Generating 4-connected graphs

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Abstract

Let $\mathcal{C} = \{C_n^2 : n \geq 5\}$ and let $\mathcal{L} = \{G : G$ is the line graph of an internally 4-connected cubic graph$\}$. A classical result of Martinov states that every 4-connected graph G can be constructed from graphs in $\mathcal{C} \cup \mathcal{L}$ by repeatedly splitting vertices. In this paper we prove that, in fact, G can be constructed from C_5^2 or C_6^2 in the same way, unless G belongs to $\mathcal{C} \cup \mathcal{L}$. Moreover, if G is nonplanar then G can be constructed from C_5^2.

1 Introduction

The purpose of this paper is to improve a classical chain theorem of Martinov [7] for 4-connected graphs. We begin by formally stating this result.

All graphs considered in this paper are simple. In particular, G/e denotes the graph obtained from G by contracting an edge e and then deleting parallel edges. For each integer $n \geq 5$, let C_n^2 be the graph obtained from an n-cycle C_n by joining vertices of distance two in the cycle. Notice that C_5^2 is K_5 and C_6^2 is the octahedron. In general, C_n^2 is 4-connected, and it is planar if and only if n is even. Let $\mathcal{C} = \{C_n^2 : n \geq 5\}$.

A cubic graph with at least six vertices is called *internally 4-connected* if its line graph is 4-connected. We remark that all such cubic graphs can be constructed from $K_{3,3}$ and the cube by repeatedly applying an operation known as "adding a handle" [2, 3, 8]. Let $\mathcal{L} = \{G : G$ is the line graph of an internally 4-connected cubic graph$\}$. The following is the chain theorem of Martinov [7].

Theorem 1.1. *For every 4-connected graph G there exists a sequence of 4-connected graphs $G_0, G_1, ..., G_n$ such that $G_0 = G$, $G_n \in \mathcal{C} \cup \mathcal{L}$, and every G_i $(i < n)$ has an edge e_i for which $G_i/e_i = G_{i+1}$.*

This result provides a very useful tool for analyzing 4-connected graphs. Under the current setting, it says that every 4-connected graph G can be reduced, within the class of 4-connected graphs, to a graph $G_n \in \mathcal{C} \cup \mathcal{L}$ by repeatedly contracting edges. If we reverse this process then the theorem tells us that every desired (usually unknown) 4-connected graph G can be constructed from a graph $G_n \in \mathcal{C} \cup \mathcal{L}$ by repeated "uncontractions". This approach is used successfully in characterizing 4-connected graphs that do not contains a minor isomorphic to the cube [5], to the octahedron [6], or to the octahedron plus an edge [4]. However, this theorem has two major defects which limit its further applications.

First, for a general 4-connected graph G, the starting graph G_n in the construction sequence could be any graph in $\mathcal{C} \cup \mathcal{L}$. What this means is that, in order to obtain G we have to consider infinitely many possible choices for G_n, and this increases the complexity of our analysis. It would be nice if we can narrow down the choices for G_n. The second defect of the theorem, which causes an even bigger problem, is that G_n could be planar even if G is nonplanar. As a consequence, in order to construct G, we have to exam many planar graphs, which often are useless for constructing G. The following is the main result of this paper, which corrects both defects. Let us call a sequence as described in Theorem 1.1 a (G, G_n)-*chain*.

Theorem 1.2. *Let G be a 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$. If G is planar then there exists a (G, C_6^2)-chain; if G is nonplanar then there exists a (G, K_5)-chain.*

As an application, we prove the following main result of [4]. Let Oct^+ denote the unique graph obtained from the octahedron by adding an edge.

Theorem 1.3. *If a 4-connected nonplanar graph G has no Oct^+-minor then $G = C_{2n+1}^2$ for some $n \geq 2$.*

Proof. Suppose the result is false. By Theorem 1.2, either there exists a (G, K_5)-chain or $G = L(H)$ for a nonplanar cubic graph H. The second case is impossible since $L(H)$ contains $L(K_{3,3})$, which contains Oct^+. The first case is impossible either because G has to contain one of the the three uncontractions of K_5, which are K_6, $K_6 \backslash e$, Oct^+, yet all of them contain Oct^+. ■

We close this section be introducing a few definitions. For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. If $X \subseteq V(G)$, let $N_G(X) = \{y \in V(G) - X : yx \in E(G)$ for some $x \in X\}$. Members of $N_G(X)$ are *neighbors* of X and the set $N_G(X)$ is the *neighborhood* of X. For any $x \in V(G)$, we will write $N_G(x)$ for $N_G(\{x\})$. As usual, $|N_G(x)|$ is the *degree* of x, which is denoted by $d_G(x)$. Let $E_G(x)$ stands for the set of edges of G that are incident with x. We will drop the subscript G if there is no need to emphasize G.

Let G be a k-connected graph. An edge e of G is said to be k-*contractible* if G/e is again k-connected. We may simply call e *contractible* if k is clear from the context. The new vertex of G/e will be denoted by $\overline{e}$. A subset T of $V(G)$ is called a *separating set* of G if $G - T$ has at least two components. A separating set with k vertices is called a k-*separator* of G. Observe that an edge xy of G is not k-contractible if and only if G has a k-separator containing both x and y.

Let T be a k-*separator* of a k-connected graph G. A T-*fragment* of G is the vertex set of a union of at least one but not all components of $G - T$. We often leave out the prefix T when we do not need to emphasize it. If A is a fragment of G then it is clear that $N(A)$ is a k-separator. Let us define $\bar{A} = V(G) - A - N(A)$. Then $\bar{A}$ is also a fragment of G with $N(A) = N(\bar{A})$. Notice that, for any $x \in A$, x has no neighbors in $\bar{A}$.

The organization of this paper is as follows. In Section 2, we establish a few lemmas on contractible edges. Then, in Section 3, we prove our key lemma. Finally, we prove Theorem 1.2 in Section 4.

2 Contractible edges

In this section we present a few lemmas on contractible edges. We first establish that every 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$ can be reduced to K_5 or the octahedron. Our proof is divided into two steps.

Lemma 2.1. *Suppose a k-connected ($k \geq 2$) graph G has a contractible edge $e = xy$ such that $d_{G/e}(\overline{e}) = k$ and the neighborhood of $\overline{e}$ does not contain $K_{2,k-2}$ as a subgraph. Then G has an edge e' such that G/e' is isomorphic to a graph obtained from G/e by adding at least one extra edge.*

Proof. Let $N_{G/e}(\overline{e}) = \{z_1, z_2, ..., z_k\} = Z$. Then $N_G(x) \subseteq Z \cup \{y\}$ and $N_G(y) \subseteq Z \cup \{x\}$. Since $d_G(x) \geq k$ and $d_G(y) \geq k$, we may assume, by adjusting the indices if necessary, that $N_G(x) \supseteq Z - \{z_1\}$ and $N_G(y) \supseteq Z - \{z_2\}$. Notice that G/xz_2 is isomorphic to the graph obtained from G/e by adding edges $z_2z_3, z_2z_4, ..., z_2z_k$ (and also possibly z_2z_1). If $e' = xz_2$ does not satisfy the lemma, then $z_2z_3, z_2z_4, ..., z_2z_k$ are all edges of G/e. This implies that $z_1z_3, z_1z_4, ..., z_1z_k$ are not all edges of G/e and thus $e' = yz_1$ satisfies the lemma. ■

Corollary 2.2. *Suppose e is an edge of a 4-connected graph G such that $G/e \in \mathcal{C} \cup \mathcal{L}$. Then, unless $G/e = C_5^2$ or C_6^2, G has an edge e' such that G/e' is isomorphic to a graph obtained from G/e by adding at least one extra edge.*

Proof. If $G/e = L(H)$, where H is an internally 4-connected cubic graph, then the neighborhood of $\overline{e}$ induces a matching since H is triangle free. Thus the result holds by Lemma 2.1. If $G/e = C_n^2$ for some $n \geq 7$, then the neighborhood of $\overline{e}$ induces a path and, again, the result holds by Lemma 2.1. ■

We also need the next three lemmas.

Lemma 2.3. *[1] If x is a vertex of a 4-connected graph G with $d(x) \geq 5$, then G has a contractible edge contained in $E(y)$ for some $y \in N(x)$.*

Lemma 2.4. *Let xy and xz be two edges of a k-connected graph G with $N(x) \subseteq N(y) \cup \{y, z\}$. If xy is k-contractible then so is xz.*

Proof. Suppose xz is non-contractible. Then G has a k-separator T containing both x and z. Notice that $y \notin T$ since xy is contractible. Let A be a T-fragment that contains y. Now, since $N(x) \subseteq N(y) \cup \{y, z\}$, we find that x has no neighbor in $\bar{A}$, contradicting the k-connectivity of G. ■

Lemma 2.5. *Let $e = xy$ be a k-contractible edge of a k-connected graph G. Let e' be an edge that belongs to both G and G/e. If e' is k-contractible in G/e but not in G, then some $z \in \{x, y\}$ has degree k and such that $N_G(z)$ contains both ends of e'.*

Proof. Let x', y' be the two ends of e' in G. Since e' is not contractible in G, G has a k-separator T that contains both x' and y'. Clearly, $\{x, y\} - T \neq \emptyset$ since e is contractible. Let A be a T-fragment with $A \cap \{x, y\} \neq \emptyset$. By symmetry, we may assume $x \in A$. If $y \in A$ then T is a k-separator of G/e with $T \supseteq \{x', y'\}$, contradicting the contractibility of e'. So we must have $y \in T$. If $|A| \geq 2$ then $T' = (T - \{y\}) \cup \{\overline{e}\}$ is a k-separator of G/e and T' contains both ends of e' in G/e. This is again a contradiction. It follows that $A = \{x\}$ and thus the Lemma holds with $z = x$. ■

3 A key lemma

Let x, y, z be three distinct vertices of a cycle C. Then C has two paths with ends x and y. We denote the vertex set of the path that contains z by $C[x, z, y]$, and we denote the vertex set of the other path by $C[x, \bar{z}, y]$. We also define $C(x, z, y) = C[x, z, y] - \{x, y\}$, $C(x, z, y] = C[x, z, y] - \{x\}$, and $C[x, z, y) = C[x, z, y] - \{y\}$. In addition, $C(x, \bar{z}, y)$, $C(x, \bar{z}, y]$, and $C[x, \bar{z}, y)$ are defined analogously. The purpose of this section is to prove the following.

Lemma 3.1. *If a 4-connected nonplanar graph G does not belong to $\mathcal{C} \cup \mathcal{L}$, then G has an edge e such that G/e remains 4-connected and nonplanar.*

Proof. It is hard to separate our proof into independent lemmas, so this proof will last till the end of this section. To make the proof easier to follow, we divide it into a sequence of claims.

By Theorem 1.1, G has at least one contractible edge. Let $e = xy$ be such an edge. Let us further assume that G/e is planar because otherwise we are done. This implies that $|V(G)| \geq 7$ because otherwise G/e would be a 4-connected planar graph on at most five vertices, which is impossible.

Let us consider the unique planar embedding of G/e. This embedding induces an embedding of $(G/e) - \overline{e}$. Notice that this embedding of $(G/e) - \overline{e}$ has a face F such that, in the planar embedding of G/e, all edges of $E_{G/e}(\overline{e})$ are embedded in F. Since G/e is 4-connected, $(G/e) - \overline{e}$ is 3-connected. It follows that F is bounded by a cycle C of $(G/e) - \overline{e}$, and this cycle contains all neighbors of $\overline{e}$. Moreover, the 3-connectivity of $(G/e) - \overline{e}$ also implies the following immediately.

Claim 1. *C is an induced cycle of G, and $B = G - (V(C) \cup \{x, y\})$ is connected.*

Let $x_1, x_2, \cdots, x_s$ be the neighbors of x (other than y), which are listed in the order they appear on C. Let $N_G(y) = \{x, y_1, y_2, \cdots, y_t\}$. For the purpose of simplifying our notation, we do not require $y_1, y_2, ..., y_t$ to be listed in a specific order. This setting creates a non-symmetry between x and y. As a result, in the following discussions, some of our statements are only made for one of x, y. We point out that these statements are still valid if we swap x and y, since x and y are indeed symmetric.

A quadruple (x_i, y_j, x_k, y_l) is said to be *crossing* if the four vertices are distinct and y_j, y_l are contained in different components of $C - \{x_i, x_k\}$.

Claim 2. *There exists a crossing quadruple.*

Proof. Suppose $\{x_1, x_2, ..., x_s\} = \{y_1, y_2, ..., y_t\}$. Since G/e is 4-connected, we must have $s \geq 4$ and thus the claim follows. Next, we assume by symmetry that $y_1 \notin \{x_1, x_2, ..., x_s\}$. Choose i such that $C(x_i, y_1, x_{i+1})$ contains no neighbors of x (in this section the indices are always taken modulo s). Since G is nonplanar, $C(x_i, \bar{y}_1, x_{i+1})$ must contain a neighbor of y and thus the claim is proved. □

When we say "y_j is contained in a crossing quadruple" we mean that there exists a crossing quadruple of the form (x_i, y_j, x_k, y_l). We need to make this clear since in general y_j could equal to some x_i.

Claim 3. *Every y_j is contained in a crossing quadruple, unless $y_j = x_r$ for some r and $N_G(y) - \{x\} \subseteq C[x_{r-1}, x_r, x_{r+1}]$. Moreover, there is at most one y_j that is not contained in any crossing quadruple.*

Proof. By Claim 1, G has crossing quadruple (x_m, y_a, x_n, y_b). If $y_j \notin \{x_m, x_n\}$ then either (x_m, y_a, x_n, y_j) or (x_m, y_j, x_n, y_b) is crossing. So we may assume that $y_j = x_m$. If $C(x_{m-1}, \bar{x}_m, x_{m+1})$ contains a neighbor y_l

of y, then $(x_{m-1}, y_j, x_{m+1}, y_l)$ is a crossing quadruple. Else $r = m$ satisfies the lemma. Finally, if in addition to y_j, vertex $y_{j'}$ is not contained in any crossing quadruple either, then the first part of the lemma implies that $y_{j'} = x_{r-1}$ or x_{r+1}, which in turns implies that $N_G(y) - \{x\} \subseteq C[x_{r-1}, \bar{x}_{r+1}, x_r]$ or $C[x_r, \bar{x}_{r-1}, x_{r+1}]$, contradicting the non-planarity of G. □

Claim 4. *If (x_i, y_j, x_k, y_l) is a crossing quadruple then G/yy_j is nonplanar.*

Proof. Assume G/yy_j is planar. Since G/yy_j is 3-connected, it has a unique planar embedding. On the other hand, since G/xy is 4-connected and planar, $G/xy/\overline{yx}y_j$ is 3-connected planar, and thus $G/xy/\overline{yx}y_j$ also has a unique planar embedding. It follows that the unique embedding of G/yy_j is obtained from the unique embedding of $G/xy/\overline{yx}y_j$ by splitting the vertex $z = \overline{\overline{yx}y_j}$ in a planar way. Clearly, $\{zx_i, zx_k, zy_l\} \subseteq E(G/xy/\overline{yx}y_j)$.

Without loss of generality, let us assume that, in the embedding of G/xy, $\overline{xy}$ is embedded in the interior of C. It follows that, in the embedding of $G/xy/\overline{yx}y_j$, edges zx_i, zx_k, zy_l are also embedded in the interior of C. This in turn implies that, in the embedding of G/yy_j, edges xx_i, xx_k, and $\overline{yy}_jy_l$ are embedded in the interior of C. However, this is impossible since (x_i, y_j, x_k, y_l) is crossing. So G/yy_j is nonplanar. □

Claim 5. *Suppose T is 4-separator of G that contains both y and some y_j. Then either $T = N(x)$ or $T = \{y, y_j, z, z'\}$ for some $z \in V(C) - \{y_j\}$ and $z' \in V(B)$.*

Proof. It is clear that $x \notin T$ since xy is contractible in G. Let A be a T-fragment of G with $x \in A$. If $N(x) = T$ then we are done. So let x have a neighbor $x^* \in A$. Since $y \in T$, y must have a neighbor $y^* \in \bar{A}$. Therefore, T separates x^* from y^*, which implies that some $z \in V(C) - \{y_j\}$ belongs to T.

It remains to show that $T \cap V(B) \neq \emptyset$. Suppose otherwise. Then $|T \cap V(C)| = 3$, and thus we may assume $T \cap V(C) = \{t_1, t_2, t_3\}$. Without loss of generality, let $y^* \in C(t_2, \bar{t}_3, t_1)$ and $x^* \in C(t_3, \bar{t}_1, t_2)$. It follows that $C(t_1, y^*, t_2) \subseteq \bar{A}$ and $C(t_2, x^*, t_3) \subseteq A$. Let $t_3^* \in \bar{A}$ be a neighbor of t_3. Since C has no chords (by Claim 1), we must have $t_3^* \notin C(t_1, y^*, t_2)$ and thus $\bar{A} - C(t_1, y^*, t_2) \neq \emptyset$. Similarly, since $T' = \{\overline{xy}, t_1, t_2, t_3\}$ is a 4-separator of G/xy and $A - \{x\}$ is a T'-fragment, we deduce that $A - \{x\} - C(t_2, x^*, t_3) \neq \emptyset$.

Since B is connected (by Claim 1), $V(B)$ is entirely contained in A or $\bar{A}$. It follows that either A or $\bar{A}$ is disjoint from $V(B)$. We only discuss the case $A \cap V(B) = \emptyset$ while the other case can be handled analogously. From $A - \{x\} - C(t_2, x^*, t_3) \neq \emptyset$ we deduce that $C(t_1, \bar{t}_2, t_3) \neq \emptyset$ and $A - \{x\} = C(t_2, x^*, t_3) \cup C(t_3, \bar{t}_2, t_1)$. Since G/xy is 4-connected and planar, and T' is a 4-separator of G/xy, $A - \{x\}$ must induce a connected graph, which implies that there is an edge between $C(t_2, x^*, t_3)$ and $C(t_1, \bar{t}_2, t_3)$. This is a contradiction (to Claim 1) since such an edge is a chord of C. Thus the claim is proved. □

Claim 6. *Suppose T is 4-separator of G such that $T = \{y, y_j, z, z'\}$ for some $z \in V(C) - \{y_j\}$ and $z' \in V(B)$. Then for any distinct $x_i, x_k \in V(C) - \{y_j\}$, $C(x_i, y_j, x_k)$ contains at least two neighbors of y.*

Proof. Let A be a T-fragment of G with $x \in A$. Let P_1, P_2 be the two paths of C with ends y_j and z. Since $T \cap V(C) = \{y_j, z\}$, for $i = 1, 2$, $V_i = V(P_i) - \{y_j, z\}$ is entirely contained in A or $\bar{A}$. Notice that x has a neighbor in A since $d(x) \geq 4$ and $xz' \notin E(G)$. It follows that $V(C) \cap A \neq \emptyset$. On the other hand, $V(C) \cap \overline{A} \neq \emptyset$ since y has a neighbor y_l in $\bar{A}$. Hence, we may assume $V_1 \subseteq A$ and $V_2 \subseteq \bar{A}$. Observe that $N(x) - \{y\} \subseteq V(P_1)$, it follows that $C(x_i, y_j, x_k)$ contains y_j and y_l for $x_i, x_k \in V(C) - \{y_j\}$. Hence, the claim holds. □

Claim 7. $|N(\{x, y\})| \geq 5$.

Proof. Suppose $|N(\{x, y\})| \leq 4$. Then $|N(\{x, y\})| = 4$ since G is 4-connected with $|V(G)| \geq 7$. Choose

$z \in N(\{x,y\})$ such that, if possible, z is adjacent to only one of x, y. Without loss of generality, we assume that z is adjacent to y and thus $z = y_j$ for some j. From Claim 3 and the way we choose z we deduce that y_j is in a crossing quadruple, which implies, by Claim 4, that G/yy_j is nonplanar. On the other hand, we have $|N(y) - N(x) - \{x\}| \le 1$ since $|N(\{x,y\})| = 4$. Now by the choice of z, we have $N(y) \subseteq N(x) \cup \{x, z\}$. It follows, by Lemma 2.4, that yy_j is contractible. Therefore, $e = yy_j$ satisfies Lemma 3.1 and thus the claim is proved. □

Claim 8. *Both $d(x) \ge 5$ and $d(y) \ge 5$ hold.*

Proof. By Claim 2, G has a crossing quadruple (x_i, y_j, x_k, y_l). Suppose the claim is false. Then we may assume by the symmetry between x and y that either $d(x) > d(y) = 4$ holds or $d(x) = d(y) = 4$ with $|\{y_j, y_l\} - N(x)| \ge |\{x_i, x_k\} - N(y)|$ holds. Since $d(y) = 4$ and thus $|N(y) \cap V(C)| = 3$, we may further assume that y_j is the only neighbor of y contained in $C(x_i, y_j, x_k)$.

By Claim 4, G has a 4-separator T containing both y and y_j. Note that $T \ne N(x)$ because otherwise $y_j \in N(x)$, implying $1 \ge |\{y_j, y_l\} - N(x)| \ge |\{x_i, x_k\} - N(y)|$, and thus $\{x_i, x_k\} \cap N(y) \ne \emptyset$. From these observation and $d(x) = |T| = 4 = d(y)$ we deduce that $N(\{x,y\}) = \{x_i, y_j, x_k, y_l\}$, which contradicts Claim 7. Therefore, by Claim 5, we must have $T = \{y, y_j, z, z'\}$ for some $z \in V(C) - \{y_j\}$ and $z' \in V(B)$. Consequently, by Claim 6, y has at least two neighbors in $C(x_i, y_j, x_k)$, contradicting the choice of y_j, which proves the claim. □

Claim 9. *Every y_j is contained in a crossing quadruple.*

Proof. Suppose there exists y_j that is not contained in any crossing quadruple. By Claim 2, there exists r such that $y_j = x_r$ and $N(y) - \{x\} \subseteq C[x_{r-1}, x_r, x_{r+1}]$. Note that $x_{r+2} \notin C[x_{r-1}, x_r, x_{r+1}]$ since $d(x) \ge 5$. Choose y_m, y_n such that $N(y) - \{x\} \subseteq C[y_m, \bar{x}_{r+2}, y_n]$. Since G is nonplanar, each of $C[x_{r-1}, \bar{x}_{r+2}, x_r)$ and $C(x_r, \bar{x}_{r+2}, x_{r+1}]$ contains one of y_m and y_n. As a result, (x_r, y_m, x_{r+2}, y_n) is crossing.

By Claim 4, xx_r is not contractible. It follows that there is a 4-separator T containing both x and x_r. Since $d(y) \ge 5$, Claim 5 implies that $T = \{x, x_r, z, z'\}$ for some $z \in V(C) - \{x_r\}$ and $z' \in V(B)$. Notice that x_r is the only neighbor of x in $C(y_m, x_r, y_n)$. This contradicts Claim 6 and thus the claim is proved. □

Claim 10. *No edge of C is contractible.*

Proof. Suppose to the contrary that $f \in E(C)$ is a contractible edge of G. By Claim 2, G has a crossing quadruple (x_i, y_j, x_k, y_l). Let H be the subgraph of G formed by edges in $E(C) \cup \{xy, xx_i, xx_k, yy_j, yy_l\}$. Then H is a subdivision of $K_{3,3}$. Note that G/f is planar because otherwise f is an edge satisfying the lemma, and thus we are done. It follows that H/f is no longer a subdivision of $K_{3,3}$. By symmetry, we may assume $f = x_i y_j$.

If $C(x_i, \bar{y}_j, y_l)$ contains a neighbor x_m of x, then $(H + xx_m)/f$ would still contain a subdivision of $K_{3,3}$, which is impossible. Hence $N(x) - \{y\} \subseteq C[x_i, y_j, y_l]$. This implies that $C(y_j, x_i, y_l)$ contains exactly one neighbor of x. However, by Claim 4, G has a 4-separator T that contains $\{x, x_i\}$ since (x_i, y_j, x_k, y_l) is crossing. By Claim 5, $T = \{x, x_i, z, z'\}$ for some $z \in V(C) - \{x_i\}$ and $z' \in V(B)$ since $d(y) \ge 5$. Now, by Claim 6, $C(y_j, x_i, y_l)$ contains at least two neighbors of x. This contradiction proves the claim. □

Now we are ready to complete the proof of Lemma 3.1. We apply Lemma 2.3 to $G' = G/xy$. By Claim 7, $d_{G'}(\overline{xy}) \ge 5$. Thus $E_{G'}(v)$ contains a contractible edge e' of G' for some $v \in N_{G'}(\overline{xy})$. By Lemma 2.5 and Claim 8, e' is contractible in G. However, by Claim 10, Claim 9, and Claim 4, no edge of $E(C) \cup E(\{x,y\})$ is contractible in G. Hence, $e' \notin E(C) \cup E_{G'}(\overline{xy})$. What this means is that $E(C) \cup \{xy, xx_i, xx_k, yy_j, yy_l\}$, for any crossing quadruple (x_i, y_j, x_k, y_l), remains a subdivision of $K_{3,3}$ when e' is contracted from G. Thus G/e' is 4-connected and nonplanar. The lemma is proved. ■

4 A proof of the main theorem

In this section we prove Theorem 1.2. Recall that a (G, H)-chain is a sequence $G_0, G_1, ..., G_n$ of 4-connected graphs such that $G_0 = G$, $G_n = H$, and every G_i $(i < n)$ has an edge e_i such that $G_i/e_i = G_{i+1}$.

Proof of Theorem 1.2. Let G be a 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$. By Theorem 1.1, there exists a (G, G_n)-chain $G_0, G_1, ..., G_n$ such that $G_n \in \mathcal{C} \cup \mathcal{L}$. We choose such a chain as follows:

(1) if G is planar, we choose the chain with as many terms as possible;

(2) if G is not planar, we choose the chain with as many nonplanar terms as possible.

If G is planar, we need to show that $G_n = C_6^2$. Suppose otherwise. By applying Corollary 2.2 to G_{n-1} and e_{n-1} we obtain an edge e'_{n-1} of G_{n-1} such that $G'_n = G_{n-1}/e'_{n-1}$ is 4-connected but G'_n does not belong to $\mathcal{C} \cup \mathcal{L}$. Now, by Theorem 1.1 again, there exists a (G'_n, G'_m)-chain $G'_n, G'_{n+1}, ..., G'_m$ with $G'_m \in \mathcal{C} \cup \mathcal{L}$. It follows that $G_0, G_1, ..., G_{n-1}, G'_n, G'_{n+1}, ..., G'_m$ is a chain contradicting the choice of (1), which proves the first part of the theorem.

If G is nonplanar, let $G_0, G_1, ..., G_k$ be all the nonplanar terms. We need to show that $k = n$ and $G_n = K_5$. If $k < n$ then $G_k \notin \mathcal{C} \cup \mathcal{L}$ since no graph in $\mathcal{C} \cup \mathcal{L}$ has a contractible edge while G_k has a contractible edge e_k. By Lemma 3.1, G_k has a contractible edge e'_k such that $G'_{k+1} = G_k/e'_k$ is nonplanar. Like in the planar case, we can extend $G_0, G_1, ..., G_k, G'_{k+1}$ to a chain that contradicts the choice of (2), which proves that $k = n$. If $G_n \neq K_5$, by applying Corollary 2.2 to G_{n-1} and e_{n-1} we obtain a contractible edge e'_{n-1} of G_{n-1} such that $G'_n = G_{n-1}/e'_{n-1}$ is nonplanar and not in $\mathcal{C} \cup \mathcal{L}$. Consequently, by Lemma 3.1, G'_n has an edge e'_n such that $G'_{n+1} = G'_n/e'_n$ is 4-connected and nonplanar. Now, once again, $G_0, G_1, ..., G_{n-1}, G'_n, G'_{n+1}$ can be extended into a chain. This contradicts the choice of (2), which completes our proof of the theorem. ■

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