Generalized Laguerre Functions and Differential Recursion Relations

Mark Davidson

Mathematics Department

Louisiana State Universtiy

collaborative work with

Gestur Olafsson

and

Genkai Zhang

The Classical Case

- **Rodrigues Formula**

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha})$$

Laguerre Polynomials

- **Rodrigues Formula**

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha})$$

- **Generating Function**

$$(1-w)^{-\alpha-1}\exp\left(\frac{xw}{w-1}\right) = \sum_{n=0}^{\infty} L_n^\alpha(x) w^n,$$

where $|w| < 1, \quad \alpha > -1$

Laguerre Polynomials

- **Rodrigues Formula**

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha})$$

- **Generating Function**

$$(1-w)^{-\alpha-1}\exp\left(\frac{xw}{w-1}\right) = \sum_{n=0}^{\infty} L_n^\alpha(x) w^n,$$

where $|w| < 1, \quad \alpha > -1$

- **Expansion Formula**

$$\begin{aligned} L_n^\alpha(x) &= \frac{1}{n!}\sum_{k=0}^{n}\frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)}\binom{n}{k}(-x)^k \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\,{}_1F_1(-n,\alpha+1;x) \end{aligned}$$

- $L_0^\alpha(x) = 1$
- $L_1^\alpha(x) = -x + \alpha + 1$
- $L_2^\alpha(x) = \frac{1}{2}(x^2 - 2(\alpha+2)x + (\alpha+1)(\alpha+2))$

The family of Laguerre polynomials is orthogonal as functions on $\mathbb{R}^+$ with respect to the inner product

$$(f|g) = \int_0^\infty f(x)\overline{g(x)}x^\alpha e^{-x}dx.$$

Differential Recursion Relations

The following are well known recursion relations.

- $(tD^2 + (\alpha - t + 1)D)L_n^\alpha(t) = -nL_n^\alpha(t)$, **(Laguerre's equation)**

Differential Recursion Relations

The following are well known recursion relations.

- $(tD^2 + (\alpha - t + 1)D)L_n^\alpha(t) = -nL_n^\alpha(t)$, **(Laguerre's equation)**
- $tDL_n^\alpha(t) = nL_n^\alpha(t) - (n+\alpha)L_{n-1}^\alpha(t)$,

Differential Recursion Relations

The following are well known recursion relations.

- $(tD^2 + (\alpha - t + 1)D)L_n^\alpha(t) = -nL_n^\alpha(t)$, **(Laguerre's equation)**
- $tDL_n^\alpha(t) = nL_n^\alpha(t) - (n+\alpha)L_{n-1}^\alpha(t)$,
- $tDL_n^\alpha(t) = (n+1)L_{n+1}^\alpha(t) - (n+\alpha+1-t)L_n^\alpha(t)$.

Differential Recursion Relations

The following are well known recursion relations.

- $(tD^2 + (\alpha - t + 1)D)L_n^\alpha(t) = -nL_n^\alpha(t)$, **(Laguerre's equation)**
- $tDL_n^\alpha(t) = nL_n^\alpha(t) - (n+\alpha)L_{n-1}^\alpha(t)$,
- $tDL_n^\alpha(t) = (n+1)L_{n+1}^\alpha(t) - (n+\alpha+1-t)L_n^\alpha(t)$.

- These kinds of equations are reminiscent of creation and annihilation operators that arise in physics and are codified in representation theory.

Differential Recursion Relations

The following are well known recursion relations.

- $(tD^2 + (\alpha - t + 1)D)L_n^\alpha(t) = -nL_n^\alpha(t)$, **(Laguerre's equation)**
- $tDL_n^\alpha(t) = nL_n^\alpha(t) - (n+\alpha)L_{n-1}^\alpha(t)$,
- $tDL_n^\alpha(t) = (n+1)L_{n+1}^\alpha(t) - (n+\alpha+1-t)L_n^\alpha(t)$.

- These kinds of equations are reminiscent of creation and annihilation operators that arise in physics and are codified in representation theory.

- In fact, such formulas are seen in a familiar family of representations of $SU(1,1)$ and $SL(2,\mathbf{R})$ called **highest weight representation.**

$$G = \mathrm{SL}(2,\mathbb{R}) \text{ and } G_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$$

$$T(\mathbb{R}^+) = \mathbb{R} + i\mathbb{R}^+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in T(\mathbf{R}^+)$. Let

$$g \cdot z = \frac{az+b}{cz+d}$$

This defines a transitive action of G on the upper half plane $T(\mathbb{R}^+)$. If K is the fixed point group for i:

$K = \{g \in G : g \cdot i = i\}$ then $K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$

and $G/K \simeq T(\mathbb{R}^+)$.

Highest Weight Representations of $SL(2, \mathbf{R})$

Let $\mathcal{H}_\alpha$ be the set of holomorphic functions of $T(\mathbb{R}^+)$ such that

$$(F \mid G) = \frac{2^\alpha}{2\pi\Gamma(\alpha)} \int_{\mathcal{H}} F(z)\overline{G(z)}\; y^{\alpha-1} dx dy < \infty.$$

This is a nonzero Hilbert space if $\alpha > 0$
For $F \in \mathcal{H}_\alpha$ we define

$$\pi_\alpha(g)F(z) = (a - bz)^{-\alpha-1} F(g^{-1} \cdot z)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$.
The formula π_α defines a unitary representation of G. It is a highest weight representation.

Some Properties of $\mathcal{H}_\alpha$

The space $\mathcal{H}_\alpha$ is a reproducing space: If

$$K(z,w) = \frac{\Gamma(\alpha+1)}{-i(z-\overline{w})^{\alpha+1}}$$

then the function

$$K_w(\cdot) = K(\cdot, w)$$

is in $\mathcal{H}_\alpha$ and

$$(F|K_w) = F(w),$$

for all $w \in T(\mathbb{R}^+)$ and $F \in \mathcal{H}_\alpha$.

The Lie algebra

$\mathfrak{sl}(2, \mathbb{C})$ = all 2×2 trace zero complex matrices.

1. $e^\circ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.
2. $e^+ = \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$
3. $e^- = \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$

Each of these are in $\mathfrak{sl}(2, \mathbb{C})$ and form a basis. Furthermore, we have

The Lie Algebra action

1. $\pi_\alpha(e^0) \cdot F(z) = i((\alpha+1)zF(z) + (1+z^2)F'(z))$.
2. $\pi_\alpha(e^+) \cdot F(z) = (\alpha+1)(\frac{z+i}{2})F(z) + \frac{(z+i)^2}{2}F'(z)$,
3. $\pi_\alpha(e^-) \cdot F(z) = (\alpha+1)(\frac{z-i}{2})F(z) + \frac{(z-i)^2}{2}F'(z)$,

The K-finite vectors

A K-finite vector is a vector $v \in \mathcal{H}_\alpha$ for which the linear span of all translates $\pi_\alpha(k)v$, $k \in K$, is finite dimensional.
Define

$$\gamma_{n,\alpha}(z) = c_{n,\alpha} \left(\frac{z-i}{z+i} \right)^n (z+i)^{-(\alpha+1)}$$

where $c_{n,\alpha} = i^{\alpha+1} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}$. Each of these functions are in $\mathcal{H}_\alpha(T(\mathbb{R}^+))$ and the collection forms an orthogonal basis of K-finite vectors.

(There is an equivalent realization of all this on the space of holomorphic functions on the unit disk, which is equivalent to the upper half plane by the Cayley transform. In this realization the K-finite vectors are of the form z^n, $n = 0, 1, \ldots$. The Cayley transform of these functions gives $\gamma_{n,\alpha}$.)

Moreover,

1. $\pi_\alpha(e^\circ) \cdot \gamma_{n,\alpha} = -(2n + \alpha + 1)\gamma_{n,\alpha}$.
2. $\pi_\alpha(e^+) \cdot \gamma_{n,\alpha} = -i(n + \alpha)\gamma_{n-1,\alpha}$,
3. $\pi_\alpha(e^-) \cdot \gamma_{n,\alpha} = -i(n + 1)\gamma_{n+1,\alpha}$,

The Restriction Principle

- For a function F defined on the upper half plane let

$$RF(t) = F(it),$$

where $t > 0$. The map R is known as the restriction map. Since the functions in $\mathcal{H}_\alpha$ are holomorphic if follows that R is injective. Let $k_a = K_{ia} \in \mathcal{H}_\alpha$.

The Restriction Principle

- For a function F defined on the upper half plane let

$$RF(t) = F(it),$$

where $t > 0$. The map R is known as the restriction map. Since the functions in $\mathcal{H}_\alpha$ are holomorphic if follows that R is injective. Let $k_a = K_{ia} \in \mathcal{H}_\alpha$.

- **Lemma**
(1) The linear span of $\{k_a : a > 0\}$ is dense in $\mathcal{H}_\alpha(T(\mathbb{R}^+))$.
(2) $Rk_a \in L^2(\mathbb{R}^+, d\mu_\alpha)$, where $d\mu_\alpha = t^\alpha \; dt$.
(3) The set $\{Rk_a : a > 0\}$ is dense in $L^2(\mathbb{R}^+, d\mu_\alpha)$.

The Restriction Principle

- For a function F defined on the upper half plane let

$$RF(t) = F(it),$$

where $t > 0$. The map R is known as the restriction map. Since the functions in $\mathcal{H}_\alpha$ are holomorphic if follows that R is injective. Let $k_a = K_{ia} \in \mathcal{H}_\alpha$.

- **Lemma**
(1) The linear span of $\{k_a : a > 0\}$ is dense in $\mathcal{H}_\alpha(T(\mathbb{R}^+))$.
(2) $Rk_a \in L^2(\mathbb{R}^+, d\mu_\alpha)$, where $d\mu_\alpha = t^\alpha \, dt$.
(3) The set $\{Rk_a : a > 0\}$ is dense in $L^2(\mathbb{R}^+, d\mu_\alpha)$.

- It follows that

$$R : \mathcal{H}_\alpha \to L^2(\mathbb{R}^+, d\mu_\alpha)$$

is densely defined and has dense range. It is easily seen to be closed. We can thus polarize RR^*:

Polarization of R

Let f be in the domain of R^*. Then

$$\begin{aligned}
RR^*f(y) &= R^*f(iy) \\
&= (R^*f \mid K(\cdot, iy))_{\mathcal{H}_\alpha} \\
&= (f \mid K(\cdot, iy))_{L^2} \\
&= \Gamma(\alpha+1)\int_0^\infty f(x)\frac{x^\alpha}{(x+y)^{\alpha+1}}\,dx \\
&= \int_0^\infty f(x)\mathcal{L}(t^\alpha e^{-ty})(x)\,x^\alpha dx \\
&= \int_0^\infty t^\alpha e^{-ty}\mathcal{L}(x^\alpha f(x))(t)\,dt \text{ (symmetry of } \mathcal{L}\text{)} \\
&= \mathcal{L}(t^\alpha \mathcal{L}(x^\alpha f(x)))(y)
\end{aligned}$$

The Laplace Transform

Define $Pf(y) = \mathcal{L}(x^\alpha f(x))(y)$. Then $P > 0$ and

$$P^2 = RR^*.$$

Therefore $P = \sqrt{RR^*}$. There is a unitary operator

$$U : L^2(\mathbb{R}^+, x^\alpha dx) \to \mathcal{H}_\alpha$$

so that $R^* = UP$: For $f \in L^2(\mathbb{R}^+, x^\alpha dx)$ and $z = iy$ we have

$$\begin{aligned} Uf(z) &= Uf(iy) = RUf(y) = Pf(y) \\ &= \int_0^\infty e^{-yt} f(t) t^\alpha \, dt \\ &= \int_0^\infty e^{izt} f(t) t^\alpha \, dt. \end{aligned}$$

Since Uf is holomorphic we obtain

Theorem The unitary map $U : L^2(\mathbb{R}^+, d\mu_\alpha) \to \mathcal{H}_\alpha(T(\mathbb{R}^+))$ is given by

$$Uf(z) = \int_0^\infty e^{izt} f(t)\, d\mu_\alpha(t).$$

Transferring the Representation π_α

The unitary operator U allows us to transfer the representation, π_α on $\mathcal{H}_\alpha$ to an equivalent representation, λ_α, on $L^2(\mathbb{R}^+, x^\alpha dx)$:

$$\pi_\alpha(g)Uf = U\lambda_\alpha(g)f.$$

Theorem Suppose $f \in L^2(\mathbb{R}^+, d\mu_\alpha)$ is twice differentiable. Then

1. $\lambda_\alpha(e^+)f(t) = \frac{-i}{2}(tD^2 + (2t + (\alpha+1))D + (t+\alpha+1))f(t)$
2. $\lambda_\alpha(e^-)f(t) = \frac{-i}{2}(tD^2 - (2t - (\alpha+1))D + (t-(\alpha+1)))f(t)$
3. $\lambda_\alpha(e^\circ)f(t) = (tD^2 + (\alpha+1)D - t)f(t)$

We define $\ell_n^\alpha(t) = \mathcal{L}^{-1}(\gamma_{n,\alpha})$

Representation implied Recursion Relations

Theorem With notation as above we have

$$\ell_n^\alpha = e^{-t} L_n^\alpha(2t).$$

Furthermore,

1. $\lambda_\alpha(e^\circ) \cdot \ell_n^\alpha(t) = -(2n+\alpha+1)\ell_n^\alpha(t)$.
2. $\lambda_\alpha(e^+) \cdot \ell_n^\alpha(t) = -i(n+\alpha)\ell_{n-1}^\alpha(t)$,
3. $\lambda_\alpha(e^-) \cdot \ell_n^\alpha(t) = -i(n+1)\ell_{n+1}^\alpha(t)$,

which we can write

1. $(tD^2 + (\alpha+1)D + (2n+\alpha+-t)\ell_n^\alpha = 0$,
2. $(tD^2 + (2t+(\alpha+1))D + (t+\alpha+1))\ell_n^\alpha = 2(n+\alpha)\ell_{n+1}^\alpha$,
3. $(tD^2 - (2t-(\alpha+1))D + (t-(\alpha+1)))\ell_n^\alpha = 2(n+1)\ell_{n-1}^\alpha$.

These formulas, in turn, imply the following recursion relations for the Laguerre polynomials L_n^α.

1. $(tD^2 + (\alpha - t + 1)D + n)L_n^\alpha(t) = 0$,
2. $tDL_n^\alpha(t) = nL_n^\alpha(t) - (n+\alpha)L_{n-1}^\alpha(t)$,
3. $tDL_n^\alpha(t) = (n+1)L_{n+1}^\alpha(t) - (n+\alpha+1-t)L_n^\alpha(t)$.

Summary

- Thus the representation theory of $Sl(2, \mathbb{R})$ encodes the classical differential recursion relations for the Laguerre polynomials.

Summary

- Thus the representation theory of $Sl(2,\mathbb{R})$ encodes the classical differential recursion relations for the Laguerre polynomials.
- The formula for the generating function falls right out of the representation theory here presented.

- Thus the representation theory of $Sl(2,\mathbb{R})$ encodes the classical differential recursion relations for the Laguerre polynomials.
- The formula for the generating function falls right out of the representation theory here presented.
- Further analysis (of a less representation nature) gives the recursion relations in the α parameter.

Faraut-Koranyi Generalized Laguerre Polynomials on Jordan Algebras

- GENERAL SETUP

Jordan Algebras

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \left\{x^2 : x \in J\right\}_\circ$: a symmetric cone.

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \{x^2 : x \in J\}_\circ$: a symmetric cone.
- Let H be connected component of the subgroup of $GL(J)$ that leaves Ω invariant.

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \{x^2 : x \in J\}_\circ$: a symmetric cone.
- Let H be connected component of the subgroup of $GL(J)$ that leaves Ω invariant.
- Let L be the fixed point subgroup of e.

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \left\{x^2 : x \in J\right\}_\circ$: a symmetric cone.
- Let H be connected component of the subgroup of $GL(J)$ that leaves Ω invariant.
- Let L be the fixed point subgroup of e.
- The H acts transitively on Ω and $\Omega = H/L$.

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \{x^2 : x \in J\}_\circ$: a symmetric cone.
- Let H be connected component of the subgroup of $GL(J)$ that leaves Ω invariant.
- Let L be the fixed point subgroup of e.
- The H acts transitively on Ω and $\Omega = H/L$.

Jordan Algebras

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \{x^2 : x \in J\}_\circ$: a symmetric cone.
- Let H be connected component of the subgroup of $GL(J)$ that leaves Ω invariant.
- Let L be the fixed point subgroup of e.
- The H acts transitively on Ω and $\Omega = H/L$.

- EXAMPLE

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \{x^2 : x \in J\}_\circ$: a symmetric cone.
- Let H be connected component of the subgroup of $GL(J)$ that leaves Ω invariant.
- Let L be the fixed point subgroup of e.
- The H acts transitively on Ω and $\Omega = H/L$.

- EXAMPLE
- $J = Herm(n)$ with product $A \circ B = \frac{1}{2}(AB + BA)$ and $e = I$

Jordan Algebras

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \left\{x^2 : x \in J\right\}_\circ$: a symmetric cone.
- Let H be connected component of the subgroup of $GL(J)$ that leaves Ω invariant.
- Let L be the fixed point subgroup of e.
- The H acts transitively on Ω and $\Omega = H/L$.

- EXAMPLE
- $J = Herm(n)$ with product $A \circ B = \frac{1}{2}(AB + BA)$ and $e = I$
- $\Omega = Herm^+(n)$

Jordan Algebras

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \{x^2 : x \in J\}_\circ$: a symmetric cone.
- Let H be connected component of the subgroup of $GL(J)$ that leaves Ω invariant.
- Let L be the fixed point subgroup of e.
- The H acts transitively on Ω and $\Omega = H/L$.

- EXAMPLE
- $J = Herm(n)$ with product $A \circ B = \frac{1}{2}(AB + BA)$ and $e = I$
- $\Omega = Herm^+(n)$
- $H = GL(n, \mathbf{C})$ acting on Ω by $g \cdot x = gxg^*$

Jordan Algebras

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \{x^2 : x \in J\}_\circ$: a symmetric cone.
- Let H be connected component of the subgroup of $GL(J)$ that leaves Ω invariant.
- Let L be the fixed point subgroup of e.
- The H acts transitively on Ω and $\Omega = H/L$.

- EXAMPLE
- $J = Herm(n)$ with product $A \circ B = \frac{1}{2}(AB + BA)$ and $e = I$
- $\Omega = Herm^+(n)$
- $H = GL(n, \mathbf{C})$ acting on Ω by $g \cdot x = gxg^*$
- $L = U(n)$

Jordan Algebras

- GENERAL SETUP
- Let J be a simple finite dimensional Euclidean Jordan Algebra with unit e.
- $\Omega = \{x^2 : x \in J\}_\circ$: a symmetric cone.
- Let H be connected component of the subgroup of $GL(J)$ that leaves Ω invariant.
- Let L be the fixed point subgroup of e.
- The H acts transitively on Ω and $\Omega = H/L$.

- EXAMPLE
- $J = Herm(n)$ with product $A \circ B = \frac{1}{2}(AB + BA)$ and $e = I$
- $\Omega = Herm^+(n)$
- $H = GL(n, \mathbf{C})$ acting on Ω by $g \cdot x = gxg^*$
- $L = U(n)$
- $Herm^+(n) \simeq GL(n, \mathbf{C})/U(n)$

Generalized Power Functions

The expansion formula for the Laguerre polynomials involve Gamma functions, a binomial coefficient, and powers of x. Each of these objects have analogues on Jordan algebras. Let J be a Euclidean Jordan algebra of dimension d. If J has rank r then there are r *principle minors*,

$$\Delta_1, \Delta_2, \ldots, \Delta_r.$$

Let $\mathbf{m} = (m_1, \ldots, m_n)$ be a multi-index of positive integer such that $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. Define

$$\Delta_{\mathbf{m}} = \Delta_1^{m_1 - m_2} \cdots \Delta_r^{m_r}.$$

Let

$$\psi_{\mathbf{m}}(x) = \int_L \Delta_{\mathbf{m}}(lx)\, dl.$$

$\psi_{\mathbf{m}}$ is a nonzero L-invariant polynomials on J of degree $|m| = m_1 + m_2 + \cdots + m_r$ and are referred to as *generalized power functions*.

The Gamma Function

The function Δ_r is the determinant function on J and usually denoted by Δ. Furthermore, if $d = \dim(J)$ then $\Delta^{\frac{-d}{r}} dx$ is the H-invariant measure of Ω.

The classical Gamma function is given by

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \, \frac{1}{t} dt.$$

The Gamma Function

The function Δ_r is the determinant function on J and usually denoted by Δ. Furthermore, if $d = \dim(J)$ then $\Delta^{\frac{-d}{r}} dx$ is the H-invariant measure of Ω.

The classical Gamma function is given by

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \, \frac{1}{t} dt.$$

For the cone Ω we have

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr} t} \Delta_{\mathbf{s}}(t) \, \Delta(t)^{-\frac{d}{r}} dt,$$

where tr is the trace operator on J and $\mathbf{s}$ is a multi-index.

The Gamma Function

The function Δ_r is the determinant function on J and usually denoted by Δ. Furthermore, if $d = \dim(J)$ then $\Delta^{\frac{-d}{r}} dx$ is the H-invariant measure of Ω.

The classical Gamma function is given by

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \, \frac{1}{t} dt.$$

For the cone Ω we have

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr} t} \Delta_{\mathbf{s}}(t) \, \Delta(t)^{-\frac{d}{r}} dt,$$

where tr is the trace operator on J and $\mathbf{s}$ is a multi-index.

The function Δ_r is the determinant function on J and usually denoted by Δ. Furthermore, if $d = \dim(J)$ then $\Delta^{\frac{-d}{r}}dx$ is the H-invariant measure of Ω.

The classical Gamma function is given by

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \, \frac{1}{t} dt.$$

For the cone Ω we have

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr} t} \Delta_{\mathbf{s}}(t) \, \Delta(t)^{-\frac{d}{r}} dt,$$

where tr is the trace operator on J and $\mathbf{s}$ is a multi-index.

The Gamma Function

The function Δ_r is the determinant function on J and usually denoted by Δ. Furthermore, if $d = \dim(J)$ then $\Delta^{\frac{-d}{r}} dx$ is the H-invariant measure of Ω.

The classical Gamma function is given by

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \, \frac{1}{t} dt.$$

For the cone Ω we have

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr} t} \Delta_{\mathbf{s}}(t) \, \Delta(t)^{-\frac{d}{r}} dt,$$

where tr is the trace operator on J and $\mathbf{s}$ is a multi-index.

Generalized Binomial Coefficient

The usual binomial coefficient can be defined by the rule

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Since $(1+x)^n$ is a polynomial of degree n it is a linear combination of $\{1, x, \ldots, x^n\}$. The coefficient of x^k thus uniquely define the binomial coefficients.

The L-invariant function $\psi_{\mathbf{m}}$ is a polynomial of degree $|\mathbf{m}|$ and the collection $\{\psi_{\mathbf{m}} : |\mathbf{m}| \leq \alpha\}$ spans the set of all L-invariant polynomials of degree $\leq \alpha$. The L-invariant polynomial $\psi_{\mathbf{m}}(e+x)$ has degree $|\mathbf{m}|$ and is thus a linear combination of terms of the form $\psi_{\mathbf{n}}$, where $|\mathbf{n}| \leq |\mathbf{m}|$. The *generalize binomial coefficients* , are thus defined such that

$$\psi_{\mathbf{m}}(e+x) = \sum_{|\mathbf{n}| \leq |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \psi_{\mathbf{n}}(x).$$

Generalized Laguerre Polynomials

Recall the classical Laguerre polynomial:

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \binom{n}{k} (-x)^k$$

Generalized Laguerre Polynomials

Recall the classical Laguerre polynomial:

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \binom{n}{k} (-x)^k$$

Faraut and Koranyi define the generalized Laguerre polynomial by the formula:

$$L_{\mathbf{m}}^\nu(x) = \sum_{|\mathbf{n}| \leq |\mathbf{m}|} \frac{\Gamma_\Omega(\nu+\mathbf{m})}{\Gamma_\Omega(\nu+\mathbf{n})} \binom{\mathbf{m}}{\mathbf{n}} \psi_{\mathbf{n}}(-x).$$

Generalized Laguerre Polynomials

Recall the classical Laguerre polynomial:

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \binom{n}{k} (-x)^k$$

Faraut and Koranyi define the generalized Laguerre polynomial by the formula:

$$L_{\mathbf{m}}^\nu(x) = \sum_{|\mathbf{n}| \le |\mathbf{m}|} \frac{\Gamma_\Omega(\nu+\mathbf{m})}{\Gamma_\Omega(\nu+\mathbf{n})} \binom{\mathbf{m}}{\mathbf{n}} \psi_{\mathbf{n}}(-x).$$

Generalized Laguerre Polynomials

Recall the classical Laguerre polynomial:

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \binom{n}{k} (-x)^k$$

Faraut and Koranyi define the generalized Laguerre polynomial by the formula:

$$L_{\mathbf{m}}^\nu(x) = \sum_{|\mathbf{n}|\le|\mathbf{m}|} \frac{\Gamma_\Omega(\nu+\mathbf{m})}{\Gamma_\Omega(\nu+\mathbf{n})} \binom{\mathbf{m}}{\mathbf{n}} \psi_{\mathbf{n}}(-x).$$

Generalized Laguerre Polynomials

Recall the classical Laguerre polynomial:

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \binom{n}{k} (-x)^k$$

Faraut and Koranyi define the generalized Laguerre polynomial by the formula:

$$L_{\mathbf{m}}^\nu(x) = \sum_{|\mathbf{n}|\le|\mathbf{m}|} \frac{\Gamma_\Omega(\nu+\mathbf{m})}{\Gamma_\Omega(\nu+\mathbf{n})} \binom{\mathbf{m}}{\mathbf{n}} \psi_{\mathbf{n}}(-x).$$

Orthogonality

- Let $L^2(\Omega, d\mu_\nu)$ be the space of square integrable functions on Ω with respect to the measure $d\mu_\nu = \Delta^{\nu - \frac{d}{r}}(x)dx$.

Orthogonality

- Let $L^2(\Omega, d\mu_\nu)$ be the space of square integrable functions on Ω with respect to the measure $d\mu_\nu = \Delta^{\nu-\frac{d}{r}}(x)dx$.
- Let $\ell^\nu_{\mathbf{m}}(x) = e^{-\operatorname{tr}(x)} L^\nu_{\mathbf{m}}(2x)$. These are the **generalized Laguerre Functions.**

- Let $L^2(\Omega, d\mu_\nu)$ be the space of square integrable functions on Ω with respect to the measure $d\mu_\nu = \Delta^{\nu - \frac{d}{r}}(x)dx$.
- Let $\ell^\nu_{\mathbf{m}}(x) = e^{-\operatorname{tr}(x)} L^\nu_{\mathbf{m}}(2x)$. These are the **generalized Laguerre Functions.**
- **THEOREM** The set

$$\{\ell^\nu_{\mathbf{m}}(x) : \mathbf{m} \geq 0\}$$

is an orthogonal basis of

$$L^2(\Omega, d\mu_\nu)^L.$$

Tube-type Domains, Hermitian Symmetric Groups, and Highest Weight Representations

Tube-type domains and Hermitian groups

Let $T(\Omega) = i\Omega + J \subset J_{\mathbf{C}}$. Let $Aut(T(\Omega))$ be the group of biholomorphic automorphisms of $T(\Omega)$ and $G = Aut(T(\Omega))_{\circ}$. If K is the fixed point group for the point $ie \in T(\Omega)$ then K is a maximal compact subgroup of G and

$$T(\Omega) = G/K.$$

The groups H and L that are associated with Ω are subgroups of G.

The groups G that can arise have been classified. Some of the groups that arise in this way are:

- $Sl(2, \mathbf{R})$
- $SU(n, n)$
- $Sp(n, \mathbf{R})$
- $SO^*(4m)$

Highest Weight Representations

In a manner analogous to the usual upper half plane we may define a Hilbert space $\mathcal{H}_\nu$ of holomorphic functions on $T(\Omega)$:

$$\mathcal{H}_\nu = \left\{ F : T(\Omega) \to \mathbb{C} : d_\nu \int_{T(\Omega)} |F(z)|^2 \, \Delta(z)^{\nu - \frac{2d}{r}} dz < \infty \right\}.$$

$\mathcal{H}_\nu$ is nonzero if and only if $\nu > 1 + a(r-1)$, where a is a constant that depends on the Jordan algebra J.

There is a unitary highest weight representation, π_ν, of G on $\mathcal{H}_\nu$ given by

$$\pi(g)F(z) = J(g^{-1}, z)^{\frac{\nu r}{2d}} F(g^{-1}z),$$

where $J(g, z)$ is the complex Jacobian of the action $g \cdot z$. This representation is a highest weight representation.

Some K-finite vectors

- The generalized power functions, $\psi_{\mathbf{m}}$, extend to $J_{\mathbf{C}}$ and their Cayley transform:

$$q_{\mathbf{m},\nu}(z) = \Delta(z+e)^{-\nu}\psi_{\mathbf{m}}\left(\frac{z-e}{z+e}\right),$$

are in $\mathcal{H}_\nu^L$. These functions play a role analogous to $\gamma_{n,\alpha}(x)$ for $SL(2,\mathbf{R})$.

Some K-finite vectors

- The generalized power functions, $\psi_{\mathbf{m}}$, extend to $J_{\mathbf{C}}$ and their Cayley transform:

$$q_{\mathbf{m},\nu}(z) = \Delta(z+e)^{-\nu}\psi_{\mathbf{m}}\left(\frac{z-e}{z+e}\right),$$

  are in $\mathcal{H}^L_\nu$. These functions play a role analogous to $\gamma_{n,\alpha}(x)$ for $SL(2,\mathbf{R})$.

- **THEOREM** The set

$$\{q_{\mathbf{m},\nu} : \mathbf{m} \geq 0\}$$

  is an orthogonal basis of $\mathcal{H}^L_\nu$.

The Restriction Principle

For F a holomorphic function on $T(\Omega)$ we define the restriction map

$$RF(x) = F(ix).$$

Then R is a densely defined, closed, and has dense image. Polarization of R^* gives

THEOREM The map,

$$\mathcal{L}_\nu(f)(z) = \int_\Omega e^{-(iz,x)} f(x)\, d\mu_\nu,$$

defines a unitary isomorphism of $L^2(\Omega, d\mu_\nu)$ onto $\mathcal{H}_\nu$.

Let λ_ν be the representation of G equivalent to π_ν via $\mathcal{L}_\nu$.

Technical difficulties

The representation theoretic interpretation of the differential recursion relations in the $SL(2, \mathbf{R})$ case relied on having explicit formulas for the action of $\pi_\nu(x)$ and how they act on $\gamma_{n,\alpha}$. Recall

1. $\pi_\alpha(e^0) \cdot F(z) = i((\alpha+1)zF(z) + (1+z^2)F'(z))$.
2. $\pi_\alpha(e^+) \cdot F(z) = (\alpha+1)(\frac{z+i}{2})F(z) + \frac{(z+i)^2}{2}F'(z)$,
3. $\pi_\alpha(e^-) \cdot F(z) = (\alpha+1)(\frac{z-i}{2})F(z) + \frac{(z-i)^2}{2}F'(z)$,

and

1. $\pi_\alpha(e^\circ) \cdot \gamma_{n,\alpha} = -(2n+\alpha+1)\gamma_{n,\alpha}$.
2. $\pi_\alpha(e^+) \cdot \gamma_{n,\alpha} = -i(n+\alpha)\gamma_{n-1,\alpha}$,
3. $\pi_\alpha(e^-) \cdot \gamma_{n,\alpha} = -i(n+1)\gamma_{n+1,\alpha}$.

The subalgebra $\mathfrak{g}_{\mathbf{C}}^{L}$

In general we can compute formulas for the operators $\pi_\nu(x)$, $x \in \mathfrak{g}_{\mathbf{C}}$, but we do not have explicit formulas for their action on $q_{\mathbf{m},\nu}$. Part of the problem arises from the fact that $\pi_\nu(x)$ does not leave $\mathcal{H}_\nu^L$ invariant for all $x \in \mathfrak{g}_{\mathbf{C}}$.

However, the subalgebra

$$\mathfrak{g}_{\mathbf{C}}^{L} = \{x \in \mathfrak{g}_{\mathbf{C}} : Ad(l)x = x, \text{ for all } l \in L\}$$

does leave $\mathcal{H}_\nu^L$ invariant and is, furthermore, a three dimensional subalgebra isomorphic to $SL(2, \mathbb{C})$.

Since $L \subset K$ the center of K, $\mathfrak{k}$, is a subset of $\mathfrak{g}_{\mathbf{C}}^{L}$. As G is a Hermitian group the center of $\mathfrak{k}$ is spanned by a single vector, X°. The operator $\operatorname{ad} X^{\circ}$ on $\mathfrak{g}_{\mathbf{C}}$ has only the eigenvalues 0, 1, -1.

- The 0-eigenspace is $\mathfrak{k}_{\mathbf{C}}$
- The $+1$-eigenspace is denoted $\mathfrak{p}^{+}$
- The -1 eigenspace is denoted $\mathfrak{p}^{-}$.

The intersection of $\mathfrak{g}_{\mathbf{C}}^{L}$ and $\mathfrak{h}$, the Lie algebra of H, is one dimensional and spanned by a single vector Z°. It turns out that $Z^{\circ} = X^{+} + X^{-}$, where $X^{+} \in \mathfrak{p}^{+}$ and $X^{-} \in \mathfrak{p}^{-}$.

LEMMA The Lie algebra $\mathfrak{g}_{\mathbf{C}}^{L}$ is spanned by X°, X^{+} and X^{-} and $\mathfrak{g}_{\mathbf{C}}^{L} \cap \mathfrak{g}$ is spanned by iX°, $Z^{\circ} = X^{+} + X^{-}$, and $i(X^{+} - X^{-})$.

THEOREM We have

$$\pi_\nu(X^\circ)q_{\mathbf{m},\nu} = (r\nu + |\mathbf{m}|)q_{\mathbf{m},\nu}$$

and

$$\begin{aligned}\pi_\nu(Z^\circ)q_{\mathbf{m},\nu} &= \sum_{j=1}^{r}\binom{\mathbf{m}}{\mathbf{m}-\mathbf{e_j}} q_{\mathbf{m}-\mathbf{e_j},\nu} \\ &- \sum_{j=1}^{r}(\nu + m_j - \frac{a}{2}(j-1))c_{\mathbf{m}}(j)q_{\mathbf{m}+\mathbf{e_j},\nu}.\end{aligned}$$

Observe that the action of Z° and $q_{\mathbf{m},\nu}$ involves both a shift upward and a shift downward in the multi-indices. But it is known that this is precisely the role of $\mathfrak{p}^-$ and $\mathfrak{p}^+$; their actions are the so-called raising and lowering operators (or creation and annihilation operators).

Since $Z^\circ = X^+ + X^-$ we have the following corollary.

COROLLARY

$$\pi_\nu(X^+)q_{\mathbf{m},\nu} = \sum_{j=1}^{r} \binom{\mathbf{m}}{\mathbf{m}-\mathbf{e_j}} q_{\mathbf{m}-\mathbf{e_j},\nu}$$

and

$$\pi_\nu(X^-)q_{\mathbf{m},\nu} = -\sum_{j=1}^{r}(\nu + m_j - \frac{a}{2}(j-1))c_{\mathbf{m}}(j)q_{\mathbf{m}+\mathbf{e_j},\nu}.$$

Transferring the action to $L^2(\Omega, d\mu_\nu)$

THEOREM

$$\mathcal{L}_\nu(\ell^\nu_{\mathbf{m}}) = \Gamma_\Omega(\nu) q_{\mathbf{m},\nu}.$$

THEOREM

$$\lambda_\nu(Z^\circ) f(x) = (\nu r + E) f(x),$$

where E is the Euler operator:
$Ef(x) = \frac{d}{dt} f(tx)|_{t=1} = \frac{d}{dt} f(\exp(tZ^\circ))|_{t=1}.$

THEOREM

- $\lambda_\nu(X^\circ)\ell^\nu_{\mathbf{m}} = (r\nu + 2|\mathbf{m}|)\ell^\nu_{\mathbf{m}}$
- $\lambda_\nu(X^+)\ell^\nu_{\mathbf{m}} = \sum_{j=1}^r \begin{pmatrix} \mathbf{m} \\ \mathbf{m} - e_j \end{pmatrix} (m_j - 1 + v - \frac{a}{2}(j-1))\ell^\nu_{\mathbf{m}-e_j}$
- $\lambda_\nu(X^-)\ell^\nu_{\mathbf{m}} = \sum_{j=1}^r c_{\mathbf{m}}(j)\ell^\nu_{\mathbf{m}+e_j}$

Differential Recursion Relations for $SU(n,n)$

For $SU(n,n)$ we have determine explicitly the formulas for the algebraic action. When applied to X°, X^+ and X^- and to the Laguerre functions we obtain the following differential recursion relations.

- $\mathrm{tr}(s\nabla\nabla + \nu\nabla - s)\ell^\nu_{\mathbf{m}} = -(r\nu + 2|\mathbf{m}|)\ell^\nu_{\mathbf{m}}$.
- $\frac{1}{2}\mathrm{tr}(s\nabla\nabla + (\nu I + 2s)\nabla + (\nu I + s))\ell^\nu_{\mathbf{m}}(s) =$
  $-\sum_{j=1}^r \begin{pmatrix} \mathbf{m} \\ \mathbf{m} - \gamma_{\mathbf{j}} \end{pmatrix} (m_j - 1 + \nu - (j-1))\ell^\nu_{\mathbf{m}-\gamma_{\mathbf{j}}}$
- $\frac{1}{2}\mathrm{tr}(s\nabla\nabla + (\nu I - 2s)\nabla + (s - \nu I))\ell^\nu_{\mathbf{m}} = -\sum_{j=1}^r c_{\mathbf{m}}(j)\ell^\nu_{\mathbf{m}+\gamma_{\mathbf{j}}}$.

Notice the similarity to the classical case:

- $(tD^2 + (\alpha+1)D - t)\ell^\alpha_n = -(2n + \alpha + 1)\ell^\alpha_n$,
- $(tD^2 + (2t + (\alpha+1))D + (t + \alpha + 1))\ell^\alpha_n = -2(n+\alpha)\ell^\alpha_{n-1}$,
- $(tD^2 - (2t - (\alpha+1))D + (t - (\alpha+1)))\ell^\alpha_n = -2(n+1)\ell^\alpha_{n+1}$.

Conclusions

Results in Other Settings

- With the unit disk playing the role of the upper half plane and the interval $(0,1)$ playing the role of the cone $\mathbb{R}^+$ we get a similar theory involving the Meixner-Pollacyk polynomials.

Results in Other Settings

- With the unit disk playing the role of the upper half plane and the interval $(0, 1)$ playing the role of the cone $\mathbb{R}^+$ we get a similar theory involving the Meixner-Pollacyk polynomials.
- Since G/K has a realization as a bounded symmetric domain the result extend.

Results in Other Settings

- With the unit disk playing the role of the upper half plane and the interval $(0,1)$ playing the role of the cone $\mathbb{R}^+$ we get a similar theory involving the Meixner-Pollacyk polynomials.
- Since G/K has a realization as a bounded symmetric domain the result extend.
- The Spherical-Fourier transform transfers these results to a space on Weyl-group invariant functions on an r dimensional space $\mathfrak{a}_{\mathbb{C}}^*$. The recursion relations take the form of difference equations.

Directions

- Related analysis suggests that such generalizations should extend to generalizations of other special functions like the Hermite and Legendre polynomials.

Directions

- Related analysis suggests that such generalizations should extend to generalizations of other special functions like the Hermite and Legendre polynomials.
- We have only dealt with the so-called scalar highest weight representations. Is there an analogue with extensions to vector valued Laguerre functions?

Directions

- Related analysis suggests that such generalizations should extend to generalizations of other special functions like the Hermite and Legendre polynomials.
- We have only dealt with the so-called scalar highest weight representations. Is there an analogue with extensions to vector valued Laguerre functions?
- The Laguerre functions defined by Faraut and Koranyi are L-invariant. How does the theory change when one considers functions that transform according to a character of L?