Test I

Name: KEY

1 (8pts). Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^2}$ if it exists, or show that the limit does not exist.

Since $f(0,y)=0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the y-axis.

However, $f(x,x)=1/3$, so $f(x,y)\to 1/3$ as $(x,y)\to(0,0)$ along the line $y=x$.

Since we have obtained different limits along different paths, the given limit does not exist.

2 (8pts). Find an equation of the tangent plane to surface $z=1+x\ln(xy-5)$ at point $(3,2)$.

$f_x = 0+1\cdot\ln(xy-5)+x\cdot\frac{y}{xy-5} \implies f_x(3,2)=0+0+6=6$

$f_y = 0+x\cdot\frac{x}{xy-5} \implies f_x(3,2)=0+9=9$

$f(3,2)=1+0=0$. So the equation is $\boxed{z-1=6(x-3)+9(y-2)}$ or $\boxed{z=6x+9y-35}$

3 (9pts). Let $w=xe^{y/z}$, $x=s-2t$, $y=st$, $z=t^s$. Find $\dfrac{\partial w}{\partial t}$ when $s=2$, $t=2$.

We apply $\dfrac{\partial w}{\partial t}=\dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}+\dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial t}$. First we have

$\dfrac{\partial w}{\partial x}=e^{y/z}$, $\dfrac{\partial w}{\partial y}=xe^{y/z}(1/z)$, $\dfrac{\partial w}{\partial z}=xe^{y/z}(-y/z^2)$; $\dfrac{\partial x}{\partial t}=-2$, $\dfrac{\partial y}{\partial t}=s$, $\dfrac{\partial z}{\partial t}=s\cdot t^{s-1}$.

When $s=t=2$, $x=2-4=-2$, $y=2\cdot 2=4$, $z=2^2=4$. So

$\dfrac{\partial w}{\partial t}=e(-2)+(-1/2)e(2)+(1/2)e(4)=-e$

4 (9pts). Find the directional derivative of $f(x,y,z)=xe^y+ye^z+ze^x$ at point $P(0,0,1)$ in the direction $\langle 1,1,-2\rangle$.

$\nabla f=\langle f_x,f_y,f_y\rangle=\langle e^y+ze^x,\ \ xe^y+e^z,\ \ ye^z+e^x\rangle=\langle 2,e,1\rangle$

$\mathbf{u}=\langle 1,1,-2\rangle/\sqrt{1^2+1^2+(-2)^2}=\langle 1,1,-2\rangle/\sqrt{6}$

So $D_uf=\nabla f\cdot\mathbf{u}=(2+e-2)/\sqrt{6}=e/\sqrt{6}$.

5 (9pts). Find all critical points of $f(x, y) = x + 4y - \ln(xy^2)$.
Then determine if they are local maxima, local minima, or saddle points.

$$f_x = 1 + 0 - \frac{y^2}{xy^2} = 1 - \frac{1}{x} = 0 \quad \Longrightarrow \quad x = 1$$

$$f_y = 0 + 4 - \frac{2xy}{xy^2} = 4 - \frac{2}{y} = 0 \quad \Longrightarrow \quad y = \tfrac{1}{2}.$$ So the only critical point is $(1, \frac{1}{2})$

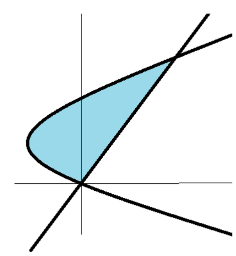
$$f_{xx} = 1/x^2 = 1, \quad f_{xy} = 0, \quad f_{yy} = 2/y^2 = 8.$$

$D = f_{xx}f_{yy} - f_{xy}^2 = 8 > 0, \quad f_{xx} = 1 > 0,$ so $f(1, \frac{1}{2}) = 3 + 2\ln 2$ has a local minimum.

6 (9pts). Calculate the iterated integral $\int_0^2 \int_0^1 ye^{2x}\, dydx$.

$$\int_0^2 \int_0^1 ye^{2x}\, dydx = \int_0^2 \frac{1}{2}y^2e^{2x}\Big|_0^1 dx = \int_0^2 \frac{1}{2}e^{2x}\, dx = \frac{1}{4}e^{2x}\Big|_0^2 = \frac{1}{4}(e^4 - 1)$$

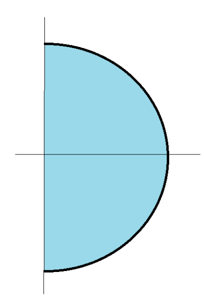
7 (9pts). Evaluate the double integral $\iint_D y dA$, where D is enclosed by curves $x = y$ and $x = y^2 - y$.



Intersection: $y = y^2 - y \implies y^2 - 2y = 0 \implies y(y-2) = 0 \implies y = 0,\ 2$

$$\iint_D ydA = \int_0^2 \int_{y^2-y}^{y} ydxdy = \int_0^2 yx\Big|_{y^2-y}^{y} dy = \int_0^2 2y^2 - y^3 dy = \frac{2}{3}y^3 - \frac{1}{4}y^4\Big|_0^2 = \frac{16}{3} - 4 = \frac{4}{3}$$

8 (9pts). Evaluate $\iint_D e^{-x^2-y^2} dA$, where D is enclosed by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis.



Using polar coordinates, we can describe D as $0 \le r \le 2, \quad -\pi/2 \le \theta \le \pi/2$

Using $x^2 + y^2 = r^2$ we have

$$\iint_D e^{-x^2-y^2} dA = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} rdrd\theta = \pi \int_0^2 e^{-r^2} rdr = \pi\, \frac{-1}{2}e^{-r^2}\Big|_0^2 = \frac{\pi}{2}(1 - e^{-4})$$

(here we use $\int_{-\pi/2}^{\pi/2} d\theta = \pi$, and $\int e^{-r^2} rdr = \frac{-1}{2}e^{-r^2}$, which can be obtained by using substitution $u = -r^2$)