Test II

Name: KEY

1. Evaluate $\iiint_E \left(\frac{x}{y} + 8z\right) dV$, where $E = \{(x,y,z)|\ 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le y\}$.

$$\iiint_E \left(\frac{x}{y} + 8z\right) dV = \int_0^1 \int_0^x \int_0^y \left(\frac{x}{y} + 8z\right) dzdydx = \int_0^1 \int_0^x \left(\frac{x}{y}z + 4z^2\right)\Big|_0^y dydx$$

$$= \int_0^1 \int_0^x (x + 4y^2) dydx = \int_0^1 \left(xy + \frac{4y^3}{3}\right)\Big|_0^x dx$$

$$= \int_0^1 \left(x^2 + \frac{4x^3}{3}\right) dx = \left(\frac{x^3}{3} + \frac{x^4}{3}\right)\Big|_0^1 = \frac{2}{3}$$

2. Evaluate $\iiint_E x\, dV$, where E is the solid in the first octant that lies under the paraboloid $z = 2 - x^2 - y^2$.

Use cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Then the paraboloid is $z = 2 - r^2$.

So $0 \le z \le 2 - r^2,\ 0 \le \theta \le \pi/2,\ 0 \le r \le \sqrt{2},\ dV = r dz dr d\theta$.

$$\iiint_E x\, dV = \int_0^{\pi/2} \int_0^{\sqrt{2}} \int_0^{2-r^2} r\cos\theta\ r dz dr d\theta = \int_0^{\pi/2} \int_0^{\sqrt{2}} (2 - r^2) r^2 \cos\theta dr d\theta$$

$$= \left(\int_0^{\pi/2} \cos\theta d\theta\right) \left(\int_0^{\sqrt{2}} (2r^2 - r^4) dr\right)$$

$$= \left(\ \sin\theta \Big|_0^{\pi/2}\right) \left(\frac{2r^3}{3} - \frac{r^5}{5}\Big|_0^{\sqrt{2}}\right) = (1 - 0)\left(\frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{5}\right) = \frac{8\sqrt{2}}{15}$$

3. Evaluate $\iiint_E z\, dV$, where E is the solid hemisphere $x^2 + y^2 + z^2 \le 4$, $z \ge 0$.

Use spherical coordinates: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, $dV = \rho^2 \sin\phi\ d\rho d\theta d\phi$.

Then the domain is : $0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/2$. So

$$\iiint_E z\, dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^2 \rho\cos\phi\ \rho^2 \sin\phi\ d\rho d\theta d\phi = \left(\int_0^{\pi/2} \cos\phi\ \sin\phi\ d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 \rho^3 d\rho\right)$$

$$= \left(\frac{1}{2}\sin^2\phi\Big|_0^{\pi/2}\right)(2\pi)\left(\frac{\rho^4}{4}\Big|_0^2\right) = \frac{1}{2} \cdot 2\pi \cdot 4 = 4\pi$$

4. Evaluate $\int_C x\sin(yz)\,ds$, where C is the line segment from $(0,0,0)$ to $(1,2,3)$.

The vector equation for C: $\langle x,y,z\rangle = \langle 0,0,0\rangle + t\ (\ \langle 1,2,3\rangle - \langle 0,0,0\rangle\ ) = t\ \langle 1,2,3\rangle,\ \ 0\le t\le 1.$
So the parametric equations for C: $x=t,\ y=2t,\ z=3t,\ \ 0\le t\le 1.$

$$\int_C x\sin(yz)\,ds = \int_0^1 t\ \sin(2t\cdot 3t)\ \sqrt{(x')^2+(y')^2+(z')^2}\ dt = \sqrt{14}\int_0^1 t\ \sin(6t^2)dt \qquad (u=6t^2,\ du=12tdt)$$

$$= \sqrt{14}\int_0^6 \sin(u)\frac{1}{12}du = \frac{\sqrt{14}}{12}(-\cos u)\Big|_0^6 = \frac{\sqrt{14}}{12}(-\cos 6+\cos 0) = \frac{\sqrt{14}}{12}(1-\cos 6)$$

5. Let $\mathbf{F}(x,y) = 2xy^2\mathbf{i} + (2x^2y+1)\mathbf{j}$.
(a) Is there a function f for which $\nabla f = \mathbf{F}$? If the answer is yes, find such an f; if the answer is no, explain why.

$P=2xy^2,\ \ Q=2x^2y+1 \ \ \Rightarrow\ \ \dfrac{\partial P}{\partial y} = 4xy$ and $\dfrac{\partial Q}{\partial x} = 4xy \ \ \Rightarrow$ yes, $\mathbf{F}$ is conservative and so f does exist.

Now we find f with $f_x = P$ and $f_y = Q$.

From $f_x = P \ \ \Rightarrow\ \ f = \int f_x dx = \int 2xy^2dx = x^2y^2 + g(y).$

This implies $f_y = \frac{d}{dy}(x^2y^2+g(y)) = 2x^2y + g'$, and thus, by $f_y = Q$, we have

$2x^2y+1 = 2x^2y+g'(y) \ \ \Rightarrow\ \ g'(y)=1 \ \ \Rightarrow\ \ g(y) = y+C \ \ \Rightarrow\ \ f(x,y) = x^2y^2+y+C$

(b) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve $y = \dfrac{2x+\sin(\pi x)}{1+x^2}$ from $(0,0)$ to $(1,1)$.

Since $\mathbf{F} = \nabla f$,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(\text{end}) - f(\text{start}) = f(1,1) - f(0,0) = (1^21^2+1+C) - (0^20^2+0+C) = 2$$

6. Evaluate line integral $\int_C (\sqrt{1+x^2} - y^2)dx + (\sqrt{1+y^2} - x^2)dy,$
where C is the rectangle from $(0,0)$ to $(2,0)$ to $(2,1)$ to $(0,1)$ to $(0,0)$.

Let D be the rectangle enclosed by C. Then $D = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le 1\}$

By Green's Theorem, $\displaystyle\int_C (\sqrt{1+x^2} - y^2)dx + (\sqrt{1+y^2} - x^2)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA =$

$$= \int_0^2 \int_0^1 (-2x + 2y)dydx = \int_0^2 (-2xy + y^2)\Big|_0^1 dx = \int_0^2 (-2x+1)dx = (-x^2 + x)\Big|_0^2 = -4 + 2 = -2$$

7. Evaluate surface integral $\iint_S (xy - x - y + z)dS$, where S is the parallelogram with parametric equations
$x = u - v, \quad y = u + v, \quad z = 1 + 2u, \quad 0 \le u \le 2, \quad 0 \le v \le 1.$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \quad \Rightarrow \quad |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{12}$$

$$\iint_S (xy - x - y + z)dS = \iint_D ((u-v)(u+v) - (u-v) - (u+v) + (1+2u))\sqrt{12}\, dA =$$

$$= \sqrt{12}\int_0^2 \int_0^1 (u^2 - v^2 + 1)dvdu = \sqrt{12}\int_0^2 \left(u^2 v - \frac{1}{3}v^3 + v\right)\Big|_0^1 du = \sqrt{12}\int_0^2 (u^2 + \frac{2}{3})\, du = \sqrt{12}\left(\frac{1}{3}u^3 + \frac{2}{3}u\right)\Big|_0^2$$

$$= 4\sqrt{12} = 8\sqrt{3}$$