Graphs and Combinatorics (2018) 34:289–312
https://doi.org/10.1007/s00373-018-1874-z

Minimal k-Connected Non-Hamiltonian Graphs

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Received: 24 April 2017 / Revised: 31 January 2018 / Published online: 14 February 2018

Abstract In this paper, we explore minimal k-connected non-Hamiltonian graphs. Graphs are said to be minimal in the context of some containment relation; we focus on subgraphs, induced subgraphs, minors, and induced minors. When $k = 2$, we discuss all minimal 2-connected non-Hamiltonian graphs for each of these four relations. When $k = 3$, we conjecture a set of minimal non-Hamiltonian graphs for the minor relation and we prove one case of this conjecture. In particular, we prove all 3-connected planar triangulations which do not contain the Herschel graph as a minor are Hamiltonian.

Keywords Hamilton cycles · Graph minors

1 Introduction

Hamilton cycles in graphs are cycles which visit every vertex of the graph. Determining their existence in a graph is an NP-complete problem and as such, there is a large body of research proving necessary and sufficient conditions. In this paper, we analyze non-Hamiltonian graphs. In particular, we consider the following general question:

The first author was supported in part by NSF Grant DMS-1500699.
The work for this paper was largely done while the second author was at Louisiana State University.

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what are the minimal k-connected non-Hamiltonian graphs? A graph is minimal if it does not contain a smaller graph with the same properties. There are many different ways one graph can be contained in another; some examples we will consider are subgraphs, induced subgraphs, minors, and induced minors. We will look for minimal non-Hamilton graphs for these different containment relations and different values of k.

Let $\{G_1, G_2, \dots\}$ be the set of k-connected non-Hamiltonian minimal graphs for some containment relation. Then note that every k-connected graph which does not contain any of $G_1, G_2, \dots$ is Hamiltonian. Another way to state results which answer the question posed in the previous paragraph is as follows. Let G be k-connected. Then all k-connected graphs contained in G, including G itself, are Hamiltonian if and only if G does not contain any of the graphs $G_1, G_2, \dots$.

A graph H is a *minor* of a graph G if it can be formed from G by contracting edges, deleting edges, and deleting vertices. A graph H is an *induced minor* of G if it can be formed from G by contracting edges and deleting vertices. We delete loops and parallel edges so all graphs are simple. A graph with more than k vertices is *k-connected* if it cannot be disconnected by deleting fewer than k vertices. Denote by $|G|$ the order of G.

We begin by looking for minimal graphs when $k = 2$ since connectivity 1 graphs are not Hamiltonian. When the containment relation is minors, the only minimal 2-connected non-Hamiltonian graph is $K_{2,3}$. This is easy to see since 2-connected $K_{2,3}$-minor-free graphs are outerplanar graphs or K_4, both of which are Hamiltonian. When the containment relation is subgraphs, it is not hard to see that the minimal 2-connected non-Hamiltonian graphs are $\theta_{a,b,c}$ for $a, b, c \geq 2$ where $\theta_{a,b,c}$ is a theta graph consisting of two distinct vertices and three internally-disjoint paths between them of lengths a, b, and c. Note that excluding all theta graphs as subgraphs is equivalent to excluding $K_{2,3}$ as a minor (or topological minor). Some progress has been made for induced subgraphs by Brousek. In [2], he completely determines all minimal 2-connected non-Hamiltonian graphs which do not contain $K_{1,3}$ (the claw) as an induced subgraph.

When the containment relation is induced minors, the minimal graphs are $K_{2,3}$ and the graph formed from $K_{2,3}$ by adding an edge between the two degree three vertices (call this graph $K_{2,3}^+$). We prove these graphs are the only two minimal 2-connected non-Hamiltonian graphs under the induced minor relation. The theorem is stated here and the proof is in Sect. 3.

Theorem 1 *Let G be a 2-connected non-Hamiltonian graph such that all 2-connected proper induced minors of G are Hamiltonian. Then G is $K_{2,3}$ or $K_{2,3}^+$.*

In [4], Chvátal and Erdős prove that for a connected graph G of order at least three, independence number $\alpha(G)$, and connectivity $\kappa(G)$, if $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian. Theorem 1 is a strengthening of this result in the case when $\kappa(G) = 2$.

The problem becomes more difficult when we move to $k = 3$. For planar graphs, the 3-connected non-Hamiltonian graph with the fewest vertices is called the Herschel graph. For nonplanar graphs, an example of a 3-connected non-Hamiltonian graph is $K_{3,4}$. Notice that both of these graphs as drawn in Fig. 1 (Herschel on the left and $K_{3,4}$ in the middle) are symmetric about a central vertical line. If we combine half of the

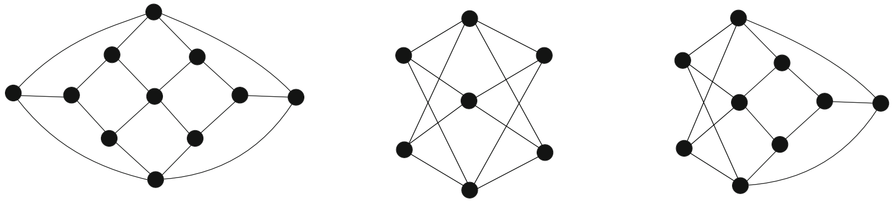


Fig. 1 (from left to right) the Herschel graph, $K_{3,4}$, and the graph Q^+

Herschel graph and half of $K_{3,4}$ at this line of symmetry, the resulting graph, call it Q^+, is also 3-connected and non-Hamiltonian. We conjecture that these three graphs are the only minimal 3-connected non-Hamiltonian graphs when the containment relation is minors.

Conjecture 1 Every 3-connected non-Hamiltonian graph contains the Herschel graph, $K_{3,4}$, or Q^+ as a minor.

In Sect. 2 we prove the following lemma:

Lemma 1 *Let G be a counterexample to Conjecture 1. Then G must be internally* 4-*connected.*

In Sect. 4 we prove the following theorem which proves Conjecture 1 for planar triangulations. Note that triangulations are necessarily 3-connected.

Theorem 2 *Let G be a non-Hamiltonian planar triangulation. Then G contains the Herschel graph as a minor.*

An anonymous referee pointed out a potential connection between our Conjecture 1 and a special case of Conjecture 1.4 of Chen, Yu, and Zhang in [3] which says that every 4-connected $K_{3,4}$-minor-free graph contains a cycle of linear length. The referee raised the question: is it true that every 4-connected $K_{3,4}$-minor-free graph is Hamiltonian? We agree with the referee that this is a very nice problem. We also recognize that our current approach may not work since obtaining all 4-connected $K_{3,4}$-minor-free graphs would be very challenging.

2 Progress Towards Conjecture 1

Before we can prove Lemma 1 we must describe a family of graphs without a certain substructure. These graphs will again be useful in Sect. 4 when we prove Theorem 2.

Another way to think of a k-vertex graph H as a minor in a graph G is as a set of k disjoint subsets of the vertices of G, $(V_1, V_2, \ldots, V_k)$. Each set V_i corresponds to a single vertex v_i of H and is called the *branch set* of v_i. The subgraph induced by V_i in G, denoted $G[V_i]$, is connected and in addition, for every edge v_iv_j in H, there is an edge between a vertex of V_i and a vertex of V_j in G. A *rooted graph H* is a graph H together with a subset of the vertices of H, $\{v_1, \ldots, v_m\}$, called the *roots* of H. We say a graph G contains a *rooted H minor at* $\{v'_1, \ldots, v'_m\}$ for some designated

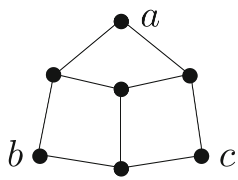


Fig. 2 The rooted graph (Q, a, b, c)

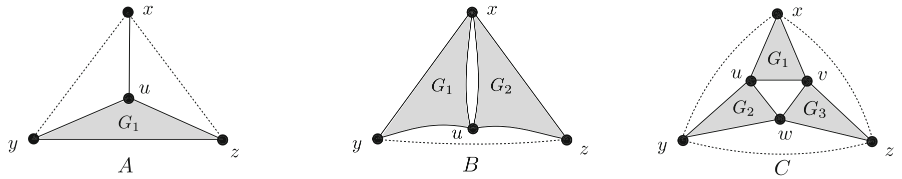


Fig. 3 Graphs in $\mathcal{P}_Q$ of Type A, B, and C

vertices $v'_1, \ldots, v'_m$ of G if G contains an H minor with v'_i in the branch set of v_i for $1 \leq i \leq m$.

For three distinct vertices a, b, c, denote by (Q, a, b, c) the rooted graph shown in Fig. 2 (a, b, c are the roots). A graph G with designated vertices x, y, z is said to be (Q, a, b, c)-free if it does not contain a rooted (Q, a, b, c) minor at $\{x, y, z\}$. Let $\mathrm{si}(G, E)$ be the graph formed from the graph G by adding all edges of the set E which are not already present in G. Let $G^+ = \mathrm{si}(G, \{xy, yz, xz\})$. Observe if we take two copies of (Q, a, b, c) and identify vertices with the same labels a, b, c, the result is the Herschel graph.

Let $\mathcal{P} = \{(G, x, y, z) : G$ is a plane graph, x, y, z are distinct vertices of the infinite face of G, and G^+ is 3-connected. Note graphs in $\mathcal{P}$ necessarily contain at least four vertices. Let $\mathcal{P}_Q = \{(G, x, y, z) : (G, x, y, z) \in \mathcal{P}$ and is (Q, a, b, c)-free$\}$. The next lemma describes graphs in the family $\mathcal{P}_Q$. The structure is shown in Fig. 3. In graphs of Type A, either $|G_1| = 3$ or (G_1, u, y, z) is a graph in $\mathcal{P}_Q$. In graphs of Type B, (G_1, x, y, u) and (G_2, x, z, u) are both in $\mathcal{P}_Q$. In graphs of Type C, either $|G_1| = 3$ and both xu and xv are edges of G_1 or (G_1, x, u, v) is a graph in $\mathcal{P}_Q$ and the same holds for G_2 and G_3. In all cases, the dotted edges may or may not be present.

The proof of the next lemma uses bridges in a graph. Let H be a proper subgraph of a graph G. Then an H-*bridge* is a subgraph of G induced by the edges of a component C of $G - V(H)$ together with the edges linking C to H. Additionally, an edge of G not in H but with both ends in H is also called a (trivial) H-bridge. The vertices of an H-bridge that are in H are the *feet* of the bridge.

Lemma 2 *Let* $(G, x, y, z) \in \mathcal{P}_Q$. *Then* (G, x, y, z) *has the form of a graph of Type A, B, or C in Fig.* 3.

Proof Let $(G, x, y, z) \in \mathcal{P}_Q$. Since (G, x, y, z) remains in $\mathcal{P}_Q$ even after deleting edges xy, yz, and xz, and graphs of Type A, B, or C remain of Type A, B, or C, respectively, even after adding edges xy, yz, and xz, we may assume G does not contain any of these edges.

Since G^+ is 3-connected, x, y, and z each have positive degree. Suppose one of x, y, z has degree one, say x, and let u be the neighbor of x. Then $u \notin \{y, z\}$ because we assume $xy, xz \notin E(G)$. Let F=si$(G - x, \{uy, yz, uz\})$. Then any k-cut ($k < 3$) of F is a k-cut of G^+; so if F is not 3-connected, then $|F| < 4$. Hence $|G| \leq 4$ so $|G| = 4$ and $G^+ \cong K_4$ which means (G, x, y, z) is a graph of Type A. Now F is 3-connected and because any rooted (Q, a, b, c) minor in $(G-x, u, y, z)$ would result in a rooted (Q, a, b, c) minor in (G, x, y, z), we can conclude $(G-x, u, y, z) \in \mathcal{P}_Q$. Again (G, x, y, z) is a graph of Type A. Henceforth we assume x, y, z all have degree at least two.

We claim G is 2-connected. If G is not connected, then the addition of edges xy, xz, yz to form G^+ would not result in a 3-connected graph since these three edges form a triangle. If G is not 2-connected, then it contains a cutvertex u. But now either u is also a cutvertex of G^+, a contradiction, or two of x, y, and z, say x and y, are in different components of $G - u$. Since x has degree at least two in G, it is in a component of order at least two in $G-u$. Now $\{x, u\}$ is a 2-cut in G^+, a contradiction. Hence G is 2-connected and the outer walk of G is an outer cycle, H. Denote by S_1 the set of interior vertices of the path between x and y on H which does not include z. Denote by S_2 the set of interior vertices of the path between x and z on H which does not include y. Denote by S_3 the set of interior vertices of the path between y and z on H which does not include x. Since $xy, yz, xz \notin E(G)$, each of S_1, S_2, S_3 is nonempty.

Suppose to start that $G-x$ does not have two disjoint paths from $S_1 \cup \{y\}$ to $S_2 \cup \{z\}$. Then by Menger's Theorem, $G-x$ has a cutvertex u separating $S_1 \cup \{y\}$ from $S_2 \cup \{z\}$. Let G_1 and G_2 be subgraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{u, x\}$. We have $u \in S_3 \cup \{y, z\}$ and in fact, $u \in S_3$ since otherwise $\{x, y\}$ or $\{x, z\}$ would be a 2-cut in G^+ separating S_1 from S_2 which contradicts the 3-connectedness of G^+. Since S_1 and S_2 are each nonempty, it follows that $|G_1|, |G_2| \geq 4$. To see that (G_1, x, y, u) and (G_2, x, u, z) are both in $\mathcal{P}_Q$, first observe that any rooted (Q, a, b, c) minor in either one would result in a rooted (Q, a, b, c) minor in (G, x, y, z). Next note that any k-cut ($k < 3$) in si$(G_1, \{xy, yu, xu\})$ would also be a k-cut of G^+. The same is true for si$(G_2, \{xz, zu, xu\})$. Hence (G_1, x, y, u) and (G_2, x, u, z) are both in $\mathcal{P}_Q$ and we conclude (G, x, y, z) is a graph of Type B.

Now $G - x$ does have two disjoint paths from $S_1 \cup \{y\}$ to $S_2 \cup \{z\}$. By planarity there must be an H-bridge with feet in both S_1 and S_2. If there is an H-bridge with feet in S_1, S_2, and S_3 then (G, x, y, z) would have a rooted (Q, a, b, c) minor. So by symmetry, each H-bridge has its feet contained in $H - S_i$ for one i. Suppose there is an H-bridge B with feet only in one of $S_1 \cup \{x, y\}$, $S_2 \cup \{x, z\}$ or $S_3 \cup \{y, z\}$, say in $S_1 \cup \{x, y\}$. Then by planarity the foot of B closest to (and possibly equal to) x along H and the foot of B closest to (and possibly equal to) y along H form a 2-cut in G^+. Thus each H-bridge has feet in $H - S_i$ for exactly one i. Denote by G_3 the union of all H-bridges with feet contained in $H - S_1$, by G_2 the union of all H-bridges with feet contained in $H - S_2$, and by G_1 the union of all H-bridges with feet contained in $H - S_3$.

Order the vertices of S_3 from y to z. Because G_3 contains a vertex of S_3 and G is planar, there is some vertex w of S_3 such that there is no path from G_2 to a vertex after w along S_3 in $G - S_2$ and additionally, there is no path from G_3 to a vertex before w along S_3 in $G - S_1$. Symmetrically, we can find such a vertex u in S_1 and v in S_2. Now consider (G_1, x, u, v). If $|G_1| = 3$, then necessarily G_1 contains edges

xu and xv since these will be edges of H. Otherwise if $|G_1| \geq 4$, we can argue that $(G_1, x, u, v) \in \mathcal{P}_Q$. To see this, first note that any rooted (Q, a, b, c) minor in (G_1, x, u, v) would result in a rooted (Q, a, b, c) minor in (G, x, y, z). Next observe that any k-cut ($k < 3$) in $\mathrm{si}(G_1, \{xu, xv, uv\})$ would also be a k-cut of G^+. Thus $\mathrm{si}(G_1, \{xu, xv, uv\})$ is 3-connected and therefore in $\mathcal{P}_Q$. Symmetric arguments apply to (G_2, y, u, w) and (G_3, z, v, w) and we conclude (G, x, y, z) is a graph of Type C. □

Let $(G, x, y, z) \in \mathcal{P}_Q$. A *loner vertex* of G is a vertex of degree one. Since G has the property that G^+ is 3-connected, only x, y, and z can be loners and the single neighbor of a loner is not in $\{x, y, z\}$. In graphs of Type A, x is a loner when edges xy and xz are not in G. Additionally y (and z) can be loners in graphs of Type A if y (or z) is a loner in G_1 and edges yx and yz (or zx and zy) are not present in G. In graphs of Type B, y and z can be loners but x cannot since even if it is a loner in G_1 and G_2, it would still have degree two. In graphs of Type C, x, y, z can all be loners.

Consider the vertex u in a graph of Type B. If u is a loner in G_1, then u cannot also be a loner in G_2 since then u would have degree two which violates the connectivity of G^+. Similarly, u, v, w cannot be loners on both sides in graphs of Type C.

Lemma 3 *Let $(G, x, y, z) \in \mathcal{P}_Q$. Then $G - x$ has a Hamilton path from y to z and when x is not a loner vertex, G also has a Hamilton path from y to z.*

Proof Suppose the lemma is false and let (G, x, y, z) be a counterexample on the smallest number of vertices. We consider cases based on whether (G, x, y, z) is of Type A, B, or C.

Suppose to start G is of Type A. If G_1 is trivial, then G^+ is K_4 and all paths exist as necessary based on whether or not x, y, z are loners. Otherwise, (G_1, u, y, z) is smaller than (G, x, y, z) so it satisfies the lemma. Additionally note u is not a loner in G_1 since then it would have degree two in G. Up to symmetry, we need to find Hamilton paths from x to y in G and $G - z$ and from y to z in G and $G - x$.

Hamilton path from x to y in $G - z$: Take a y to u path missing z in G_1 and add ux.

Hamilton path from x to y in G: If z is a loner, then this path does not exist. So z is not a loner in G. If z is also not a loner in G_1, then take a path from y to u in G_1 including z and add ux. If z is a loner in G_1, then the edge zx must be in G. Take a path from y to z including u in G_1 and add the edge zx.

Hamilton path from y to z in $G - x$: Take a path from y to z including u in G_1.

Hamilton path from y to z in G: If x is a loner, this path does not exist. So x is not a loner and without loss of generality, we have the edge yx. Take a path from u to z in G_1 missing y and add the edges yx and xu.

Now G is of Type B. (G_1, x, y, u) and (G_2, x, u, z) both have fewer vertices than G so they satisfy the lemma. Note u cannot be a loner in both G_1 and G_2. Up to symmetry, we need to find Hamilton paths from x to y in G and $G - z$ and from y to z in G and $G - x$.

Hamilton path from x to y in $G - z$: Join a path in G_2 from x to u missing z with a path in G_1 from u to y missing x.

Hamilton path from x to y in G: If z is a loner, then this path does not exist. So z is not a loner in G. If z is also not a loner in G_2, then combine a path from x to u

including z in G_2 with a path from u to y missing x in G_1. If z is a loner in G_2, then the edge zy exists in G. Join a path from x to u missing y in G_1 with a path from u to z missing x in G_2 and the edge zy.

Hamilton path from y to z in $G - x$: Join a path from y to u missing x in G_1 with a path from u to z missing x in G_2.

Hamilton path from y to z in G: Without loss of generality, suppose u is not a loner in G_1. Join a path from y to x including u in G_1 with a path from x to z missing u in G_2.

Finally suppose G is of Type C. Note G_1 either has order three and includes edges xu and xv or $(G_1, x, u, v) \in \mathcal{P}_Q$ and contains fewer vertices than G so G_1 has all Hamilton paths as described in the lemma. Symmetric statements are true for G_2 and G_3. Up to symmetry, we need to find Hamilton paths from x to y in G and $G - z$.

Hamilton path from x to y in $G - z$: The vertex u cannot be degree one in both G_1 and G_2 so without loss of generality, assume it has degree at least two in G_1. If $(G_3, z, v, w) \in \mathcal{P}_Q$ or if $|G_3| = 3$ and $vw \in E(G_3)$, join three paths: from x to v including u in G_1, from v to w missing z in G_3, and from w to y missing u in G_2. If $|G_3| = 3$ and $vw \notin E(G_3)$, then v must be degree at least two in G_1 and w must be degree at least two in G_2 so the following paths exist and we can join them: from x to u including v in G_1 and from u to y including w in G_2.

Hamilton path from x to y in G: If z is a loner, then this path does not exist. So z is not a loner in G. The vertex u cannot be degree one in both G_1 and G_2 so without loss of generality, assume it has degree at least two in G_1. If z is also not a loner in G_3, then we join three paths: from x to v including u in G_1, from v to w including z in G_3, and from w to y missing u in G_2.

Now z is a loner in G_3 and without loss of generality, yz is an edge of G. Assume first that $(G_2, y, u, w) \in \mathcal{P}_Q$ or $|G_2| = 3$ and $uw \in E(G_2)$. The vertex v cannot have degree one in both G_1 and G_3. If v has degree at least two in G_1, then we join three paths with the edge zy: a path from x to u including v in G_1, a path from u to w missing y in G_2, and a path from w to z missing v in G_3. Otherwise v has degree at least two in G_3 and we join these three paths with the edge zy: a path from x to u missing v in G_1, a path from u to w missing y in G_2, and a path from w to z including v in G_3. Now assume $|G_2| = 3$ and $uw \notin E(G_2)$. Then u must have degree at least two in G_1 and w must have degree at least two in G_3. We join the following paths with the edge zy: a path from x to v including u in G_1 and a path from v to z including w in G_3. □

Note that if $(G, x, y, z) \in \mathcal{P}_Q$ and G is a planar triangulation, then G has no loner vertices.

Corollary 1 *Let G be a planar triangulation with x, y, z vertices of the outer face and let $(G, x, y, z) \in \mathcal{P}_Q$. Then both G and $G - x$ have Hamilton paths from y to z.*

A *3-separation* of a graph G is a pair (G_1, G_2) of edge-disjoint non-spanning subgraphs of G such that $G_1 \cup G_2 = G$ and $|G_1 \cap G_2| = 3$. In a 3-separation, the set $V(G_1 \cap G_2)$ is a 3-*cut*.

A 3-connected planar graph G is *weakly* 4-*connected* if for every 3-cut S of G, $G - S$ has only two components and contains an isolated vertex. A weakly 4-connected graph is *internally* 4-*connected* if every 3-cut is an independent set. We use the following lemma in the proof of Lemma 1.

Lemma 4 (Seymour [9]) *Let e be an edge of a* 3*-connected graph G of order at least five. Then either G/e is obtained from a 3-connected graph by adding parallel edges or $G - e$ is obtained from a* 3*-connected graph by subdividing edges.*

Proof (Proof of Lemma 1) Let G be a minor-minimal counterexample. If G is weakly 4-connected but not internally 4-connected, then G must have a cubic vertex x such that x is contained in a triangle xyz. By Lemma 4, since G/yz is not formed from a 3-connected graph by adding parallel edges because deleting the multiple edge incident with x results in x having degree two, $G - yz$ must be formed from a 3-connected graph by subdividing edges. Hence $G - yz$ is not 3-connected. It follows that at least one of y and z is also cubic; without loss of generality, say y is cubic. Let x' and y' be the neighbors of x and y, respectively, not in triangle xyz. If $x' = y'$, then $\{x', z\}$ is a 2-cut, a contradiction. Now $|G| < 7$ because otherwise $\{x', y', z\}$ is a 3-cut separating x and y from the rest of the graph which contradicts the weakly 4-connectedness of G. If $|G| = 5$, then $G \cong W_4$ and is Hamiltonian, a contradiction. If $|G| = 6$, the sixth vertex z' is adjacent to both x' and y' and the graph contains a Hamilton cycle, a contradiction.

Thus G is not weakly 4-connected so it contains a 3-separation (H_1, H_2) such that $\min\{|H_1|, |H_2|\} > 4$. Let $\{x, y, x\} = V(H_1 \cap H_2)$ and let $A_1, \ldots, A_k$ be the components of $G - \{x, y, z\}$. Because G is $K_{3,4}$-minor-free, $k \leq 3$. For each i, let G_i be obtained from G by deleting all components A_j with $j \neq i$ and then adding a new vertex v_i adjacent to x, y, and z. There are three possibilities for each G_i:

Type 1: G_i is not planar. It follows from (2.4) of [8] that G_i has a subgraph G_i' such that G_i' is a subdivision of $K_{3,3}$ and v_i is a cubic vertex of G_i'.
Type 2: G_i is planar and has a subgraph G_i' such that G_i' is a subdivision of the cube and v_i is a cubic vertex of G_i'.
Type 3: $(G_i - v_i, x, y, z) \in \mathcal{P}_Q$.

Suppose to start that $k = 3$. If G_1, G_2, G_3 are all of Type 3, then by Lemma 3, there is a Hamilton path P_1 in $G_1 - \{v_1, y\}$ from x to z, a Hamilton path P_2 in $G_2 - \{v_2, x\}$ from z to y, and a Hamilton path P_3 in $G_3 - \{z, v_3\}$ from y to x. Now $P_1 \cup P_2 \cup P_3$ is a Hamilton cycle in G, a contradiction. If some G_i is of Type 1, then G contains $K_{3,4}$. Thus some G_i must be of Type 2, but then G contains Q^+.

Now $k = 2$. If G_1 and G_2 are both of Type 1, then G contains $K_{3,4}$. If G_1 and G_2 are both of Type 2, then G contains the Herschel graph as a minor. If one of G_i and G_j is of Type 1 and the other is of Type 2, then G contains Q^+ as a minor. Now we have argued that G has a 3-separation (H_1, H_2) with $|H_1|, |H_2| > 4$ and such that (H_1, x, y, z) is of Type 3. We choose such a 3-separation with $|H_1|$ as large as possible. We claim H_2 has a Hamilton path with both ends in $\{x, y, z\}$. Then by this claim and Lemma 3, we can find a Hamilton cycle in G. This contradiction will complete the proof.

If one of x, y, z, say x, has only one neighbor x' in H_2, then $\{x', y, z\}$ defines a 3-separation $(H_1 + xx', H_2 - x)$ of G. Note that $(H_1 + xx', x', y, z)$ belongs to $\mathcal{P}_Q$ so by the maximality of H_1, $|H_2| = 5$. Let w be the fifth vertex of H_2. Then to avoid any 2-cuts in H_2 that would also be 2-cuts in G, we must have edges wy, wz, and at least one of $x'y$ or $x'z$. Now $xx'ywz$ (or $xx'zwy$) is a desired Hamilton path. Now we assume that x, y, z all have degree at least two in H_2. Then G_2 is 3-connected. Note G_2 is a proper minor of G so by the minimality of G, G_2 has a Hamilton cycle and thus H_2 has the required Hamilton path. □

3 Minimal Graphs Under the Induced Minor Relation

Here we prove Theorem 1.

Proof (Proof of Theorem 1) Let G be a 2-connected non-Hamiltonian graph such that all 2-connected proper induced minors of G are Hamiltonian. We will show that G contains an induced minor H isomorphic to $K_{2,3}$ or $K_{2,3}^+$. Since both of these graphs are non-Hamiltonian, it will follow that G actually is isomorphic to H. The proof follows through a sequence of claims.

Claim 1 We can assume for any 2-cut $\{x, y\}$ of G, $G-\{x, y\}$ has only two components.

If $G - \{x, y\}$ has three or more components, contract three of the components each down to a single vertex u_1, u_2, and u_3. These vertices, together with x and y, result in a $K_{2,3}$ or $K_{2,3}^+$ induced minor.

Claim 2 We can assume for any 2-cut $\{x, y\}$ of G, $G - \{x, y\}$ has an isolated vertex.

Suppose G has two induced subgraphs H_1 and H_2 such that $V(H_1) \cap V(H_2) = \{x, y\}$, and $|H_1|, |H_2| \geq 4$. Let H_1^+ be the graph formed from H_1 by adding a vertex z adjacent to both x and y. Then H_1^+ is a 2-connected proper induced minor of G, so H_1^+ has a Hamilton cycle. Thus H_1 has a Hamilton path P_1 from x to y. Symmetrically we can find a Hamilton path P_2 in H_2 from x to y. Then $P_1 \cup P_2$ is a Hamilton cycle in G, a contradiction.

Claim 3 There exists a vertex z in G such that $G - z$ is 2-connected.

If G is 3-connected, take any vertex of G as z. If G is not 3-connected, then there exists a 2-cut $\{x, y\}$ of G because otherwise $G \cong K_3$ and is Hamiltonian. By Claim 2, $G - \{x, y\}$ has an isolated vertex, call it z. Note that vertex z has degree two in G. We claim $G' = G - z$ is 2-connected. Suppose not and let w be a cutvertex of G'. Then $\{z, w\}$ is a 2-cut of G. By Claim 2 and without loss of generality, x is an isolated vertex of $G - \{z, w\}$ so x has degree two in G. Now either $|G| = 4$ and $zxwy$ is a Hamilton cycle, a contradiction, or $\{w, y\}$ is a 2-cut. When $\{w, y\}$ is a 2-cut, to avoid a contradiction with Claim 2, we must have $|G| = 5$. Then $zxwuy$ is a Hamilton cycle where u is the fifth vertex, a contradiction. This completes the proof of Claim 3.

Take z as in Claim 3. $G - z$ is a 2-connected proper induced minor of G so it has a Hamilton cycle C. Fix a forward direction on C. Denote by u^- the vertex directly before a vertex u on C and u^+ the vertex directly after when traveling in the forward direction. Denote by uCv the subpath of C from u to v. Let u and v be the two neighbors of z which are on C. If u and v are consecutive along C, form a Hamilton cycle in G from C by replacing uv with uzv. Hence u and v are not consecutive and furthermore u^- and v^- are not neighbors of z. If u^- and v^- are adjacent, we can form a Hamilton cycle in G by replacing u^-Cv with $u^-v^- \cup uCv^- \cup uzv$. Thus u^- and v^- are not adjacent and G contains $K_{2,3}$ or $K_{2,3}^+$ as an induced minor. □

4 A Minimal Graph Under the Minor Relation

The containment relation throughout this section is minors. We say a graph is *H-minor-free* if it does not contain H as a minor. In this section, we will prove Theorem 2 which can be restated as follows: every Herschel-minor-free planar triangulation is Hamiltonian. In this formulation, it can be seen as a Hamiltonicity result for 3-connected graphs on the plane. Other results of this type include a proof by Ellingham et. al. that 3-connected planar $K_{2,5}$-minor-free graphs are Hamiltonian [5]. Additionally, Jackson and Yu show planar triangulations with restrictions on the pieces of a certain decomposition of the graph are Hamiltonian [6].

The proof of Theorem 2 uses 3-sums. A *3-sum* of two graphs G_1 and G_2 is a new graph formed by identifying a triangle in G_1 with a triangle in G_2 (and deleting multiple edges). It is clear that every triangulation can be expressed as 3-sums of K_4 and 4-connected triangulations. Thus we begin the proof of Theorem 2 by determining all 4-connected Herschel-minor-free planar triangulations; the result is stated in the next subsection. In moving from 4-connected graphs to 3-connected graphs, we first look at weakly 4-connected graphs. Theorem 4 in Sect. 4.2 proves Theorem 2 in the case when G is weakly 4-connected. We prove Theorem 2 in general in Sect. 4.3. For graphs on fewer than 14 vertices, we verify our results by computer.

4.1 4-Connected Herschel-Minor-Free Planar Triangulations

Let $\mathcal{G}_k$ denote the set of 4-connected Herschel-minor-free planar triangulations on k vertices. To describe these graphs, we use four families of graphs shown in Fig. 4 and denoted by A_n, B_n, C_n, and $D_{m,n}$. The figure includes two different drawings of the family $D_{m,n}$; the first is a planar embedding and will be used for reference in later proofs and the second is included to illustrate the full symmetries of the family. For A_n, B_n, and C_n, the parameter n is the number of interior vertices along the xy-path shown in bold in the figures. So A_n has $n+7$ vertices and B_n and C_n each have $n+8$ vertices. For $D_{m,n}$, the parameter n again is the number of interior vertices along the xy-path shown in bold and the parameter m is the number of interior vertices in the vertical path between the other two marked vertices in the figure; $D_{m,n}$ has $m+n+4$ vertices. Observe that $D_{m,1}$ is the double wheel, $D_{2,2} = A_1$, and $D_{2,3} = C_1$. These four families together with 11 small graphs, denoted $\Gamma_1, \ldots, \Gamma_{11}$ shown in Fig. 5, make up the 4-connected Herschel-minor-free planar triangulations.

Theorem 3 *4-connected Herschel-minor-free planar triangulations are precisely $\Gamma_1, \ldots, \Gamma_{11}$ and A_n, B_n, C_n, $D_{m,n}$ for $m, n \geq 1$.*

Because the Herschel graph has 11 vertices, $\mathcal{G}_k$ for $k \leq 10$ is precisely the set of 4-connected planar triangulations on k vertices. These graphs can be found by taking the duals of internally 4-connected cubic planar graphs on up to 16 vertices which are

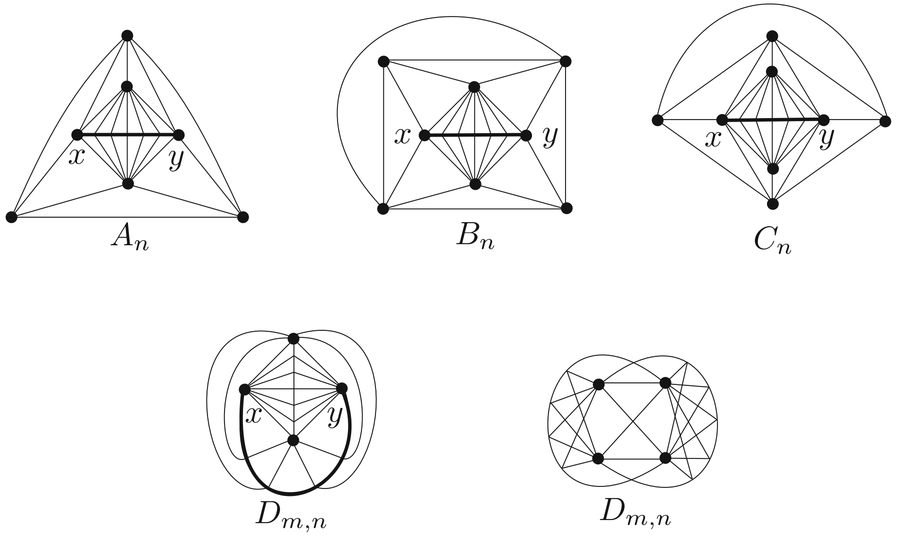


Fig. 4 Families of graphs A_n, B_n, C_n, and two drawings of $D_{m,n}$

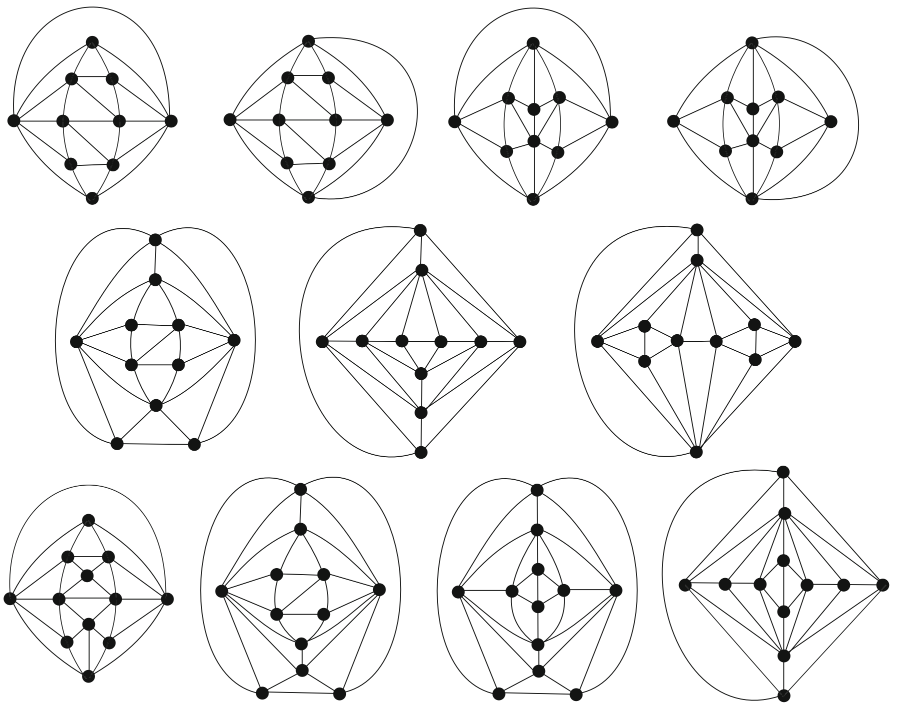


Fig. 5 Graphs Γ_1 through Γ_{11} (from left to right, top to bottom)

listed by Kotzig in [7]. We use his result to find $\mathcal{G}_k$ for $k \leq 10$. For $k \geq 11$, we use another result from Kotzig in [7]. He shows that all 4-connected planar triangulations can be obtained from the octahedron by repeatedly splitting vertices where a *vertex split* of a vertex v of a planar triangulation is defined as follows. Delete the vertex v and add two new adjacent vertices u and w. Add new edges with one end in $\{u, w\}$ and the other in $N_G(v)$ so that the resulting graph is a planar triangulation and u and w each have degree at least four. By Kotzig, vertex splits of 4-connected planar triangulations are 4-connected planar triangulations. Observe that the outcome of this operation is not necessarily unique.

Brinkmann et. al. strengthened Kotzig's result by showing that 4-connected planar triangulations can be generating from the octahedron only using vertex splits in which at least one of the two new vertices created has degree four or five [1]. In our analysis, we never need to consider a split in which both new vertices have degree at least six. Therefore, the two results are of the same strength in our application. We choose to use Kotzig's result because otherwise we would have to add an unnecessary step of checking the degrees of the split vertices.

To find $\mathcal{G}_k$, we consider all graphs resulting from vertex splits of graphs in $\mathcal{G}_{k-1}$. Then we remove all graphs containing a Herschel minor. For $k \leq 13$, we use computer to generate all vertex splits and to check for a Herschel minor. We state the results separately for $\mathcal{G}_k$, $k \leq 13$, in the following lemma.

Lemma 5 *The sets of graphs $\mathcal{G}_k$ for $6 \leq k \leq 13$ are as follows:*

$\mathcal{G}_6 = \{D_{1,1}\}$
$\mathcal{G}_7 = \{D_{1,2}\}$
$\mathcal{G}_8 = \{D_{1,3}, D_{2,2}\}$
$\mathcal{G}_9 = \{A_2, B_1, D_{1,4}, D_{2,3}\}$
$\mathcal{G}_{10} = \{A_3, B_2, C_2, D_{1,5}, D_{2,4}, D_{3,3}, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$
$\mathcal{G}_{11} = \{A_4, B_3, C_3, D_{1,6}, D_{2,5}, D_{3,4}, \Gamma_5, \Gamma_6, \Gamma_7\}$
$\mathcal{G}_{12} = \{A_5, B_4, C_4, D_{1,7}, D_{2,6}, D_{3,5}, D_{4,4}, \Gamma_8, \Gamma_9, \Gamma_{10}, \Gamma_{11}\}$
$\mathcal{G}_{13} = \{A_6, B_5, C_5, D_{1,8}, D_{2,7}, D_{3,6}, D_{4,5}\}$

We follow the same process to prove Theorem 3, this time generating and checking all vertex splits by hand.

Proof (Proof of Theorem 3) By Lemma 5, the result holds for graphs on at most 13 vertices. It remains to show that for $k \geq 14$, $\mathcal{G}_k = \{A_{k-7}, B_{k-8}, C_{k-8}\} \cup \{D_{i,k-i-4} : i = 1, \ldots, \lfloor (k-4)/2 \rfloor\}$. We will start with a general graph A_n, B_n, C_n, and $D_{m,n}$ and verify that all splits of each vertex result in either a desired graph or a Herschel minor.

Splits of A_n: Consider the graph A_n with vertices labeled as in Fig. 6. Vertices given the same number are symmetric so we do not consider them separately. Denote by P the path induced by the unlabeled vertices. Let v stand for any vertex of this path. While not all vertices of P are symmetric, as we will see they behave similarly regarding vertex splits so we generally do not consider them separately. Note that we are splitting vertices of graphs with at least 13 vertices so P contains at least 6 vertices.

When we split a vertex, the result must still be a planar 4-connected triangulation. Thus the two new vertices created must have two common neighbors among the

Fig. 6 A_n with labeled vertices

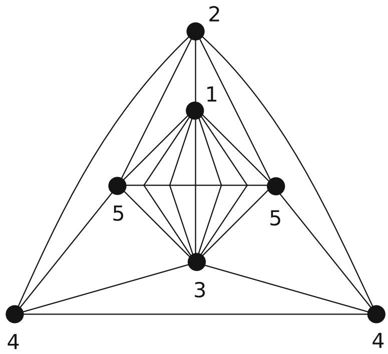
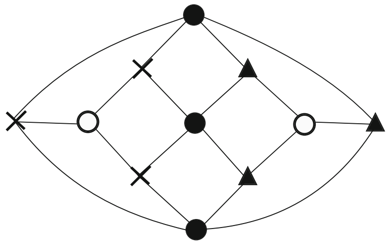


Fig. 7 Herschel graph with labeled vertices



neighbors of the original vertex, and the two common neighbors cannot be adjacent. We can list the two common neighbors as a way of distinguishing between different splits of a vertex. For example, the neighbors of 2 are 1,5,4,4, and 5. Up to symmetry, the splits of 2 are then 14, 45, and 55. We use the notation 14 to mean vertices 1 and 4 are the two common neighbors of the two vertices created after the split. Here with 45 and in general for any xy, we assume the vertices listed are a nonadjacent pair (since in this case there is an adjacent 45 pair). In each of the figures showing Herschel minors, the vertices of the Herschel graph are marked as in Fig. 7. All edges required for the minor are shown but the graph may contain additional edges. There also may be additional vertices of the graph and these will always be additional vertices of P which have been contracted in the figures. We now consider splits of A_n.

Splits of vertices of P either give A_{n+1} or the Herschel minor shown on the left of Fig. 8. There are five splits of 3 to consider: $5v$, vv, 45, $4v$, and 55. For splits $5v$ or vv, the result has the Herschel minor shown on the left of Fig. 8. For splits 45 or $4v$, the result has the Herschel minor shown on the right of Fig. 8. Finally for split 55, the result is isomorphic to C_{n+1}.

There are four splits of 1 to consider: $2v$, $5v$, vv, and 55. For splits $2v$, $5v$, and vv, the result has the Herschel minor shown on the left of Fig. 9. For split 55, the result has the Herschel minor on the right of Fig. 9.

As already mentioned, the splits of 2 are 14, 54, and 55. Splits 54 and 55 both result in the Herschel minor shown on the right of Fig. 9. Split 14 results in a graph isomorphic to B_{n+1}.

There are two splits of 4: 23 and 45. Split 23 results in a variation of the Herschel minor shown on the right of Fig. 9. To see this, delete the edge between the two

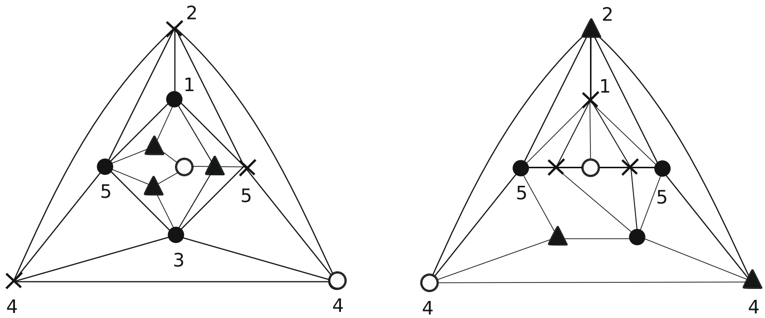


Fig. 8 Herschel graph in splits of A_n

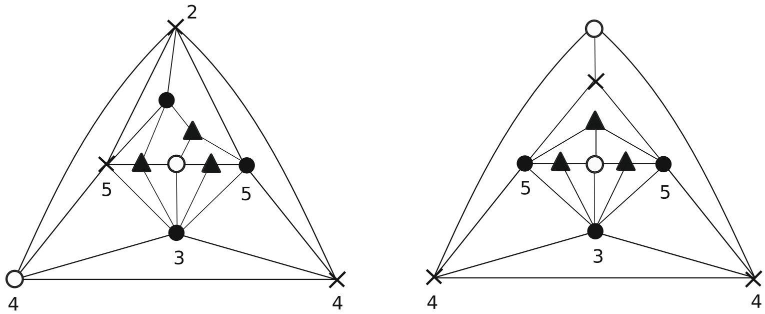


Fig. 9 Herschel graph in splits of A_n

vertices labeled 4 and add an edge between the top vertex and the vertex labeled 3. The Herschel minor remains unchanged. Split 45 results in the Herschel minor shown in the right of Fig. 8.

Finally consider splits of 5. There are five to consider: $2v$, 23, 13, 14, and $v4$ where v is now specifically the interior vertex of P adjacent to 5. Split $2v$ and split $2v$ of vertex 1 result in isomorphic graphs. Split 23 and split 23 of vertex 4 result in isomorphic graphs. Split 13 results in a graph isomorphic to A_{n+1}. Split 14 and split 14 of vertex 2 result in isomorphic graphs. Split $v4$ and split $v4$ of vertex 3 result in isomorphic graphs.

Splits of B_n: Consider the graph B_n with vertices labeled as on the left of Fig. 10. As with A_n, vertices given the same label are symmetric. Again denote by P the path induced by the unlabeled vertices. Here we assume P contains at least 5 vertices. Let v refer to a general vertex of P. Splits of v either result in B_{n+1} or contain the Herschel minor shown on the right of Fig. 10.

In the remaining figures, any dotted edges are contracted to form the Herschel minor. Up to symmetry, there are seven splits of 1: $3v$, $2v$, $4v$, vv, 24, 44, and 34. Split $4v$ for any vertex v of P results in the Herschel minor shown on the left of Fig. 11. Split $3v$ separates into two cases based on the choice of v. If 3 and v are not at distance two, then we have the Herschel minor shown on the left of Fig. 11. Note the minor does not use the edge 33 so vertices 2 and 3 are symmetric and this Herschel minor also results

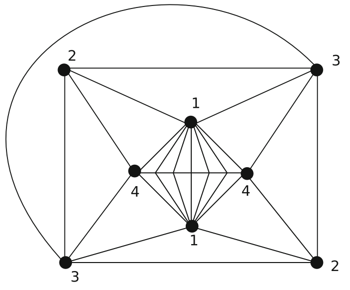


Fig. 10 Vertex labels of B_n and a vertex split

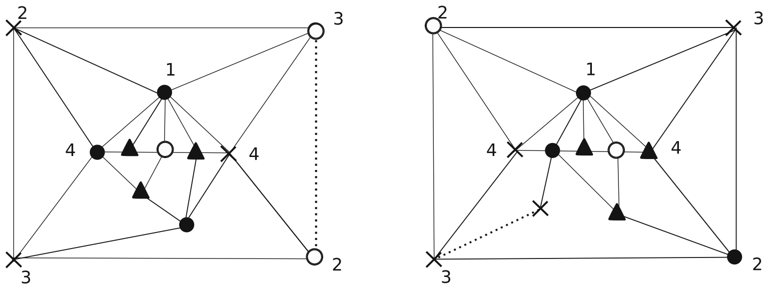


Fig. 11 Vertex splits of B_n

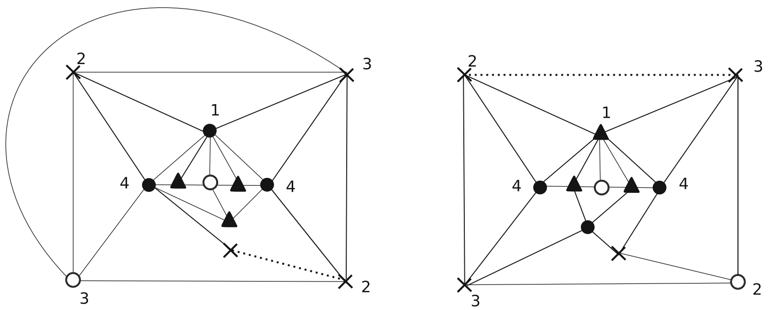


Fig. 12 Vertex splits of B_n

from split $2v$ where 2 and v are not at distance two. For split $3v$ (and symmetrically $2v$) where 3 and v are at distance two, we have the Herschel minor shown on the right of Fig. 11. Split vv results in the Herschel minor shown on the left of Fig. 11 where one common neighbor v has been contracted to the vertex 4 on the left. Splits 24 and 44 result in the Herschel minor shown on the left of Fig. 12. Finally split 34 results in the Herschel minor shown on the right of Fig. 12.

There are two splits of 2: 34 and 13. Both result in the Herschel minor shown on the right of Fig. 12.

There are five splits of 4: 13, 12, 11, $3v$, and $2v$ where v is specifically the vertex of P adjacent to 4. Split 13 results in the Herschel minor shown on the right of Fig. 12. Note that since this minor does not use the edge 33, it also results from the split 12.

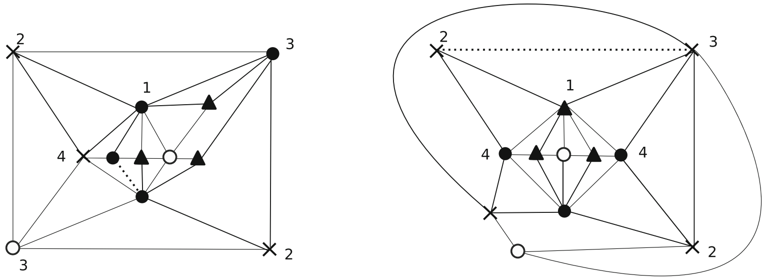


Fig. 13 Vertex splits of B_n

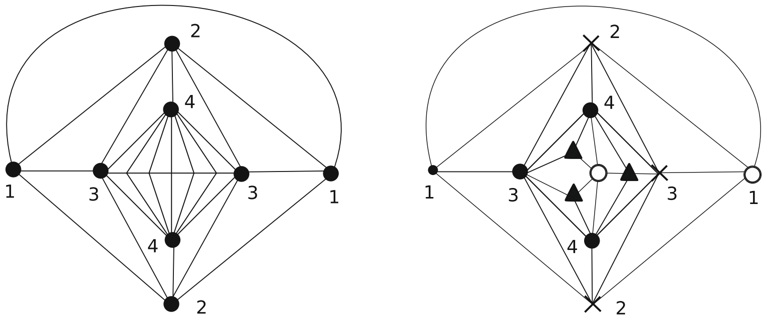


Fig. 14 Vertex labels of C_n and a vertex split of C_n

Split 11 results in B_{n+1}. Splits $3v$ and $2v$ both result in the Herschel minor shown on the left of Fig. 13 (note edge 33 is not used so vertices 2 and 3 are symmetric).

Finally, there are five splits of 3 to consider: 21, 22, 24, 34, and 31. Splits 21, 22, and 24 all result in a Herschel minor which can be seen by swapping the labels of vertices 2 and 3 on the right of Fig. 12. Note this minor does not use the edge 33 so vertices 2 and 3 are symmetric. To see how this represents these splits of 3, swap the labeling of vertices 2 and 3; then we are splitting the vertex 3 on the bottom right. Split 34 also results in the Herschel minor shown on the right of Fig. 12. In the figure, the vertex 3 on the top right is the one being split. To see this, we must adjust the labels. This vertex and the vertex labeled 2 on the bottom right are the new vertices created in the split. The unlabeled vertex adjacent to this vertex 2 instead plays the role of 2. Split 31 results in the Herschel minor shown on the right of Fig. 13.

Splits of C_n: Consider the graph C_n with vertices labeled as on the left of Fig. 14. As with A_n and B_n, vertices given the same label are symmetric. Again denote by P the path induced by the unlabeled vertices. Here we assume P contains at least 5 vertices. Let v refer to a general vertex of P. Splits of v either result in C_{n+1} or contain the Herschel minor shown on the right of Fig. 14.

There are four splits of 4: $2v$, $3v$, vv, and 33. Splits $2v$, $3v$, and vv result in the Herschel minor shown on the left of Fig. 15 where for vv, one of the common vertices v has been contracted to 3. Split 33 results in the Herschel minor shown on the right of Fig. 15.

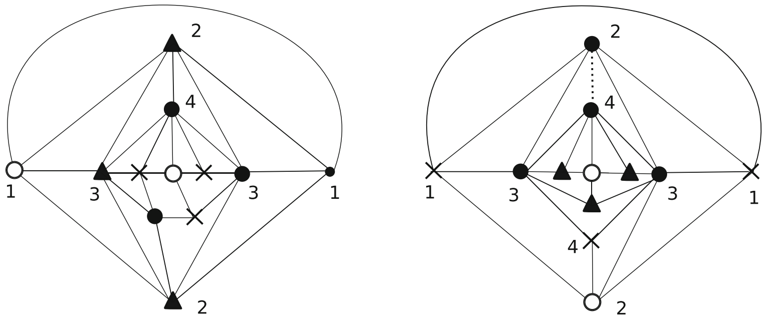


Fig. 15 Vertex splits of C_n

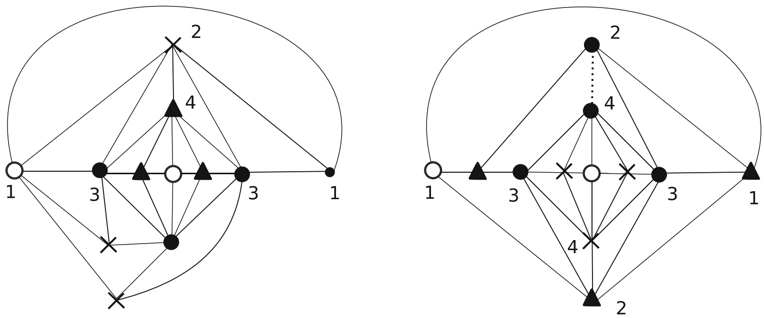


Fig. 16 Vertex splits of C_n

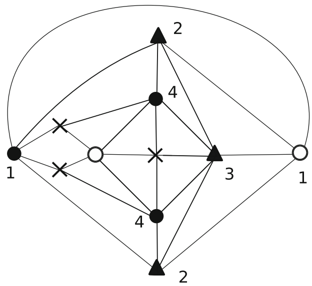
Up to symmetry, there are three splits of 2: 13, 33, and 14. Splits 13 and 33 result in the Herschel minor shown on the right of Fig. 15. Split 14 results in the Herschel minor shown on the left of Fig. 16.

There are two splits of 1: 13 and 22. Split 13 results in the Herschel minor shown on the right of Fig. 15. To see this, relabel the lower vertex labeled 4 with label 2. Then the vertex labeled 2 on the bottom and the vertex labeled 1 on the right are the two new vertices resulting from the split. Split 22 results in the Herschel minor shown on the right of Fig. 16.

Up to symmetry, there are six splits of 3: 44, 14, 22, 24, $2v$, and $1v$ where v here is the vertex of P adjacent to 3. Split 44 results in C_{n+1}. Split 14 results in the Herschel minor shown on the left of Fig. 16. Splits 22 and 24 result in the Herschel minor shown on the right of Fig. 16. Split $2v$ results in the Herschel minor shown on the left of Fig. 15. Finally split $1v$ results in the Herschel minor shown in Fig. 17.

Splits of $D_{m,n}$: First we consider $D_{1,n}$, the double wheel. There are two types of vertices to consider: rim vertices and center vertices. Rim vertices have two possible splits. One results in $D_{1,n+1}$ and the other results in $D_{2,n}$. In splitting a center vertex, the two common neighbors of the resulting vertices are both rim vertices. If they are distance two apart on the rim, then the result is $D_{2,n}$. If they are distance three apart, then the result is A_{n-1}. Finally, if they are at least distance four apart, then the result has the Herschel minor shown in Fig. 18.

Fig. 17 Vertex splits of C_n



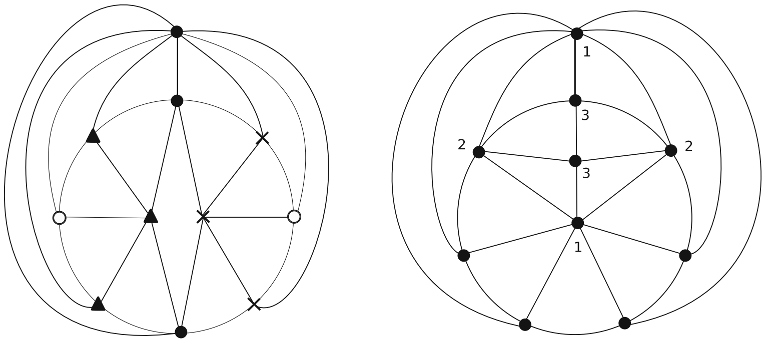


Fig. 18 A vertex split of $D_{1,n}$ and vertex labels of $D_{2,n}$

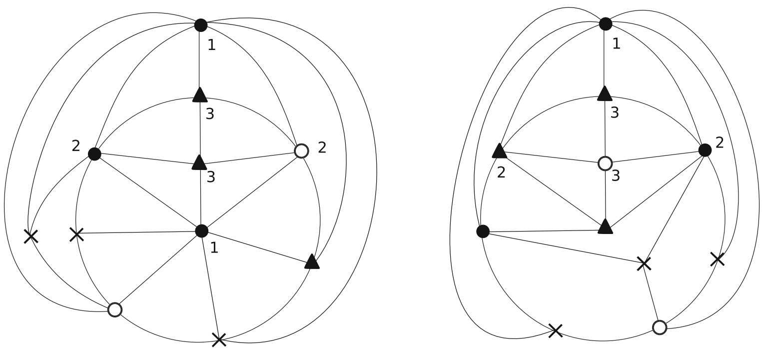


Fig. 19 Vertex splits of $D_{2,n}$

Next consider splits of $D_{2,n}$ with vertices labeled as on the right in Fig. 18. Vertices given the same label are symmetric. Again denote by P the path induced by the unlabeled vertices. Here we assume P contains at least 7 vertices. Let v refer to a general vertex of P. Splits of v either result in $D_{2,n+1}$ or have the Herschel minor shown on the left of Fig. 19.

Up to symmetry, there are four splits of 1: 22, $2v$, $3v$, and vv. Split 22 results in $D_{3,n}$. Split $2v$ results in the Herschel minor shown on the right of Fig. 19. Split $3v$ separates further into two cases. If v is adjacent to vertex 2, then split $3v$ results in

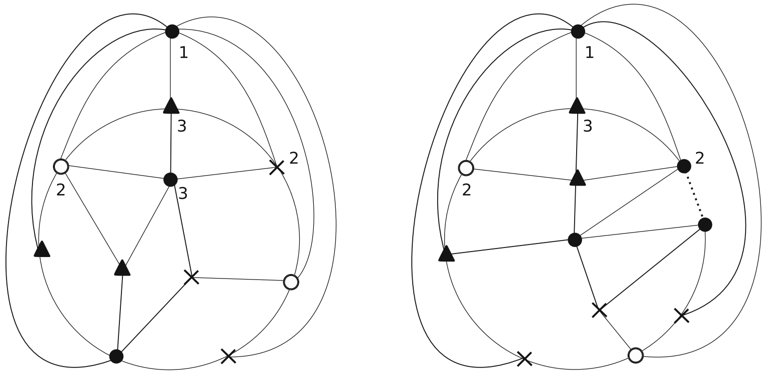


Fig. 20 Vertex splits of $D_{2,n}$

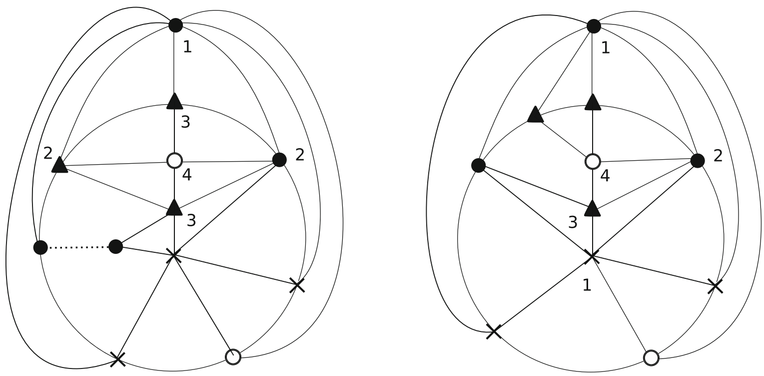


Fig. 21 Vertex splits of $D_{3,n}$

B_{n-1}. Otherwise split $3v$ results in the Herschel minor shown on the left of Fig. 20. Split vv results in the Herschel minor shown on the right of Fig. 20.

Up to symmetry, there are three splits of 2: 11, 13, and $3v$. Split 11 results in $D_{2,n+1}$. Split 13 results in A_n. Split $3v$ results in B_{n-1}. There are two splits of 3: 22 and 13. Split 22 results in a graph isomorphic to $D_{3,n}$ and split 13 results in a graph isomorphic to A_n.

Finally consider splits of $D_{3,n}$. Label the vertices as for $D_{2,n}$ with an additional vertex 4 created by subdividing the edge 33 and adding both edges 24. Now the path P will have at least 6 vertices instead of 7. All splits of 1 which resulted in a Herschel minor in $D_{2,n}$ will also result in a Herschel minor here since none of the minors required P to have length at least 7. Split 22 of 1 now results in $D_{4,n}$ instead of $D_{3,n}$. Split $3v$ where v is the vertex of P adjacent to 2 now results in the Herschel minor shown on the left of Fig. 21.

Up to symmetry, there are six splits of 2: 13, 14, 11, $3v$, $4v$ and 33. Splits 13 and 14 result in the Herschel minor shown on the right of Fig. 21. Split 11 results in $D_{3,n+1}$. Splits $3v$ and $4v$ result in the Herschel minor shown on the left of Fig. 21. Split 33 results in C_{n-1}.

There are two splits of 3. Split 22 results in $D_{4,n}$ and split 14 results in the Herschel minor shown on the right of Fig. 21. There are also two splits of 4. Split 22 results in $D_{4,n}$ (and is the same as split 22 of 3) and split 33 results in C_{n-1} (and is the same as split 33 of 2). Splits of vertices v along P either result in $D_{4,n+1}$ or the Herschel minor shown on the left of Fig. 19.

Consider $D_{m,n}$ with $m, n \geq 4$. Label vertices 1 and 2 as in $D_{m,n}$ for $m \leq 3$. All splits of 1 and 2 which resulted in Herschel minors for $D_{3,n}$ will again result in Herschel minors here. Split 11 of 2 now results in $D_{m,n+1}$ and split 22 of 1 now results in $D_{m+1,n}$. All splits of vertices along the path between the two vertices labeled 1 either result in $D_{m+1,n}$ or the Herschel minor shown on the right of Fig. 21. All splits of vertices v along the path P either result in $D_{m,n+1}$ or have the Herschel minor shown on the left of Fig. 19. □

4.2 Weakly 4-Connected Graphs

To move from 4-connected graphs to 3-connected graphs, we begin by adding degree three vertices into triangular faces. Lemma 6 and Theorem 4 show that if we begin with a Herschel-minor-free planar triangulation, then the result either contains a Herschel minor or is Hamiltonian. As in the previous section, the lemma states the small cases verified by computer and the theorem states the general result. Deleting degree three vertices from a weakly 4-connected triangulation results in K_4 or a 4-connected triangulation. Hence Theorem 4 proves Theorem 2 in the case when G is weakly 4-connected.

For any set T of triangles of a graph G, let G^T be the graph obtained from G by adding a new vertex v_t for each $t \in T$ and edges from v_t to all three vertices of t. We say *divide* t to indicate this process of adding v_t and say the end result is that t has been *divided*. In a 4-connected planar triangulation, all triangles are necessarily facial triangles.

Theorem 4 *Let G be a 4-connected Herschel-minor-free planar triangulation and let T be a set of triangles of G. Then either G^T contains a Herschel minor or G^T is Hamiltonian.*

For a 4-connected Herschel-minor-free planar triangulation G, let T^* denote the set of all triangles of G. The steps of our proof will be as follows:

1. Identify a set $\mathcal{T}$ of subsets of T^* such that G^T contains a Herschel minor for all $T \in \mathcal{T}$.
2. Determine $\mathcal{S}$, the set of all maximal subsets S of T^* such that $T \nsubseteq S$ for all $T \in \mathcal{T}$.
3. Verify that G^S is Hamiltonian for all $S \in \mathcal{S}$.

If we complete these three steps, then G satisfies the conclusion of the theorem. Each of these steps is easily performed by computer for small graphs, say graphs on at most 11 vertices. Sets of triangles which satisfy the conditions of Step 1 can be generated by computer from which we can determine $\mathcal{S}$ and then verify Step 3. Note there may be more than one possible set $\mathcal{T}$ which satisfies the conditions of Step 1 but it is enough to pick any one set and verify Steps 2 and 3 to complete the proof.

Table 1 Sets of triangles in A_4, B_3, C_3, $D_{3,4}$, and $D_{2,5}$ which satisfy the conditions of Step 1

Graph	
A_4	$\{ade\}, \{xcd\}, \{yce\}, \{bxv_1, cxv_1\}, \{bv_1v_2, cv_1v_2\}, \{bv_2v_3, cv_2v_3\}, \{bv_3v_4, cv_3v_4\}, \{bv_4y, cv_4y\}$
B_3	$\{abe\}, \{bef\}, \{acx\}, \{bcy\}, \{xde\}, \{ydf\}, \{cxv_1, dxv_1\}, \{cv_1v_2, dv_1v_2\}, \{cv_2v_3, dv_2v_3\}, \{cv_3y, dv_3y\}$
C_3	$\{acx\}, \{ayd\}, \{cxf\}, \{ydf\}, \{acd\}, \{cdf\}, \{bxv_1, cxv_1\}, \{bv_1v_2, cv_1v_2\}, \{bv_2v_3, cv_2v_3\}, \{bv_3y, cv_3y\}$
$D_{3,4}$	$\{xau_1, yau_1\}, \{xu_1u_2, yu_1u_2\}, \{xu_2u_3, yu_2u_3\}, \{xu_3b, yu_3b\}, \{axv_1, bxv_1\}, \{av_1v_2, bv_1v_2\}, \{av_2v_3, bv_2v_3\}, \{av_3v_4, bv_3v_4\}, \{av_4y, bv_4y\}$
$D_{2,5}$	$\{xau_1, yau_1\}, \{xu_2b, yu_2b\}, \{axv_1, bxv_1\}, \{av_1v_2, bv_1v_2\}, \{av_2v_3, bv_2v_3\}, \{av_3v_4, bv_3v_4\}, \{av_4v_5, bv_4v_5\}, \{av_5y, bv_5y\}$

Theorem 4 for graphs on at most 11 vertices and for the gamma graphs is entirely done by computer, so we state the result separately in the following lemma.

Lemma 6 *Let G be a 4-connected Herschel-minor-free planar triangulation with $|G| \le 11$ or $G = \Gamma_i$ for some $i \in \{1, \ldots, 11\}$. Let T be a set of triangles of G. Then either G^T contains a Herschel minor or G^T is Hamiltonian.*

In Table 1 we list sets of triangles for each of the graphs in the four families of 4-connected Herschel-minor-free planar triangulations on 11 vertices (except $D_{1,6}$). By computer, we verified that each of these sets satisfies the conditions of Step 1 for the set $\mathcal{T}$. Additionally we verified Steps 2 and 3 by computer for this set $\mathcal{T}$. These sets will be used to extend the result to larger graphs when we prove Theorem 4. For the families A_n, B_n, and C_n, denote by $v_1, \ldots, v_n$ the interior vertices along the path from x to y and label the remaining vertices of the graphs as in Fig. 22. For $D_{m,n}$, denote by $v_1, \ldots, v_n$ the interior vertices along the path from x to y, label a and b as in Fig. 23, and denote by $u_1, \ldots, u_m$ the interior vertices along the path from a to b. We break the family $D_{m,n}$ into three cases: $D_{m,n}$ for $m \ge 3$ and $n \ge 4$, $D_{2,n}$ for $n \ge 5$, and $D_{1,n}$ for $n \ge 6$. $D_{1,n}$ is handled in the proof without using a computer so we do not state a choice for $\mathcal{T}$ here.

Proof (Proof of Theorem 4) As in the previous lemma, let T^* denote the set of all triangles of G. We follow the same three steps as in Lemma 6.

To find choices for $\mathcal{T}$, we extend the sets for smaller graphs found by computer. Next we determine $\mathcal{S}$ and finally we verify Step 3.

Begin with the family A_n. Since A_4 is a minor of A_n for all $n \ge 4$, it is not hard to see that for a general A_n, the set $\mathcal{T} = \{\{ade\}, \{xcd\}, \{yce\}, \{bxv_1, cxv_1\}, \{bv_1v_2, cv_1v_2\}, \ldots, \{bv_{n-1}v_n, cv_{n-1}v_n\}, \{bv_ny, cv_ny\}\}$ satisfies the conditions of Step 1. Now each set S in $\mathcal{S}$ contains the set $\{axd, abx, aby, aye, cde\}$ and exactly one triangle from each of the pairs of triangles which share an edge along the xy-path. To see that G^S is Hamiltonian for all $S \in \mathcal{S}$, consider the picture on the left of Fig. 22. It shows a Hamilton cycle in A_n with certain loops added. There is a loop on the edge cd, for example, which indicates that if we divide triangle cde with a vertex u, then we can replace cd in the Hamilton cycle with the path cud to have a Hamilton cycle in the new

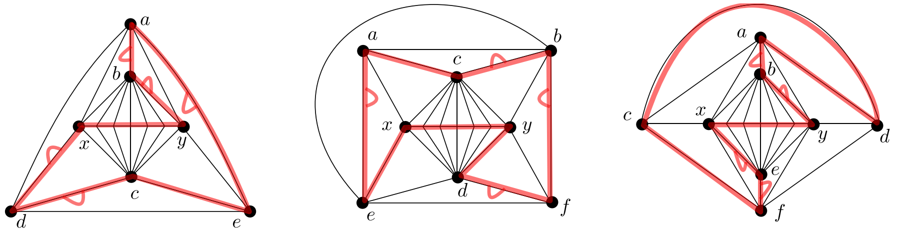


Fig. 22 Hamilton cycles in A_n, B_n, and C_n

graph. Loops on edges xd, ab, by, and ae indicate how to extend the Hamilton cycle if we divide triangles axd, abx, aby, and aye, respectively. For pairs of triangles along the xy-path, since at most one of these can be divided without creating a Herschel minor, the Hamilton cycle shown in the figure can be extended to either one of the divided triangles.

Next look at the family B_n. Since B_3 is a minor of B_n for all $n \geq 3$, it is not hard to see that for a general B_n, the set $\mathcal{T} = \{\{abe\}, \{bef\}, \{acx\}, \{bcy\}, \{xde\}, \{ydf\}, \{cxv_1, dxv_1\}, \{cv_1v_2, dv_1v_2\}, \ldots, \{cv_{n-1}v_n, dv_{n-1}v_n\}, \{cv_ny, dv_ny\}\}$ satisfies the conditions of Step 1. Each set S in $\mathcal{S}$ contains the set $\{abc, axe, byf, def\}$ and exactly one triangle from each of the pairs of triangles which share an edge along the xy-path. As with A_n, the picture in the middle of Fig. 22 demonstrates that G^S is Hamiltonian for all $S \in \mathcal{S}$. Loops on the edges bc, ae, bf, and df demonstrate how to extend the cycle into divided triangles abc, axe, byf, and def, respectively. Again we can also extend into one triangle from each of the pairs of triangles along the xy-path.

Now consider C_n. Since C_3 is a minor of C_n for all $n \geq 3$, it is not hard to see that for a general C_n, the set $\mathcal{T} = \{\{acx\}, \{ayd\}, \{cxf\}, \{ydf\}, \{acd\}, \{cdf\}, \{bxv_1, cxv_1\}, \{bv_1v_2, cv_1v_2\}, \ldots, \{bv_{n-1}v_n, cv_{n-1}v_n\}, \{bv_ny, cv_ny\}\}$ satisfies the conditions of Step 1. Each set S in $\mathcal{S}$ contains the set $\{abx, aby, xef, yef\}$ and exactly one triangle from each of the pairs of triangles which share an edge along the xy-path. As with A_n and B_n, the picture on the right of Fig. 22 demonstrates that G^S is Hamiltonian for all $S \in \mathcal{S}$. Loops on the edges ab, by, xe, and ef demonstrate how to extend the cycle into divided triangles abx, aby, xef, and yef, respectively. Again we can also extend into one triangle from each of the pairs of triangles along the xy-path.

We break the family $D_{m,n}$ into three cases: $D_{m,n}$ for $m \geq 3$ and $n \geq 4$, $D_{2,n}$ for $n \geq 5$, and $D_{1,n}$ for $n \geq 6$. Since $D_{3,4}$ is a minor of $D_{m,n}$ for all $m \geq 3, n \geq 4$, it is not hard to see that for a general $D_{m,n}$ (with $m \geq 3, n \geq 4$), the set $\mathcal{T} = \{\{xau_1, yau_1\}, \ldots, \{xu_mb, yu_mb\}, \{axv_1, bxv_1\}, \ldots, \{av_ny, bv_ny\}\}$ satisfies the conditions of Step 1. Since every triangle of the graph is contained in some $T \in \mathcal{T}$, each set S in $\mathcal{S}$ contains exactly one triangle from each of the pairs of triangles listed in $\mathcal{T}$.

The picture on the left of Fig. 23 demonstrates that G^S is Hamiltonian for all $S \in \mathcal{S}$. The Hamilton cycle shown includes the path $au_1u_2 \ldots u_mb$ so it can be extended into one triangle from each of the pairs of triangles whose shared edges are the edges of this path. Similarly, the Hamilton cycle also includes the path $xv_1v_2 \ldots v_ny$ so it can be extended in the same way here.

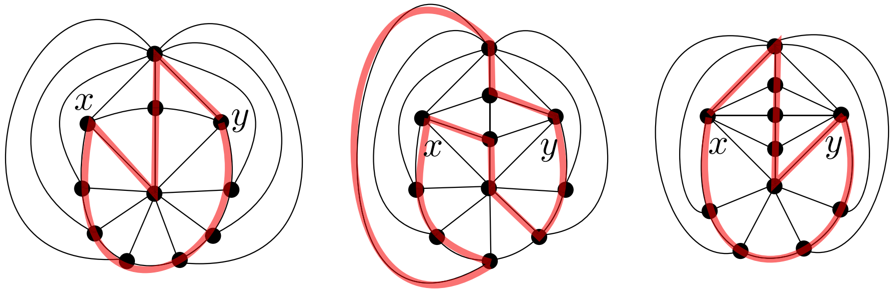


Fig. 23 Hamilton cycles in $D_{m,n}$ and $D_{2,n}$

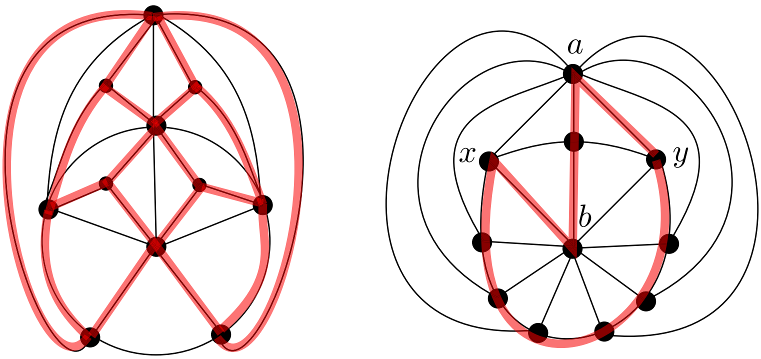


Fig. 24 A Herschel subgraph and a Hamilton cycle in $D_{1,n}$

Since $D_{2,5}$ is a minor of $D_{2,n}$ for all $n \geq 5$, it is not hard to see that for a general $D_{2,n}$, the set $\mathcal{T} = \{\{xau_1, yau_1\}, \{xu_2b, yu_2b\}, \{axv_1, bxv_1\}, \ldots, \{av_ny, bv_ny\}\}$ satisfies the conditions of Step 1. Each set S in $\mathcal{S}$ contains the set $\{xu_1u_2, yu_1u_2\}$ and one triangle from each of the pairs of triangles listed in $\mathcal{T}$. The picture on the right of Fig. 23 demonstrates that G^S is Hamiltonian for all $S \in \mathcal{S}$. Loops on the edges u_1y and xu_2 indicate how the Hamilton cycle can be extended if u_1u_2y and u_1u_2x are divided, respectively. Edges of the Hamilton cycle along the path $xv_1 \ldots v_ny$ can be extended into one triangle from the pairs of triangles listed in $\mathcal{T}$. The edge v_3v_4 is not included in the cycle so loops on bv_4 and av_3 indicate how the cycle can be extended into bv_3v_4 and av_3v_4, respectively. Finally edges au_1 and u_2b can be extended into either xau_1 or yau_2 or either xu_2b or yu_2b, respectively.

The graph $D_{1,n}$ is isomorphic to the double wheel. If we let $x = v_0$, $y = v_{n+1}$ and $u_1 = v_{n+2}$, then all triangles have the form av_iv_{i+1} or bv_iv_{i+1} (indices taken modulo $n+2$). Call a pair of triangles of the form av_iv_{i+1} and bv_iv_{i+1} a *rim pair*. The picture on the left in Fig. 24 shows that if we divide both triangles of two adjacent rim pairs in $D_{1,n}$, then the resulting graph has a Herschel minor (a Herschel subgraph in fact). Dividing both triangles of any two rim pairs in graphs $D_{1,n}$ where $n \geq 3$ will give this graph and hence the Herschel graph as a minor. Thus for $D_{1,n}$, $n \geq 3$, each set S in $\mathcal{S}$ will contain two triangles from one rim pair and one triangle from the remaining rim pairs. The picture on the right of Fig. 24 shows a Hamilton cycle that can be extended into every divided triangle of each set S in $\mathcal{S}$. □

4.3 Proof of Theorem 2

Proof (Proof of Theorem 2) Suppose the theorem is not true and let G be a Herschel-minor-free non-Hamiltonian planar triangulation on the fewest vertices. Then G must have connectivity 3 since all 4-connected planar graphs are Hamiltonian [10]. We consider two cases based on 3-separations of G.

First assume G contains a 3-separation (G_1, G_2) such that $|G_i| > 4$ for $i = 1, 2$. Let $V(G_1) \cap V(G_2) = \{x, y, z\}$. We may assume G_2 does not contain a rooted (Q, a, b, c) minor at $\{x, y, z\}$. Let $G'_1 = G/(G_2 - G_1)$. Then G'_1 has fewer vertices than G and is a 3-connected Herschel-minor-free planar triangulation so G' has a Hamilton cycle C. Let v be the new vertex of G'_1 which results after contracting edges of $G_2 - G_1$. Without loss of generality, C contains the edges xv and vy. Let $G'_2 = \text{si}(G_2, \{xy, yz, xz\})$. Then G'_2 is a Herschel-minor-free planar triangulation and is rooted (Q, a, b, c)-free so by Corollary 1, $G'_2 - z$ contains a Hamilton path P from x to y. Form a new cycle C' from C by replacing xvy with P. Now C' is a Hamilton cycle in G, a contradiction.

Now for every 3-separation (G_1, G_2) of G, either $|G_1| = 4$ or $|G_2| = 4$. Then $G = H^T$ for some 4-connected Herschel-minor-free planar triangulation H and some set T of facial triangles of H. Then by Theorem 4, G is Hamiltonian, a contradiction. □

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