Bonds with Parity Constraints

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Abstract

Given a connected graph $G = (V, E)$ and three even-sized subsets A_1, A_2, A_3 of V, when does V have a partition (S_1, S_2) such that $G[S_i]$ is connected and $|S_i \cap A_j|$ is odd for all $i = 1, 2$ and $j = 1, 2, 3$? This problem arises in the area of integer flow theory and has theoretical interest in its own right. The special case when $|A_1| = |A_2| = |A_3| = 2$ has been resolved by Chakravarti and Robertson, and the general problem can be rephrased as a problem on binary matroids that asks if a given triple of elements is contained in a circuit. The purpose of this paper is to present a complete solution to this problem based on a strengthening of Seymour's theorem on triples in matroid circuits.

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1 Introduction

Graphs considered in this paper may have multiple edges but contain no loops. Let $G = (V, E)$ be a graph. For each $A \subseteq V$, we use $G[A]$ to denote the subgraph of G induced by A and write $\bar{A} := V - A$. For each $B \subseteq \bar{A}$, let $[A, B]$ denote the set of edges of G with one end in A and the other in B. We call $[A, B]$ a *bond* of G if $\emptyset \neq A \neq V$, $B = \bar{A}$, and both $G[A]$ and $G[B]$ are connected. By a *quadruple* we mean a connected graph $G = (V, E)$ together with three even-sized subsets A_1, A_2, A_3 of V, and we denote it by $\langle G, \mathcal{A}\rangle$, where $\mathcal{A} = \{A_1, A_2, A_3\}$. We say that $\langle G, \mathcal{A}\rangle$ is *feasible* if G has a bond $[S_1, S_2]$ such that $|S_i \cap A_j|$ is odd, for all $i = 1, 2$ and $j = 1, 2, 3$, and *infeasible* otherwise. The *bond problem* is to decide whether a given quadruple $\langle G, \mathcal{A}\rangle$ is feasible. A bond with the desired property is called a *feasible solution* to the problem or to the quadruple.

In [4], Chakravarti and Robertson obtained a complete solution to the bond problem for the case $|A_1| = |A_2| = |A_3| = 2$, where they assumed that A_1, A_2, A_3 are the ends of three edges e_1, e_2, e_3. Their theorem, when restricted to a 4-connected graph G with $|A_1 \cup A_2 \cup A_3| \geq 4$, asserts that $\langle G, \mathcal{A}\rangle$ is infeasible if and only if G has a plane representation in which e_1, e_2, e_3 are contained in a facial cycle. This theorem implies the following result of Jung [1]: Let G be a 4-connected graph and let s_1, t_1, s_2, t_2 be four distinct vertices of G. Then G contains no disjoint paths from s_i to t_i, $i = 1, 2$, respectively, if and only if G has a plane representation in which vertices s_1, s_2, t_1, t_2 occur on a facial cycle in cyclic order. To see this implication, let $A_1 := \{s_1, s_2\}$, $A_2 := \{s_2, t_1\}$, and $A_3 := \{t_1, t_2\}$. It is then a routine matter to check that $\langle G, \mathcal{A}\rangle$ is feasible if and only if G has disjoint paths from s_i to t_i, $i = 1, 2$, respectively. For a complete solution to this disjoint paths problem, see Seymour [7], Shiloach [3, 8], and Thomassen [11].

In addition to its theoretical interest, the bond problem has an interesting application in integer flow theory: A subdivision of K_4 (the complete graph with four vertices) is called a *fully odd* K_4 if each of the six edges of the K_4 is subdivided into a path of odd length. As conjectured by Toft [9] and proved independently by Thomassen [10] and Zang [12], every graph containing no fully odd K_4 is 3-colorable. With the same motivation as Tutte's 3-, 4-, and 5-flow conjectures, we strongly believe that the dual of this theorem (conjectured by C.Q. Zhang) also holds; that is, every 2-edge-connected graph with no fully odd K_4-partition admits a nowhere-zero 3-flow, where a *fully odd* K_4*-partition* of a graph $G = (V, E)$ is a partition (V_1, V_2, V_3, V_4) of V such that $G[V_i]$ is connected for each $1 \leq i \leq 4$, and that $|[V_i, V_j]|$ is odd for each pair $1 \leq i < j \leq 4$. A crucial step in our proof of this 3-flow conjecture is to characterize all infeasible quadruples.

The present paper is concerned with the bond problem in its general setting. As we shall see in Section 2, this problem can be rephrased as a problem on binary matroids that asks if a given triple of elements is contained in a circuit, and the later problem has been resolved by Seymour [6]. The purpose of this paper is to present a structural characterization of all infeasible quadruples based on a strengthening of Seymour's theorem on triples in matroid circuits. Since no commonly used graph operations correspond directly to reductions used by Seymour to lift matroid connectivity, and since specializations of Seymour's operations on matroids to graphs involve edge contractions, which cannot be employed in our investigation of the aforementioned nowhere-zero 3-flow conjecture, a large portion of this paper will be devoted to further development and refinement of Seymour's work so as to fulfill our needs.

We introduce some notions before presenting our results. A quadruple $\langle G, \mathcal{A}\rangle$ is called *trivial* if some $A_i = \emptyset$, and is called *cyclic* if $A_1 \Delta A_2 \Delta A_3 = \emptyset$ and *acyclic* otherwise, where Δ stands for the symmetric difference operator. We write $V(\mathcal{A}) = A_1 \cup A_2 \cup A_3$. Clearly, a trivial quadruple is infeasible, and so is a cyclic quadruple (in which, for any bond $[S, \bar{S}]$, the sum $\sum_{i=1}^{3} |S \cap A_i|$ is even as every vertex in $V(\mathcal{A})$ contributes 2 to the sum; thus some $|S \cap A_i|$ must be even, showing the infeasibility of the quadruple). We say that $\mathcal{A}$ is *linked* by a cycle C if there are edge-disjoint paths $P_{i,j}$ with positive length ($i = 1, 2, 3$ and $j = 1, 2, ..., t_i$, where $t_i \geq 1$) of C such that, for $i = 1, 2, 3$, paths $P_{i,1}$, $P_{i,2}$, ..., P_{i,t_i} are vertex-disjoint and A_i precisely consists of all ends of these paths. Clearly, a necessary condition for $\mathcal{A}$ to be linked by a cycle C is that C contains all vertices in $V(\mathcal{A})$ and each $|A_i|$ ($i = 1, 2, 3$) is a positive even number. We point out that $\langle G, \mathcal{A} \rangle$ is infeasible if G has a plane representation in which $\mathcal{A}$ is linked by a facial cycle C. For suppose that $[S_1, S_2]$ is a feasible solution. Then $|[S_1, S_2] \cap E(C)| \leq 2$ since C is a facial cycle. Consequently, there exists $i \in \{1, 2, 3\}$ such that $P_{i,1}$, $P_{i,2}$, ..., P_{i,t_i} are all disjoint from $[S_1, S_2]$. Therefore, each of these paths is contained in either $G[S_1]$ or $G[S_2]$, which implies that $|A_i \cap S_1|$ and $|A_i \cap S_2|$ are even, contradicting the feasibility of $[S_1, S_2]$.

We shall demonstrate in Section 5 that the bond problem can be easily reduced to the situation when the given graph is 2-connected. We shall also use three other reductions to simplify our problem. These operations are illustrated in Figure 1 and will be formally defined in Section 5. We remark that all these reductions preserve the feasibility/infeasibility of a quadruple.

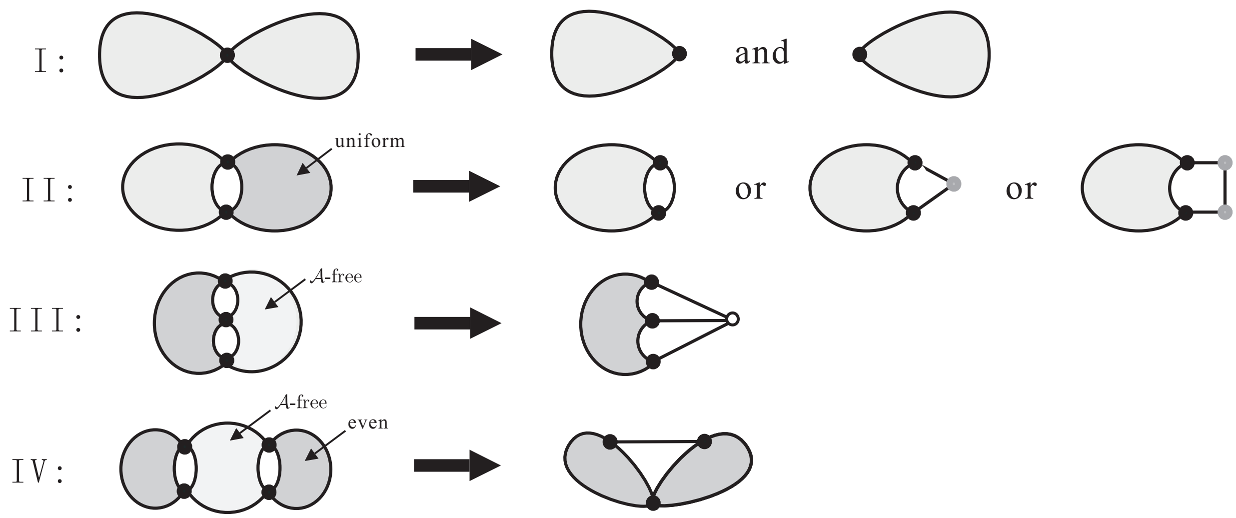

Figure 1: Reductions I, II, III, and IV.

Theorem 1.1 *Let $\langle G, \mathcal{A} \rangle$ be a quadruple. Then one of the following statements hold:*

(i) $\langle G, \mathcal{A} \rangle$ *is feasible;*

(ii) $\langle G, \mathcal{A} \rangle$ *admits one of reductions I-IV;*

(iii) $\langle G, \mathcal{A} \rangle$ *is trivial or cyclic;*

(iv) G *has a plane representation in which* $\mathcal{A}$ *is linked by a facial cycle.*

Since reductions I-IV preserve feasibility/infeasibility, and since conditions (iii) and (iv) imply infeasibility, Theorem 1.1 completely characterizes feasible quadruples. In particular, if G is 4-connected, this theorem says that $\langle G, \mathcal{A} \rangle$ is infeasible if and only if (iii) or (iv) holds.

In our investigation of the aforementioned 3-flow problem, we have observed that the presence of fully odd K_4-partitions in a given graph G depends, to a large extent, on the locations of the vertices in $V(\mathcal{A})$. Therefore, we are in need of a transparent global structural description of G. As both reductions II and IV involve edge contractions, Theorem 1.1 cannot immediately be applied to integer flow problems as the whereabouts of the vertices of $V(\mathcal{A})$ are lost under edge contractions and nowhere-zero-3-flows are not preserved under edge expansions. (To the best of our knowledge, no simple edge contractions have been used successfully to deal with integer flow problems to date.) Therefore, we made a serious effort to improve Theorem 1.1.

In Section 8 we define *weakly linkable* quadruples. Roughly speaking, $\langle G, \mathcal{A}\rangle$ is weakly linkable if G is 2-connected and, modulo certain small separations, G has a plane representation in which $\mathcal{A}$ is linked by a facial cycle. This is the same type of condition as the one used by Seymour [7] in his solution to the 2-disjoint paths problem.

The main result of this paper is the following.

Theorem 1.2 *A quadruple on a* 2*-connected graph is infeasible if and only if it is trivial, cyclic, or weakly linkable.*

We point out that Seymour's theorem on triples in matroid circuits [6] plays an important role in the proof of Theorem 1.1. However, the derivation of Theorem 1.2 relies only on Theorem 1.1 and requires more efforts on exploiting graph structures. What we have to do is to describe the precise locations where the reductions are performed, which makes Theorem 1.2 much stronger than Theorem 1.1.

We conclude this section with some more terminology. Let $G = (V, E)$ be a graph. For $X \subseteq V \cup E$, we use $G\backslash X$ to denote the graph obtained from G by deleting elements of X from G. If $x \in V \cup E$, we write $G\backslash x$ instead of $G\backslash\{x\}$. Let $X \subseteq E$. We denote by $G[X]$ the subgraph of G induced by edges in X, and by G/X the graph obtained from G by contracting all edges in X. For $H = G\backslash X$, we consider $H + X$ as G. We also write G/x instead of $G/\{x\}$, and write $H + x$ instead of $H + \{x\}$ if $X = \{x\}$.

2 The matroid formulation

In this section we show how to rephrase the bond problem as a matroid problem. We refer the reader to Oxley [2] for basic matroid theory. Let $\langle G, \mathcal{A}\rangle$ be a quadruple, where $G = (V, E)$ and $\mathcal{A} = \{A_1, A_2, A_3\}$. For each $U \subseteq V$, let χ_U be the characteristic vector of U (with length $|V|$), which is considered as a vector over $GF(2)$.

We first explain how G defines a binary matroid. For each edge $e = xy$ of G, let $\chi_e = \chi_{\{x,y\}}$. Then it is routine to verify that a nonempty set C of edges forms a simple cycle in G if and only if $\sum_{e\in C} \chi_e = 0$ and $\sum_{e\in C'} \chi_e \neq 0$ for any nonempty proper subset C' of C. This means that simple cycles of G are precisely circuits (minimal dependent sets) of the binary matroid represented by vectors $\{\chi_e : e \in E\}$. Similarly, spanning forests of G are precisely bases (maximal independent sets) of this matroid, which, denoted by $\mathcal{M}(G)$, is known as the *graphic matroid* of G. To extend $\mathcal{M}(G)$, we require the following lemma.

Lemma 2.1 *Let $T \subseteq V$ and $F \subseteq E$. Then χ_T is spanned by vectors in $\{\chi_e : e \in F\}$ if and only if $T \subseteq V(G[F])$ and every component of $G[F]$ contains exactly an even number of vertices in T.*

Proof. Note that χ_T is spanned by vectors in $\{\chi_e : e \in F\}$ if and only if there exists $F' \subseteq F$ such that $\chi_T = \sum_{e\in F'} \chi_e$, if and only if there exists $F' \subseteq F$ such that the odd-degree vertices of

$G[F']$ are precisely those in T (in the literature such an F' is called a *T-join* of $G[F]$). Thus the lemma follows (see, for instance, (29.1) of Schrijver [5]). ∎

Taking the even-sized subsets A_1, A_2, A_3 of V into account, we reserve the symbol M for the binary matroid represented by the vectors in the set $\{\chi_{A_1}, \chi_{A_2}, \chi_{A_3}\} \cup \{\chi_e : e \in E\}$. Since G is a connected graph, from Lemma 2.1 we deduce that χ_{A_i} is spanned by $\{\chi_e : e \in E\}$ for $i = 1, 2, 3$. Hence, with $r(\cdot)$ denoting the rank function of M, we obtain

$$r(M) = r(\mathcal{M}(G)) = r(E) = |V| - 1. \tag{2.1}$$

To simplify our notation, let us think of the element set of M as $A \cup E$ (where A consists of a_1, a_2, and a_3 corresponding to χ_{A_1}, χ_{A_2}, and χ_{A_3}, respectively), instead of the set of vectors.

Lemma 2.2 *The quadruple $\langle G, \mathcal{A}\rangle$ is feasible if and only if M has a cocircuit containing A.*

Proof. Our proof is based on Proposition 2.1.16 of Oxley [2], which asserts that

Cocircuits of a matroid are precisely the minimal sets that meet every basis. (2.2)

To prove the "only if" part, let $[S_1, S_2]$ be a bond of G such that $|S_i \cap A_j|$ is odd for all $i = 1, 2$ and $j = 1, 2, 3$ and set $F := [S_1, S_2]$. We aim to show that $D := A \cup F$ is a cocircuit of M. For this purpose, let B be a basis of M. If $A \cap B \neq \emptyset$, then $D \cap B \neq \emptyset$; if $A \cap B = \emptyset$, then B forms a spanning tree of G, which implies $D \cap B = F \cap B \neq \emptyset$. Next, for any proper subset D' of D, we need to find a basis B' with $D' \cap B' = \emptyset$. If $D' \not\supseteq F$, then G has a spanning tree T (obtained from $(G \backslash F) \cup (F - D')$) which is disjoint from D'. In this case, by (2.1), we can take $B' = T$. If $D' \supseteq F$, we may assume that $a_1 \notin D'$. Let $J \subseteq E - F$ such that J forms a spanning forest of $G \backslash F$, which has exactly two components (whose vertex sets are S_1 and S_2, respectively). If $J \cup \{a_1\}$ is dependent, then χ_{A_1} is spanned by $\{\chi_e : e \in J\}$. It follows from Lemma 2.1 that every component of J has an even number of vertices in A_1, contradicting the definition of $[S_1, S_2]$. Therefore $J \cup \{a_1\}$ is independent in M, which, in combination of (2.1), implies that $J \cup \{a_1\}$ is a basis of M, so we can take $B' = J \cup \{a_1\}$. Hence, by (2.2), D is a cocircuit of M.

To see the "if" part, suppose M has a cocircuit of the form $D = A \cup F$ with $F \subseteq E$. We propose to show that F is a bond of G of the form $F = [S_1, S_2]$ such that $|S_i \cap A_j|$ is odd for all $i = 1, 2$ and $j = 1, 2, 3$, which implies that the quadruple $\langle G, \mathcal{A}\rangle$ is feasible. Indeed, for every spanning tree T of G, since T is a basis of M by (2.1), we have $T \cap F = T \cap D \neq \emptyset$. As D is a cocircuit of M, for any proper subset F' of F, there exists a basis T' of M such that $(A \cup F') \cap T' = \emptyset$. It follows that T' is a spanning tree of G with $T' \cap F' = \emptyset$. We can thus conclude from (2.2) that F is a cocircuit of $\mathcal{M}(G)$ and hence is a bond of G, denoted by $[S_1, S_2]$. It remains to verify that both $|S_1 \cap A_j|$ and $|S_2 \cap A_j|$ are odd for every $1 \le j \le 3$. Since D is a cocircuit of M, by (2.2), M has a basis B such that $B \cap (D - \{a_j\}) = \emptyset$; that is, $B \subseteq (E - F) \cup \{a_j\}$. It follows that B consists of a_j and a spanning forest J of $G \backslash F$ (which has two components with vertex sets S_1 and S_2, respectively). Since a_j is not spanned by J, we deduce from Lemma 2.1 that both $|S_1 \cap A_j|$ and $|S_2 \cap A_j|$ are odd. ∎

3 Application of Seymour's theorem

The result established by Seymour in [6] is actually stronger than what he stated in the paper. In this section we extract this stronger version, which will be used in the proof of Theorem 1.1.

Let N be a matroid on $E(N)$ and let (X, Y) be a partition of $E(N)$. The *order* of (X, Y) is defined as

$$o(X, Y) = r_N(X) + r_N(Y) - r_N(N) + 1, \tag{3.1}$$

where $r_N(\cdot)$ denotes the rank in N. Let k be a positive integer with $o(X, Y) \leq k$. Then (X, Y) is called

- a *k-separation* if $\min\{|X|, |Y|\} \geq k$;
- an *internal k-separation* if $\min\{|X|, |Y|\} \geq k + 1$; and
- a *vertical k-separation* if $\max\{r_N(X), r_N(Y)\} < r_N(N)$.

We say that N is *k-connected* (resp. *internally k-connected, vertically k-connected*) if N has no k'-separation (resp. *internal k'-separation, vertical k'-separation*) for any $k' < k$. It is well known that a matroid is k-connected (resp. internally k-connected) if and only if its dual is k-connected (resp. internally k-connected). However, the dual of a vertically k-connected matroid is not necessarily vertically k-connected. By a *k-circuit* (resp. *k-cocircuit*) in N we mean a circuit (resp. cocircuit) in N of cardinality k. A 3-circuit is also called a *triangle*.

In the rest of this section, N is a binary matroid and $A = \{a_1, a_2, a_3\}$ consists of three specified elements of N. The following lemmas were proved by Seymour [6].

Lemma 3.1 ((2.3) of [6]) *If N is 3-connected and vertically 4-connected, then N is internally 4-connected.*

Lemma 3.2 ((2.9) of [6]) *Suppose N is 3-connected and has no vertical 3-separation (X, Y) with $A \subseteq X$. Suppose a_1, a_2 are not contained in any triangle in N. Let a'_3 be a new element and let N' be the unique binary matroid on $E(N) \cup \{a'_3\}$ such that $N' \backslash a'_3 = N$ and $\{a_1, a_2, a'_3\}$ is a triangle in N'. Then N' is 3-connected and has no vertical 3-separation (X, Y) with $A \subseteq X$. Moreover, N has a circuit containing A if and only if N' has a circuit containing A.*

Lemma 3.3 ((2.10) of [6]) *Suppose N is 3-connected and has no vertical 3-separation (X, Y) with $A \subseteq X$. If $A - \{a_i\}$ is contained in a triangle for all $i = 1, 2, 3$, then N is vertically 4-connected.*

Lemma 3.4 ((3.11) of [6]) *Suppose N is 3-connected and internally 4-connected. Then A is not contained in any circuit of N if and only if either A is a cocircuit or $N = \mathcal{M}(H)$ for a graph H, such that members of A are edges in H incident with a common vertex.*

Let us now use the above lemmas to derive a statement slightly stronger (see Lemma 3.1) than Lemma 3.4.

Lemma 3.5 *Suppose N is 3-connected and has no vertical 3-separation (X, Y) with $A \subseteq X$. Then A is not contained in any circuit of N if and only if either A is a cocircuit or $N = \mathcal{M}(H)$ for a graph H, such that members of A are edges in H incident with a common vertex.*

Proof. The "if" part is obvious. So we proceed to the "only if" part. Suppose N has no circuit containing A. Let us apply the procedure described in Lemma 3.2 to all pairs $\{a_1, a_2\}$, $\{a_2, a_3\}$, and $\{a_1, a_3\}$. Then we obtain a matroid N' that is 3-connected and has no vertical 3-separation (X, Y) with $A \subseteq X$. Moreover, $A - \{a_i\}$ is contained in a triangle in N' for all $i = 1, 2, 3$, while A is contained in no circuit in N'. By Lemma 3.3, N' is vertically 4-connected and hence, using Lemma 3.1, internally 4-connected. From Lemma 3.4 we see that A is a cocircuit of N' or

$N' = \mathcal{M}(H')$ for a graph H', such that members of A are edges in H' incident with a common vertex. Let $Z = E(N') - E(N)$. Then $N = N' \backslash Z$. In the first case, A contains a cocircuit of N, so A itself is a cocircuit of N because $|A| = 3$ and N is 3-connected. In the second case, $N = \mathcal{M}(H') \backslash Z = \mathcal{M}(H' \backslash Z)$, which proves that $H = H' \backslash Z$ has the desired property. ∎

In our application, we shall use a modified version of the preceding lemma. We say that two elements in a 2-cocircuit (resp. 2-circuit) of a matroid are *in series* (resp. *parallel*) with each other, and that a matroid N_1 is a *series-extension* of a matroid N_2 if N_1 is obtained from N_2 by adding elements (possibly none), each of which is in series with an element of N_2; this, in terms of dual matroids, amounts to that N_1^* is obtained from N_2^* by adding elements (possibly none), each of which is in parallel with an element of N_2^*.

Corollary 3.6 *Suppose N is a series-extension of a 3-connected matroid N' with $A \subseteq E(N')$, and N has no vertical 3-separation (X, Y) with $A \subseteq X$. Then A is not contained in any circuit of N if and only if either A is a cocircuit of N or $N = \mathcal{M}(H)$ for a graph H, such that members of A are edges in H incident with a common vertex.*

Proof. Notice that N' has no vertical 3-separation (X, Y) with $A \subseteq X$, as N has no such separations. Since N is a series-extension of a 3-connected matroid N' and $A \subseteq E(N')$, the following statements hold: (i) A is contained in a circuit of N if and only if A is contained in a circuit of N'; (ii) A is a cocircuit of N if and only if A is a cocircuit of N'; and (iii) $N = \mathcal{M}(H)$ for a graph H, such that members of A are edges in H incident with a common vertex if and only if $N' = \mathcal{M}(H')$ for a graph H', such that members of A are edges in H' incident with a common vertex. Thus the corollary follows instantly from Lemma 3.5. ∎

We remark that assumptions in Corollary 3.6 in fact imply that N is 3-connected, except that each $a_i \in A$ could be in a series class of size two.

We also need a characterization of graphs that have the same matroid. Suppose G is obtained from disjoint graphs G_1 and G_2 by identifying $u_1, v_1 \in V(G_1)$ with $u_2, v_2 \in V(G_2)$, respectively. Then the graph obtained from G_1, G_2 by identifying u_1 with v_2, and v_1 with u_2 is called a *twist* of G. It is not difficult to see that if one graph is obtained from another graph by a sequence of twisting operations, then the two graphs have the same matroid. The following theorem of Whitney (5.3.1 of [2]) asserts the converse.

Lemma 3.7 *Two 2-connected graphs have the same matroid if and only if one can be obtained from the other by a sequence of twisting operations.*

The following is another fact we will use. We omit its proof since it follows immediately from the definition of twist.

Lemma 3.8 *Suppose G has a plane representation with a facial cycle C. If G' is a twist of G, then G' also has a plane representation with C as a facial cycle.*

Now let us restrict our attention to the quadruple $\langle G, \mathcal{A} \rangle$ and the matroid M discussed in Section 2, the binary matroid represented by vectors in the set $\{\chi_{A_1}, \chi_{A_2}, \chi_{A_3},\} \cup \{\chi_e : e \in E(G)\}$. Applying Corollary 3.6 with respect to M^*, we obtain the following result.

Lemma 3.9 *Suppose $\langle G, \mathcal{A} \rangle$ is nontrivial and G is 2-connected. If M^* is a series-extension of a 3-connected matroid containing A, and M^* has no vertical 3-separation (F_1, F_2) with $A \subseteq F_1$, then $\langle G, \mathcal{A} \rangle$ is infeasible if and only if it is cyclic or G has a plane representation in which $\mathcal{A}$ is linked by a facial cycle.*

Proof. In view of Lemma 2.2, $\langle G, \mathcal{A} \rangle$ is infeasible if and only if A is not contained in any circuit of M^*; this, by Corollary 3.6, is equivalent to saying that one of the following statements holds:

(i) A is a circuit of M;
(ii) $M = \mathcal{M}^*(H)$ for a graph H, such that a_1, a_2, a_3 are edges v_1v_0, v_2v_0, v_3v_0 in H, respectively, with a common vertex v_0.

Since $\langle G, \mathcal{A} \rangle$ is nontrivial, the following are equivalent:

(i) $\Leftrightarrow$ $\chi_{A_1} + \chi_{A_2} + \chi_{A_3} = 0$ over $GF(2)$ $\Leftrightarrow$ $A_1 \Delta A_2 \Delta A_3 = \emptyset$ $\Leftrightarrow$ $\langle G, \mathcal{A} \rangle$ is cyclic.

It remains to prove that (ii) is equivalent to

(iii) G has a plane representation in which $\mathcal{A}$ is linked by a facial cycle C.

Suppose (ii) holds. Without loss of generality, we assume that H has no isolated vertices. Since $\mathcal{M}(H) = M^*$ is connected, by Proposition 4.1.8 of [2], H is 2-connected. It follows that, for $i = 1, 2, 3$, edges in H that are incident with v_i form a minimal edge cut $\{a_i\} \cup E_i$, where $E_i = [\{v_i\}, V(H) - \{v_i\}] - \{a_i\}$. Since $\mathcal{M}^*(H/A) = \mathcal{M}^*(H) \backslash A = M \backslash A = \mathcal{M}(G)$, matroid $\mathcal{M}(H/A)$ is both graphic and cographic, which implies, by Theorem 5.2.2 of [2], that H/A is planar. Let us consider H/A as a plane graph and let $(H/A)^*$ be its geometric dual. Since G is 2-connected, $\mathcal{M}((H/A)^*) = \mathcal{M}^*(H/A) = \mathcal{M}(G)$ is connected and thus $(H/A)^*$ is 2-connected. Note that all edges in $E_1 \cup E_2 \cup E_3$ are incident with a common vertex of H/A, hence $E_1 \cup E_2 \cup E_3$ is contained in a facial cycle of $(H/A)^*$. By applying Lemma 3.7 and Lemma 3.8 to graphs $(H/A)^*$ and G we deduce that G has a plane representation such that $E_1 \cup E_2 \cup E_3$ is contained in a facial cycle C. In addition, we also deduce from the 2-connectivity of $(H/A)^*$ that H/A has no loops and hence the three sets E_1, E_2, and E_3 are pairwise disjoint.

Since $\{a_i\} \cup E_i$ is a cocircuit of $\mathcal{M}(H)$, it is a circuit of M, implying $\chi_{A_i} + \sum_{e \in E_i} \chi_e = 0$ and hence A_i is the set of all odd-degree vertices in the graph $G[E_i]$. Recall that $G[E_i]$ is a subgraph of facial cycle C, so $G[E_i]$ consists of some paths $P_{i,1}$, $P_{i,2}$, ..., P_{i,t_i} (with positive lengths) on C whose ends are precisely vertices in A_i for $i = 1, 2, 3$, where $t_i \geq 1$ as $A_i \neq \emptyset$. Since E_1, E_2, and E_3 are pairwise disjoint, it is routine to check that $\mathcal{A}$ is linked by C. Hence (ii) $\Rightarrow$ (iii).

To show the reverse implication we will use the following simple fact whose proof we omit. *Let N_1, N_2 be two binary matroids on the same ground set F. Suppose $f \in F$ and $Z \subseteq F - \{f\}$ such that $N_1 \backslash f = N_2 \backslash f$, and $\{f\} \cup Z$ is a circuit in both N_1 and N_2. Then $N_1 = N_2$.*

Assuming (iii), there exist edge-disjoint paths $P_{i,j}$ of C ($i = 1, 2, 3$ and $j = 1, 2, \ldots, t_i$, with $t_i \geq 1$) such that, for $i = 1, 2, 3$, paths $P_{i,1}$, $P_{i,2}$, ..., P_{i,t_i} are vertex-disjoint and their ends form A_i. Set $E_i := \cup_{j=1}^{t_i} E(P_{i,j})$ for $i = 1, 2, 3$. Then the three sets E_1, E_2, and E_3 are pairwise disjoint. For $i = 1, 2, 3$, E_i satisfies $\chi_{A_i} + \sum_{e \in E_i} \chi_e = 0$ (over $GF(2)$) and E_i is a minimal set with this property, which means that $\{a_i\} \cup E_i$ is a circuit of M. On the other hand, we consider the dual graph G^* of G. Since C is a facial cycle of G, there exists a vertex v^* of G^* such that edges incident with v^* are precisely those in $E(C)$. Let H be obtained from G^* by replacing v^* with a claw consisting of edges $a_i = v_0v_i$ ($i = 1, 2, 3$) in a way that edges in E_i are incident with v_i ($i = 1, 2, 3$) and edges in $E(C) - (E_1 \cup E_2 \cup E_3)$ are incident with v_0. Graph H is well defined because E_1, E_2, and E_3

are pairwise disjoint. Clearly, $\mathcal{M}^*(H)\backslash A = \mathcal{M}^*(H/A) = \mathcal{M}(G) = M\backslash A$. In addition, since G is 2-connected and each $E_i \neq \emptyset$, H must also be 2-connected. As a result, each $\{a_i\} \cup E_i$ is a circuit of $\mathcal{M}^*(H)$. Now the simple fact mentioned in the last paragraph implies $\mathcal{M}^*(H) = M$, which proves (iii) $\Rightarrow$ (ii). ∎

4 Separations in graphs

Throughout this section, we assume that $G = (V, E)$ is a 2-connected graph and $\langle G, \mathcal{A}\rangle$ is a nontrivial quadruple. Let M be the matroid as defined in Section 2 and let $r(\cdot)$ be the rank function of M. For any $F \subseteq E$, let $c(F)$ denote the number of components of $G[F]$. Since the graphic matroid $\mathcal{M}(G)$ is a restriction of M to E, we have $r(F) = |V(G[F])| - c(F)$.

For any partition (E_1, E_2) of E, it follows from (3.1) and the submodular inequality of the rank function that

$$o(E_1, E_2) = r(E_1) + r(E_2) - r(E) + 1 \geq 1. \tag{4.1}$$

For $i = 1, 2$, let $V_i = \emptyset$ if $E_i = \emptyset$, else let $G_i := G[E_i]$, let $V_i := V(G_i)$, and let $G_i^1, G_i^2, ..., G_i^{t_i}$ be all the components of G_i, where $t_i = c(E_i)$.

Lemma 4.1 *Let $V_0 := V_1 \cap V_2$ and $k := o(E_1, E_2)$. Then $k = |V_0| - t_1 - t_2 + 2$. Furthermore,*

(i) $k = 1$ iff $E_1 = \emptyset$ or $E_2 = \emptyset$;

(ii) $k = 2$ iff $t_1 = t_2 = t = |V_0|/2$ and $|V(G_i^j) \cap V_0| = 2$ for all $i = 1, 2$ and $j = 1, 2, ..., t$;

(iii) $k = 3$ iff $|V(G_i^j) \cap V_0| = 2$ for all $i = 1, 2$ and $j = 1, 2, ..., t_i$, except $|V(G_i^j) \cap V_0| = 4$ for exactly one G_i^j or except $|V(G_i^j) \cap V_0| = 3$ for exactly two G_i^j.

Proof. By definition, $r(E_i) = |V_i| - t_i$ for $i = 1, 2$. Using (4.1) we obtain

$$k = (|V_1| - t_1) + (|V_2| - t_2) - (|V| - 1) + 1 = |V_0| - t_1 - t_2 + 2,$$

as desired. Equivalently,

$$t_1 + t_2 = |V_0| + 2 - k. \tag{4.2}$$

We claim that

$$\textit{if } E_1 \neq \emptyset \neq E_2, \textit{ then } |V(G_i^j) \cap V_0| \geq 2 \textit{ for } i = 1, 2 \textit{ and } j = 1, 2, ..., t_i. \tag{4.3}$$

To justify this, note that $G_i^1, G_i^2, ..., G_i^{t_i}$ are pairwise vertex-disjoint for $i = 1, 2$ and each has at least one edge. Since G is 2-connected, G_i^j ($i = 1, 2$ and $j = 1, 2, \ldots, t_i$) contains at least two vertices from G_{3-i}, which clearly belong to V_0. Thus (4.3) holds. It follows that

$$\textit{if } E_1 \neq \emptyset \neq E_2, \textit{ then } 0 < t_i \leq |V_0|/2 \textit{ for } i = 1, 2. \tag{4.4}$$

From (4.2) and (4.4), we conclude (i) instantly. Next, $k = 2$ if and only if $t_1 = |V_0|/2 = t_2$. In view of (4.3), we obtain (ii). Finally, using $\sum_{j=1}^{t_i} |V(G_i^j) \cap V_0| = |V_0|$ ($i = 1, 2$) we deduce from (4.2) that: $k = 3$ if and only if $2 = 2|V_0| - 2(t_1 + t_2) = \sum_{i=1}^{2} \sum_{j=1}^{t_i} (|V(G_i^j) \cap V_0| - 2)$ if and only if the graph structures are as described in (iii) (by (4.3)). ∎

In what follows, we consider a partition (F_1, F_2) of $E \cup A = E(M)$ and its *restriction* (E_1, E_2) to E, where $E_i = F_i \cap E$ for $i = 1, 2$. From (2.1), (3.1) and (4.1) we see that

$$r(F_i) \leq r(E) \textit{ for } i = 1, 2 \textit{ and } o(F_1, F_2) = r(F_1) + r(F_2) - r(E) + 1 \geq o(E_1, E_2) \geq 1. \tag{4.5}$$

Lemma 4.2 *M is connected.*

Proof. First, since G is 2-connected, $M|E = \mathcal{M}(G)$ is connected. Second, by (2.1), E is a spanning set of M. Finally, since $\langle G, \mathcal{A}\rangle$ is nontrivial, no element of A is a loop of M, which proves that M is connected. ∎

Lemma 4.3 *If M is not a parallel-extension of 3-connected matroid containing A, then one of the following holds:*

(i) $A_i = A_j$ for some $1 \le i < j \le 3$;

(ii) E has a partition (E_1, E_2) with $o(E_1, E_2) = 2 \le \min\{r(E_1), r(E_2)\}$, such that for each h in $\{1, 2, 3\}$, there exists $i = i(h) \in \{1, 2\}$ for which $A_h \subseteq V_i$ and $|A_h \cap V(G_i^j)|$ is even for all $1 \le j \le t_i$.

Proof. Depending on the structure of M, we distinguish between two cases.

Case 1. M is a parallel-extension of a 3-connected matroid. In this case, the hypothesis of the lemma allows us to assume that a_i and a_j are in parallel with each other for some $1 \le i < j \le 3$. It follows that $A_i = A_j$.

Case 2. M is not a parallel-extension of a 3-connected matroid. In particular, M is not 3-connected. By Lemma 4.2, M admits a 2-separations (F_1, F_2) with $o(F_1, F_2)= 2$. We claim that

$$M \text{ has a 2-separation } (F_1, F_2) \text{ with } \min\{r(F_1), r(F_2)\} \ge 2 = o(F_1, F_2). \tag{4.6}$$

Otherwise, for every 2-separation (F_1, F_2) of M, we have $\min\{r(F_1), r(F_2)\} = 1$, which means either F_1 or F_2 consists of parallel elements. Let $si(M)$ be the simple matroid associated with M (cf. page 46 of Oxley [2]). Then $si(M)$ would have no 2-separations and hence is 3-connected. Clearly, we may assume that $si(M)$ contains A, for otherwise a_i and a_j would be in parallel with each other for some $1 \le i < j \le 3$ and hence (i) holds. Therefore M is a parallel-extension of 3-connected matroid $si(M)$ containing A, contradicting the hypothesis of the lemma. So (4.6) is established.

Let (F_1, F_2) be as exhibited in (4.6) and let (E_1, E_2) be the restriction of (F_1, F_2) to E. We propose to show that (E_1, E_2) is as desired. Indeed, from (4.5) and (4.6) we see that $\max\{r(E_1), r(E_2)\} \le \max\{r(F_1), r(F_2)\} \le r(E)-1$, so $E_1 \ne \emptyset \ne E_2$ and hence $o(E_1, E_2) \ge 2$ by Lemma 4.1(i). In view of (4.5) and (4.6), we further obtain $o(E_1, E_2) = 2$. It follows from (4.1) and (4.5) that $r(E_i) = r(F_i) \ge 2$ for $i = 1, 2$. Hence χ_{A_h} $(h = 1, 2, 3)$ is spanned by $\{\chi_e : \; e \in E_i\}$ if $a_h \in F_i$, we can thus deduce (ii) from Lemma 2.1. ∎

A subset of V or a subgraph of G is called *uniform* in a quadruple $\langle G, \mathcal{A}\rangle$ if its intersection with $V(\mathcal{A}) = A_1 \cup A_2 \cup A_3$ is a subset of $A_1 \cap A_2 \cap A_3$, or a subset of $A_h - (A_i \cup A_j)$, or a subset of $(A_h \cap A_i) - A_j$ for some permutation h, i, j of $1, 2, 3$.

Let us now proceed to vertical 3-separations in M^*. By definition, a partition (F_1, F_2) of $E \cup A$ is a vertical 3-separation in M^* if and only if $r^*(F_1) + r^*(F_2) - r^*(E \cup A) + 1 = o(F_1, F_2) \le 3$ and $r^*(F_i) < r^*(M)$ for $i = 1, 2$. Since $r^*(X) = |X| - r(M) + r((E \cup A) - X)$ for each $X \subseteq E \cup A$, by (2.1), it is easy to verify that the latter holds if and only if

$$r(F_1) + r(F_2) \le |V| + 1, \text{ and } r(F_i) < |F_i| \text{ for } i = 1, 2. \tag{4.7}$$

In the following lemma we use notation introduced in Lemma 4.1.

Lemma 4.4 *If M is a parallel-extension of a 3-connected matroid containing A, (F_1, F_2) is a vertical 3-separation of M^* with $A \subseteq F_1$, and (E_1, E_2) is the restriction of (F_1, F_2) to E, then $|V(G_2^\ell)| \le |E(G_2^\ell)|$ for some $1 \le \ell \le t_2$. Moreover, one of the following holds:*

(i) $A_1 \Delta A_2 \Delta A_3 = \emptyset$ or $A_i = A_j$ for some $1 \le i < j \le 3$;

(ii) $o(E_1, E_2) = 2$ and $V_2 - V_0$ is uniform;

(iii) $o(E_1, E_2) = 3$, $A_1 \cup A_2 \cup A_3 \subseteq V_1$, and $|A_i \cap V(G_1^j)|$ is even for every $1 \le i \le 3$ and $1 \le j \le t_1$.

Proof. Since $A \subseteq F_1$, we have $E_2 = F_2 \neq \emptyset$ by (4.7). Therefore

$$\sum_{\ell=1}^{t_2}(|V(G_2^\ell)| - 1) = \sum_{\ell=1}^{t_2} r(E(G_2^\ell)) = r(E_2) = r(F_2) < |F_2| = |E_2| = \sum_{\ell=1}^{t_2} |E(G_2^\ell)|,$$

which implies $|V(G_2^\ell)| \le |E(G_2^\ell)|$ for some $1 \le \ell \le t_2$.

Since $A_i \neq \emptyset$ for $i = 1, 2, 3$, we have $r(A) \ge 1$. If $r(A) < |A| = 3$, then (i) holds. So we assume hereafter that $r(A) = |A| = 3$.

Put $k := o(E_1, E_2)$. Recall that $k \ge 1$. If $k = 1$, then $E_1 = \emptyset$ by Lemma 4.1(i). So $F_1 \subseteq A$ and hence $F_1 = A$ by hypothesis. It follows from (4.7) that $r(A) < |A|$, contradicting the preceding assumption. So $k \ge 2$. By (4.5), we obtain $2 \le k \le o(F_1, F_2) \le 3$. Observe that

$$\text{if } o(F_1, F_2) = 2, \text{ then } \min\{r(F_1), r(F_2)\} = 1. \tag{4.8}$$

Otherwise, $r(F_i) \ge 2$ for $i = 1, 2$. Consequently, as $r(M) \ge r(A) \ge 3$, M has a separation (F_1', F_2') with $o(F_1', F_2') = 2$ and $r(F_i') \ge 2$ $(i = 1, 2)$, and such that every parallel class of M is completely contained in either F_1' or F_2'. Let $si(M)$ be a 3-connected matroid such that M is its parallel-extension (see the hypothesis). Then the restriction of (F_1', F_2') to $si(M)$ would be a 2-separation of $si(M)$, contradicting its 3-connectivity.

Let us now consider two cases.

Case 1. $o(F_1, F_2) = 2$. In this case, $k = 2$. Since $r(F_1) \ge r(A) = 3$, by (4.8) we obtain $r(E_2) = r(F_2) = 1$. It follows that $|V_2| = 2$. As the 2-connectivity of G implies $|V_0| = |V_1 \cap V_2| \ge 2$, we obtain $V_2 - V_0 = \emptyset$. Thus (ii) holds now.

Case 2. $o(F_1, F_2) = 3$. In this case, combining (4.1), (4.5), and the equality $r(F_2) = r(E_2)$, we obtain $r(F_1) = r(E_1) + 3 - k$. If $k = 3$, then $r(F_1) = r(E_1)$. Hence χ_{A_i} $(i = 1, 2, 3)$ is spanned by $\{\chi_e : e \in E_1\}$. By Lemma 2.1, we thus obtain (iii). If $k = 2$, then $r(F_1) = r(E_1) + 1$. By symmetry we may assume the existence of a spanning forest B of G_1 such that $\{a_1\} \cup B$ is a basis of F_1. Consequently, a_i $(i = 2, 3)$ is spanned by $\{a_1\} \cup B$, which implies that $A_i - V_1 = \emptyset$ or $A_i - V_1 = A_1 - V_1$. Therefore (ii) holds, completing the proof. ∎

5 Reductions

In this section, we introduce four reductions which reduce the input quadruple in the bond problem to "smaller" ones. We begin with several definitions and facts on graph separations which will be used to prove reducibility and feasibility of a quadruple.

Let H be a graph. A *separation* of H is a pair (X, Y) of subgraphs of H with $V(X) \cup V(Y) = V(H)$, $E(X) \cup E(Y) = E(H)$, $E(X) \cap E(Y) = \emptyset$, and $E(X) \neq \emptyset \neq E(Y)$. If, in addition, $|V(X) \cap V(Y)| = k$, then (X, Y) is called a *k-separation* at $V(X) \cap V(Y)$.

The following simple fact ensures the existence of bonds in the most general sense.

Lemma 5.1 *Let H be a connected graph and let X_1, X_2 be nonempty disjoint subsets of $V(H)$. Then H has a bond $[Y_1, Y_2]$ with $X_i \subseteq Y_i$ for $i = 1, 2$ if and only if there exist vertex-disjoint connected subgraphs H_1 and H_2 of H such that $X_i \subseteq V(H_i)$ for $i = 1, 2$.*

It is well known that in a 2-connected graph that is not a triangle, every single edge can be either deleted or contracted so that the resulting graph remains 2-connected. This fact can be used to prove the following statement.

Lemma 5.2 *Let H be a 2-connected graph and let X_1, X_2 be nonempty subsets of $V(H)$. Then H has a bond $[Y_1, Y_2]$ of H such that $|X_i \cap Y_1| = 1$ for $i = 1, 2$.*

Proof. We apply induction on $E(H)$. Since the base case when H is a triangle is trivial, we proceed to the induction step, and assume that the assertion holds for all 2-connected graphs H' with $|E(H')| < |E(H)|$. If there exists $v \in X_1 \cap X_2$, then $Y_1 = \{v\}$ yields a desired bond of H. So we also assume $X_1 \cap X_2 = \emptyset$. Let $e = u_1u_2$ be an edge of H with $u_1 \in X_1$ and $u_2 \notin X_1$. Recall that either $H \backslash e$ or H/e is 2-connected as H is not a triangle.

If $H \backslash e$ is 2-connected, then the induction hypothesis guarantees the existence of a bond $[Y_1, Y_2]$ of $H \backslash e$, which also defines a bond of H, such that $|X_i \cap Y_1| = 1$ for $i = 1, 2$. So we assume that $H' = H/e$ is 2-connected. If $u_2 \in X_2$, setting $Y_1 = \{u_1, u_2\}$ yields a bond of H as desired. So we assume $u_2 \notin X_2$. Let $v \in V(H')$ be the vertex resulted from the contraction of e, and let $X_1' := (X_1 - \{u_1\}) \cup \{v\}$ and $X_2' := X_2$. By induction hypothesis, there is a bond $[Y_1', Y_2']$ of H' with $|X_i' \cap Y_1'| = 1$ for $i = 1, 2$. Define $Y_1 := Y_1'$ if $v \notin Y_1'$ and $Y_1 := (Y_1' - \{v\}) \cup \{u_1, u_2\}$ otherwise, and put $Y_2 := V(H) - Y_1$. Clearly, $[Y_1, Y_2]$ is a bond of H as desired. ■

Let $\langle G, \mathcal{A} \rangle$ be a nontrivial quadruple, where $G = (V, E)$, and let M be the matroid as defined in Section 2. For each edge $e = xy \in E$, it is easy to verify that M/e corresponds to the quadruple $\langle G/e, \mathcal{A}/e \rangle$ in the same way as M does to $\langle G, \mathcal{A} \rangle$, where, letting $z \in V(G/e)$ be resulted from contracting e, the triple $\mathcal{A}/e = \{A_1', A_2', A_3'\}$ satisfies $A_i' = (A_i - \{x, y\}) \cup \{z\}$ if $|A_i \cap \{x, y\}| = 1$ and $A_i' = A_i - \{x, y\}$ otherwise for $i = 1, 2, 3$. In particular,

$$A_i' \textit{ is an even-sized subset of } A_i \cup \{z\} \textit{ for } i = 1, 2, 3. \tag{5.1}$$

Moreover, a feasible solution $[S_1', S_2']$ of the quadruple $\langle G/e, \mathcal{A}/e \rangle$ with $z \in S_1'$ gives a feasible solution $[V - S_2', S_2'] = [(S_1' - \{z\}) \cup \{x, y\}, S_2']$ of $\langle G, \mathcal{A} \rangle$. Thus if $\langle G/e, \mathcal{A}/e \rangle$ is feasible, then so is $\langle G, \mathcal{A} \rangle$. This simple fact can be extended to a quadruple obtained via a sequence of contractions. Let $\langle G/\emptyset, \mathcal{A}/\emptyset \rangle = \langle G, \mathcal{A} \rangle$. For any $e \in F \subseteq E$, we define the *contraction* of $\langle G, \mathcal{A} \rangle$ (with respect to F), inductively, as $\langle G, \mathcal{A} \rangle / F = \langle G/F, \mathcal{A}/F \rangle = \langle G/F, \mathcal{A}/(F - \{e\})/e \rangle$. It is straightforward to verify that the result is independent of the order of the contractions. So the feasibility of $\langle G/F, \mathcal{A}/F \rangle$ implies the feasibility of $\langle G, \mathcal{A} \rangle$, though the reverse is not necessarily true. The combination of this fact with Lemma 5.1 instantly gives the following.

Lemma 5.3 *Let $H = (V', E')$ be a subgraph of G with $V(\mathcal{A}) \subseteq V'$, and let $F \subseteq E'$. If $\langle H, \mathcal{A} \rangle / F$ is feasible, then so is $\langle G, \mathcal{A} \rangle$.*

Next we formally define reductions I–IV (illustrated in Section 1) and show that they preserve the feasibility/infeasibility of quadruples. We point out that, when applied to a quadruple $\langle G, \mathcal{A} \rangle$, these reductions produce new quadruples $\langle G', \mathcal{A}' \rangle$ such that G' is *simpler* than G, meaning that

$2|E(G')|-|V(G')| < 2|E(G)|-|V(G)|$. In most cases, G' is a proper minor of G, which leads to the inequality. The reduced quadruples $\langle G', \mathcal{A}'\rangle$ will also maintain the *evenness* of $\mathcal{A}' = \{A_1', A_2', A_3'\}$, meaning that $|A_i'|$ is always even for $i = 1, 2, 3$. In fact, the evenness in all reductions, except for reduction II-2 (defined below), is guaranteed by the fact that $\mathcal{A}' = \mathcal{A}/F$ for some $F \subseteq E(G)$.

Suppose that (G_1, G_2), where $G_i = (V_i, E_i)$, $i = 1, 2$, is a 1-separation of G. Clearly, G_1 and G_2 are both simpler than G since they are proper minors of G. We say that $\langle G_i, \mathcal{A}_i\rangle = \langle G, \mathcal{A}\rangle/E_{3-i}$, $i = 1, 2$, are obtained from $\langle G, \mathcal{A}\rangle$ by a *reduction I*.

Lemma 5.4 *Let $\langle G_i, \mathcal{A}_i\rangle$, $i = 1, 2$, be obtained from $\langle G, \mathcal{A}\rangle$ by reduction I. Then $\langle G, \mathcal{A}\rangle$ is feasible if and only if $\langle G_i, \mathcal{A}_i\rangle$ is feasible, for some $i \in \{1, 2\}$.*

Proof. The sufficiency follows from Lemma 5.3. To see the necessity, let $[S_1, S_2]$ be a feasible solution to $\langle G, \mathcal{A}\rangle$. By symmetry, we may assume that the only common vertex of V_1 and V_2 belongs to S_1. Since $G[S_2]$ is connected, S_2 must be a subset of V_1 or V_2, say V_2. Thus $[S_1 \cap V_2, S_2]$ is in fact a bond of G_2, which, by the definition of contraction, is a feasible solution to $\langle G_2, \mathcal{A}_2\rangle$. ∎

Note that the bond problem is trivial when G is 2-connected and $A_1 \cap A_2 \cap A_3 \neq \emptyset$, because $[\{u\}, V - \{u\}]$ is obviously a feasible solution for any $u \in A_1 \cap A_2 \cap A_3$. This simple observation will be used repeatedly in this paper. The remaining reductions II, III, and IV deal with 2-connected graphs. As we shall see, they all maintain 2-connectedness of graphs.

Suppose that G is 2-connected and has a 2-separation (G_1, G_2) with $G_i = (V_i, E_i)$, $i = 1, 2$, such that $V_2 - V_1$ is uniform. Then $\langle G, \mathcal{A}\rangle$ *admits* a *reduction II* (see Figure 2 below) if one of the following occurs:

(II-1) $G_2 = v_1uv_2$ is a path of length 2, $u \notin A_1 \cap A_2 \cap A_3$, and $\{v_1, u\} = A_i$ for some $i \in \{1, 2, 3\}$: Let $\langle G', \mathcal{A}'\rangle = \langle G, \mathcal{A}\rangle/uv_2$.

(II-2) G_2 is not a path of length at most three with $V_2 - V_1 \subseteq V(\mathcal{A})$: Let G' be obtained from G by replacing G_2 with a path P between vertices in $V_1 \cap V_2$ which is of length one if $|V(\mathcal{A}) - V_1| = 0$, of length two if $|V(\mathcal{A}) - V_1|$ is odd, and of length three if $|V(\mathcal{A}) - V_1| > 0$ is even. For $i = 1, 2, 3$, let $A_i' := A_i$ if $A_i \subseteq V_1$ and $A_i' := (A_i \cap V_1) \cup (V(P) - V_1)$ otherwise. Set $\mathcal{A}' := \{A_1', A_2', A_3'\}$.

In both cases, it is straightforward to verify the evenness for $\mathcal{A}'$ and the fact that G' is simpler than G. We will say that $\langle G', \mathcal{A}'\rangle$ is obtained from $\langle G, \mathcal{A}\rangle$ by a *reduction II based on (G_1, G_2) in $\langle G, \mathcal{A}\rangle$*.

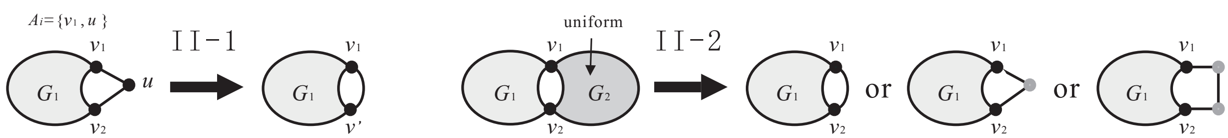

Figure 2: Reduction II based on 2-separation (G_1, G_2) at $\{v_1, v_2\}$.

Lemma 5.5 *Let $\langle G', \mathcal{A}'\rangle$ be obtained from $\langle G, \mathcal{A}\rangle$ by a reduction II. Then $\langle G, \mathcal{A}\rangle$ is feasible if and only if $\langle G', \mathcal{A}'\rangle$ is feasible.*

Proof. In the case of (II-1), the sufficiency is implied by Lemma 5.3, and the necessity follows from the fact that any feasible solution to $\langle G, \mathcal{A}\rangle$ can be expressed as $[S_1, S_2]$ such that $v_1 \in S_1$

and $\{u, v_2\} \subseteq S_2$ because some A_i equals $\{v_1, u\}$, and u has degree 2 in G and does not belong to $A_1 \cap A_2 \cap A_3$. In the rest of the proof, we assume (II-2). Let $P = v_1 v_2 \ldots v_h$ be the path used to replace G_2 in the reduction II-2, where $h \in \{2, 3, 4\}$ and $\{v_1, v_h\} = V_1 \cap V_2$. By the reduction, $P \backslash \{v_1, v_h\}$ is uniform in $\langle G', \mathcal{A}' \rangle$.

Observe that G' is also 2-connected, and that $A_1 \cap A_2 \cap A_3 = \emptyset$ if and only if $A'_1 \cap A'_2 \cap A'_3 = \emptyset$. Hence $A_1 \cap A_2 \cap A_3 \neq \emptyset$ or $A'_1 \cap A'_2 \cap A'_3 \neq \emptyset$ implies the feasibilities of both $\langle G, \mathcal{A} \rangle$ and $\langle G', \mathcal{A}' \rangle$. It remains to consider the case when $A_1 \cap A_2 \cap A_3 = \emptyset = A'_1 \cap A'_2 \cap A'_3$.

First, assume that $\langle G, \mathcal{A} \rangle$ has a feasible solution $[S_1, S_2]$. Since $A_1 \cap A_2 \cap A_3 = \emptyset$, uniform $V_2 - \{v_1, v_h\}$ must be disjoint from at least one of A_1, A_2, A_3, which implies $S_i \cap V_1 \neq \emptyset$ for $i = 1, 2$. If $V_2 - \{v_1, v_h\} \subseteq S_i$ for some $i \in \{1, 2\}$, then $S_i \cap \{v_1, v_h\} \neq \emptyset$, so $[S_{3-i}, V(G') - S_{3-i}]$ is a feasible solution to $\langle G', \mathcal{A}' \rangle$. Hence, we may assume that $V_2 - \{v_1, v_h\} \not\subseteq S_i$ for $i = 1, 2$, and further by symmetry that $v_1 \in S_1$ and $v_h \in S_2$. Let $S'_1 := (S_1 \cap V_1) \cup \{v_2\}$ if $|(S_1 - V_1) \cap V(\mathcal{A})|$ is odd, and let $S'_1 := S_1 \cap V_1$ otherwise. Let $S'_2 := V(G') - S'_1$. Since $G[S_i \cap V_1]$ is connected for $i = 1, 2$, $[S'_1, S'_2]$ is a bond of G'. Moreover, since $V_2 - \{v_1, v_h\}$ is uniform, the placement of v_2 (as an element of $V(P) - V_1$ set by II-2) implies that $[S'_1, S'_2]$ is a feasible solution to $\langle G', \mathcal{A}' \rangle$.

Next, assume that $\langle G', \mathcal{A}' \rangle$ has a feasible solution $[S'_1, S'_2]$. Since $A'_1 \cap A'_2 \cap A'_3 = \emptyset$, the uniform $P \backslash \{v_1, v_h\}$ contains neither S'_1 nor S'_2. Furthermore, if some S'_i, $i = 1$ or 2, contains $V(P) - \{v_1, v_h\}$, then it contains v_1 or v_h or both, and $[S'_{3-i}, V - S'_{3-i}]$ is a feasible solution of $\langle G, \mathcal{A} \rangle$. So we assume $V(P) - \{v_1, v_h\} \not\subseteq S'_i$ for $i = 1, 2$, which implies $h = 4$ and allows us to assume $\{v_1, v_2\} \subseteq S'_1$ and $\{v_3, v_4\} \subseteq S'_2$. Since P has length 3, by (II-2) there exists $v \in V(\mathcal{A}) - V_1 \subseteq V_2 - \{v_1, v_4\}$. As (G_1, G_2) is a 2-separation of the 2-connected graph G at $\{v_1, v_4\}$, the graph $H = G_2 + v_1 v_4$ is 2-connected and thus contains a v-v_1 path P_1 and a v-v_4 path P_2 with $V(P_1) \cap V(P_2) = \{v\}$. Observe that $P_1 \backslash v$ and $P_2 \backslash v$ are vertex-disjoint subgraphs of the connected graph $H \backslash v$. Lemma 5.1 guarantees the existence of a bond $[Y_1, Y_2]$ of $H \backslash v = (G_2 + v_1 v_4) \backslash v$ such that $V(P_i \backslash v)$ $(\subseteq Y_i)$ contains a neighbor of v for $i = 1, 2$. Hence both $[Y_1 \cup \{v\}, Y_2]$ and $[Y_1, \{v\} \cup Y_2]$ are bonds of G_2. Since $v \in V(\mathcal{A})$, there exists $Y \in \{Y_1 \cup \{v\}, Y_1\}$ such that $|(Y - \{v_1\}) \cap V(\mathcal{A})|$ is odd, $v_1 \in Y$, and $v_4 \notin Y$. Thus $|(Y - \{v_1\}) \cap A_i| \equiv |\{v_2\} \cap A'_i| \equiv |S'_1 \cap V(P \backslash \{v_1, v_4\}) \cap A'_i| \pmod 2$ for $i = 1, 2, 3$. Setting $S_1 := (S'_1 \cap V_1) \cup Y$ yields a feasible solution $[S_1, V - S_1]$ to $\langle G, \mathcal{A} \rangle$. ∎

Suppose G is 2-connected and has a 3-separation (G_1, G_2), where $G_i = (V_i, E_i)$ is connected for $i = 1, 2$, $V_1 \cap V_2 = \{v_1, v_2, v_3\}$, and $V(\mathcal{A}) \subseteq V_1$. In addition, either $G_2 = v_1 v_2 v_3 v_1$ is a triangle or $|E_2| > 3$ and some component of $G_2 \backslash \{v_1, v_2, v_3\}$ is adjacent to all v_i for $i = 1, 2, 3$. Then $\langle G, \mathcal{A} \rangle$ *admits* a *reduction III based on* (G_1, G_2), which reduces the bond problem to one on $\langle G', \mathcal{A} \rangle$, where G' is obtained from G_1 by adding a vertex v_0, called the *center*, and adding three edges $v_0 v_1$, $v_0 v_2$, and $v_0 v_3$. Again, it is routine to verify that G' is simpler than G.

Lemma 5.6 *Let $\langle G', \mathcal{A} \rangle$ be obtained from $\langle G, \mathcal{A} \rangle$ by a reduction III. Then $\langle G, \mathcal{A} \rangle$ is feasible if and only if $\langle G', \mathcal{A} \rangle$ is feasible.*

Proof. When G_2 is a triangle, it is routine to check that a solution $[S'_1, S'_2]$ to $\langle G', \mathcal{A} \rangle$ gives rise to a solution $[S'_1 - \{v_0\}, S'_2 - \{v_0\}]$ to $\langle G, \mathcal{A} \rangle$. When G_2 is not a triangle, observe that $\langle G', \mathcal{A} \rangle$ is a contraction of $\langle H, \mathcal{A} \rangle$ for a subgraph H of G with $V(\mathcal{A}) \subseteq V_1 \subseteq V(H) \cap V(G')$, so the sufficiency follows instantly from Lemma 5.3.

To see the necessity, we assume that $\langle G, \mathcal{A} \rangle$ has a feasible solution $[S_1, S_2]$ with $\{v_1, v_2\} \subseteq S_2$. From the 2-connectivity of G, it is easy to see that $[S_1 - (V_2 - \{v_3\}), (V_2 - \{v_3\}) \cup S_2]$ is also a feasible solution to $\langle G, \mathcal{A} \rangle$. Hence $[S_1 \cap V_1, \{v_0\} \cup (S_2 \cap V_1)]$ is a feasible solution to $\langle G', \mathcal{A} \rangle$. ∎

Suppose G is 2-connected. We say that $\langle G, \mathcal{A}\rangle$ *admits* a *reduction IV* (see Figure 3) if G has a 4-separation $(G_1 \cup G_2, G_3)$ at $\{v_1, v_2, v_3, v_4\}$, where $G_i = (V_i, E_i)$ with $E_i \neq \emptyset$ for $i = 1, 2, 3$, $V_1 \cap V_2 = \emptyset$, $V(\mathcal{A}) \subseteq V_1 \cup V_2$, and $(G_j, G_3 \cup G_{3-j})$ is a 2-separation of G at $\{v_{2j-1}, v_{2j}\}$ for $j = 1, 2$. Moreover, G_3 contains vertex-disjoint paths P_1 from v_1 to v_3 and P_2 from v_2 to v_4, such that either

(IV-1) $E_3 = \{v_1v_3, v_2v_4\}$, $A_h = \{v_1, v_3\}$ for some $h \in \{1, 2, 3\}$, and $|A_i \cap V_j|$ is even for every $i \in \{1, 2, 3\} - \{h\}$ and $j = 1, 2$; or

(IV-2) $|A_i \cap V_j|$ is even for all $i = 1, 2, 3$ and $j = 1, 2$.

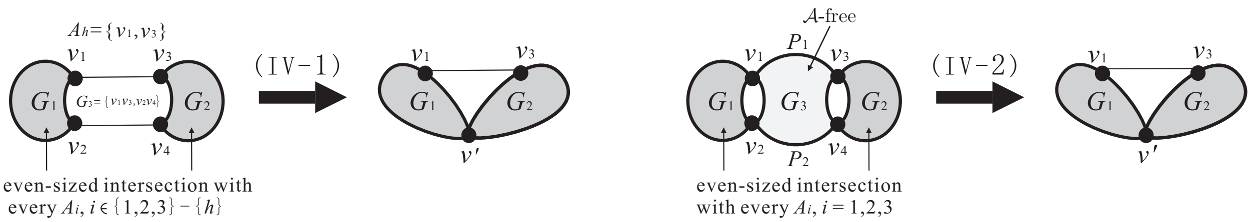

Figure 3: Reduction IV based on even 4-separation $(G_1 \cup G_2, G_3)$ at $\{v_1, v_2, v_3, v_4\}$.

We call $(G_1 \cup G_2, G_3)$ an *even* 4-separation, and say that $\langle G', \mathcal{A}'\rangle = \langle G_1 \cup G_2 \cup P_2 + v_1v_3, \mathcal{A}\rangle / E(P_2)$ is obtained from $\langle G, \mathcal{A}\rangle$ by a *reduction IV based on* $(G_1 \cup G_2, G_3)$. Observe that $G' = (V', E')$ contains the edge v_1v_3 and a vertex v' to which $E(P_2)$ is contracted. Once again, G' is simpler than G as G' is a proper minor of G.

Lemma 5.7 *Let $\langle G', \mathcal{A}'\rangle$ be obtained from $\langle G, \mathcal{A}\rangle$ by a reduction IV. Then $\langle G, \mathcal{A}\rangle$ is feasible if and only if $\langle G', \mathcal{A}'\rangle$ is feasible.*

Proof. The sufficiency follows immediately from Lemma 5.3 as $\langle G', \mathcal{A}'\rangle = \langle G_1 \cup P_1 \cup P_2 \cup G_2, \mathcal{A}\rangle / (E(P_1 \backslash v_1) \cup E(P_2))$. To verify the necessity, let $[S_1, S_2]$ be a feasible solution to $\langle G, \mathcal{A}\rangle$ and let $G' = (V', E')$. In the case of (IV-1), we deduce from $A_h = \{v_1, v_3\}$ that $v_1v_3 \in [S_1, S_2]$, which allows us to assume $v_1 \in S_1$ and $v_3 \in S_2$. From the evenness of $|A_i \cap V_j|$ for every $i \in \{1, 2, 3\} - \{h\}$ and $j = 1, 2$, we see that $v_2v_4 \notin [S_1, S_2]$, for otherwise, $V_1 \subseteq S_1$ and $V_2 \subseteq S_2$, a contradiction. By symmetry we may assume $\{v_2, v_4\} \subseteq S_1$. Then $[S_2, V' - S_2]$ is a feasible solution to $\langle G', \mathcal{A}'\rangle$.

It remains to consider the case of (IV-2). If $\{v_2, v_4\}$ is contained in one of S_1 and S_2, say S_2, in view of the edge $v_1v_3 \in E'$, we see that $S_1' = (S_1 \cap V_1) \cup (S_1 \cap V_2)$ induces a connected subgraph of G' and further that $[S_1', V' - S_1']$ is a feasible solution to $\langle G', \mathcal{A}'\rangle$. Thus we may assume $v_2 \in S_1$ and $v_4 \in S_2$. Since $V(\mathcal{A}) \subseteq V_1 \cup V_2$, $|A_i \cap S_j|$ is odd, and $|A_i \cap V_j|$ is even for all $i = 1, 2, 3$, $j = 1, 2$, by symmetry we may assume $S_1 \cap V_2 \neq \emptyset$. Since $(G_1 \cup G_3, G_2)$ is a 2-separation of G at $\{v_3, v_4\}$, the connectivities of $G[S_1]$ and $G[S_2]$ imply that $v_3 \in S_1$ and that $[S_1 \cap V_2, S_2 \cap V_2]$ is a bond of G_2. As $|A_i \cap V_2|$ is even, we obtain

$$|S_1 \cap V_2 \cap A_i| \equiv |S_2 \cap V_2 \cap A_i| \pmod 2, \textit{ for } i = 1, 2, 3. \tag{5.2}$$

If $S_2 \cap V_1 = \emptyset$, then, for $i = 1, 2, 3$, the cardinality of $S_2 \cap V_2 \cap A_i = S_2 \cap A_i$ is odd. By (5.2), $|S_1 \cap V_2 \cap A_i|$ is also odd and thus $[S_1 \cap V_2, V' - (S_1 \cap V_2)]$ is a feasible solution to $\langle G', \mathcal{A}'\rangle$. If $S_2 \cap V_1 \neq \emptyset$, then, similarly, $v_1 \in S_2$ and $[S_1 \cap V_1, S_2 \cap V_1]$ a bond of G_1. Put $S_1' := (S_2 \cap V_1) \cup (S_1 \cap V_2)$. Then the edge $v_1v_3 \in E'$ ensures that $[S_1', V' - S_1']$ is a bond of G'. Moreover, as $S_1' \subseteq V$, we have

$|S_1' \cap A_i'| = |S_1' \cap A_i| = |(S_2 \cap V_1) \cap A_i| + |(S_1 \cap V_2) \cap A_i|$. Since $V(\mathcal{A}) \subseteq V_1 \cup V_2$, it follows from (5.2) that $|S_1' \cap A_i'| \equiv |S_2 \cap V_1 \cap A_i| + |S_2 \cap V_2 \cap A_i| = |S_2 \cap A_i| \equiv 1 \pmod 2$ for $i = 1, 2, 3$, so $[S_1', V' - S_1']$ is a feasible solution to $\langle G', \mathcal{A}' \rangle$. ■

6 Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. Let $\langle G, \mathcal{A} \rangle$ be a nontrivial quadruple and let M be the matroid as defined in Section 2. Recall that M^* is a series-extension of a 3-connected matroid if and only if M is a parallel-extension of a 3-connected matroid. We say that the quadruple $\langle G, \mathcal{A} \rangle$ is *reducible* if it admits one of reductions I–IV and *irreducible* otherwise.

The following lemma essentially asserts that the hypothesis of Lemma 3.9 is satisfied if $\langle G, \mathcal{A} \rangle$ is irreducible.

Lemma 6.1 *Suppose quadruple $\langle G, \mathcal{A} \rangle$ is nontrivial, acyclic, and irreducible. If $A_i \neq A_j$ for all $1 \leq i < j \leq 3$, then G is 2-connected, M^* is a series-extension of a 3-connected matroid containing A, and M^* has no vertical 3-separation (F_1, F_2) with $A \subseteq F_1$.*

Proof. Clearly, G is 2-connected as no reduction I applies to $\langle G, \mathcal{A} \rangle$. In this proof the same notation set in Section 4 will be used. We first prove that

(1) M^* is a series-extension of a 3-connected matroid containing A.

Assuming the contrary, then M is not a parallel-extension of a 3-connected matroid containing A. Thus Lemma 4.3 guarantees the existence of a partition (E_1, E_2) of E with $o(E_1, E_2) = 2 \leq \min\{r(E_1), r(E_2)\}$, such that for each $h \in \{1, 2, 3\}$, there exists an $i \in \{1, 2\}$ for which

$$A_h \subseteq V_i, \text{ and } |A_h \cap V(G_i^j)| \text{ is even for all } 1 \leq j \leq t_i. \tag{6.1}$$

It follows from Lemma 4.1(ii) that $t_1 = t_2 = t = |V_0|/2$ and $|V(G_i^j) \cap V_0| = 2$ for $i = 1, 2$ and $j = 1, 2, ..., t$, where $V_0 = V_1 \cap V_2$ and $G_i = (V_i, E_i) := G[E_i]$ is the disjoint union of its components $G_i^1, G_i^2, \ldots, G_i^t$ for $i = 1, 2$. Symmetry allows us to assume that $A_1 \cup A_2 \subseteq V_1$. Thus $V(\mathcal{A}) - V_1 \subseteq A_3 - (A_1 \cup A_2)$ and hence $V_2 - V_0$ is uniform. Since $\langle G, \mathcal{A} \rangle$ admits no reduction II, from the definition we deduce that each G_2^j $(1 \leq j \leq t)$ is a path with all internal vertices (if any) belonging to $A_3 - (A_1 \cup A_2)$. Observe that G_2 is not a path of length two, for otherwise its internal vertex belongs to $A_3 - (A_1 \cup A_2)$ and hence $A_3 \subseteq V_2$ by (6.1), which also implies that A_3 consists of two adjacent vertices in G_2. So a reduction II-1 applies to $\langle G, \mathcal{A} \rangle$, a contradiction. Let us show that

$$\text{Every } G_1^j \ (1 \leq j \leq t) \text{ is incident with two disjoint edges } f_j \text{ and } g_j \text{ in } E_2. \tag{6.2}$$

Otherwise, $t = 1$ and G_2 is a path of length at most two linking the two vertices in V_0. Hence either G_2 is a path of length exactly two, contradicting the preceding observation, or $|E_2| = 1$, contradicting the inequality $r(E_2) \geq 2$. So (6.2) follows. Moreover,

$$\text{Each } G \backslash \{f_j, g_j\} \text{ has precisely two components } H_1^j \text{ and } H_2^j, \text{ with } H_1^j = G_1^j. \tag{6.3}$$

To justify this, let G' be the graph obtained from G by replacing each G_1^j $(1 \leq j \leq t)$ with an edge e_j between the two vertices in $V(G_1^j) \cap V_0$. Then G' is a Hamiltonian cycle, because G is 2-connected and each G_2^j $(1 \le j \le t)$ is a path. It follows that $G'\backslash\{f_j, g_j\}$ has precisely two components, one of which consists of e_j only. So (6.3) holds.

It follows from (6.3) that each pair $\{f_j, g_j\}$ defines a 4-separation $(H_1^j \cup H_2^j, H_3^j)$ of G, where $H_3^j = G[\{f_j, g_j\}]$. By (6.1) and the assumption that $A_1 \cup A_2 \subseteq V_1$, we see that A_1 and A_2 both have even-sized intersections with $V(G_1^j)$. If A_3 has an even-sized intersection with some $V(G_1^j)$, then $(H_1^j \cup H_2^j, H_3^j)$ would be an even 4-separation satisfying (IV-2). So a reduction IV applies to $\langle G, \mathcal{A}\rangle$. This contradiction implies that A_3 has an odd-sized intersection with every $V(G_1^j)$.

From (6.1) and (6.2) we deduce that $A_3 \subseteq V_2$, $|A_3 \cap V(G_2^j)|$ is even, and $A_3 \cap V(G_1^j)$ consists of precisely one vertex in V_0 for each $1 \le j \le t$. Suppose $f_1 = a_1b_1$ and $g_1 = c_1d_1$, with $\{a_1, c_1\} \subseteq V_0 \cap V(G_1^1)$. Renaming the edges if necessary, we assume $a_1 \in A_3$ and $c_1 \notin A_3$. Observe that $b_1 \in A_3$, for otherwise, let $f_1 \in E(G_2^\ell)$ for some ℓ. Since G_2^ℓ is a path with all internal vertices (if any) belonging to $A_3-(A_1\cup A_2)$, we have $V(G_2^\ell) = \{a_1, b_1\}$, contradicting the fact that $|A_3\cap V(G_2^\ell)|$ is even. Next, $b_1 \in V_0$, for otherwise, b_1 has degree two and is incident with precisely two edges f_1 and f_1'. Since each G_2^j $(1 \le j \le t)$ is a path with all internal vertices (if any) belonging to $A_3 - (A_1 \cup A_2)$, we see from (6.1) and the fact $c_1 \notin A_3$ that f_1' is incident with neither c_1 nor d_1. Thus $\{f_1', g_1\}$ defines an even 4-separation satisfying (IV-2) and hence a reduction IV applies to $\langle G, \mathcal{A}\rangle$, a contradiction again.

Without loss of generality, we assume that $f_1 = f_2$. Recalling the statements established in the preceding paragraph, we have $A_3 \cap V(G_1^2) = \{b_1\}$. Observe that $g_1 \ne g_2$, for otherwise, $\{f_1, g_1\}$ defines an even 4-separation $(G_1^1 \cup G_1^2, H_3^1)$ satisfying (IV-1) and hence a reduction IV applies to $\langle G, \mathcal{A}\rangle$, a contradiction. If g_1 and g_2 are disjoint, then $\{g_1, g_2\}$ defines an even 4-separation $(H_1 \cup H_2, H_3)$ satisfying (IV-2), with $H_1 = (G_1^1 \cup G_1^2) + f_1$, $H_2 = G\backslash V(H_1)$ and $H_3 = G[\{g_1, g_2\}]$, so a reduction IV applies to $\langle G, \mathcal{A}\rangle$. This contradiction implies that d_1 is the common end of g_1 and g_2 and $d_1 \in V_2 - V_0$. Since $|A_3 \cap V(G_1^j)| = 1$ for $j = 1, 2$ and $\{a_1, b_1\} \subseteq A_3$, the ends of g_1 and g_2 in V_0 are outside A_3. Let G_2^ℓ be the component of G_2 containing g_1 and g_2. Note that G_2^ℓ consists of g_1 and g_2 only. Since $|A_3 \cap V(G_2^\ell)|$ is even, we have $d_1 \notin A_3$ and hence $d_1 \notin V(\mathcal{A})$. It follows that $(G\backslash d_1, G_2^\ell)$ is a 2-separation satisfying (II-2) and hence a reduction II-2 applies to $\langle G, \mathcal{A}\rangle$. This contradiction proves (1).

It remains to verify that

(2) M^* has no vertical 3-separation (F_1, F_2) with $A \subseteq F_1$.

Assume, on the contrary, that (F_1, F_2) is a vertical 3-separation of M^* with $A \subseteq F_1$. Let (E_1, E_2) be the restriction of (F_1, F_2) to E and let $G_i = (V_i, E_i) := G[E_i]$ for $i = 1, 2$. Put $V_0 := V_1 \cap V_2$. By Lemma 4.4, we have

$$\textit{Some component } H_2 \textit{ of } G_2 \textit{ contains a cycle.} \tag{6.4}$$

Let $H_1 := G\backslash(V(H_2) - V_0)$ and $\{v_1, v_2, \ldots, v_h\} := V(H_1) \cap V(H_2) \subseteq V_0$. The assumption and (1) imply that Lemma 4.4 (ii) or (iii) holds. Thus, by Lemma 4.1(ii) and (iii), we have $2 \le h \le 4$. Let us now consider two cases.

Case 1. Lemma 4.4(ii) holds; that is, $o(E_1, E_2) = 2$ and $V_2 - V_0$ is uniform. In this case, $h = 2$ by Lemma 4.1(ii) and $H_2\backslash\{v_1, v_2\}$ is uniform. In view of (6.4), a reduction II-2 applies to $\langle G, \mathcal{A}\rangle$ based on (H_1, H_2), a contradiction.

Case 2. Lemma 4.4(iii) holds; that is, $o(E_1, E_2) = 3$, $V(\mathcal{A}) \subseteq V_1$, and $|A_i \cap V(G_1^j)|$ is even for every $1 \le i \le 3$ and $1 \le j \le t_1$.

When $h = 2$, clearly $\langle G, \mathcal{A}\rangle$ admits a reduction II-2 based on (H_1, H_2) in which H_2 is replaced by an edge v_1v_2.

When $h = 3$, if $H_2 = v_1v_2v_3v_1$ is a triangle, or some component of $H_2\backslash\{v_1, v_2, v_3\}$ is adjacent to all of v_1, v_2, and v_3 then, by (6.4), G admits a reduction III based on (H_1, H_2); else, by symmetry and the 2-connectivity of G, we may assume the existence of a 2-separation (H_1', H_2') of G at $\{v_1, v_2\}$ such that $H_1' \supset H_1$, $H_2' \subseteq H_2\backslash v_3$, and $|E(H_2')| > 1$, yielding a reduction II-2 for $\langle G, \mathcal{A}\rangle$ based on (H_1', H_2').

When $h = 4$, by Lemma 4.1, we see that G_i^j contains exactly two vertices from V_0 for all $i = 1, 2$ and $j = 1, 2, ..., t_i$, except for the one denoted by H_2. It is easy to see that H_1 consists of all G_i^j with $G_i^j \neq H_2$. Moreover, if we replace (in H_1) each such G_i^j with an edge between the vertices of $V(G_i^j) \cap V_0$, then in the resulting graph all vertices have degree two, except for v_1, v_2, v_3, v_4 which have degree one. Hence H_1 consists of two components J_1 and J_2 with $|V(J_j) \cap \{v_1, v_2, v_3, v_4\}| = 2$ for $j = 1, 2$, $V(\mathcal{A}) \subseteq V(J_1) \cup V(J_2)$, and $|A_i \cap V(J_j)|$ is even for all $i = 1, 2, 3$ and $j = 1, 2$. Since G is 2-connected, there are two disjoint paths between $V(J_1) \cap \{v_1, v_2, v_3, v_4\}$ and $V(J_2) \cap \{v_1, v_2, v_3, v_4\}$, which are fully contained in H_2. Thus $\langle G, \mathcal{A}\rangle$ admits a reduction IV-2 based on $(J_1 \cup J_2, H_2)$.

So $\langle G, \mathcal{A}\rangle$ is reducible in all subcases, this contradiction completes the proof of (2) and hence of the lemma. ■

Proof of Theorem 1.1. Suppose $\langle G, \mathcal{A}\rangle$ is nontrivial, acyclic, and irreducible. If $A_i \neq A_j$ for all i, j with $1 \leq i \neq j \leq 3$, then the assertion follows instantly from Lemma 6.1 and Lemma 3.9. In the opposite case, symmetry allows us to assume that $A_1 = A_2$. Since $\langle G, \mathcal{A}\rangle$ is irreducible, G is 2-connected. Thus Lemma 5.2 guarantees the existence of a bond $[S_1, S_2]$ of G such that $|S_1 \cap A_2| = |S_1 \cap A_3| = 1$. Clearly, $[S_1, S_2]$ is a feasible solution to $\langle G, \mathcal{A}\rangle$. ■

7 More on infeasible quadruples

In this section we prove three more lemmas that will be used in proving Theorem 1.2. The first is a corollary of Theorem 1.1.

Lemma 7.1 *Suppose $\langle G, \mathcal{A}\rangle$ is nontrivial and acyclic. If G has a plane representation in which $V(\mathcal{A})$ is contained in a facial cycle C, then $\langle G, \mathcal{A}\rangle$ is infeasible if and only if $\mathcal{A}$ is linked by C.*

Proof. Let us first show that $\langle G, \mathcal{A}\rangle$ is feasible if and only if $\langle C, \mathcal{A}\rangle$ is feasible. To justify this, note that the "if" part follows instantly from Lemma 5.3. To establish the "only if" part, let $[S_1, S_2]$ be a feasible solution to $\langle G, \mathcal{A}\rangle$. Since C is a facial cycle and $[S_1, S_2]$ is bond, $G[S_i] \cap C$ $(i = 1, 2)$ is either empty or a path in C or the whole C. On the other hand, since $V(\mathcal{A}) \subseteq V(C)$ and $|S_i \cap A_j|$ is odd for all $i = 1, 2$ and $j = 1, 2, 3$, we deduce that $S_i \cap V(C)$ $(i = 1, 2)$ induces a path on C having an odd-sized intersection with each of A_1, A_2, A_3. It follows that $[S_1 \cap V(C), S_2 \cap V(C)]$ is a feasible solution to $\langle C, \mathcal{A}\rangle$, as desired.

It remains to prove that $\langle C, \mathcal{A}\rangle$ is infeasible if and only if $\mathcal{A}$ is linked by C. The "if" part was proved in Section 1 so we only need to show the "only if" part. Suppose $\langle C, \mathcal{A}\rangle$ is infeasible. Let C' be a disjoint copy of C and let H be the cubic planar graph obtained from $C \cup C'$ by adding a perfect matching linking the corresponding vertices. It follows from what we proved in the last paragraph that $\langle H, \mathcal{A}\rangle$ is infeasible. Note that $|V(C)| \geq 4$, for otherwise $A_1 \cap A_2 \cap A_3 \neq \emptyset$, which would mean that $\langle C, \mathcal{A}\rangle$ is feasible. Therefore, H is triangle-free and 3-connected. It is routine to verify that $\langle H, \mathcal{A}\rangle$ is irreducible. By Theorem 1.1, $\mathcal{A}$ is linked by C. ■

It is worthy of noting that contractions in reductions I, II-1, and IV-1 might reduce a nontrivial and acyclic instance $\langle G, \mathcal{A}\rangle$ to a trivial or cyclic one. When this happens, although the reduction confirms the infeasibility of $\langle G, \mathcal{A}\rangle$, it only provides us with information on $\mathcal{A}$ and it loses all information on G. Since we want to understand the structure of a nontrivial acyclic infeasible quadruple, we wish to keep a quadruple that way after each reduction. The following lemma says that this is possible. This result is a strengthening of Theorem 1.1 when G is 2-connected.

Lemma 7.2 *Suppose quadruple $\langle G, \mathcal{A}\rangle$ is nontrivial, acyclic, and infeasible. If G is 2-connected, then $\langle G, \mathcal{A}\rangle$ can be reduced by reductions II, III, IV to a nontrivial and acyclic quadruple $\langle G', \mathcal{A}'\rangle$, such that G' is 2-connected and has a plane representation in which $\mathcal{A}'$ is linked by a facial cycle.*

Remark. Since any reduction of a trivial or cyclic quadruple remains trivial or cyclic, respectively, this lemma also implies that when $\langle G, \mathcal{A}\rangle$ is reduced to $\langle G', \mathcal{A}'\rangle$, all intermediate quadruples are nontrivial, acyclic, and infeasible as well. This observation also follows from the proof below.

Proof. Suppose the lemma is false. We consider a counterexample $\langle G, \mathcal{A}\rangle$ with $2|E(G)|-|V(G)|$ as small as possible. By Theorem 1.1, $\langle G, \mathcal{A}\rangle$ admits one of reductions II-IV, which we denote by π. It follows from Lemmas 5.5–5.7 that the result, $\langle G', \mathcal{A}'\rangle$, of applying π is infeasible. Moreover, since G' is 2-connected and is simpler than G (meaning $2|E(G')|-|V(G')| < 2|E(G)|-|V(G)|$), we deduce from the minimality of $\langle G, \mathcal{A}\rangle$ that $\langle G', \mathcal{A}'\rangle$ is either trivial or cyclic. Let $\mathcal{A} = \{A_1, A_2, A_3\}$.

Note that π is not reduction II-2 or III, because otherwise from their definitions it is clear that $\langle G', \mathcal{A}'\rangle$ would be both nontrivial and acyclic. Therefore, we may assume that reductions II-2 and III do not apply to $\langle G, \mathcal{A}\rangle$, and thus π must be reduction II-1 or IV.

Suppose π is reduction II-1, based on a 2-separation (G_1, G_2) of G at $\{v_1, v_2\}$, where $G_2 = v_1uv_2$ and $\{v_1, u\} = A_i$ for some i, say $i = 1$. Since $\langle G', \mathcal{A}'\rangle$ is trivial or cyclic, either $\{u, v_2\} = A_2$ or A_3, or $\{u, v_2\} \supseteq A_1\Delta A_2\Delta A_3$. It is a routine matter to check that $G_1\backslash\{v_1, v_2\}$ is uniform in all these cases. Using our assumption that II-2 does not apply to $\langle G, \mathcal{A}\rangle$ we conclude that G_1 is a path and thus G is a cycle, contradicting the fact that G is a counterexample (by Lemma 7.1).

It remains to consider the case when π is reduction IV. Suppose the reduction is based on 4-separation $(G_1 \cup G_2, G_3)$ of G at $\{v_1, v_2, v_3, v_4\}$, where G_3 contains disjoint paths P_1 from v_1 to v_3 and P_2 from v_2 to v_4. In the subcase of (IV-1), some $A_h = \{v_1, v_3\}$, say $h = 1$. Since $\langle G', \mathcal{A}'\rangle$ is trivial or cyclic, either $\{v_2, v_4\} = A_2$ or A_3, or $\{v_2, v_4\} \supseteq A_1\Delta A_2\Delta A_3$. An argument similar to what was used in the last case shows that both G_1 and G_2 would be paths, and G would be a cycle, leading to a contraction. In the subcase of (IV-2), the parity condition implies that neither $\{v_1, v_3\}$ nor $\{v_2, v_4\}$ can be A_h for any $h = 1, 2, 3$. Let $i \in \{1, 2\}$ such that $\{v_i, v_{i+2}\} \not\supseteq A_1\Delta A_2\Delta A_3$ and let us assume that π is performed such that v_i and v_{i+2} get identified. Then $\langle G', \mathcal{A}'\rangle$ is both nontrivial and acyclic, a contradiction, which proves the lemma. ∎

In the end of this section we introduce a simplification of reduction IV-2, which will be useful in proving Theorem 1.2. Let $\langle G, \mathcal{A}\rangle$ be a quadruple and $(G_1 \cup G_2, G_3)$ be a 4-separation of G that satisfies all requirements in the definition of reduction IV-2. Then *reduction IV'* (see Figure 4 below) reduces $\langle G, \mathcal{A}\rangle$ to quadruple $\langle G', \mathcal{A}\rangle$, instead of $\langle G', \mathcal{A}\rangle/v_2v_4$, where $G' = G_1 \cup G_2 + \{v_1v_3, v_2v_4\}$. Note that G' is a proper minor of G (and thus is simpler than G) as long as $|E(G_3)| > 2$.

Clearly, reduction IV' maintains the 2-connectivity of a graph, as well as the nontriviality and acyclicity of a quadruple (since $\mathcal{A}$ remains unchanged). In addition, we also have the following.

Lemma 7.3 *Let $\langle G', \mathcal{A}\rangle$ be obtained from $\langle G, \mathcal{A}\rangle$ by a reduction IV'. Then $\langle G, \mathcal{A}\rangle$ is feasible if and only if $\langle G', \mathcal{A}\rangle$ is feasible.*

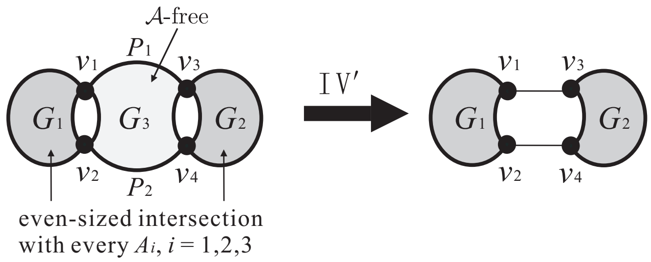

Figure 4: Reductions IV′.

Proof. Note that both $\langle G, \mathcal{A}\rangle$ and $\langle G', \mathcal{A}\rangle$ can be reduced to the same $\langle H, \mathcal{B}\rangle$ by reductions IV. Using Lemma 5.7 twice, we deduce that $\langle G, \mathcal{A}\rangle$ is feasible if and only if $\langle H, \mathcal{B}\rangle$ is feasible if and only if $\langle G', \mathcal{A}\rangle$ is feasible. ∎

8 Weakly linkable quadruples

The purpose of this section is to establish Theorem 1.2, which provides a global structure for infeasible quadruples that are nontrival and acyclic. We begin with a few definitions. Let $G = (V, E)$ be a graph, $G_1 = (V_1, E_1)$ be its subgraph, X be the set of vertices in V_1 that are not incident with any edge in $E - E_1$, and $H = (U, F)$ be a graph with $F \cap E = \emptyset$ and $U \cap V = V_1 - X$. Then $(G\backslash(X \cup E_1)) \cup H$ is the result of *substituting* G_1 *with* H. Let $\langle G, \mathcal{A}\rangle$ be a quadruple with $\mathcal{A} = \{A_1, A_2, A_3\}$. A *triad* is a subgraph of G with three edges vx, vy, vz such that x, y, z are distinct, v is not in $V(\mathcal{A})$, and v has degree three in G. A path in G is an $\mathcal{A}$-*path* if its set of internal vertices X satisfies: $\emptyset \neq X \subseteq V(\mathcal{A})$, X is uniform, and every $x \in X$ has degree two in G. Let C be a cycle in G. A C-*rectangle* is a 4-cycle $v_1v_2v_4v_3v_1$ (not $v_1v_2v_3v_4v_1$) such that v_1v_2, v_3v_4 are not in C, v_1v_3, v_2v_4 are in C, and v_1v_3, v_2v_4 form a bond of G that separates $V(G)$ into V_1, V_2 with $|A_i \cap V_j|$ even for all $i = 1, 2, 3$ and $j = 1, 2$ (cf. labels in Figure 4).

A quadruple $\langle G, \mathcal{A}\rangle$ is *linkable* by a cycle C if G has a plane representation in which C is a facial cycle and $\mathcal{A}$ is linked by C. We call $\langle G, \mathcal{A}\rangle$ *weakly linkable* if there is a quadruple $\langle G', \mathcal{A}'\rangle$ that is linkable by a cycle C', in which there exist a set $\mathcal{R}$ of C'-rectangles, a set $\mathcal{P}$ of $\mathcal{A}'$-paths (they have to be in C'), a set $\mathcal{T}$ of triads, and a set $\mathcal{F}$ of edges (which are considered as single edge subgraphs), all being mutually edge-disjoint, such that $\langle G, \mathcal{A}\rangle$ is obtained by

(i) substituting each Z in $\mathcal{F} \cup \mathcal{T}$ with a graph H_Z for which $H_Z\backslash V(Z)$ has a component that is adjacent to all (two or three) vertices in $V(H_Z) \cap V(Z)$;

(ii) substituting each $R = v_1v_2v_4v_3v_1$ in $\mathcal{R}$ with a graph H_R for which there are two vertex disjoint paths, one from v_1 to v_3 and one from v_2 to v_4;

(iii) substituting each P in $\mathcal{P}$ with a graph H_P for which $H_P + uv$ is 2-connected, where u, v are ends of P; in this case, if $X = V(P) - \{u, v\}$, we also choose nonempty $Y \subseteq V(H_P) - \{u, v\}$ with $|Y| \equiv |X| \pmod 2$, and, for each A_i that meets X, we replace A_i with $(A_i - X) \cup Y$.

We will call $(\langle G', \mathcal{A}'\rangle, C', \{H_Q : Q \in \mathcal{Q}\})$, where $\mathcal{Q} = \mathcal{R} \cup \mathcal{P} \cup \mathcal{T} \cup \mathcal{F}$, a *certificate* for $\langle G, \mathcal{A}\rangle$.

Remark 1. It is clear from this definition that, by reversing the constructions, every weakly linkable quadruple on a 2-connected graph can be reduced to a linkable quadruple by reductions II-2, III, and IV'. Since linkable quadruples are infeasible, we deduce from Lemmas 5.5, 5.6, 7.3, and

7.1 that weakly linkable quadruples on 2-connected graphs are infeasible. Conversely, Theorem 1.2 asserts that every nontrivial acyclic infeasible quadruple on a 2-connected graph is weakly linkable.

Remark 2. If A_i consists of the two ends of edge e_i $(i = 1, 2, 3)$ and $e_1e_2e_3$ is a path of length three, then $\langle G, \mathcal{A}\rangle$ is nontrivial and acyclic. Assuming that G is 2-connected, we deduce from Theorem 1.2 that $\langle G, \mathcal{A}\rangle$ is infeasible if and only if it is weakly linkable. Note that in any certificate we must have $\mathcal{R} \cup \mathcal{P} = \emptyset$, thus $\langle G, \mathcal{A}\rangle$ is infeasible if and only if, "up to" 2- and 3-separations, G has a plane representation such that e_1, e_2, e_3 are contained in a facial cycle. As discussed in the introduction, this is exactly Seymour's solution on the 2-linkage problem [7].

The remainder of this section is a proof of Theorem 1.2. We begin with a lemma.

Lemma 8.1 *Suppose G is 2-connected and $\langle G, \mathcal{A}\rangle$ is weakly linkable. Then there exists a certificate $(\langle G', \mathcal{A}'\rangle, C', \{H_Q : Q \in \mathcal{Q}\})$ such that each H_Q is 2-connected.*

Proof. Choose a certificate $(\langle G', \mathcal{A}'\rangle, C', \{H_Q : Q \in \mathcal{Q}\})$ such that

(1) $\sigma(\mathcal{Q}) = \sum\{|E(H_Q)| : Q \in \mathcal{Q}\}$ is minimized, and

(2) subject to (1), $|\mathcal{Q}|$ is maximized.

Suppose that some H_Q is not 2-connected. Let $B = V(H_Q) \cap V(G')$. Note that $|B| = 2$, 3, or 4. We first prove that H_Q is connected. Suppose otherwise. If $|B| < 4$, then a component of H_Q contains at most one vertex from B. Since G is 2-connected, this component must consist of a single vertex in B, which is impossible by constructions (i) and (iii). Thus $|B| = 4$ and Q is a C'-rectangle $R = v_1v_2v_4v_3v_1$ as defined in construction (ii). It follows that H_Q consists of two components H_1 and H_2, which contain $\{v_1, v_3\}$ and $\{v_2, v_4\}$, respectively. By Lemma 7.1, $\langle G' \backslash \{v_1v_2, v_3v_4\}, \mathcal{A}'\rangle$ remains linkable by C'. Thus we get a better certificate (with $|\mathcal{Q}|$ bigger) by deleting R from $\mathcal{R}$ and adding $e_1 = v_1v_3$ and $e_2 = v_2v_4$ to $\mathcal{F}$ with $H_{e_i} = H_i$ $(i = 1, 2)$. This contradiction shows that H_Q is connected.

Now we assume that H_Q admits a 1-separation (H_1, H_2) over a cut vertex z. If $|B| = 2$, instead of simulating entire H_Q with one path Q we simulate H_1 and H_2 with two paths. In other words, we substitute Q in G' with a new path Q_1zQ_2, where the lengths of Q_1, Q_2 are determined by reduction II-2 (as in reducing H_1, H_2), and we also modify $\mathcal{A}'$ accordingly. Let $H_{Q_i} = H_i$ $(i = 1, 2)$. We delete Q from $\mathcal{Q}$ and add Q_i $(i = 1, 2)$ to $\mathcal{Q}$, provided that H_{Q_i} has three or more vertices (so it can be used in constructions (i) and (iii)). By Lemmas 5.5 and 7.1, it is not difficult to see that we get a new certificate with either a smaller $\sigma(\mathcal{Q})$ (if some H_i has only two vertices) or a bigger $|\mathcal{Q}|$, a contradiction. The argument for the case $|B| = 3$ is almost identical so we omit the detail.

If $|B| = 4$, then Q is a C'-rectangle $R = v_1v_2v_4v_3v_1$. From construction (ii) we may assume (by renaming vertices of R if necessary) one of the following holds:

- $v_1, v_3 \in V(H_1 - z)$ and $v_2, v_4 \in V(H_2 - z)$;
- $v_1 \in V(H_1 - z)$ and $v_2, v_3, v_4 \in V(H_2 - z)$;
- $z = v_3$, $v_1 \in V(H_1)$, and $v_2, v_4 \in V(H_2)$.

As before, in the first case we simulate H_1, H_2 by two triads; in the second case we simulate H_1, H_2 by an edge and a rectangle, respectively; in the third case we simulate H_1, H_2 by an edge and a triad, respectively. It is possible that we want to simulate H_i by a triad but it does not satisfy the requirement in construction (i). In this case H_i can be simulated by three edges. It is also possible that we want to simulate a graph by an edge yet the graph does not satisfy the requirement in construction (i). This can only happen when the graph has only two vertices. In this case we may leave the graph in G' and we do not need to simulate it. In all these cases, it is straightforward to verify that we end up with a new certificate with either a smaller $\sigma(\mathcal{Q})$ or a bigger $|\mathcal{Q}|$, a contradiction, which completes the proof. ■

Proof of Theorem 1.2. The "if" part is given by Remark 1 above. To prove the "only if" part we assume that the result is false. Namely, there exists a nontrivial acyclic infeasible quadruple $\langle G^*, \mathcal{A}^*\rangle$ on a 2-connected graph that is not weakly linkable. We choose such a counterexample with $2|E(G^*)| - |V(G^*)|$ as small as possible. By Lemma 7.2, there exists π, a reduction II, III, or IV, such that applying π to $\langle G^*, \mathcal{A}^*\rangle$ results in a nontrivial acyclic infeasible quadruple $\langle G, \mathcal{A}\rangle$. Since G is 2-connected and is simpler than G^* (meaning $2|E(G)| - |V(G)| < 2|E(G^*)| - |V(G^*)|$), the minimality of $\langle G^*, \mathcal{A}^*\rangle$ implies that $\langle G, \mathcal{A}\rangle$ is weakly linkable. Let $(\langle G', \mathcal{A}'\rangle, C', \{H_Q : Q \in \mathcal{Q}\})$ be a certificate, where $\mathcal{Q} = \mathcal{R} \cup \mathcal{P} \cup \mathcal{T} \cup \mathcal{F}$. In the following we consider all possibilities for π and we deduce a contradiction in every case by showing that $\langle G^*, \mathcal{A}^*\rangle$ is weakly linkable.

Case 1a. π is reduction II-1. Let $\langle G, \mathcal{A}\rangle = \langle G^*, \mathcal{A}^*\rangle/u^*v_2$, where u^* has degree two in G^*. Let v_1 be the other neighbor of u^* in G^* and let u be the new vertex in G. Since $\{u^*, v_1\} = A_i^*$ for some i, we may assume $i = 1$. It follows that $A_1 = \{u, v_1\}$ in $\mathcal{A}$.

We first consider the subcase when $A_1 \subseteq V(H_Q) - V(C') := U$, for some $Q \in \mathcal{Q}$. This can only happen when $Q \in \mathcal{P}$. It follows that U is uniform in $\langle G, \mathcal{A}\rangle$, $U \cap V(\mathcal{A}) = A_1$ and $|(V(G) - U) \cap A_i|$ is even ($i = 1, 2, 3$). Let x_1, x_2 be the ends of Q and let H_Q^* be the subgraph of G^* obtained by uncontracting u back to u^*v_2. By the 2-connectivity of G^* we may assume that H_Q^* has disjoint paths from x_i to v_i ($i = 1, 2$), respectively. Since $A_1 = \{v_1, u\} \subseteq U$, $Q = x_1v_1ux_2$. Let G'' and C'' be obtained from G' and C', respectively, by substituting Q with a new path $x_1v_1u^*v_2x_2$. Let A_i'' ($i = 1, 2, 3$) be obtained from A_i' by deleting internal vertices of Q and adding v_1, v_2, or u^* according to if they belong to A_i^*. Notice that $\langle G', \mathcal{A}'\rangle = \langle G'', \mathcal{A}''\rangle/u^*v_2$. By Lemmas 5.5 and 7.1, $\langle G'', \mathcal{A}''\rangle$ is linkable by C''. Let $G''' = G'' + \{x_1x_2, v_1v_2\}$. Then $R = x_1v_1v_2x_2x_1$ is a C''-rectangle, as $|(V(G) - U) \cap A_i|$ is even ($i = 1, 2, 3$). Moreover, by Lemma 7.1, $\langle G''', \mathcal{A}''\rangle$ is linkable. Now we see that $\langle G^*, \mathcal{A}^*\rangle$ is weakly linkable since it has a certificate $(\langle G''', \mathcal{A}''\rangle, C'', \{H_S : S \in \mathcal{Q}'\})$, where $\mathcal{Q}' = (\mathcal{Q} - \{Q\}) \cup \{R\}$ and $H_R = H_Q^* - u^*$.

The next subcase is when some $H_Q - V(C')$ contains exactly one vertex from A_1. Using the same argument as we used in the previous subcase we can see that $\langle G^*, \mathcal{A}^*\rangle$ is weakly linkable. The only difference is that, instead of replacing $Q \in \mathcal{P}$ with a rectangle R we need to replace Q with a triad T, or, in a degenerate case, with an edge (which would be added to $\mathcal{F}$).

The above two subcases imply that $A_1 \subseteq V(G')$, and thus $A_1 \subseteq V(C')$. We claim that we may assume $e = uv_1 \in E(G')$. Suppose otherwise, that e belongs to some H_Q. Then $G' + e$ is planar (e can be drawn along an uv_1-path of Q) and $\langle G' + e, \mathcal{A}'\rangle$ remains linkable by C' (by Lemma 7.1). In addition, if $Q \in \mathcal{R}$ then $H_Q \backslash e$ satisfies the requirement in construction (ii), because the required disjoint paths in H_Q do not contain e, which follows from the definition of a rectangle and the fact that A_1 consists of the two ends of e. The same reasons also imply that Q remains a C'-rectangle in $\langle G' + e, \mathcal{A}'\rangle$. Therefore, $(\langle G' + e, \mathcal{A}'\rangle, C', \{H_S \backslash e : S \in \mathcal{Q}\})$ is also a certificate for $\langle G, \mathcal{A}\rangle$, and thus the claim is proved.

We further claim that we may assume $e \in E(C')$. Suppose otherwise. Let C_1', C_2' be the two uv_1-paths of C'. Since $\mathcal{A}$ is linked by C' and $|A_1| = 2$, one of C_i', say C_1', satisfies $V(C_i') \supseteq V(\mathcal{A})$. In addition, since C' is a facial cycle and e is a chord, $\{u, v_1\}$ defines a 2-separation (G_1', G_2') of G' with G_i' contains C_i' ($i = 1, 2$). By flipping G_2' it is clear that G' can be redrawn so that $C'' = C_1' + e$ is a facial cycle. By Lemma 7.1, $\langle G', \mathcal{A}'\rangle$ is also linkable by C''. Now it is routine to verify that $(\langle G', \mathcal{A}'\rangle, C'', \{H_Q : Q \in \mathcal{Q}\})$ is also a certificate for $\langle G, \mathcal{A}\rangle$, and so the claim is proved.

Let G'' and C'' be obtained by uncontracting u^*v_2 in G' and C', respectively. For $i = 1, 2, 3$, let A_i'' be obtained from $A_i' - \{u\}$ by adding u^* and/or v_2, according to if they belong to A_i^*. Then $\langle G', \mathcal{A}' \rangle = \langle G'', \mathcal{A}'' \rangle / u^*v_2$. By Lemmas 5.5 and 7.1, $\langle G'', \mathcal{A}'' \rangle$ is linkable by C''. Moreover, it is straightforward to verify that $(\langle G'', \mathcal{A}'' \rangle, C'', \{H_Q : Q \in \mathcal{Q}\})$ is a certificate for $\langle G^*, \mathcal{A}^* \rangle$, which completes the proof for Case 1a.

In the rest of the proof we assume that

$$\textit{the result of any reduction II-1 in } \langle G^*, \mathcal{A}^* \rangle \textit{ is either trivial or cyclic.} \tag{8.1}$$

We also assume that the certificate of $\langle G, \mathcal{A} \rangle$ is chosen according to Lemma 8.1.

Case 1b. π is reduction II-2. Suppose the reduction is applied to 2-separation (G_1, G_2) of G^* such that G_2 is substituted by a path P (so $G = G_1 \cup P$). If some edge of P belongs to some H_Q, Lemma 8.1 implies that the entire P is a subgraph of H_Q. Let H_Q^* be obtained from H_Q by substituting P with G_2. It follows that replacing H_Q with H_Q^* results in a certificate for $\langle G^*, \mathcal{A}^* \rangle$, a contradiction. Therefore, P is a subgraph of G' and is edge disjoint from all Q in $\mathcal{Q}$. Now it is clear that adding P to $\mathcal{P}$ and letting $H_P = G_2$ again results in a certificate for $\langle G^*, \mathcal{A}^* \rangle$, a contradiction.

Since reduction II-2 does not make a quadruple trivial or cycle, we assume in the following that

$$\langle G^*, \mathcal{A}^* \rangle \text{ admits no reduction II-2.} \tag{8.2}$$

Case 2. π is reduction III. Suppose the reduction is applied to 3-separation (G_1, G_2) of G^* such that G_2 is substituted by a triad T (so $G = G_1 \cup T$). If edges of T do not belong to any H_Q, then adding T to $\mathcal{T}$ and letting $H_T = G_2$ would result in a certificate for $\langle G^*, \mathcal{A}^* \rangle$, a contradiction. So we assume that some H_Q (which is 2-connected) contains an edge of T. By the choice of our certificate, H_Q contains at least two edges from T. We claim that we can get a certificate (which may not satisfy the conclusion of Lemma 8.1 any more) such that the entire T is a subgraph of H_Q.

Let vx, vy be two edges of T that are in H_Q and let $e = vz$ be the other edge of T. If e is also in H_Q, then we do not need to do anything. So we assume that e is not in H_Q. Since elements in $\{H_Q : Q \in \mathcal{Q}\}$ are edge-disjoint, by Lemma 8.1, e is not in H_S for any $S \in \mathcal{Q}$, and v is incident only with e and some edges of Q. Let G'' be obtained from $G' - v$ by joining z with all neighbors of v, except for z (so G'' is isomorphic to G'/e). Let C'' be the facial cycle of G'' that corresponds to C'. Let Q' be obtained from Q by the same operation. Since $v \notin V(\mathcal{A}')$, if Q is a triad, an $\mathcal{A}'$-path, or a C'-rectangle, then Q' is a triad, an $\mathcal{A}'$-path, or a C''-rectangle, respectively. In addition, $\langle G'', \mathcal{A}' \rangle$ is linkable by C''. Therefore, we have a desired certificate if we replace G' with G'' and Q with Q', and if we take $H_{Q'} = H_Q + e$.

Let the entire T be a subgraph of H_Q. Let H_Q^* be obtained from H_Q by substitute T with G_2. Then it is easy to see that replacing H_Q with H_Q^* results in a certificate for $\langle G^*, \mathcal{A}^* \rangle$, a contradiction.

Case 3a. π is reduction IV-1. Let v_1, v_2, v_3, v_4 and G_1, G_2 be as in the definition of IV-1 and let $A_1 = \{v_1, v_3\}$. Let $\langle G, \mathcal{A} \rangle = \langle G^*, \mathcal{A}^* \rangle / v_2v_4$ and let v' be the new vertex. We also consider G_1, G_2 as subgraphs of G, where we rename v_2, v_4 with v'.

If v_1v_3 does not belong to any H_Q, then, by Lemma 8.1, every H_Q is a subgraph of G_1 or G_2. Therefore, the uncontraction $\langle G'', \mathcal{A}'' \rangle$ of v_2v_4 in $\langle G', \mathcal{A}' \rangle$ is well defined. Let $C'' \subseteq G''$ be the facial cycle corresponds to C'. Then it is easy to see that $(\langle G'', \mathcal{A}'' \rangle, C'', \{H_Q : Q \in \mathcal{Q}\})$ is a certificate for $\langle G^*, \mathcal{A}^* \rangle$, a contradiction.

Next we assume that v_1v_3 belongs to some H_Q. By Lemma 8.1, H_Q also contains v'. Let $B = V(H_Q) \cap V(G')$. We claim that $|B| = 2$. If $|B| = 3$, then Q is a triad. Note that $A_1 \subseteq B$ since, by construction (i), $V(H_Q) - B$ is disjoint from $V(\mathcal{A})$. Hence $V(C') \supseteq A_1$. It follows that $C' - q$, where q is the center of triad Q, contains a v_1v_3-path, and thus $G \backslash E(H_Q)$ contains a v_1v_3-path. This path avoids edge v_1v_3 and so it contains v'. Consequently, $B = \{v_1, v_3, v'\}$, which is impossible since no component in $H_Q \backslash B$ (or even in $G \backslash B$) is adjacent to all vertices in B. If $|B| = 4$, then Q is a C'-rectangle $u_1u_2u_4u_3u_1$, where u_1u_3 is in C'. Again $A_1 \subseteq B$ and thus $V(C') \supseteq A_1$. The parity condition on rectangles implies that $A_1 = \{u_1, u_2\}$ or $\{u_3, u_4\}$. Therefore, construction (ii) implies that $G \backslash v_1v_3$ has a cycle (obtained by extending C') that contains A_1, which is impossible. Thus the claim is proved.

From $|B| = 2$ we can deduce that G_1 or G_2 (say G_2) is a subgraph of H_Q. Otherwise, since $|V(G_1) \cap V(G_2)| = 1$ and, by the 2-connectivity of G, $V(G_i) - V(H_Q)$ $(i = 1, 2)$ has at least two neighbors in $V(G_i) \cap V(H_Q)$, it would follow that $G \backslash V(H_Q)$ has at least three neighbors in $V(H_Q)$, contradicting the fact $|B| = 2$. Notice from $|B| = 2$ that $H_Q \backslash B$ is uniform in $\langle G, \mathcal{A} \rangle$. Thus $v_3 \in A_1 \cap V(H_Q \backslash B)$ implies $V(H_Q \backslash B) \cap V(\mathcal{A}) \subseteq A_1$. It follows that in G^*, $G_2 \backslash \{v_3, v_4\}$ does not contain any vertex in $V(\mathcal{A})$, which, by (8.2), implies $V(G_2) = \{v_3, v_4\}$. We may assume $B \neq \{v_1, v'\}$ because otherwise H_Q would be the triangle $v_1v_3v'v_1$ and Q would be the path v_1v_3v', which would mean that deleting Q from $\mathcal{Q}$ and adding v_1v' to G' result in a certificate for $\langle G, \mathcal{A} \rangle$ that satisfies the requirement in the first subcase of Case 3a (where v_1v_3 does not belong to H_Q for any $Q \in \mathcal{Q}$), and thus we would be done.

By the parity condition in IV-1, $|A_1^* \cap \{v_3, v_4\}| = |\{v_3\}| = 1$ but $|A_i^* \cap \{v_3, v_4\}|$ is even $(i = 2, 3)$. It follows that $\langle G^*, \mathcal{A}^* \rangle / v_3v_4$ is acyclic, as the new vertex belongs to exactly one member of $\mathcal{A}^*/v_3v_4$. By (8.1), $\langle G^*, \mathcal{A}^* \rangle / v_3v_4$ must be trivial, and so there exists $h \in \{2, 3\}$ such that $A_h^* = \{v_3, v_4\}$. Since $B \neq \{v_1, v'\}$, either v_1 or v' is in $H_Q \backslash B$. However, since $H_Q \backslash B$ is uniform in $\langle G, \mathcal{A} \rangle$, both possibilities contradict $A_h^* = \{v_3, v_4\}$ because in the first case $v_1 \in A_h^*$ and in the second case $v' \notin V(\mathcal{A})$, which leads to $v_2 \in A_h^*$. This contradiction settles Case 3a.

Case 3b. π is reduction IV-2. Let v_1, v_2, v_3, v_4, v' and G_1, G_2, G_3 be as in the definition of IV-2. Similar to Cases 3a, G_1, G_2 are also considered as subgraphs of G, where we rename v_2, v_4 with v'. We first claim that $G_i \backslash V(G_3)$ meets $V(\mathcal{A}^*)$, for $i = 1, 2$. Suppose the claim is false for, say, $i = 2$. By (8.2), $V(G_2) = \{v_3, v_4\}$. Then, by the parity condition in IV-2, $(G_2 \cup G_3) \backslash \{v_1, v_2\}$ is uniform in $\mathcal{A}^*$, which implies, by (8.2) again, that v_1v_3, v_2v_4 are the only edges of G_3 and $v_3, v_4 \in V(\mathcal{A}^*)$. If $v'v_3$ does not belong to any H_Q then it belongs to C', as $v_3 \in V(\mathcal{A}^*)$. It is easy to see that uncontracting v_2v_4 in $\langle G', \mathcal{A}' \rangle$ leads to a certificate for $\langle G^*, \mathcal{A}^* \rangle$, a contradiction. So $v'v_3$ belongs to some H_Q. By Lemma 8.1, this H_Q also contains v_1v_3. It follows that v_3 belongs to $H_Q \backslash V(G')$, which implies $Q \in \mathcal{P}$, as $v_3 \in V(\mathcal{A})$. Let H_Q^* be obtained by the uncontraction of v_2v_4 in H_Q. Since $\{v_3, v_4\}$ is uniform in $\mathcal{A}^*$ and $H_Q \backslash V(G')$ is uniform in $\mathcal{A}$, $H_Q^* \backslash V(G')$ must be uniform in $\mathcal{A}^*$. Thus replacing H_Q with H_Q^* results in a certificate for $\langle G^*, \mathcal{A}^* \rangle$. This contradiction proves the claim.

If v_1v_3 is not in any H_Q, then, by Lemma 8.1, every H_Q is a subgraph of G_1 or G_2. It follows that v' and v_1v_3 belong to C'. Moreover, uncontracting v_2v_4 in $\langle G', \mathcal{A}' \rangle$ and adding edges v_1v_2, v_3v_4 would create a rectangle $R = v_1v_2v_4v_3v_1$, and thus a certificate for $\langle G^*, \mathcal{A}^* \rangle$, where we take $H_R = G_3$. This contradiction implies that v_1v_3 belongs to some H_Q. By Lemma 8.1, this H_Q also contains v'. Let H_Q^* be obtained from H_Q by putting G_3 back. That is, $H_Q^* = G^*[E(G^*) - E(G \backslash E(H_Q))]$.

If some G_i, say $i = 2$, is a subgraph of H_Q, then, since $G_2 \backslash V(G_3)$ meets $V(\mathcal{A}^*)$, as claimed above, Q must be a path in $\mathcal{P}$ and $H_Q \backslash V(G')$ is uniform in $\mathcal{A}$. Since $V(G_2 \backslash v') \subseteq V(H_Q \backslash V(G'))$, $V(G_2 \backslash v_4)$ is uniform in $\mathcal{A}^*$. By the parity condition in IV-2, $|A_i^* \cap V(G_2)|$ must be even for all

$i = 1, 2, 3$. Hence $V(G_2)$ is uniform in $\mathcal{A}^*$, which in turn implies that $H_Q^* \backslash V(G')$ is uniform in $\mathcal{A}^*$, contradicting (8.2).

For $i = 1, 2$, let $H_i = G_i[E(G_i) - E(H_Q)]$ be subgraphs of G. Since $v_1v_2 \in E(H_Q)$ and $v' \in V(H_Q)$, the 2-connectivity of G implies $|V(H_i) \cap V(H_Q)| \geq 2$ for $i = 1, 2$. It follows from $V(H_1) \cap V(H_2) \subseteq \{v'\}$ that $|V(H_Q) \cap V(G')| \geq 3$. Thus $H_Q \backslash V(G')$ is disjoint from $V(\mathcal{A})$, and so $H_Q^* - V(H_1 \cup H_2) - \{v_2, v_4\}$ is disjoint from $V(\mathcal{A}^*)$. If $|V(H_Q) \cap V(G')| = 3$, then $v' \in V(H_Q) \cap V(G')$ and $H_Q^* - V(H_1 \cup H_2) = H_Q^* - V(H_1 \cup H_2) - \{v_2, v_4\}$ is disjoint from $V(\mathcal{A}^*)$. In this case triad $Q = \{ux, uy, uv'\}$ can be converted into a rectangle $R = xyv_4v_2x$, which leads to a certificate for $\langle G^*, \mathcal{A}^* \rangle$ if we take $H_R = H_Q^*$. If $|V(H_Q) \cap V(G')| = 4$, the parity conditions in IV-2 and C'-rectangle imply that $\{v_2, v_4\}$ is disjoint from $V(\mathcal{A}^*)$ and thus $H_Q^* \backslash V(H_1 \cup H_2)$ is disjoint from $V(\mathcal{A}^*)$. In this case by replacing H_Q with H_Q^* we get a certificate for $\langle G^*, \mathcal{A}^* \rangle$. This completes Case 3b and also the proof of Theorem 1.2. ■

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