A characterization of graphs with no octahedron minor

Guoli Ding
Mathematics Department, Louisiana State University, Baton Rouge, LA 70803

Submitted: June 3, 2010
Revised: August 12, 2012

Abstract

It is proved that a graph does not contain an octahedron minor if and only if it is constructed from $\{K_1, K_2, K_3, K_4\} \cup \{C^2_{2n-1} : n \geq 3\}$ and five other internally 4-connected graphs by 0-, 1-, 2-, and 3-sums.

1 Introduction

All graphs considered in this paper are simple. For any two graphs G and H, G is called *H-free* if no minor of G is isomorphic to H. In graph theory, many important problems are about H-free graphs. For instance, Hadwiger's Conjecture [3], made in 1943, states that every K_n-free graph is $n-1$ colorable. This conjecture is still open for $n \geq 7$ and the main difficulty for attacking the conjecture is that we don't have enough structural information on K_n-free graphs. A closely related problem is a conjecture of Jorgensen [6], which says that every 6-connected K_6-free graph is obtained from a planar graph by adding a new vertex and arbitrary edges between this vertex and the planar graph. This conjecture does provide structural information on K_6-free graphs and it immediately implies Hadwiger's Conjecture for the case $n = 6$. However, this conjecture is also open. Recently, Kawarabayashi et. al. [7] proved Jorgensen's conjecture for sufficiently large graphs. Since Jorgensen's conjecture is already hard, to completely characterize K_6-free graphs is an even harder problem.

There are only a handful of graphs H for which H-free graphs have been completely characterized. At this point, for each 3-connected graph H with at most eleven edges, H-free graphs are characterized [2]. These characterizations include classical results on K_4-, $K_{3,3}$-, Prism-, and K_5-free graphs. For 3-connected graphs with twelve or more edges, there are only two results: Robertson characterized V_8-free graphs (see Lemma 3.2) and Maharry characterized cube-free graphs [8]. In this paper, we consider the octahedron, a graph also known as $K_{2,2,2}$, which is shown in Figure 1.1. The octahedron has twelve edges, like V_8 and the cube, and it is only three edges away from K_6. Results in this paper will be used in a subsequent paper to characterize graphs without $K_{1,1,2,2}$ (octahedron plus an edge) as a minor, which will be the first result on excluding a thirteen-edge graph. Our hope is to go one step further by charactering $K_6\backslash e$-free graphs, which could eventually lead to a characterization of K_6-free graphs.

According to a classical result of Halin and Jung [4], no 4-connected planar graph is octahedron-free (or oct-free for short). Maharry [9] sharpened this fact by showing that a 4-connected graph is oct-free if and only if the graph is isomorphic to C^2_{2n-1} for some integer $n \geq 3$, where C^2_{2n-1} is obtained from an odd cycle

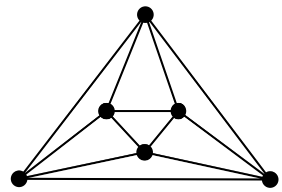

Figure 1.1: The Octahedron

C_{2n-1} by joining every pair of vertices of distance two in the cycle. Note that $C_5^2 = K_5$. In this paper we strengthen Maharry's result by completely characterizing all oct-free graphs.

For $k = 0, 1, 2, 3$, we define *k-sum* as follows. Let G_1, G_2 be disjoint graphs with more than k vertices. The *0-sum* of G_1, G_2 is their disjoint union; a *1-sum* of G_1, G_2 is obtained by identifying one vertex of G_1 with one vertex of G_2; a *2-sum* of G_1, G_2 is obtained by identifying an edge of G_1 with an edge of G_2, where the common edge might be deleted after the identification; a *3-sum* of G_1, G_2 is obtained by identifying a triangle of G_1 with a triangle of G_2, where some of the three common edges might be deleted after the identification. In this paper we denote the Petersen graph by P_{10}.

Theorem 1.1 *A graph is oct-free if and only if it is constructed by 0-, 1-, 2-, and 3-sums starting from graphs in* $\{K_1, K_2, K_3, K_4\} \cup \{C_{2n-1}^2 : n \geq 3\} \cup \{L_4', L_5, L_5', L_5'', P_{10}\}$.

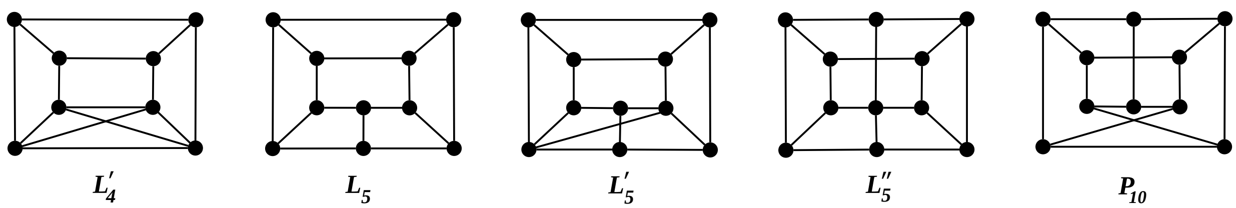

Figure 1.2: Five basic oct-free graphs

Since 4-connected graphs are not k-sums ($k \leq 3$) of any two graphs, Maharry's result follows from Theorem 1.1 immediately. We remark that our proof does not assume Maharry's result.

Let $k \geq 0$ be an integer. A *k-separation* of a graph G is a pair (G_1, G_2) of induced subgraphs of G such that $E(G_1) \cup E(G_2) = E(G)$, $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) - V(G_2) \neq \emptyset$, $V(G_2) - V(G_1) \neq \emptyset$, and $|V(G_1) \cap V(G_2)| = k$. If G is 3-connected with five or more vertices, then we call G *internally 4-connected* if for every 3-separation (G_1, G_2) of G, at least one of G_1, G_2 is isomorphic to $K_{1,3}$. As pointed out by a referee, a byproduct of our proof is Theorem 1.2, an unpublished result of Thomas, which characterizes internally 4-connected oct-free graphs.

For any integer $n \geq 3$, let V_{2n} be the cubic graph that consists of a cycle $v_1 v_2 ... v_{2n} v_1$ and a matching $v_1 v_{n+1}, v_2 v_{n+2}, ..., v_n v_{2n}$, which are called *rungs* of V_{2n}. Note that $V_6 = K_{3,3}$. Let $S_{30} = (X, Y, E)$ be the unique bipartite graph with $|X| = 5$ and $|Y| = 10$ such that every $y \in Y$ has degree three and no two vertices in Y have the same set of neighbors. Let Γ denote the graph in Figure 1.3.

Theorem 1.2 *An internally 4-connected graph G is oct-free if and only if either $G = V_{2n}$ or C_{2n-1}^2, for some $n \geq 3$, or G is a minor of P_{10}, L_5'', Γ, or S_{30}.*

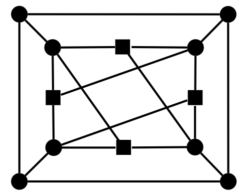

Figure 1.3: Graph Γ

As we will see in the next section that, if G is a k-sum ($k = 0, 1, 2, 3$) of two oct-free graphs, then G is also oct-free. Therefore, to prove Theorem 1.1, we only need to determine all oct-free graphs that are not k-sums ($k = 0, 1, 2, 3$) of any two oct-free graphs. Graphs with this property will be called *prime* graphs. Using this language, Theorem 1.1 can be restated as: all prime graphs belong to the set $\{K_1, K_2, K_3, K_4\} \cup \{C^2_{2n-1}\colon n \geq 3\} \cup \{L'_4, L_5, L'_5, L''_5, P_{10}\}$. In fact, it is not difficult to verify that these are precisely all prime graphs. In other words, this set is the unique minimal set for which Theorem 1.1 holds.

To prove Theorem 1.1, we first establish in Section 2 that all prime graphs are internally 4-connected, except for K_1, K_2, K_3, K_4. Thus, from that point on, we will only need to consider internally 4-connected graphs. It should be pointed out that not all internally 4-connected oct-free graphs are prime graphs. For instance, V_8 is internally 4-connected and oct-free, but it is a 3-sum of K_5 and three copies of K_4.

The remainder of the proof of Theorem 1.1 is divided into three main parts. In Section 3 we prove that K_n ($n = 1, 2, 3, 4, 5$), C^2_7, and L_5 are the only V_8-free prime graphs; in Section 5 we prove that graphs C^2_{2n-1} ($n \geq 6$) are the only prime graphs having a V_{10} minor; in Section 6 and Section 7 we prove that C^2_9, L'_4, L'_5, L''_5, and P_{10} are the only V_{10}-free prime graphs having a V_8 minor. The main tool we use in dealing with the last two parts is a splitter theorem of Johnson and Thomas [5], which is stated in Section 4. Using this result, we are able to generate all internally 4-connected oct-free graphs. Due to the intensity of the case analysis, some cases in the last part are handled by a computer (a detailed explanation of the computation can be found in [1]). Finally, we prove Theorem 1.1 and Theorem 1.2 in section 8.

In this paper, unless otherwise specified, "G contains H" means that G contains a minor isomorphic to H. We will also say that "G is H" if G is isomorphic to H. For any $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced on S. A *non-edge* of G is an edge of the complement of G. If $e_1, ..., e_k$ are non-edges of G, then $G + e_1 + ... + e_k$ is the graph obtained from G by adding edges $e_1, ..., e_k$.

2 k-sums

The goal of this section is to show that oct-free graphs are precisely 0-, 1-, 2-, 3-sums of K_1, K_2, K_3, K_4, and internally 4-connected oct-free graphs. This result will allow us to focus on internally 4-connected graphs. Recall that a graph is 4-*connected* if it has more than four vertices and has no k-separations for any $k \leq 3$. In fact, as shown in the next theorem, the result we are aiming for holds not only for the octahedron, but also for a general 4-connected graph.

Theorem 2.1 *Let H be a 4-connected graph. Then H-free graphs are precisely 0-, 1-, 2-, 3-sums of graphs in $\mathcal{H} = \{K_1, K_2, K_3, K_4\} \cup \{$internally 4-connected H-free graphs$\}$.*

The essence of this result is not new, but the author was not able to find a reference that contains this result explicitly. For the completeness of this paper, we include a proof, which is divided into two lemmas.

Let (G_1, G_2) be a k-separation ($k \le 3$) of a graph G. For $i = 1, 2$, let G_i^+ be obtained from G_i by adding all edges between any two non-adjacent vertices in $V(G_1) \cap V(G_2)$. Let G_1', G_2' be disjoint graphs that are isomorphic to G_1^+, G_2^+, respectively. Then it is straightforward to verify that G is (isomorphic to) a k-sum of G_1' and G_2'. To simplify our notation, we will simply say that G is a k-sum of G_1^+ and G_2^+.

Lemma 2.2 *Let (G_1, G_2) be a k-separation ($k \le 3$) of G. Then the following hold.*
(i) G is a k-sum of G_1^+ and G_2^+;
(ii) if $k \le 2$ and G is k-connected, then both G_1^+ and G_2^+ are minors of G;
(iii) if G is 3-connected and neither G_1 nor G_2 is $K_{1,3}$, then both G_1^+ and G_2^+ are minors of G.

Proof. Conclusion (i) follows from the above discussion. Conclusion (ii) is clear if $k = 0$ or 1. In fact, in these two cases, $G_1^+ = G_1$ and $G_2^+ = G_2$, and they are induced subgraphs of G. To settle the case for $k = 2$, choose $v_i \in V(G_i)\backslash V(G_j)$ ($\{i, j\} = \{1, 2\}$). Since G is 2-connected, it has a cycle C containing both v_1, v_2. Let P_i ($i = 1, 2$) be the path obtained by restricting C to G_i. Then $G_1 \cup P_2$ and $G_2 \cup P_1$ contain G_1^+ and G_2^+, respectively. Finally, we prove (iii). By symmetry, we only need to consider G_1^+. Since G is 3-connected, G_2 must be connected. Moreover, if G_2 is a tree, then all its leaves are in $V(G_1) \cap V(G_2)$ and all vertices in $V(G_2)\backslash V(G_1)$ have degree three or more. But these imply $G_2 = K_{1,3}$, which is impossible, so G_2 must have a cycle C. Let $v \in V(G_1)\backslash V(G_2)$ and let P_1, P_2, P_3 be three vertex-disjoint (except at v) paths of G from v to C. Then it is easy to see that C can be contracted (along paths $P_1 \cap G_2$, $P_2 \cap G_2$, $P_3 \cap G_2$) to $V(G_1) \cap V(G_2)$ and thus G_1^+ is a minor of G. ∎

Lemma 2.3 *Let $0 \le k \le 3$ and let H be a graph with no k'-separations for any $k' \le k$. If G is a k-sum of two H-free graphs, then G is also H-free.*

Proof. Since a k-sum always leads to a k-separation in the resulting graph, we may assume that G has a k-separation (G_1, G_2) such that both G_1^+ and G_2^+ are H-free. Suppose that, for a contradiction, H is a minor of G. Since vertices of H are obtained from contracting connected subgraphs of G, there must exist a set $\{W_v : v \in V(H)\}$ of pairwise disjoint subsets of $V(G)$ and a set $\{f_e : e \in E(H)\}$ of edges of G such that

(i) $G[W_v]$ is connected, for every $v \in V(H)$, and
(ii) if $e = uv \in E(H)$, then f_e is between W_u and W_v.

That is, H is modeled by $\{W_v\}$ and $\{f_e\}$. Let $V_0 = V(G_1) \cap V(G_2)$ and let Z be the set of vertices v of H with $W_v \cap V_0 \neq \emptyset$. Then $|Z| \le |V_0| = k$. We first observe that there do not exist vertices $v_1, v_2 \in V(H)$ with $W_{v_1} \subseteq V(G_1)\backslash V_0$ and $W_{v_2} \subseteq V(G_2)\backslash V_0$. Suppose otherwise. Since V_0 separates W_{v_1} and W_{v_2} in G, Z separates v_1 and v_2 in H, which implies that H has a k'-separation with $k' = |Z| \le k$, contradicting our assumption on H. Therefore, by symmetry, we may assume that $W_v \subseteq V(G_1)$, for all $v \in V(H)\backslash Z$. For each $v \in V(H)$, let $W_v' = W_v \cap V(G_1^+)$. Since $G_1^+[V_0]$ is a clique, each $G_1^+[W_v']$ remains to be connected. Then it is straightforward to verify that an H-minor can be obtained from G_1^+ by contracting each $G_1^+[W_v']$, which is a contradiction and thus G has to be H-free. ∎

Remark 2.4 In the above proof we did not use the assumption $k \le 3$. What we used was that a k-sum of two graphs is obtained by identifying a clique of one graph with a clique of the other and then deleting some common edges. Since this definition of k-sum (which is indeed the one used in the literature) is valid for all k, the above proof in fact shows that Lemma 2.3 holds for all $k \ge 0$.

Proof of Theorem 2.1. Let G be H-free. If G is not in $\mathcal{H}$, then G has a k-separation (G_1, G_2) for which either $k < 3$ or $k = 3$ and such that neither G_1 nor G_2 is $K_{1,3}$. By choosing the smallest such k we deduced that G is k-connected. Then, by Lemma 2.2, G is a k-sum of G_1^+, G_2^+, which are minors of G. It follows that G is a k-sum of H-free graphs. Clearly, repeating this argument shows that all H-free graph is are 0-, 1-, 2-, 3-sums of graphs in $\mathcal{H}$. Conversely, Lemma 2.3 immediately implies that 0-, 1-, 2-, 3-sums of graphs in $\mathcal{H}$ are H-free. Thus the theorem is prove. ∎

The following similar result can be proved similarly using the last two lemmas. The proof is omitted.

Theorem 2.5 *If H is k-connected $(k = 1, 2, 3)$, then H-free graphs are precisely 0-, ..., $(k-1)$-sums of $K_1, ..., K_k$ and k-connected H-free graphs.*

3 V_8-free graphs

In proving Theorem 1.1, as explained in the introduction, oct-free graphs are divided into three classes, depending on if they contain V_8 or V_{10}. We first consider V_8-free graphs. The goal of this section is to establish the following.

Lemma 3.1 *All V_8-free prime graphs belong to $\{K_1, K_2, K_3, K_4, K_5, C_7^2, L_5\}$.*

Our proof is based on an unpublished result of Robertson (see 1.4 of [5]) that characterizes V_8-free graphs. A *double-wheel* is obtained from a cycle by adding two adjacent vertices and making both of them adjacent to all vertices on the cycle. An *alternating double-wheel* is obtained from a cycle $v_1v_2...v_{2n}v_1$ by adding two adjacent vertices u_1, u_2 and adding edges u_iv_j for all i, j of the same parity.

Lemma 3.2 *Every internally 4-connected V_8-free graph belongs to one of the following five families: planar graphs, graphs on seven or fewer vertices, double-wheels and alternating double-wheels, graphs with four vertices meeting all edges, and the line graph of $K_{3,3}$.*

We will need three results for analyzing graphs listed in Lemma 3.2. The first is Theorem 9 of [10].

Lemma 3.3 *Let G be a 3-connected oct-free planar graph such that for any 3-separation (G_1, G_2) of G, at least one of G_1, G_2 has only four vertices. Then either $G = L_5$ or G is a subgraph of Q.*

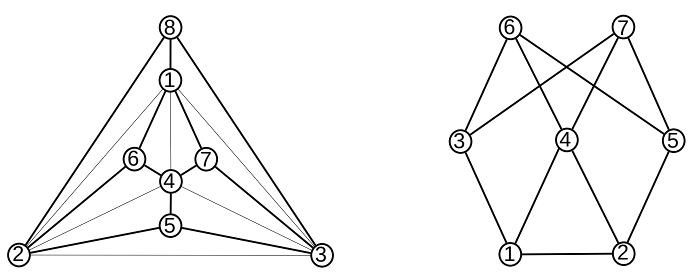

Figure 3.1: Graphs Q and $K_{3,3}^{\nabla}$

The next is a characterization of $K_{3,3}^{\nabla}$-free graphs [2].

Lemma 3.4 *Every 3-connected $K_{3,3}^{\nabla}$-free graph belongs to one of the following three families: planar graphs, graphs with six or fewer vertices, graphs with three vertices meeting all edges.*

The following is a simple observation that follows immediately from the definition of internally 4-connected graphs. It is listed here because we will use it several times.

Lemma 3.5 *If v is a cubic vertex of an internally 4-connected graph, then v is not in a triangle.*

Now we begin to analyze graphs listed in Lemma 3.2. Let L_4 denote the cube.

Lemma 3.6 *If G is an internally 4-connected oct-free planar graph, then G is L_4 or L_5.*

Proof. Note that G satisfies all assumptions in Lemma 3.3. Thus $G = L_5$ or G is a subgraph of Q. Clearly, we only need to consider the second case. Since G has at least five vertices and Q has only four non-cubic vertices, G must contain a cubic vertex of Q. By symmetry, let 8 be a vertex of G. Then 1, 2, 3 are also vertices of G. From Lemma 3.5 we deduce that 12, 13, and 23 are not edges of G. Since vertex 1 has degree at least three in G, at least one of 6 and 7 is in G, which implies that 14 is not an edge of G. It follows that 1 is a cubic vertex of G and both 6 and 7 are in G. Now it is easy to see that G consists of the heavy edges, which means $G = L_4$. ∎

Lemma 3.7 *(i) The only internally 4-connected graph on five vertices is K_5;*
(ii) The only internally 4-connected oct-free graph on six vertices is $K_{3,3}$;
(iii) The only internally 4-connected oct-free graph on seven vertices is C_7^2.

Proof. Conclusion (i) follows from Lemma 3.5 immediately. Next, let G be an internally 4-connected oct-free graph on six vertices. If the minimum degree of G is at least four, then the complement of G is a matching, which implies that G contains the octahedron since the complement of the octahedron is a perfect matching. Thus G has a cubic vertex v. By Lemma 3.5, the three neighbors of v are pairwise non-adjacent, which implies that they are also cubic and not in any triangle. Consequently, G is $K_{3,3}$ and so (ii) is proved.

It remains to consider the case that G has seven vertices. By Lemma 3.6 we may assume that G is nonplanar. Then we deduce from Lemma 3.4 that $K_{3,3}^{\nabla}$ is a spanning subgraph of G. Since G is internally 4-connected, $\{3, 4, 5\}$ does not separate $\{1, 2\}$ from $\{6, 7\}$ in G, so G must have an edge from $\{1, 2\}$ to $\{6, 7\}$. By symmetry, we assume that $H = K_{3,3}^{\nabla} + 16$ is a subgraph of G. By Lemma 3.5, vertex 2 is not cubic in G, so 2 is adjacent to 3, 6, or 7. Similarly, 3 is adjacent to 2, 4, or 5. Note that $H + 23$, $H + 26 + 34$, $H + 26 + 35$, and $H + 27 + 34$ contain the octahedron (by contracting 57, 57, 47, and 56 respectively). Thus $H + 27 + 35 = C_7^2$ is a subgraph of G. Now there is only one way (up to isomorphism) to add an edge to C_7^2 and the new graph contains $H + 23$ and thus also the octahedron. Thus $G = C_7^2$, as required. ∎

For $t = 4, 5$, let $K_{4,t}^-$ be obtained from $K_{4,t}$ by deleting a matching of size $t-1$. Note that $K_{4,4}^-$ is exactly the alternating double wheel on eight vertices.

Lemma 3.8 *Let G be internally 4-connected, V_8-free, and oct-free. Then G belongs to $\{K_5, K_{3,3}, C_7^2, L_4, L_5, K_{4,4}^-, K_{4,5}^-\}$.*

Proof. By Lemma 3.2 we need to consider five cases. If G is planar or G has at most seven vertices, then the result follows from Lemma 3.6 and Lemma 3.7. If G is a double-wheel or an alternating double-wheel with $|V(G)| \geq 8$, then either $G = K_{4,4}^-$ or G contains the octahedron. Note that the line graph of $K_{3,3}$ contains the octahedron. Thus we only need to consider the case when G has a set X of four vertices that meet all edges of G. Let $Y = V(G) - X$ and let Y_3, Y_4 consist of vertices of Y of degree 3 and 4, respectively. Since G is internally 4-connected, no two vertices in Y_3 have the same neighbors. Observe that deleting two non-incident edges from $K_{4,4}$ results in a graph that contains the octahedron. Thus we deduce from $|Y| \geq 4$ that $|Y_4| \leq 1$. Now, by Lemma 3.5, $G = L_4$ if $|Y_4| = 0$, and $G = K_{4,4}^-$ or $K_{4,5}^-$ if $|Y_4| = 1$. ■

Proof of Lemma 3.1. Let G be a V_8-free prime graph. If G is not internally 4-connected, by Theorem 2.1, G is K_1, K_2, K_3, or K_4. Assume G is internally 4-connected. Then G is one of the graphs listed in Lemma 3.8. Note that $K_{3,3}$ is a 3-sum of three copies of K_4, L_4 is a 3-sum of five copies of K_4 (see Q in Figure 3.1), and $K_{4,t}^-$ $(t = 4, 5)$ is a 3-sum of K_5 and $t - 1$ copies of K_4. Thus $G \in \{K_5, C_7^2, L_5\}$, as required. ■

4 Splitter theorems

Before we start analyzing graphs that contain V_8, we present two splitter theorems, which are the main tools that we will use. The first is a result of Seymour [11] (see 1.2 of [5]). A graph H is a *proper minor* of a graph G if H is a minor of G and $|E(H)| < |E(G)|$.

Lemma 4.1 *Let H be a simple 3-connected proper minor of a 3-connected simple graph G, where H is not a wheel. Then G has a simple 3-connected minor J such that $H = J\backslash e$ or J/e, for some edge e of J.*

The following is our first application of this theorem in this paper. As always, the reverse of a contraction is called a *split* of a vertex. If a vertex v of H is split, note that the two new vertices do not have a common neighbor and they both have degree at least three, which imply that v has degree at least four in H.

Lemma 4.2 *Let G be a 3-connected oct-free graph. If $n \geq 4$ is an integer such that C_{2n-1}^2 is a minor of G while V_{2n} is not, then $G = C_{2n-1}^2$.*

Proof. Suppose $G \neq C_{2n-1}^2$. By Lemma 4.1, G has a minor J which is obtained from C_{2n-1}^2 by adding an edge or splitting a vertex. If $J = C_{2n-1}^2 + e$, then J contains $C_7^2 + e$, which contains the octahedron (by Lemma 3.7(iii)), contradicting our assumption. Thus J is obtained from C_{2n-1}^2 by splitting a vertex. Since all vertices of C_{2n-1}^2 are symmetric, we may focus on any one particular vertex v. As illustrated in Figure 4.1, there are three possible splits at v. Note that the first two contain V_{2n} (illustrated with heavy edges) and the last one contain the octahedron (by contracting the heavy paths labeled 1, 2, 3, and 5, respectively). Again, both cases contradict the assumptions of the lemma and thus we must have $G = C_{2n-1}^2$. ■

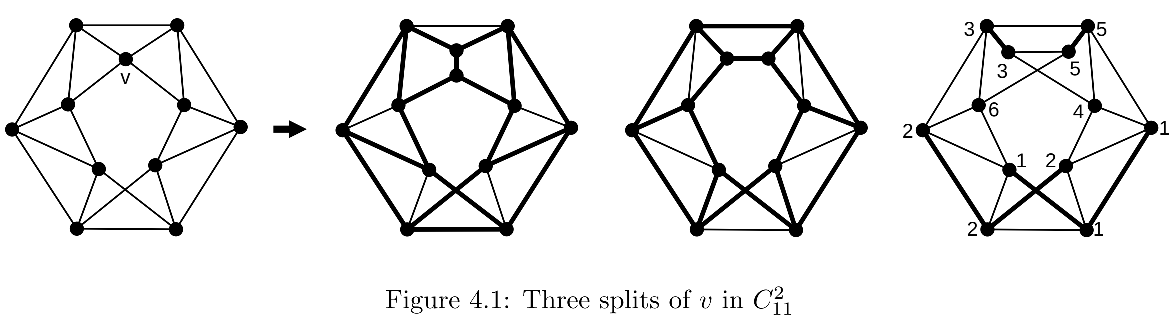

Figure 4.1: Three splits of v in C_{11}^2

We need some definitions for the next splitter theorem. Let $H = (V, E)$ be a graph. If $v \in V$ has degree three and two of its neighbors are joined by an edge $e \in E$, then we call e a *violating edge* and (v, e) a *violating pair* of H. Notice that if H is internally 4-connected yet $H + e$ is not, then (v, e) is a violating pair for some cubic vertex v of H. For any $v \in V$ and $e \in E$, graph $H + (v, e)$ is obtained by first subdividing e with a new vertex u and then adding an edge uv. If $v_1, ..., v_k \in V$, then $H + (v_1, ..., v_k)$ is obtained by adding a new vertex u and making it adjacent to all v_i.

Suppose H is internally 4-connected. An *addition extension* of H is either $H + e$, where e is a non-edge and $H + e$ is internally 4-connected, or $H + e_1 + e_2$, where e_1, e_2 are non-edges, $H + e_1$ has one or two violating pairs, and e_2 is adjacent to every v for which (v, e_1) is a violating pair of $H + e_1$. Note that e_1 and e_2 are not symmetric under our definition, which means that we need to consider all ordered pairs (e_1, e_2) when generating all addition extensions.

If $v_1v_2v_3v_4v_1$ is a cycle in H, where v_1, v_2, v_3, v_4 all have degree three, then $H + (v_1, v_2, v_3, v_4)$ is a *quadrangular extension* of H. If $v_1v_2v_3v_4v_5v_1$ is a cycle of H such that v_2 and v_5 have degree three and that v_1 is adjacent to neither v_3 nor v_4, then $H + (v_1, v_3v_4)$ is a *pentagonal extension* of H. If v_1, v_2, v_3 are distinct vertices of H such that no vertex of H of degree three has neighbors v_1, v_2, v_3, yet every pair from $\{v_1, v_2, v_3\}$ has a common neighbor of degree three, then $H + (v_1, v_2, v_3)$ is a *hexagonal extension* of H. These three operations are illustrated in the figure below (dashed lines indicate non-edges).

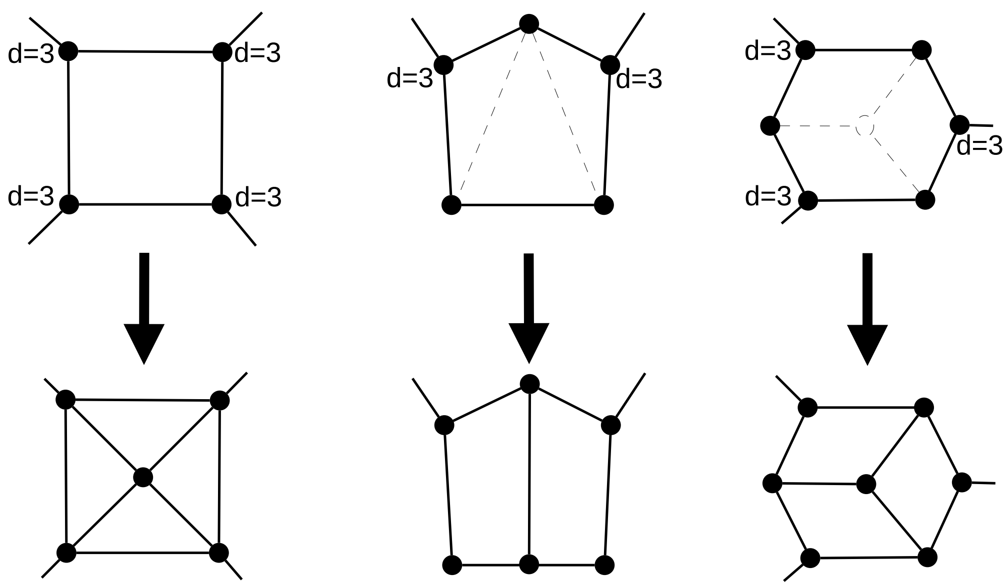

Figure 4.2: quadrangular, pentagonal, and hexagonal extensions

Lemma 4.3 *If G is a quadrangular extension of an internally 4-connected graph H then G contains the octahedron.*

Proof. Let C be the 4-cycle over which the extension takes place. Since H is internally 4-connected and all vertices on C are cubic, $H - V(C)$ must be connected. Clearly, $G/E(H - V(C))$ is the octahedron. ■

We will need the main result (2.2) from [5]. The original theorem is very long, which covers many cases that do not concern us. In the following we state a restricted version of the theorem for graphs containing V_8 but not containing the octahedron. We also point out that our definition of addition extension is less restrictive than that of [5]. In fact, every addition extension in the sense of [5] contains one of our addition extensions as a spanning subgraph. Thus our modification does not affect the validity of the theorem.

Lemma 4.4 *Let H and G be internally 4-connected oct-free graphs, such that V_8 is a minor of H, H is a proper minor of G, and G is not V_{2n} for any n. Moreover, if $H = V_{2n}$ for some n, then C^2_{2n+1} is not a minor of G. Then G has a minor J such that*
(i) J is an addition extension of H,
(ii) J is a pentagonal or hexagonal extension of H, or
(iii) J is a split of H.

5 Extensions of large V_{2n}

In this section we consider oct-free graphs that contain V_{10}. Our main result is the following.

Lemma 5.1 *If G is a prime graph containing V_{10}, then $G \in \{C^2_{2n-1} : n \geq 6\}$.*

We divide the proof into a few lemmas. Let the vertices of V_{2n} be $v_1, v_2, ..., v_{2n}$, ordered as in the definition of V_{2n} given in the introduction.

Lemma 5.2 *All the following graphs contain the octahedron.*
(i) $V_{10} + v_1v_4$;
(ii) $V_{10} + v_1v_5 + v_6v_{10}$;
(iii) $V_8 + v_1v_3 + v_2v_5$;
(iv) $V_8 + v_1v_3 + v_2v_4$;
(v) $V_8 + v_1v_3 + v_5v_7$.

Proof. An octahedron minor can be obtained by contracting $\{v_2v_3,\ v_5v_{10},\ v_6v_7,\ v_8v_9\}$, $\{v_1v_2,\ v_3v_4,\ v_7v_8,\ v_9v_{10}\}$, $\{v_4v_8, v_6v_7\}$, $\{v_5v_6, v_7v_8\}$, and $\{v_2v_6, v_4v_8\}$, respectively. ∎

Lemma 5.3 *Let G be an internally 4-connected oct-free graph that contains V_{2n} for some $n \geq 5$. If G does not contain C^2_{2n+1}, then either $G = V_{2n}$ or $n = 5$ with G containing V^+_{10}.*

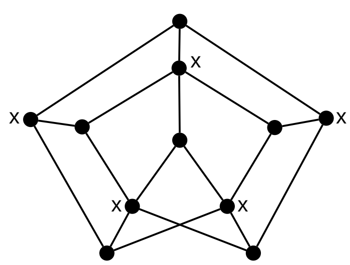

Figure 5.1: V^+_{10} – the unique hexagonal extension of V_{10}

Proof. Suppose $G \neq V_{2n}$. Then G has a minor J as described in Lemma 4.4. We need to show that $n = 5$ and $J = V^+_{10}$. We first prove that J is not an addition extension. Suppose J is obtained from V_{2n} by adding one edge e_1 or two edges e_1, e_2. Symmetry allows us to assume that e_1 is incident with v_1. By Lemma 5.2(i), we may also assume that the other end of e_1 is v_3 or v_n. If $e_1 = v_1v_3$, then (v_2, e_1) is the unique violating pair of $H + e_1$ and so a second edge e_2 must be added at v_2. However, by Lemma 5.2(iii-iv), this is impossible and thus $e_1 = v_1v_n$. Since (v_{n+1}, e_1) and (v_{2n}, e_1) are the only two violating pairs of $H + e_1$, we must have $e_2 = v_{n+1}v_{2n}$, which, by Lemma 5.2(ii), is again impossible. So J is not an addition extension.

Since V_{2n} has no vertices of degree four or more, J is not a split of V_{2n}. Since V_{2n} has no 5-cycles, J is not a pentagonal extension either. As a result, J has to be a hexagonal extension. There are two types of 6-cycles in V_{2n}: those that are formed by three consecutive rungs $v_iv_{i+1}v_{i+2}v_{n+i+2}v_{n+i+1}v_{n+i}$ and those of the form $v_iv_{i+1}v_{i+2}v_{i+3}v_{i+4}v_{i+5}$, which only occurs when $n = 5$. In a 6-cycle of the first type, say $i = 1$, both triples $\{v_1, v_3, v_{n+2}\}$ and $\{v_2, v_{n+3}, v_{n+1}\}$ fail to satisfy the the requirements in the definition of hexagonal extension (v_2 has neighbors $\{v_1, v_3, v_{n+2}\}$ and v_{n+2} has neighbors $\{v_2, v_{n+3}, v_{n+1}\}$), so J is not a hexagonal extension over such a cycle. Over a 6-cycle of the second type, there is only one possible extension (up to isomorphism), which is V_{10}^+. Therefore, $n = 5$ and G contains V_{10}^+, as required. ∎

Recall that $S_{30} = (X, Y, E)$ is the unique bipartite graph with $|X| = 5$ and $|Y| = 10$ such that every $y \in Y$ has degree three and no two vertices in Y have the same set of neighbors. Note that V_{10}^+ is an induced subgraph of S_{30}, where X consists of the five vertices that are labeled with x in Figure 5.1. Let $\mathcal{S}$ be the set of (exactly seven) induced subgraphs of S_{30} that contain V_{10}^+ as an induced subgraph. Observe that each member of $\mathcal{S}$ is obtained from V_{10} by repeatedly adding cubic vertices such that their neighbors are different from that of any other cubic vertices. Consequently, all graphs in $\mathcal{S}$ are internally 4-connected.

Lemma 5.4 *If G an internally 4-connected oct-free graph that contains V_{10}^+ then G belongs to $\mathcal{S}$.*

Proof. For the convenience of doing induction we prove the following modified statement:

(*) if G contains $H \in \mathcal{S}$ then $G \in \mathcal{S}$.

The lemma follows from (*) immediately with $H = V_{10}^+$. Suppose (*) does not hold. We choose the largest $H \in \mathcal{S}$ such that (*) is false. Then G has a minor J as described in Lemma 4.4. We prove that $J \in \mathcal{S}$, which contradicts the maximality of H and thus (*) will be proved.

Since H is bipartite, it has no 5-cycles and thus J is not a pentagonal extension.

Let X, Z be the two color classes of H, and let $Z_1 \subseteq Z_2 \subseteq Z$ be such that the subgraphs H_1 and H_2 of H induced on $X \cup Z_1$ and $X \cup Z_2$ are V_{10} and V_{10}^+, respectively. Moreover, let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Z_1 = \{z_1, z_2, z_3, z_4, z_5\}$ such that $x_1z_1x_2z_2...x_5z_5x_1$ is the special cycle of V_{10}. Observe that

(†) the neighborhood of each $z \in Z - Z_1$ is one of the five consecutive sets $\{x_1, x_2, x_3\}$, $\{x_2, x_3, x_4\}$, $\{x_3, x_4, x_5\}$, $\{x_4, x_5, x_1\}$, $\{x_5, x_1, x_2\}$.

Let us consider edge additions. We first claim that no addition extension of H can use an edge incident with a vertex in Z_1. Notice that the bipartite complement of V_{10} is a 10-cycle. Thus all non-edges of H_1 between X and Z_1 are symmetric. Therefore, by Lemma 5.2(i), adding any edge between Z_1 and $X \cup (Z - Z_1)$ will create an octahedron minor. Now we consider adding z_1z_2 or z_1z_3, which are the only two remaining cases (up to isomorphism). Suppose z_1z_2 is added. If x_2 is adjacent to a vertex in $Z - Z_1$, then an octahedron minor can be obtained by Lemma 5.2(iii-iv). If x_2 is not adjacent to any vertex in $Z - Z_1$, then, by (†), x_4 and x_5 have a common neighbor in $Z - Z_1$, which, by Lemma 5.2(v), also leads to an octahedron minor. Next, suppose z_1z_3 is added. From (†) we can see that at least one of the three pairs $\{x_1, x_2\}$, $\{x_1, x_4\}$, $\{x_3, x_4\}$ is a subset of the neighborhood of a vertex in $Z - Z_1$, which implies, by Lemma 5.2(ii-iii), an octahedron minor, and thus the claim is proved.

Observe that there is only one way (up to isomorphism) to add a degree-four vertex to V_{10} that results in a bipartite graph (see Figure 5.2). Moreover, this graph contains the octahedron (by contracting the heavy edges). In other words, adding any edge between X and $Z - Z_1$ creates an octahedron minor. This observation and the last claim imply that no addition extension can use any edge incident with any vertex in Z. Since all vertices in Z are cubic and any two vertices in X have a common neighbor in Z, it follows that every addition extension of H has to use an edge incident with a vertex in Z. Therefore, J is not an addition extension.

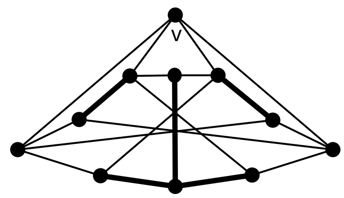

Figure 5.2: Adding a degree-four vertex v to V_{10}

Suppose J is obtained by splitting a vertex v. Since all vertices in Z are cubic, v must belong to X. Let v_1, v_2 be the two new vertices. Since v has three neighbors in Z_1, we may assume that v_1 has two or more neighbors in Z_1. If v_1 has exactly two neighbors in Z_1, let y' be the third neighbor of v in Z_1 and let $y'' \in Z - Z_1$ be a neighbor of v_2. Then contracting v_2y' and v_2y'' leads to the graph in Lemma 5.2(i), which is impossible. So v_1 has three neighbors in Z_1. In this case, contracting all edges incident with v_2, except for v_1v_2, will lead to the graph in Figure 5.2, which is again impossible. So J is not a split of H.

Finally, suppose J is obtained by a hexagonal extension. Notice that X has only two cubic vertices in H_2, so X has at most two cubic vertices in H. It follows that every 6-cycle contains a vertex in X of degree at least four. Thus every hexagonal extension is a graph in $\mathcal{S}$, as required. ∎

Now we summarize the last two lemmas.

Lemma 5.5 *If an internally 4-connected oct-free graph G contains V_{10}, then $G \in \mathcal{S}$ or G is V_{2n} or C^2_{2n+1} for some $n \geq 5$.*

Proof. Let us choose the largest n such that V_{2n} is a minor of G. Since G contains V_{10}, the required n does exist and $n \geq 5$. If G contains C^2_{2n+1}, by the choice of n we deduce from Lemma 4.2 that $G = C^2_{2n+1}$. Assume G does not contain C^2_{2n+1}. It follows from Lemma 5.3 and Lemma 5.4 that $G = V_{2n}$ or $G \in \mathcal{S}$. ∎

Proof of Lemma 5.1. Observe that every graph in $\mathcal{S}$ is a 3-sum of K_5 and copies of K_4. Moreover, V_{2n} ($n \geq 5$) is a 3-sum of $C^2_{1+2\lfloor n/2 \rfloor}$ and copies of K_4. Thus the result follows from Lemma 5.5. ∎

6 Extensions of L'_4

The main tool we use is Lemma 4.4, which is very powerful in general. However, as pointed out in [5], its weakness is that it may produce a graph that is not internally 4-connected. If that occurs, the new graph does not satisfy the requirements of the lemma and thus the lemma cannot be applied anymore. This unfortunate situation indeed happens to graph L'_4 (illustrated in Figure 1.2), so a separate treatment is necessary. There is a different result in [5] that is applicable in this situation. Since the proof of that result has not been published yet, and since we do not need the whole strength of the result, we prove a much simpler lemma to deal with L_4'. In a very weak sense, this lemma can be considered as a simplified version of the aforementioned result.

Let (v, e) be a violating pair. Let $e = v_1v_2$, $e_1 = vv_1$, and $e_2 = vv_2$. A (v, e)-*split* is a split at v_i ($i = 1$ or 2) such that e and e_i no longer share an endpoint (and thus (v, e) is no longer a violating pair).

Lemma 6.1 *Let v be a cubic vertex of a graph $H = (V, E)$ and let v_1, v_2, v_3 be the three neighbors of v. Suppose $|V| \geq 6$, $e = v_1v_2 \in E$, and v_3 is cubic. If H is a minor of an internally 4-connected graph G, then G has a minor J such that either J is a (v, e)-split, or $J = H + uv$ or $H + (u, e)$ for some $u \in V - \{v, v_1, v_2, v_3\}$.*

Proof. Since H is a minor of G, there exist vertex-disjoint trees T_u ($u \in V$) of G and internally vertex-disjoint paths P_f ($f \in E$) of G such that if $f = u_1u_2$ then the ends of P_f are in T_{u_1} and T_{u_2}, respectively. Each path P_f will be called an *edge-path*. We choose trees T_u so that they have as few vertices as possible (by possibly making edge-paths longer). As a result, the following clearly holds:

(*) *every leaf of each T_u is an end of at least two edge-paths.*

Moreover, $|V(T_v)| = |V(T_{v_3})| = 1$ since v and v_3 are cubic. If P_e and some P_{vv_i} ($i = 1, 2$) are vertex-disjoint, then, by (*), G contains a (v, e)-split at v_i. Else the three paths P_e, P_{vv_1}, P_{vv_2} form a cycle since $|V(T_v)| = 1$. For $i = 1, 2, 3$, let x_i be the end of P_{vv_i} in $V(T_{v_i})$. Note that $V(T_{v_3}) = \{x_3\}$. Let Q be the union of paths P_e, P_{vv_1}, P_{vv_2}, and P_{vv_3}. Observe that G has at least two vertices not in Q since $|V| \geq 6$. Therefore, the internal 4-connectivity of G implies that $G - \{x_1, x_2, x_3\}$ has a path R from $Q - \{x_1, x_2, x_3\}$ to $H' - V(Q)$, where H' is the union of T_u and P_f, over all $u \in V$ and all $f \in E$.

Let z be the end of R in $H' - V(Q)$. We first consider the case $z \in V(T_{v_1} \cup T_{v_2})$. By symmetry, let $z \in V(T_{v_1}) - \{x_1\}$. If x_1 is a leaf of T_{v_1} and P_e, P_{vv_1} are the only two edge-paths ended at x_1, then R can used to reduce the size of T_{v_1} (while keeping the size of all other T_u), which is impossible. So either x_1 is not a leaf of T_{v_1} or at least three edge-paths are ended at x_1. In both cases, R can be used to produce a (v, e)-split at v_1. Therefore, we may assume that $z \notin V(T_{v_1} \cup T_{v_2})$, and thus z can be contracted to T_u for some $u \in V - \{v, v_1, v_2, v_3\}$. Now it is easy to see that G contains either $H + vu$ (if $V(R) \cap V(P_e) = \emptyset$) or $H + (u, e)$ (if $V(R) \cap V(P_e) \neq \emptyset$). ∎

We remark that the assumption "v_3 is cubic" in the last lemma can be removed by introducing two more output graphs. We choose to prove this simple version since it is enough for our application. It should be pointed out that graphs generated from this lemma do not have to be internally 4-connected. However, since the input graph H is not required to be internally 4-connected, this lemma is more applicable than Lemma 4.4, which makes it more useful in certain circumstances. In particular, even though the next result does not follow from Lemma 4.4, it can be proved by repeatedly using Lemma 6.1.

Lemma 6.2 *If an internally 4-connected oct-free graph G properly contains L_4' then G contains either L_5' or G_{1015a} (illustrated in Figure 1.2 and Figure 7.2, respectively).*

Proof. Let L_4' be labeled as in Figure 6.1 (the second graph). We first observe that there are three ways (up to isomorphism) of adding an edge to it: $L_4' + 16$, $L_4' + 17$, and $L_4' + 68$. All these additions contain the octahedron, shown by contracting $\{37, 58\}$, $\{48, 56\}$, and $\{15, 37\}$, respectively. By Lemma 4.1, G has a minor H obtained from L_4' by splitting a vertex. Clearly, there are only two ways (up to isomorphism) to make the split, which are the first and third graphs in Figure 6.1.

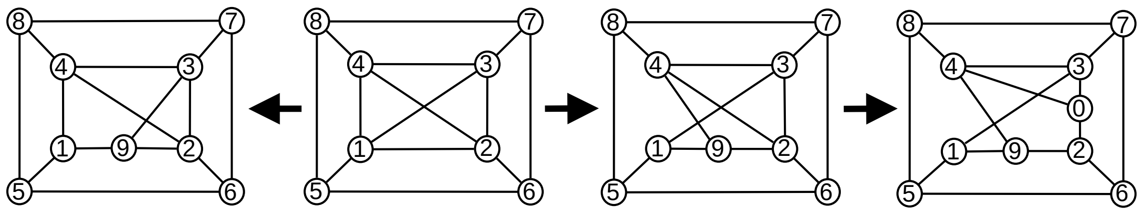

Figure 6.1: How L_4' is expanded

Let H be the first graph in Figure 6.1. Note that H has a unique violating pair $(9,23)$. By Lemma 6.1, G has a minor J obtained from H by one of three types of extensions. If J is obtained by adding an edge $u9$, then $u \in \{4,5\}$ since J contains $L_4' + e$ for other choices of u. For these two choices of u, J also contains the octahedron, shown by contracting $\{15,26,78\}$ and $\{14,26,78\}$, respectively. If $J = H + (u,23)$, then $u = 4$ since J contains $L_4' + e$ for other choices of u. It follows that $J \backslash \{24,34\} = G_{1015a}$. Finally, we assume that J is a $(9,23)$-split. At vertex 2, there are two ways to make the split: $\{3,4\}$-$\{6,9\}$ or $\{3,6\}$-$\{4,9\}$ (representing the corresponding partition of neighbors of vertex 2). In the first case $J\backslash 34 = G_{1015a}$; in the second case $J/\{15,78,2'6,2''9\}$ contains the octahedron, where $2', 2''$ are the two new vertices obtained by splitting 2. At vertex 3, there are also two ways to make the split: $\{2,4\}$-$\{7,9\}$ or $\{2,7\}$-$\{4,9\}$. In the first case $J\backslash 24 = G_{1015a}$ and in the second case $J = L_5'$.

Next, let H be the third graph in Figure 6.1. Our proof is almost identical to that of the last case, except for an extra outcome H_1, which is the fourth graph in Figure 6.1. Again, H has only one violating pair $(9,24)$ and G has a minor J obtained from H by one of three types of extensions. If $J = H + u9$, then $u \in \{3,5\}$ and J contains the octahedron, shown by contracting $\{15,26,78\}$ and $\{13,26,78\}$, respectively. If $J = H + (u,24)$, then $u = 3$ and $J\backslash 23$ is H_1. At vertex 2, there are two $(9,24)$-splits: $\{3,4\}$-$\{6,9\}$ or $\{3,9\}$-$\{4,6\}$. In the first case J is H_1 and in the second case J contains the octahedron, shown by contracting $\{15,78,2'6,2''9\}$. At vertex 4, there are two $(9,24)$-splits: $\{2,3\}$-$\{8,9\}$ or $\{2,8\}$-$\{3,9\}$. In the first case J is (isomorphic to) H_1 and in the second case J contains the octahedron, shown by contracting $\{15,67,4'8,4''9\}$.

Finally, we repeat the same argument on H_1, which has a unique violating pair $(0,34)$. Let J be obtained from H_1 by one of three types of extensions. If $J = H_1 + u0$, then $u \in \{1,6,9\}$ and J contains the octahedron, shown by contracting $\{15,26,49,78\}$, $\{19,29,37,58\}$, and $\{15,26,37,58\}$, respectively. If $J = H_1 + (u,34)$, then $u \in \{1,9\}$. In the first case $J/19\backslash\{13,49\}$ is G_{1015a}; in the second case $J/\{04,15,26,78,9v\}$ contains the octahedron, where v is the vertex subdividing 34. At vertex 3, there are two $(0,34)$-splits: $\{0,7\}$-$\{1,4\}$ or $\{0,1\}$-$\{4,7\}$. In the first case $J/19\backslash 49$ is G_{1015a}; in the second case $J/\{19,26,3'0,3''7,58\}$ contains the octahedron. At vertex 4, there are also two $(0,34)$-splits: $\{0,8\}$-$\{3,9\}$ or $\{0,9\}$-$\{3,8\}$. In the first case $J/15\backslash 13$ is G_{1015a} and in the second case $J/15$ is L_5'. Now the proof is complete. ∎

7 Extensions of V_8

This is the last and the most difficult part of our proof. Our main result in this section is the following.

Lemma 7.1 *If G is a V_{10}-free prime that contains V_8, then $G \in \{C_9^2, L_4', L_5', L_5'', P_{10}\}$.*

Through out this section, let G be an internally 4-connected oct-free and V_{10}-free graph. We will not repeatedly state this assumption when we state our lemmas in this section. In the following, we will generate many small graphs. We choose to name these graphs with four digits that correspond to the number of vertices and edges of the graph. For instance, G_{1222} (see Figure 7.4) has 12 vertices and 22 edges. We will also use a suffix a, b, c, etc. to distinguish different graphs of the same size. As before, we prove Lemma 7.1 by proving a sequence of lemmas.

Lemma 7.2 *If G is C_9^2-free and G properly contains V_8, then G must contain L_4', G_{0914a}, or G_{0914b}.*

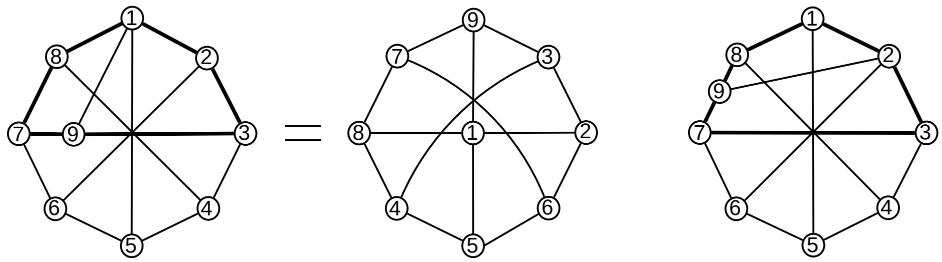

Figure 7.1: G_{0914a} and G_{0914b}

Proof. After verifying the requirements of Lemma 4.4 with $H = V_8$, we see that G contains a minor J as described in Lemma 4.4. Since V_8 is cubic, J is not a split of V_8. By Lemma 5.2(iii-iv), the only addition extension of V_8 is L_4'. As discussed in the proof of Lemma 5.3, the only 6-cycles of V_8 are formed by three consecutive rungs, which do not satisfy the requirements for a hexagonal extension and thus J is not a hexagonal extension. Finally, V_8 has only one type of 5-cycles and over which G_{0914a} and G_{0914b} are the only two possible pentagonal extensions. The lemma is proved. ■

Lemma 7.3 *If G properly contains H, one of the three output graphs in Lemma 7.2, then G contains P_{10}, L_5', or one of the following graphs.*

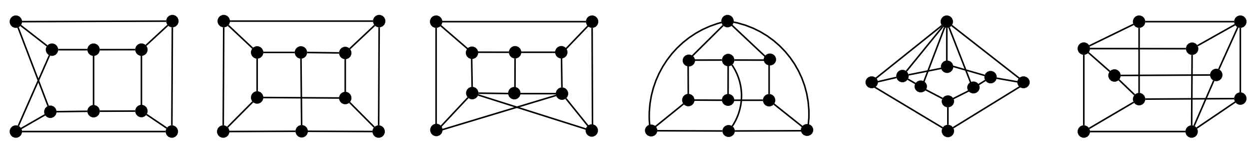

Figure 7.2: G_{1015a}, G_{1015b}, G_{1016a}, G_{1016b}, G_{1016c}, and G_{1017}

Proof. If $H = L_4'$, then the result follows from Lemma 6.2. Suppose $H = G_{0914a}$ and J is the graph determined in Lemma 4.4. By Lemma 5.2(iii-iv) and symmetry, it is not difficult to see that $H + u2$ contains the octahedron for every non-neighbor u of 2. Therefore, by symmetry again, $H + uv$ contains the octahedron for all non-edges uv with $v \in B := \{2, 5, 8, 9\}$. On the other hand, for every non-edge uv with neither end in B, u and v have a common neighbor in B, which is a set of cubic vertices. It follows that, for any addition extension of H, either the first or the second edge is incident with a vertex in B, which implies that J is not an addition extension.

If J is obtained by splitting a vertex, then the vertex is 1, which can be split in two ways (up to isomorphism) resulting graphs P_{10} and G_{1015b}.

There are two types of 6-cycles, represented by 126548 and 345679. The first does not satisfy the requirements for a hexagonal extension and the second leads to $H + 03 + 05 + 07$, which contains the octahedron, shown by contracting $\{05, 26, 39, 48\}$. Thus J is not a hexagonal extension

For $w = 1, 2, 3$, it is routine to check that if uv is an edge and there is a 5-cycle of the form $wxuvyw$ if and only if (w, uv) is one of the following: $(1, 34)$, $(1, 67)$, $(2, 45)$, $(2, 79)$, $(3, 15)$, $(3, 18)$, $(3, 56)$, $(3, 78)$, $(3, 67)$. By symmetry, there are five (up to isomorphism) pentagonal extensions. Now it is straightforward to verify that $H + (1, 34) = G_{1016c}$, $H + (3, 67) = G_{1016b}$, and $H + (2, 45)$, $H + (3, 15)$, $H + (3, 56)$ contain the octahedron, by contracting $\{04, 39, 56, 78\}$, $\{05, 26, 48, 79\}$, $\{05, 26, 48, 79\}$, respectively. The case analysis for $H = G_{0914a}$ is complete.

Let $H = G_{0914b}$ and let J be an extension of H as described in Lemma 4.4. The way we determine J is similar to that in the last case, so we only state the conclusions (also see Remark below). First, all addition extensions contain the octahedron, so J is not an addition extension. If J is obtained by splitting vertices, then J is G_{1015a} or G_{1015b}. Up to isomorphism, there are eight pentagonal extensions, four of which contain the octahedron and the other four are L_5', G_{1016a}, G_{1016b}, and G_{1016c}. Finally, among all three hexagonal extensions, two contain the octahedron and the last is G_{1017}. ■

Remark. The rest of the proof is pretty much the same as our analysis of G_{0914a}. As we have seen, there are basically two steps: generating all the extensions according to Lemma 4.4 (or Lemma 4.1 in a few cases) and deciding if an extension is oct-free. Both steps are straightforward, yet the amount of work is tremendous. The author wrote a simple program in Mathematica that is used to perform the case checking. The program does three things: generating all extensions, testing for isomorphisms, which eliminates duplications, and checking for octahedron minors. Since the detail is lengthy and dull, only final outcomes are reported in the following proofs. For the completeness of the proof, a supplement [1] is prepared, which contains more details on our program and our computation.

It turns out that, among all operations listed in Lemma 4.4, addition extension is the one that creates the most graphs. To cut down the number of cases, we make the following observation. Suppose $H + e_1 + e_2$ is an addition extension of H. If $H + e_1$ or $H + e_2$ contains the octahedron, then there is no need to determine $H + e_1 + e_2$. This observation motivates the following definition. Let F be the set of non-edges of H. Let F_1 consist of members e of F such that either $H + e$ is internally 4-connected (that is, $H + e$ is an addition extension of H) or $H + e$ contains the octahedron. Let $F_2 = F \backslash F_1$. Then a *partial addition* of H is either a graph $H + e$ with $e \in F_1$ or an addition extension $H + e_1 + e_2$ of H with $e_1, e_2 \in F_2$. Clearly, to prove that all addition extensions of H contain the octahedron, it is enough to prove that all partial additions of H contain the octahedron, which is exactly what we will do in the following proofs.

Lemma 7.4 *If G properly contains H, one of the eight output graphs in Lemma 7.3, then G contains L_5'' or one of the following graphs.*

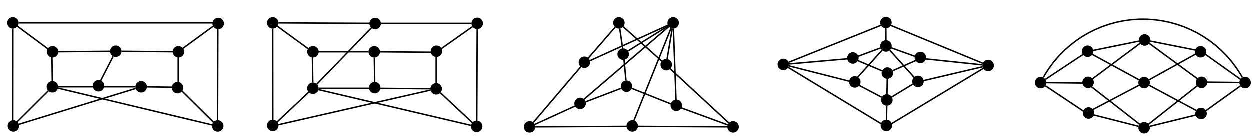

Figure 7.3: G_{1117}, G_{1118a}, G_{1118b}, G_{1119a}, and G_{1119b}

Proof. In this and the next two proofs, extensions are counted up to isomorphism. Suppose $H = P_{10}$. There is only one way to add an edge to H, which results in an octahedron minor. On the other hand, H is cubic so none of its vertices can be split. By Lemma 4.1, G does not properly contain H.

Suppose $H = L_5'$. There are sixteen ways of adding an edge to H and all of them contain the octahedron. By Lemma 4.1, some split of H is a minor of G. There are three ways of splitting vertices of H, two of which contain the octahedron and the third is L_5''.

For the remaining six graphs, we will use Lemma 4.4. Our proofs will consist of summaries on different extensions of H. Let $H = G_{1015a}$. Then H admits no splits since H is cubic. There are thirteen partial additions and two hexagonal extensions, all of which contain the octahedron. There are six pentagonal extensions, five contain the octahedron and the last is G_{1117}.

Let $H = G_{1015b}$. Then H admits no splits since H is cubic. There are six partial additions and two hexagonal extensions, all of which contain the octahedron. There are six pentagonal extensions, four contain the octahedron and the other two are L_5'' and G_{1117}.

Let $H = G_{1016a}$. There are eleven partial additions and they all contain the octahedron. There are three splits, two contain the octahedron and the other is G_{1117}. Among the six pentagonal extensions, four contain the octahedron, one is V_{10}^+, which contains V_{10}, and the last is G_{1118a}. There is only one hexagonal extension, which is G_{1119a}.

Let $H = G_{1016b}$. There are twelve partial additions and they all contain the octahedron. There are five splits, three contain the octahedron and the other two are L_5'' and G_{1117}. Among the six pentagonal extensions, four contain the octahedron, one is V_{10}^+, which contains V_{10}, and the last is G_{1118a}. There are three hexagonal extensions, one contains the octahedron and the other two are G_{1119a} and G_{1119b}.

Let $H = G_{1016c}$. There are seven partial additions, all of which contain the octahedron. There are four splits, three contain the octahedron and the last is G_{1117}. Among the five pentagonal extensions, three contain the octahedron and the other two are G_{1118a} and G_{1118b}. There are two hexagonal extensions, one contains the octahedron and the other is G_{1119a}.

Let $H = G_{1017}$. There are six partial additions and two splits, all of which contain the octahedron. There is no hexagonal extension and the only pentagonal extension is G_{1119a}. ∎

Lemma 7.5 *If G properly contains H, one of the six output graphs in Lemma 7.4, then G contains Γ (see Figure 1.3), G_{1221}, or G_{1222}.*

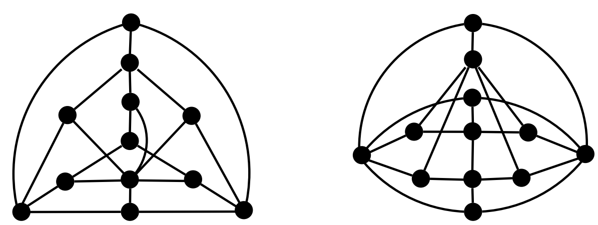

Figure 7.4: G_{1221} and G_{1222}

Proof. Suppose $H = L_5''$. There are twenty-one ways of adding an edge and two ways of splitting a vertex. All these graphs contain the octahedron, and thus, by Lemma 4.1, G does not properly contain H.

Let $H = G_{1117}$. There twenty partial additions, two splits, and nine pentagonal extensions, all of which contain the octahedron. There are six hexagonal extensions, five contain the octahedron and the other is Γ.

Let $H = G_{1118a}$. There is no pentagonal extension since H is bipartite. There are sixteen partial additions and eight splits, all of which contain the octahedron. There are four hexagonal extensions, two contain V_{10}, one contains the octahedron, and the last is G_{1221}.

Let $H = G_{1118b}$. There is no pentagonal extension since H is bipartite. There are three partial additions and four splits, all of which contain the octahedron. There are two hexagonal extensions, one contains the octahedron and the other is G_{1221}.

Let $H = G_{1119a}$. There are seventeen partial additions and they all contain the octahedron. There are eleven splits, ten of which contain the octahedron and the last one is Γ. There are four pentagonal extensions, two contain the octahedron, one contains V_{10}, and the other is G_{1221}. Finally, there is a unique hexagonal extension, which is G_{1222}.

Let $H = G_{1119b}$. There are five partial additions and four splits, all of which contain the octahedron. The only pentagonal extension contains V_{10} and the only hexagonal extension is G_{1222}. ∎

Lemma 7.6 *If G contains H, one of the three output graphs in Lemma 7.5, then $G = H$.*

Proof. Suppose $H = \Gamma$. There are nine partial additions, two splits, one pentagonal extension, and one hexagonal extension. All these graphs contain the octahedron. Thus Lemma 4.4 implies $G = H$.

Let $H = G_{1221}$. There is no pentagonal extension since H is bipartite. There are eleven partial additions and nine splits, all of which contain the octahedron. There is a unique hexagonal extension, which contains V_{10}. Therefore, G does not contain H properly and so $G = H$.

Let $H = G_{1222}$. There is no hexagonal extension since no 6-cycle satisfies the requirement for a hexagonal extension. There are seven partial additions and five splits, all of which contain the octahedron. There is a unique pentagonal extension, which contains V_{10}. Again, G does not contain H properly and so $G = H$. ∎

Remark. The unique hexagonal extension of G_{1221} and the unique pentagonal extension of G_{1222} are the same graph, which is an induced subgraph of S_{30}.

Now we summarize results proved so far in this section. Let us write $G_1 \leq G_2$ is G_1 is a minor of G_2.

Lemma 7.7 *Let G be an internally 4-connected, oct-free, and V_{10}-free graph that contains V_8. Then*
(i) G is one of V_8, C_9^2, L_4', P_{10}, L_5', L_5'', Γ, and graphs in figures 7.1-7.4;
(ii) $V_8 \leq G_{0914a} \leq G_{1016c} \leq G_{1118b} \leq G_{1221}$, $G_{0914b} \leq G_{1017} \leq G_{1119a} \leq G_{1222}$, $L_4' \leq G_{1015a} \leq G_{1117}$, $G_{1016a} \leq G_{1118a} \leq G_{1221} \leq S_{30}$, $G_{1016b} \leq G_{1119b} \leq G_{1222} \leq S_{30}$, $G_{1015b} \leq G_{1117} \leq \Gamma$, $L_5' \leq L_5''$.

Proof. (i). If G contains C_9^2, by Lemma 4.2, G is C_9^2. Suppose G does not contain C_9^2. Then the result follows from lemmas 6.2 and 7.2-7.6.
(ii). The result follows from the analysis of these graphs. ∎

Proof of Lemma 7.1. Since S_{30} is a 3-sum of K_5 and copies of K_4, all its minors must be 3-sums of graphs with at most five vertices. It follows that graphs in Lemma 7.7(ii) that are minors of S_{30} are not primes. Observe that Γ is a 3-sum of L_4' and four copies of K_4, where each of the square vertex belongs to a K_4 (see Figure 1.3), so Γ is not a prime. This decomposition of Γ is also inherited by its minors G_{1117}, G_{1015a}, and G_{1015b}, so they are not prime either. Thus, by Lemma 7.7, all primes are in $\{C_9^2, L_4', L_5', L_5'', P_{10}\}$, as required. ∎

8 Proof of main theorems

In this section we prove results stated in the Section 1. Our proofs are basically summaries of what we have shown.

Proof of Theorem 1.1. By Theorem 2.1, we only need to determine all prime graphs. Then the result follows from Lemmas 3.1, 5.1, and 7.1. ∎

Proof of Theorem 1.2. Let us assume that G is not V_{2n} or C^2_{2n-1}, for any $n \geq 3$. If G contains V_{10}, by Lemma 5.5, G is a minor of S_{30}. Suppose G is V_{10}-free. If G contains V_8, by Lemma 7.7, G is a minor of S_{30}, L''_5, P_{10} or Γ. Finally, assume G is V_8-free. Since $L_4 \leq L_5 \leq L''_5$ and $K^-_{4,4} \leq K^-_{4,5} \leq S_{30}$, the result follows from Lemma 3.8. ∎

Acknowledgment. The author would like to thank the two anonymous referees for their constructive comments and suggestions. This research is supported in part by NSA grant H98230-10-1-0186 and NSF grant DMS-1001230.

References

[1] G. Ding, A characterization of graphs with no octahedron minor (Supplement), www.math.lsu.edu/˜ ding/octahedron-supplement.docx.

[2] G. Ding and C. Liu, Excluding a small minor, accepted by *Discrete Applied Mathematics*.

[3] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierteljschr. Naturforsch. Ges. Zürich* 88 (1943) 133 - 143

[4] R. Halin and H.A. Jung. Uber minimalstrukturen von graphen, insbesondere von n-fach zusammenhangenden graphen, *Math. Ann.* 152 (1963) 75-94.

[5] T. Johnson and R. Thomas, Generating Internally Four-Connected Graphs, *Journal of Combinatorial Theory, Series B* 85 (2002) 2158.

[6] L. Jorgensen, Contraction to K_8, *Journal of Graph Theory* 18 (1994) 431 - 448.

[7] K. Kawarabayashi, S. Norine, R. Thomas, and P. Wollan, K_6 minors in large 6-connected graphs, *Preprint*, www.math.princeton.edu/˜ snorin/papers/k6large.pdf.

[8] J. Maharry, A characterization of graphs with no cube minor, *Journal of Combinatorial Theory* Series B 80 (2000) 179 - 201.

[9] J. Maharry, An excluded minor theorem for the octahedron, *Journal of Graph Theory* 31 (1999) 95-100.

[10] T. Politof and A. Satyanarayana, Minors of quasi 4-connected graphs, *Discrete Mathematics* 126 (1994) 245-256.

[11] P. Seymour, Decomposition of regular matroids, *Journal of Combinatorial Theory*, Series B 28 (1980) 305-359.