T = T_\lambda$:

$$T(g)F(z) = J(g^{-1}, z)^{-1}F(g^{-1}z).$$

$(T, \mathcal{H}(\mathcal{D}, V_\lambda))$ is call the **Geometric Realization** of $T = T_\lambda$.

Intertwining Operators

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Now suppose $\pi = \pi_\lambda$ is any irreducible unitary highest weight representation on $\mathbb{H} = \mathbb{H}_\lambda$. Then there is a unitary operator

$$\Theta : \mathbb{H} \to \mathcal{H}(\mathcal{D}, V)$$

which intertwines the representations. The operator has the following characterization:

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Theorem [Fabec,Davidson 1995]: Let $z \in \mathfrak{p}_+$. Let $v \in V$ and formally define

$$q_z v = \sum_{n=0}^{\infty} \frac{\pi(\bar{z})^n}{n!} v.$$

Then $q_z : V \to \mathbb{H}$ converges in $\mathbb{H}$ if and only if $z \in \mathcal{D}$. Furthermore,

$$\Theta F(z) = q_z^* F.$$

In other words,

$$(\Theta F(z) \mid v) = (F \mid q_z v),$$

for all $z \in \mathcal{D}$, $v \in V$, and $F \in \mathbb{H}$.

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Generating Functions: revisited

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Theorem [Davidson 2004]: Suppose

$$\Theta : L^2(X, W, d\mu) \to \mathcal{H}(\mathcal{D}, V)$$

is a unitary operator between equivalent highest weight representations. Then the kernel $K = K^\Theta$ associated with Θ satisfies

$$K(z, \cdot)^* v = q_z v = q_z v = \sum_{n=0}^{\infty} \frac{\pi(\bar{z})^n}{n!} v.$$

Laguerre Functions

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The group $SL(2,\mathbb{R})$ acts on $L^2(\mathbf{R}^+, x^\alpha dx)$ by a unitary highest weight representation and is equivalent to the geometric realization $\mathcal{H}_\alpha(\mathcal{D})$ with intertwining operator $\Theta = C \circ \mathcal{L} : L^2(\mathbb{R}^+, x^\alpha dx) \to \mathcal{H}_\alpha(\mathcal{D})$ given by by

$$\Theta f(z) = (1-z)^{-(\alpha+1)} \int_{\mathbb{R}^+} e^{-\frac{1+z}{1-z}t} f(t)\, t^\alpha dt.$$

- $\mathfrak{p}_+ = \mathbb{C}E^+$ and for $f \in L^2(\mathbf{R}^+, x^\alpha dx)$ we have

$$\pi(\overline{E^+})f(t) = (-tD^2 + (2t-\alpha-1)D + (\alpha-1-t))f(t).$$

- $$\pi(\overline{E^+})\ell_n^\alpha(t) = (n+1)\ell_{n+1}^\alpha(t).$$

- Inductively,

$$\frac{\pi(\overline{E^+})^n}{n!}\ell_0^\alpha(t) = \ell_n^\alpha(t).$$

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- $$\frac{\pi(\overline{zE^+})^n}{n!}\ell_0^\alpha(t) = \ell_n^\alpha(t)\overline{z}^n.$$

- $$K(z,\cdot)^*\ell_0^\alpha = q_z\ell_0^\alpha = \sum_{n=0}^\infty \frac{\pi(\overline{zE^+})^n}{n!}\ell_0^\alpha(t) = \sum_{n=0}^\infty \ell_n^\alpha(t)\overline{z}^n.$$

- Complex conjugation connects this back to the classical formula:

$$(1-z)^{-(\alpha+1)}e^{-\frac{1+w}{1-w}t} = \sum_{n=0}^\infty \ell_n^\alpha(t)z^n.$$