Baire Category Theorem

Recall that a set \( A \in \mathbb{R} \) is called \( G_\delta \) if it can be written as a countable intersection of open sets; i.e.
\[
A = \cap O_n,
\]
where each \( O_n \) is open. A set \( B \) is called a \( F_\sigma \) if it can be written as a countable union of closed sets; i.e.
\[
B = \cup F_n,
\]
where each \( F_n \) is closed.

What sets are \( G_\delta \) and what sets are \( F_\sigma \)? This is not an easy question to answer in general.

**Problem 1.** Show that the rationals \( \mathbb{Q} \) is a \( F_\sigma \) and the irrationals is a \( G_\delta \).

Can the rationals also be expressed as a \( G_\delta \)? Questions such as these dance around an idea about the size of sets. This set of notes discusses this issue in a more general context and culminates with the Baire Category Theorem that has far reaching implications in analysis.

Most of these notes consist of a sequence of easy problems that you should work out. I have tried to keep them simple and concise.

Let’s begin with the problem I asked about: The rationals \( \mathbb{Q} \), is it a \( G_\delta \)? The answer is no! Consider the following problems that lead to a proof.

**Problem 2.** Suppose \( \{O_n\} \) is a countable collection of open sets and \( E = \cap O_n \). Show that we can write \( E = \cap G_n \) where \( G_n \) is open and \( G_{n+1} \subset G_n \). (Hint: Let \( G_1 = O_1, G_2 = O_1 \cap O_2, \) etc.)

**Problem 3.** If \( O \) is a non empty open set there is a closed interval \( J = [a, b], a < b \) with \( J \subset O \).

**Definition 4.** A set \( E \) is **dense** in \( \mathbb{R} \) if \( \hat{E} = \mathbb{R} \).

**Problem 5.** A set \( E \) is dense in \( \mathbb{R} \) if and only if \( E \cap I \neq \emptyset \) for all non empty open intervals \( I \).

**Problem 6.** If \( A \subset B \) and \( A \) is dense so it \( B \).

**Problem 7.** Suppose \( O_1 \) and \( O_2 \) are dense and open. Then \( O_1 \cap O_2 \) is dense and open.

**Problem 8.** Suppose \( \{O_n\} \) is a countable collection of open dense sets. Then \( \cap O_n \) is not empty.
Proof. First, note that density is essential here. For example, \( O_n = (0, \frac{1}{n}) \) has empty intersection.

By Problems 2 and 7 (and the hint for problem 1) we can assume that \( O_{n+1} \subset O_n \) for all \( n \). Let \( J_1 = [a_1, b_1], a_1 < b_1 \), be a closed set in \( O_1 \). Since \( O_2 \) is open and dense, \( O_2 \cap (a_1, b_1) \) is open and non empty. Let \( J_2 = [a_2, b_2], a_2 < b_2 \), be a closed set in \( O_2 \cap (a_1, b_1) \). Then \( J_2 \subset J_1 \). Since \( O_3 \) is open and dense, \( O_3 \cap (a_2, b_2) \) is open and non empty. Let \( J_3 = [a_3, b_3], a_3 < b_3 \), be a closed set in \( O_3 \cap (a_2, b_2) \). Then \( J_3 \subset J_2 \subset J_1 \). We continue in this way to create a decreasing sequence of closed sets \( \{J_n\} \). By Exercise 3.1 p33 of Richardson \( \cap J_n \neq \emptyset \). (This proposition is grounded in the completeness of the real line.) But since \( J_n \subset O_n \) it follows that \( \cap O_n \neq \emptyset \).

Remark 9. The proof of the Baire Category theorem that we prove below contains ideas that are similar to the proof given above. Moreover, the Baire Category theorem implies that the intersection of dense open sets is not only not empty but dense, and in fact ‘very’ dense.

Problem 10. The rationals \( \mathbb{Q} \) is not a \( G_\delta \).

Proof. Here is an outline. Fill in the gaps. We proceed by assuming \( \mathbb{Q} \) is a \( G_\delta \). Suppose \( \mathbb{Q} = \cap O_n \). Show the following:

1. We may assume \( O_{n+1} \subset O_n \).
2. Each \( O_n \) is dense.
3. Let \( q_1, q_2, \ldots \) be an enumeration of the rationals. Define \( G_n = O_n \setminus \{q_n\} \). Then \( G_n \) is open and dense.
4. \( \cap G_n = \emptyset \), a contradiction.

Nowhere dense sets

We now turn to a slightly more general setting that the one above.

Definition 11. A set \( E \) is said to be nowhere dense in \( \mathbb{R} \) if for every interval \( I \) there is a subinterval \( J \subset I \) such that

\[ J \cap E = \emptyset. \]
In a certain sense a nowhere dense set is ‘small’ in that its complement has lots of open intervals in it. But beware, a nowhere dense set can have uncountably many elements in it. We’ll get to that later. For now consider the following examples.

**Problem 12.** Show the following sets are nowhere dense.

1. A finite set
2. The set \( \{ x_n \} \) if \( \lim x_n = x \in \mathbb{R} \).
3. The set of natural numbers \( 1, 2, 3, \ldots \).
4. The complement of an open dense set.

**Problem 13.** Show that the rationals, \( \mathbb{Q} \), is not nowhere dense.

**Problem 14.** Show that any subset of a nowhere dense set is nowhere dense.

**Problem 15.** If \( A \) and \( B \) are nowhere dense show that \( A \cup B \) is nowhere dense.

**Problem 16.** A finite union of nowhere dense sets is nowhere dense.

**Problem 17.** Give an example of a set which is a countable union of nowhere dense sets that is not nowhere dense.

**Problem 18.** If \( A \) is nowhere dense then \( \bar{A} \), the complement of \( A \), is dense.

**Problem 19.** Give an example of a dense set whose complement is not nowhere dense.

**Problem 20.** A set \( A \) is nowhere dense if and only if \( \bar{A} \) contains no open intervals.

**Some Definitions**

**Definition 21.**

1. We say that a set is of **first category** if it is the countable union of nowhere dense sets.
2. We say a set is of **second category** if it is not first category.
3. We say a set is **residual** if it is the complement of a set of first category.

Agreed, these terms are not very inspiring. They are historical and here to stay. As we will see first category sets are ‘small’ and second category sets are ‘big’. For this reason, some have said that a first category set should be called **meager**. I will not use this term. The definition of residual does not, a priori, exclude the possibility
that it be of first category. However, the Baire category theorem that follows will imply that this is not the case: residual sets are second category.

Keep in mind the following basic example. The set of rationals, $\mathbb{Q}$, is of first category because if $q_1, q_2, \ldots$ is an enumeration of $\mathbb{Q}$ then $\mathbb{Q} = \bigcup \{q_k\}$ and each $\{q_k\}$ is nowhere dense. The set of irrationals is residual. Both sets are dense in $\mathbb{R}$ but the irrationals is much larger as we will see below.

**Problem 22.** Any countable set is of first category.

**Problem 23.** A countable union of first category sets is first category.

**Problem 24.** A countable intersection of residual sets is residual. (Hint: this follows from the previous problem.)

**Theorem 25** (The Baire Category Theorem). A residual set is dense.

**Proof.** The basic idea of the proof is very similar to that of Problem 8. Let $R$ be a residual set. Then $\tilde{R}$ is a countable union of nowhere dense sets:

$$\tilde{R} = \bigcup E_n,$$

where $E_n$ is nowhere dense. We thus have that

$$R = \cap \tilde{E}_n.$$

To show $R$ is dense we must show that the intersection of $R$ with any open interval is nonempty. Let $I$ be an open interval. Since $E_1$ is nowhere dense there is an open nonempty interval $I_1 \subset I$ such that

$$I_1 \cap E_1 = \emptyset.$$

Let $J_1 = [a_1, b_1]$ be a closed subset of $I_1$, with $a_1 < b_1$. Then $J_1 \cap E_1 = \emptyset$ and therefore $J_1 = J_1 \cap \tilde{E}_1 \subset \tilde{E}_1$ Since $E_2$ is nowhere dense there is a subinterval $I_2$ of $(a_1, b_1)$ such that

$$I_2 \cap E_2 = \emptyset.$$

Let $J_2 = [a_2, b_2]$ be a closed subset of $I_2$ with $a_2 < b_2$. Then $J_2 \cap E_2 = \emptyset$ and therefore $J_2 = J_2 \cap \tilde{E}_2 \subset \tilde{E}_2$. Furthermore, $J_2 \subset J_1$. We continue inductively to create a sequence of closed sets $\{J_n\}$ such that $J_{n+1} \subset J_n$, $J_n \subset I$, and $J_n \subset \tilde{E}_n$. By construction we have $\cap J_n \subset I \cap R$. By Proposition 16 in Chapter 2 of Royden $\cap J_n \neq \emptyset$ and this implies $I \cap R \neq \emptyset$. Therefore $\tilde{R}$ is dense. 

\[\square\]
Remark 26. This Theorem says that residual sets are ‘large’ dense sets. Remember from Problem 24 the countable intersection of residual sets is residual and hence dense by the Baire Category Theorem. This is not a property of all dense sets, for example:

$$\mathbb{Q} \cap (\mathbb{Q} + \sqrt{2}) = \emptyset.$$ 

Neither set is residual although both are dense.

Problem 27. Show that $\mathbb{R}$ is not the countable union of nowhere dense sets.

Let’s tidy up one loose end.

**Proposition 28.** A residual set is of second category.

*Proof.* Suppose $A$ is a residual set and assume it is first category. We will arrive at a contradiction. Then $A$ and $\mathring{A}$ are first category, i.e. they each are the countable union of nowhere dense sets. But $A \cup \mathring{A} = \mathbb{R}$ so $\mathbb{R}$ is the union of nowhere dense sets. A contradiction by Problem 27. \qed

Remark 29. Returning to our original problem: Can the set of rationals be a $G_\delta$? We can use the Baire Category Theorem to establish this easily: Suppose $\mathbb{Q} = \cap O_n$ with each $O_n$ open. Then $\mathbb{Q} \subset O_n$ and therefore $O_n$ is dense. This implies that $\mathring{O}_n$ is nowhere dense, which implies $O_n$ is residual, which implies $\mathbb{Q}$ is residual, a contradiction.