Packing Feedback Arc Sets in Tournaments Exactly

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Abstract

Let $T = (V, A)$ be a tournament with a nonnegative integral weight $w(e)$ on each arc e. A subset F of arcs is called a *feedback arc set* (FAS) if $T \backslash F$ contains no cycles (directed). A collection $\mathcal{F}$ of FAS's (with repetition allowed) is called an *FAS packing* if each arc e is used at most $w(e)$ times by the members of $\mathcal{F}$. The purpose of this paper is to give a characterization of all tournaments with the property that, for every nonnegative integral weight function w defined on A, the minimum total weight of a cycle is equal to the maximum size of an FAS packing.

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1 Introduction

Let $G = (V, A)$ be a digraph with a nonnegative integral weight $w(e)$ on each arc e. A subset F of arcs is called a *feedback arc set* (FAS) of G if $G \backslash F$ contains no cycles (directed). The *FAS problem* is to find an FAS of G with minimum total weight, which can be naturally formulated as an integer program. One approach to this NP-hard problem is to consider its linear programming (LP) relaxation and explore integrality properties satisfied by its constraints. Let M be the cycle-arc incidence matrix of G, let $\pi(G)$ denote the linear system $Mx \geq \mathbf{1}$, $x \geq \mathbf{0}$, and let P denote the polyhedron defined by $\pi(G)$. We call P *integral* if it is the convex hull of all integral vectors contained in P. As is well known, P is integral iff the minimum in the LP-duality equation

$$\min\{w^T x : Mx \geq \mathbf{1},\ x \geq \mathbf{0}\} = \max\{y^T \mathbf{1} : y^T M \leq w^T,\ y \geq \mathbf{0}\}$$

has an integral optimal solution, for every nonnegative integral vector w for which the optimum is finite. If, instead, the maximum in the equation satisfies this property, then the system $\pi(G)$ is called *totally dual integral* (TDI). We say that G is *cycle ideal* (CI) if P is an integral polyhedron, and that G is *cycle Mengerian* (CM) if $\pi(G)$ is a TDI system. As shown by Edmonds and Giles [19], total dual integrality implies primal integrality, so every CM digraph is CI and hence being CM can be more intuitively stated in terms of a minimax relation. A collection $\mathcal{C}$ of cycles (with repetition allowed) is called a *cycle packing* of G if each arc e is used at most $w(e)$ times by the members of $\mathcal{C}$. Let $\nu_w(G)$ be the maximum size of a cycle packing, and let $\tau_w(G)$ be the minimum total weight of an FAS. Then G is CM iff $\nu_w(G) = \tau_w(G)$ for all nonnegative integral weight functions w defined on A. Note that a characterization of CI and CM digraphs can yield not only beautiful mathematical theorems but also a polynomial-time algorithm for the FAS problem on such digraphs, by a general theorem of Grötschel, Lovász, and Schrijver [21], so the study of these digraphs has both great theoretical interest and practical value. Initiated in the early 1960s [43], it has inspired many minimax theorems in combinatorial optimization, such as Lucchesi and Younger [31], Seymour [38, 39], Geelen and Guenin [22], Guenin [23, 24], Guenin and Thomas [25], Cai, Deng, and Zang [9], and Ding, Xu, and Zang [17, 18]. Despite tremendous research efforts, only some special classes of CI and CM digraphs [4, 5, 9, 11, 12, 23, 25, 31, 39] have been identified to date, and a complete characterization seems extremely hard to obtain.

A digraph G is called a *tournament* if there is precisely one arc between any two vertices in G. The FAS problem remains NP-hard even when the input digraph G is a tournament; see Alon [3] and Charbit, Thomassé, and Yeo [14]. As this special version also arises in a rich variety of applications, it has been studied extensively from the combinatorial, statistical, and algorithmic points of view, and thus has produced a vast body of literature. In [32], Mathieu and Schudy devised a polynomial time approximation scheme (PTAS) for the FAS problem on tournaments. Ailon, Charikar, and Newman [2] developed approximation algorithms with small constant approximation factors for the FAS problem on tournaments. Bessy et al. [7] showed that the problem of determining if a tournament has a cycle packing and a feedback arc set of the same size is NP-complete, and the problem of packing arc-disjoint cycles in tournaments is fixed-parameter tractable. Applegate, Cook, and McCormick [4] and Barahona, Fonlupt, and Mahjoub [5] independently proved that every tournament with five vertices is CM, thereby confirming a conjecture posed by both Barahona and Mahjoub [6] and Jünger [27]. We call a tournament *Möbius-free* if it contains none of $K_{3,3}$, $K'_{3,3}$, M_5, and M_5^* depicted in Figure 1 as a subgraph; these four Möbius ladders are actually the only obstructions to CI and CM tournaments.

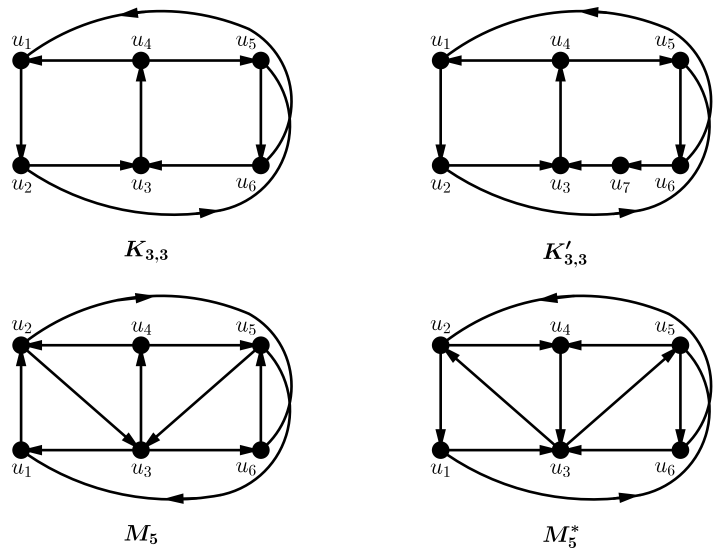

Figure 1. Forbidden Structures

Theorem 1.1. (Chen et al. [11, 12]) *For a tournament T, the following statements are equivalent:*

(i) T is Möbius-free;

(ii) T is cycle ideal; and

(iii) T is cycle Mengerian.

Minimax relations in combinatorial optimization often appear in pairs. Given a minimax relation, a common practice in this field is to establish its blocker version. For example, a graph is perfect iff its complement is perfect, as shown by Lovász [29]. The blocker version of the famous max-flow min-cut theorem is a Fulkerson theorem (see [37]), which asserts that the maximum size of s-t-cut packing equals to the minimum length of an s-t-path. The blocker version of Edmonds' disjoint arborescence theorem is Fulkerson's optimum arborescence theorem (see [37]). At this point a natural question to ask is: When does the minimax relation on packing and covering FAS's in tournaments hold?

Let $G = (V, A)$ and w be as given at the beginning of this section. We use N to denote the FAS-arc incidence matrix of G. A collection $\mathcal{F}$ of FAS's (with repetition allowed) is called an *FAS packing* of G if each arc e is used at most $w(e)$ times by the members of $\mathcal{F}$. Let $\lambda_w(G)$ be the maximum size of an FAS packing, and let $\mu_w(G)$ be the minimum total weight of a cycle (directed). Clearly, $\lambda_w(G) \le \mu_w(G)$; this inequality, however, need not hold with equality in general. We say that G is *FAS ideal* (FASI) if $Nx \ge \mathbf{1}$, $x \ge \mathbf{0}$ defines an integral polyhedron, and that G is *FAS Mengerian* (FASM) if $Nx \ge \mathbf{1}$, $x \ge \mathbf{0}$ is a TDI system. Again, by the aforementioned Edmonds-Giles theorem [19], G is FASM iff $\lambda_w(G) = \mu_w(G)$ for every nonnegative integral weight function w defined on A. Since feedback arc sets are a type of combinatorial objects involving global structural properties, they are not so easily visualized as cycles and hence are more difficult to manipulate. Thus it is no surprise that packing FAS's in a digraph is harder than packing cycles.

The origin of FASM digraphs can be traced back to 1976, when Lucchesi and Younger [31] proved their min-max theorem on packing dicuts. For an algorithmic proof of this theorem, see Frank [26]. We introduce some notions before proceeding. For each $U \subseteq V$, let $\delta(U)$ denote the set of all arcs between U and $V \backslash U$, and let $\delta^+(U)$ (resp. $\delta^-(U)$) denote the set of arcs from U to $V \backslash U$ (resp. from $V \backslash U$ to U) in G. A *dicut* is a set of arcs of the form $\delta^+(U)$ for some subset U of V with $\emptyset \neq U \neq V$ and with $\delta^-(U) = \emptyset$, which is also denoted by $(U, V \backslash U)$. A *dijoin* is a set of arcs that intersects every dicut. We can then define both *dicut packing* and *dijoin packing* in a similar way to cycle packing. The Lucchesi-Younger theorem [31] states that the maximum size of dicut packing is equal to the minimum total weight of a dijoin for all weight functions w. Edmonds and Giles [19] conjectured that the assertion remains true if we swap the terms dicut and dijoin; that is, the maximum size of dijoin packing is also equal to the minimum total weight of a dicut for all weight functions w. This conjecture has been confirmed for several classes of digraphs such as source-sink connected digraphs [20, 35] and series-parallel digraphs [28]. The assertion of the general conjecture, however, was refuted by Schrijver [34]; more counterexamples have been found by Cornuéjols and Guenin [15] and Williams and Guenin [41]. Despite this, Woodall [42] strongly believed that the unweighted version of the Edmonds-Giles conjecture holds true. Motivated by this conjecture, Chudnovsky et al. [16], Mészáros [33], and Abdi, Cornuéjols, and Zlatin [1] have obtained several results on disjoint dijoins.

When restricted to a plane digraph, dicut and dijoin are dualized to cycle and feedback arc set, respectively. Thus the above Edmonds-Giles conjecture can be rephrased as saying that every planar digraph is FASM (a counterexample is the dual of Schrijver's digraph [34]), and Woodall's conjecture amounts to saying that the maximum number of disjoint feedback arc sets is equal to the length of a shortest cycle.

The purpose of this paper is to establish the blocker version of Theorem 1.1.

Theorem 1.2. *For a tournament T, the following statements are equivalent:*

(i) T is Möbius-free;

(ii) T is FAS ideal; and

(iii) T is FAS Mengerian.

Corollary 1.3. *A tournament is cycle Mengerian iff it is FAS Mengerian iff it is Möbius-free.*

The reader is referred to [10] (resp. [13]) for a structural characterization of all undirected graphs (resp. tournaments) with the min-max relation on packing and covering feedback vertex sets and the corresponding blocker version [18, 17] (resp. [9]).

The remainder of this paper is organized as follows: In Section 2, we present a global structural description of Möbius-free strong tournaments. In Section 3, we establish the minimax relation on packing and covering FAS's in Möbius-free strong tournaments other than F_1 and G_1 (to be shown in Figures 4 and 5). In Section 4, we give a computer-assisted proof of the minimax relation on G_1, thereby completing the whole proof.

2 Global Structure

Our proof of Theorem 1.1 [11, 12] relies heavily on a structural description of Möbius-free strong tournaments, which continues to play an important role in the characterization of FAS Mengerian tournaments.

Let us recall some terminology and notation introduced in [11]. Let $G = (V, A)$ be a digraph with a nonnegative integral weight $w(e)$ on each arc e. We use $|G|$ to denote the total number of vertices in G. For each $v \in V$, we use $G\backslash v$ to denote the digraph arising from G by deleting vertex v, and use $d_G^+(v)$ and $d_G^-(v)$ to denote the out-degree and in-degree of v, respectively. We call v a *near-sink* of G if its out-degree is one, and call v a *near-source* if its in-degree is one. For simplicity, an arc $e = (u, v)$ of G is also denoted by uv. Arc e is called *special* if u is a near-sink or v is a near-source of G. For each $U \subseteq V$, we use $G[U]$ to denote the subgraph of G induced by U. Recall that G is called *weakly connected* if its underlying undirected graph is connected, and is called *strongly connected* or *strong* if each vertex is reachable from every other vertex. Clearly, a weakly connected digraph G is strong iff G has no dicut. A dicut (X, Y) is called *trivial* if $|X| = 1$ or $|Y| = 1$. Furthermore, a weakly connected digraph G is called *internally strong* if every dicut of G is trivial, and is called *internally 2-strong* (*i*2*s*) if G is strong and $G\backslash v$ is internally strong for every vertex v.

Let $T_i = (V_i, A_i)$ be a tournament, with $|V_i| \geq 3$ for $i = 1, 2$. We say that T_1 is *smaller* than T_2 if $|V_1| < |V_2|$. Suppose that (a_1, b_1) is a special arc of T_1 with $d_{T_1}^+(a_1) = 1$ and (b_2, a_2) is a special arc of T_2 with $d_{T_2}^-(a_2) = 1$. The *1-sum* of T_1 and T_2 over (a_1, b_1) and (b_2, a_2) is the tournament arising from the disjoint union of $T_1\backslash a_1$ and $T_2\backslash a_2$ by identifying b_1 with b_2 (the resulting vertex is denoted by b) and adding all arcs from $T_1\backslash\{a_1, b_1\}$ to $T_2\backslash\{a_2, b_2\}$. We call b the *hub* of the 1-sum. See Figure 2 for an illustration. Note that if T_i is strong and $|V_i| = 3$ for $i = 1$ or 2, then T_i is a triangle (a directed cycle of length three), and thus $T = T_{3-i}$.

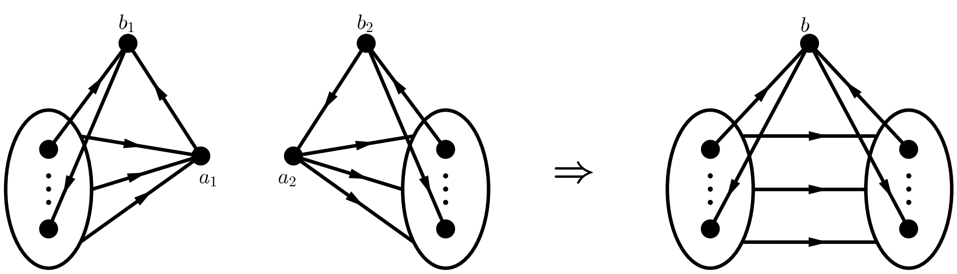

Figure 2. 1-sum of T_1 and T_2.

In our original definition of 1-sum [11, 12], we assume that $T_i = (V_i, A_i)$ is strong for $i = 1, 2$; this assumption is removed here just for more convenience. The lemma below asserts that these two definitions are equivalent when restricted to a strong tournament T.

Lemma 2.1. *Suppose a strong tournament T is a 1-sum of two tournaments T_1 and T_2. Then the following statements hold:*

(i) Both T_1 and T_2 are strong; and

(ii) Both T_1 and T_2 are sub-tournaments of T.

As the proof is completely straightforward, we omit it here. Let (X_1, X_2) be the dicut of $T \backslash b$ as shown in Figure 2. Observe that any out-neighbor of b in X_1 can be taken as a_2 and any in-neighbor of b in X_2 can be taken as a_1 in the 1-sum (such neighbors are available as T is strong). Furthermore, T_i is the subtournament of T induced by $X_i \cup \{b, a_i\}$ for $i = 1, 2$. The following lemma (see Lemma 2.2 in [11]) states that being Möbius-free is closed under taking 1-sums.

Lemma 2.2. *Suppose a strong tournament T is a 1-sum of two tournaments T_1 and T_2. Then T is Möbius-free iff both T_1 and T_2 are Möbius-free.*

Let C_3 (resp. F_0) denote the strong tournament with three (resp. four) vertices (see Figure 3), let F_1, F_2, F_3, F_4, F_5 be the five tournaments depicted in Figure 4, and let G_1, G_2, G_3 be the three tournaments shown in Figure 5. We reserve the symbols

$$\mathcal{T}_0 = \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$$

and

$$\mathcal{T}_1 = \{C_3, F_0, F_2, F_3, F_4, G_2, G_3\} = \mathcal{T}_0 \backslash \{F_1, G_1\}.$$

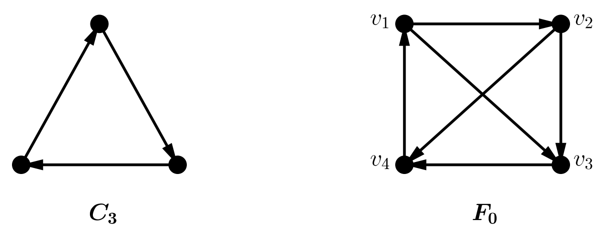

Figure 3. Strong tournaments with three or four vertices.

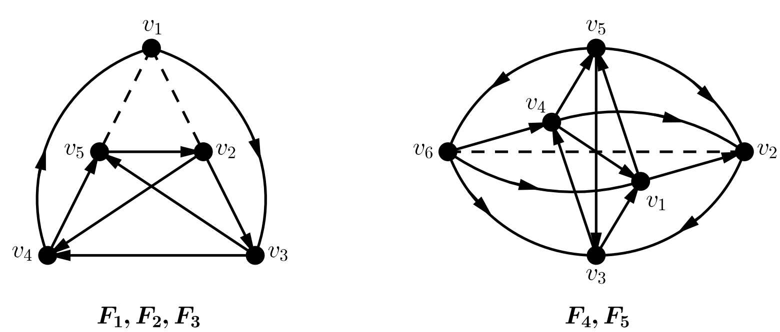

Figure 4. $v_1v_2, v_5v_1 \in F_1$; $v_2v_1, v_1v_5 \in F_2$; $v_2v_1, v_5v_1 \in F_3$; $v_6v_2 \in F_4$; $v_2v_6 \in F_5$.

In [11] we have obtained the following structural descriptions of Möbius-free tournaments.

Theorem 2.3. (Chen et al. [11]) *Let $T = (V, A)$ be an i2s tournament with $|V| \geq 3$. Then T is Möbius-free iff $T \in \mathcal{T}_0$.*

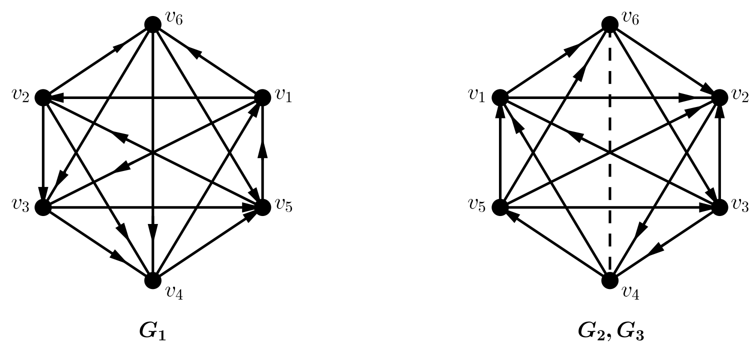

Figure 5. $v_6v_4 \in G_2$ and $v_4v_6 \in G_3$.

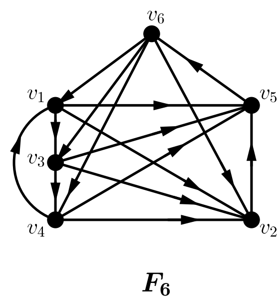

Figure 6. A minimal tournament involved in Lemma 2.5

Theorem 2.4. (Chen et al. [11]) *Let $T = (V, A)$ be a Möbius-free strong tournament with $|V| \geq 3$. Then either $T \in \{F_1, G_1\}$ or T can be obtained by repeatedly taking 1-sums starting from the tournaments in $\mathcal{T}_1$.*

Let F_6 be the tournament depicted in Figure 6 and let

$$\mathcal{T}_2 = \{F_0, F_2, F_3, F_4, F_6, G_2, G_3\}.$$

Then $\mathcal{T}_2 = (\mathcal{T}_1 \backslash \{C_3\}) \cup \{F_6\}$. Lemma 2.4 in [12] states that if a Möbius-free strong tournament T is a 1-sum of two smaller strong tournaments T_1 and T_2 such that T_2 is minimal (with respect to vertex set inclusion), then $T_2 \in \mathcal{T}_2$. From Lemma 2.1, we see that the "strong" condition imposed on T_1 and T_2 can be removed.

Lemma 2.5. *Let $T = (V, A)$ be a Möbius-free strong tournament. Suppose T is a 1-sum of two smaller tournaments T_1 and T_2 such that T_2 is minimal (with respect to vertex set inclusion). Then $T_2 \in \mathcal{T}_2$.*

Notice that every tournament in $\mathcal{T}_0$ has a near-sink or a near-source, except F_1 and G_1. So the above three results imply the following.

Corollary 2.6. *Let $T = (V, A)$ be a Möbius-free strong tournament, with $T \notin \{C_3, F_1, G_1\}$. Then T can be constructed from a tournament in $\{F_0, F_2, F_3, F_4, G_2, G_3\}$ by repeatedly taking 1-sums with tournaments in $\mathcal{T}_2$.*

So far we have exhibited some local structural properties satisfied by Möbius-free strong tournaments. Due to the global nature of feedback arc sets, we need a description of global structures of Möbius-free strong tournaments in order to establish the desired minimax relation. Let $\mathcal{Q}$ consist of all tournaments G whose vertex set can be partitioned into $U_0, U_1, \ldots, U_k$ for some integer $k \geq 0$, such that $|U_0| = 1$, $G[U_i]$ is either a singleton or a triangle for $1 \leq i \leq k$, and the arcs between U_i and U_j are all directed to U_j for $1 \leq i < j \leq k$. Let v be the vertex in U_0. We call v the *center* of G, call $H_i = G[U_i \cup \{v\}]$ a *building block* of G centered at v for $1 \leq i \leq k$, and call H_1 (resp. H_k) the *leftmost* (resp. *rightmost*) building block of G.

Theorem 2.7. *Let $T = (V, A)$ be a strong tournament other than F_1 and G_1. Then T is Möbius-free iff it satisfies the following description:*

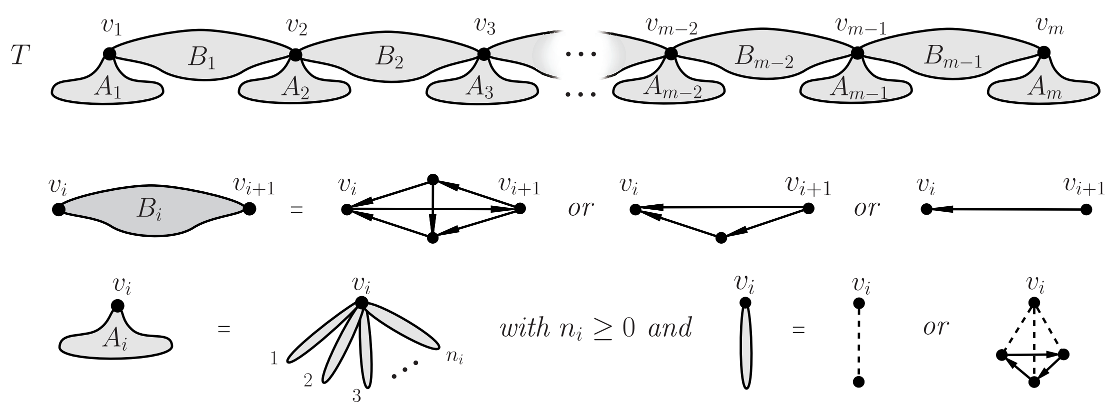

Figure 7. Global structure

where $m \geq 1$ (undirected/dotted edges in the following can be directed arbitrarily), and all other arcs (that are not drawn) are directed from "left" to "right". Furthermore, v_1 has an out-neighbor in the leftmost building block of A_1, and v_m has an in-neighbor in the rightmost building block of A_m.

Note that in Figure 7 by from "left" to "right" we mean from vertices on the left to those on the right. Besides, each A_i contains v_i and each B_i contains both v_i and v_{i+1}.

Let $\mathcal{P}$ denote the class of all strong tournaments T described in the above theorem. We call $A_1, A_2, \ldots, A_m$ *vertical blocks* of T, call $B_1, B_2, \ldots, B_{m-1}$ *horizontal blocks* of T, and call $v_1, v_2, \ldots, v_m$ the *join vertices* of T. Clearly, each vertical block A_i of T belongs to $\mathcal{Q}$. We reserve the symbols $A_{i,1}, A_{i,2}, \ldots, A_{i,n_i}$ for the building blocks of A_i centered at v_i from left to right, where $n_i \geq 0$.

Let us prove four technical lemmas before presenting a proof of Theorem 2.7.

Lemma 2.8. *Every tournament in $\{C_3\} \cup \mathcal{T}_2$ belongs to $\mathcal{P}$.*

Proof. The statement holds trivially for C_3. As shown in Figure 8 (where the missing arcs are all directed from left to right), F_0 can be expressed in two ways, with $m = 2$ and $m = 1$, respectively; F_3 and F_4 can be expressed with $m = 2$, while F_2, F_6, G_2 and G_3 can be expressed with $m = 1$. ∎

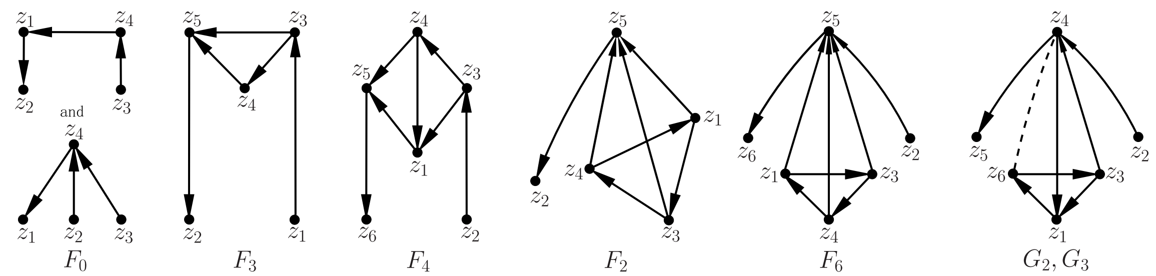

Figure 8. Tournaments in $\mathcal{T}_2$

Lemma 2.9. *Let G be a strong tournament on five vertices with a near-sink or a near-source. Then G is F_2 or F_3 or a 1-sum of two copies of F_0.*

Proof. Since G is a tournament on five vertices, it is Möbius-free. If G is $i2s$, then $G \in \{F_1, F_2, F_3\}$ by Theorem 2.3 and hence G is F_2 or F_3, because F_1 contains no near-sink nor near-source. So we assume that G is not $i2s$. By definition, $G\backslash v$ has a dicut (X, Y) with $|X| = |Y| = 2$ for some vertex v. Since G is strong, there exist a vertex a_1 in Y and a vertex a_2 in X, such that both (a_1, v) and (v, a_2) are arcs of G. Let T_1 be the sub-tournament of G induced by the vertex subset $X \cup \{a_1, v\}$ and let T_2 be the sub-tournament of G induced by the vertex subset $Y \cup \{a_2, v\}$. Then T_1 and T_2 are two copies of F_0 and G is the 1-sum of T_1 and T_2 over arcs (a_1, v) and (v, a_2). Let b_1 (resp. b_2) be the vertex in $Y - \{a_1\}$ (resp. $X - \{a_2\}$). Depending on the directions of arcs between v and $\{b_1, b_2\}$, we have four cases to consider, each of which is straightforward and yields a 1-sum that contains a near-sink or a near-source. ∎

In what follows, $\mathcal{R}_5$ is the set of all strong tournaments on five vertices with a near-sink or a near-source, and F_6^* arises from F_6 by reversing the direction of each arc.

Lemma 2.10. *Let $G = (V, A)$ be a strong tournament in $\mathcal{Q}$ with at least three vertices. Then either $G \in \{C_3, F_0\}$ or G can be obtained by repeatedly taking 1-sums starting from tournaments in $\{F_0, F_6, F_6^*, G_2, G_3\} \cup \mathcal{R}_5$, such that the hubs of these sums are always the center of G.*

Proof. Let v be the center of G and let $H_1, H_2, \ldots, H_k$ be the building blocks of G centered at v, where the arcs between $H_i\backslash v$ and $H_j\backslash v$ are all directed to H_j for $1 \le i < j \le k$. We proceed by induction on k. If $k = 1$, then $G = F_0$ and hence the statement holds trivially. So we assume that $k \ge 2$ and set $X_i := V(H_i)$ for $1 \le i \le k$.

We first assume that $|X_1| = 4$. Let a_1 be an in-neighbor of v in $G\backslash X_1$ and a_2 be an out-neighbor of v in H_1 (such a_1 and a_2 exist, as G is strong). Let T_1 and T_2 be the strong sub-tournaments of G induced by $X_1 \cup \{a_1\}$ and $(V\backslash X_1) \cup \{v, a_2\}$, respectively. Then G is the 1-sum of T_1 and T_2 over arcs (a_1, v) and (v, a_2). Note that $T_1 \in \mathcal{R}_5$ and $T_2 \in \mathcal{Q}$. By induction hypothesis, either $T_2 \in \{C_3, F_0\}$ or T_2 can be obtained by repeatedly taking 1-sums starting from tournaments in $\{F_0, F_6, F_6^*, G_2, G_3\} \cup \mathcal{R}_5$, such that the hubs of these sums are always v. Clearly, $G = T_1$ when $T_2 = C_3$. Therefore G can be obtained by repeatedly taking 1-sums starting from tournaments in $\{F_0, F_6, F_6^*, G_2, G_3\} \cup \mathcal{R}_5$, such that the hubs of these sums are always the center of G.

Next, we assume that $|X_1| = 2$. If $k = 2$, then G is either a C_3 or a strong tournament on five vertices with a near-source, so the desired statement holds trivially. Thus we may further assume that $k \geq 3$. Let a_1 be an in-neighbor of v in $G\backslash(X_1 \cup X_2)$ and a_2 be an out-neighbor of v in $H_1 \cup H_2$ (such a_1 and a_2 exist, as G is strong). Let T_1 and T_2 be the strong sub-tournaments of G induced by $X_1 \cup X_2 \cup \{a_1\}$ and $(V\backslash(X_1 \cup X_2)) \cup \{v, a_2\}$, respectively. From Figure 8 we see that $T_1 \in \{F_0, F_6, F_6^*, G_2, G_3\}$. (Note that $T_1 = F_6^*$ if $|X_2| = 4$ and v is a source of H_2.) Furthermore, $T_2 \in \mathcal{Q}$ and G is the 1-sum of T_1 and T_2 over arcs (a_1, v) and (v, a_2). By the induction hypothesis, either $T_2 \in \{C_3, F_0\}$ or T_2 can be obtained by repeatedly taking 1-sums starting from tournaments in $\{F_0, F_6, F_6^*, G_2, G_3\} \cup \mathcal{R}_5$, such that the hubs of these sums are always v. Clearly, $G = T_1$ when $T_2 = C_3$. Therefore G can be obtained by repeatedly taking 1-sums starting from tournaments in $\{F_0, F_6, F_6^*, G_2, G_3\} \cup \mathcal{R}_5$, such that the hubs of these sums are always the center of G. ∎

Lemma 2.11. *Every tournament in $\mathcal{Q}$ is Möbius-free.*

Proof. Let G be a tournament in $\mathcal{Q}$. To prove that G is Möbius-free, it suffices to consider the case when G is strong, because Möbius ladders exhibited in Figure 1 are all strong. Thus, by Lemma 2.10, either $G \in \{C_3, F_0\}$ or G can be obtained by repeatedly taking 1-sums starting from tournaments in $\{F_0, F_6, F_6^*, G_2, G_3\} \cup \mathcal{R}_5$, such that the hubs of these sums are always the center of G.

By Theorem 2.3, C_3, F_0, F_2, G_2 and G_3 are all Möbius-free. Since F_6 is a 1-sum of F_2 and F_0 (with hub v_6; see Figure 6), it is also Möbius-free by Lemma 2.2 and hence so is F_6^*. Therefore each tournament in $\{F_0, F_6, F_6^*, G_2, G_3\} \cup \mathcal{R}_5$ is Möbius-free. It follows from Lemma 2.2 that G is Möbius-free. ∎

Now we are ready to establish the main result of this section.

Proof of Theorem 2.7. Let us first show the "if" part. Let $T = (V, A)$ be a strong tournament in $\mathcal{P}$ as described in Figure 7, with vertical blocks $A_1, A_2, \ldots, A_m$, horizontal blocks $B_1, B_2, \ldots, B_{m-1}$, and join vertices $v_1, v_2, \ldots, v_m$; subject to this, we assume that m is minimum (the choices of A_i and B_i may not be unique). This assumption implies that

(1) $|A_1| \geq 2$. Furthermore, $|A_1| \geq 3$ if $|B_1| \leq 3$, for otherwise, let A_2' be the sub-tournament of T induced by all vertices in $A_1 \cup A_2 \cup B_1$. Then T can be depicted as in Figure 7, with vertical blocks $A_2', A_3, \ldots, A_m$ and horizontal blocks $B_2, B_3, \ldots, B_{m-1}$, contradicting the minimality assumption on m.

Similarly, $|A_m| \geq 2$. Since $A_i \in \mathcal{Q}$, by Lemma 2.11 we have

(2) A_i is Möbius-free for $1 \leq i \leq m$.

We propose to show, by induction on $m+n$, that T is Möbius-free, where $n = |V|$. If $m = 1$, then $T = A_1$, so the statement is true by (2). If $n \leq 5$, trivially the statement holds. Thus we may assume that $m \geq 2$ and $n \geq 6$.

Consider the case when $|A_1| = 2$. Now $|B_1| = 4$ by (1). Besides, we may assume that there are at least two vertices outside $A_1 \cup B_1$, for otherwise, $T = F_4$ (see Figure 8), which is Möbius-free by Theorem 2.3. Let a_1 be an in-neighbor of v_2 outside $A_1 \cup B_1$ and let a_2 be an out-neighbor of v_2 in B_1. Let T_1 be the sub-tournament of T induced by all vertices in $V(A_1 \cup B_1) \cup \{a_1\}$ and let T_2 be the sub-tournament of T induced by all vertices outside $V(A_1 \cup B_1)\backslash\{v_2, a_2\}$. Then T_1 is F_4 (see Figure 8), T_2 is a tournament in $\mathcal{P}$ with $m-1$ vertical blocks, and T is the 1-sum of T_1 and T_2 over arcs (a_1, v_2) and (v_2, a_2). By induction hypothesis, T_2 is Möbius-free and hence so is T by Lemma 2.2.

It remains to consider the case when $|A_1| \geq 3$. Let a_1 be an in-neighbor of v_1 outside A_1 and let a_2 be an out-neighbor of v_1 in A_1 (such a_1 and a_2 exist, as T is strong). Let T_1 be the sub-tournament of T induced by all vertices in $V(A_1) \cup \{a_1\}$ and let T_2 be the sub-tournament of T induced by all vertices outside $A_1 \backslash \{a_2, v_1\}$. Note that $T_i \in \mathcal{P}$, $4 \leq |T_i| < n$ for $i = 1, 2$, and T is the 1-sum of T_1 and T_2 over arcs (a_1, v_1) and (v_1, a_2). By induction hypothesis, T_i is Möbius-free for $i = 1, 2$ and hence so is T by Lemma 2.2. This establishes the "if" part.

Let us now proceed to the "only if" part. Let $T = (V, A)$ be a strong Möbius-free tournament other than F_1 and G_1. We aim to show, by induction on $n = |V|$, that $T \in \mathcal{P}$. If T is $i2s$, then $T \in \{C_3\} \cup \mathcal{T}_2$ by Theorem 2.3 and hence $T \in \mathcal{P}$ by Lemma 2.8. So we assume that T is not $i2s$. Then T a 1-sum of two smaller tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2), such that $T_2 \in \mathcal{T}_2$ by Lemma 2.5. Keep in mind that a_i and b_i are two vertices of T_i for $i = 1, 2$.

By induction hypothesis, T_1 is as described in Figure 7, with vertical blocks $A_1, A_2, \ldots, A_m$, horizontal blocks $B_1, B_2, \ldots, B_{m-1}$, and join vertices $v_1, v_2, \ldots, v_m$; subject to this, we assume that m is minimum. This assumption implies that

(3) $|A_1| \geq 2$ and $|A_m| \geq 2$ (see (1) for an argument).

If $m = 1$, then $T_1 = A_1 \in \mathcal{Q}$. Since a_2 is a near-source of T_2, (b_2, a_2) is the leftmost arc of the corresponding tournament shown in Figure 8. Thus the 1-sum of T_1 and a tournament in $\{F_0, F_2, F_6, G_2, G_3\}$ belongs to $\mathcal{Q}$, and the 1-sum of T_1 and a tournament in $\{F_3, F_4\}$ belongs to $\mathcal{P}$ with two vertical blocks. Hence $T \in \mathcal{P}$, as desired. So we assume that $m \geq 2$. Since a_1 is a near-sink of T_1, it belongs to $B_{m-1} \cup A_m$. If $a_1 \in V(B_{m-1} \backslash v_m)$, then $|B_{m-1}| = 2$ or 3 and $V(A_m) = \{v_m\}$, contradicting (3). If $a_1 = v_m$, then B_{m-1} consists of only one arc $(v_m, v_{m-1}) = (a_1, b_1)$ and v_m is a sink of A_m. Thus we can combine A_{m-1}, B_{m-1} and A_m to form a new A'_{m-1} and depict T_1 as in Figure 7, with vertical blocks $A_1, A_2, \ldots, A_{m-2}, A'_{m-1}$ and horizontal blocks $B_1, B_2, \ldots, B_{m-2}$, contradicting the minimality assumption on m. So $a_1 \in V(A_m \backslash v_m)$. Let $A_{m,1}, A_{m,2}, \ldots, A_{m,n_m}$ be the building blocks of A_m centered at v_m. Again, since a_1 is a near-sink of T_1, we obtain

(4) a_1 is contained in $(A_{m,n_m-1} \cup A_{m,n_m}) \backslash v_m$.

For simplicity, in the remainder of this proof, we frequently define B_m, A_{m+1}, etc. in terms of vertex sets only. For example, by $B_m = \{b_1, v_m\}$ we mean that B_m is the tournament with vertex set $\{b_1, v_m\}$. By (4), a_1 is either contained in A_{m,n_m-1} or A_{m,n_m}. Depending on the location of a_1, we consider two cases.

Case 1. a_1 is contained in A_{m,n_m-1}. Then A_{m,n_m-1} consists of only one arc (v_m, a_1) and A_{m,n_m} consists of only one arc (b_1, v_m) (as T_1 is strong by Lemma 2.1). If $T_2 \neq F_4$ (possibly $T_2 = F_3$; see Figure 8), then $T \in \mathcal{P}$ with the join vertices $v_1, \ldots, v_m, v_{m+1} := b_1$ and with new blocks $A_m := A_m \backslash \{a_1, b_1\}$, $B_m = \{b_1, v_m\}$, and $A_{m+1} = T_2 \backslash a_2$; if $T_2 = F_4$ (see Figure 8), then T is in $\mathcal{P}$ with join vertices $v_1, \ldots, v_m, v_{m+1} := b_1, v_{m+2} := z_3$ and with new blocks $A_m := A_m \backslash \{a_1, b_1\}$, $B_m = \{b_1, v_m\}$, $A_{m+1} = \{b_1\}$, $B_{m+1} = \{b_1, z_1, z_3, z_4\}$, and $A_{m+2} = \{z_2, z_3\}$.

Case 2. a_1 is contained in A_{m,n_m}. Depending on $|A_{m,n_m}|$, we distinguish between two subcases.

Subcase 2.1. $|A_{m,n_m}| = 2$. Now A_{m,n_m} consists of arc (a_1, v_m) only and $b_1 = v_m$. If $T_2 \in \{F_0, F_2, F_6, G_2, G_3\}$, where F_0 corresponds to $m = 1$ in Figure 8, then $T \in \mathcal{P}$ with join vertices $v_1, v_2, \ldots, v_m$ and with new block A_m equal to the sub-tournament of T induced by all vertices in $(A_m \backslash a_1) \cup (T_2 \backslash \{a_2, b_2\})$. If $T_2 = F_0$ corresponds to $m = 2$ in Figure 8, then $T \in \mathcal{P}$

with join vertices $v_1, \ldots, v_m, v_{m+1} := z_4$ and with new blocks $B_m = \{z_4, z_1\}$, $A_m := A_m \backslash a_1$, and $A_{m+1} = \{z_3, z_4\}$. If $T_2 = F_3$, then $T \in \mathcal{P}$ with join vertices $v_1, \ldots, v_m, v_{m+1} := z_3$ and with new blocks $B_m = \{z_3, z_4, z_5\}$, $A_m := A_m \backslash a_1$, and $A_{m+1} = \{z_1, z_3\}$. If $T_2 = F_4$, then $T \in \mathcal{P}$ with join vertices $v_1, \ldots, v_m, v_{m+1} := z_3$ and with new blocks $B_m = \{z_1, z_3, z_4, z_5\}$, $A_m := A_m \backslash a_1$, and $A_{m+1} = \{z_2, z_3\}$.

Subcase 2.2. $|A_{m,n_m}| = 4$. Now $A_{m,n_m} \backslash v_m$ is a triangle $a_1 b_1 c_1 a_1$. Since a_1 is a near-sink, (v_m, a_1) is an arc of T_1. Since T_1 is strong, at least one of the two arcs between v_m and $\{b_1, c_1\}$ is directed to v_m.

Suppose (b_1, v_m) and (c_1, v_m) are two arcs of T_1. If $T_2 \neq F_4$ (possibly $T_2 = F_3$; see Figure 8), then $T \in \mathcal{P}$ with join vertices $v_1, \ldots, v_m, v_{m+1} := b_1$ and with new blocks $A_m := A_m \backslash \{a_1, b_1, c_1\}$, $B_m = \{b_1, c_1, v_m\}$, and $A_{m+1} = T_2 \backslash a_2$. If $T_2 = F_4$ (see Figure 8), then $T \in \mathcal{P}$ with join vertices $v_1, \ldots, v_m, v_{m+1} := b_1, v_{m+2} := z_3$ and with new blocks $A_m := A_m \backslash \{a_1, b_1, c_1\}$, $B_m = \{b_1, c_1, v_m\}$, $A_{m+1} = \{b_1\}$, $B_{m+1} = \{b_1, z_1, z_3, z_4\}$, and $A_{m+2} = \{z_2, z_3\}$.

So we assume that exactly one of the two arcs between v_m and $\{b_1, c_1\}$ is directed to v_m.

When (b_1, v_m) and (v_m, c_1) are two arcs of T_1, we see that if $T_2 \neq F_4$ (possibly $T_2 = F_3$; see Figure 8), then $T \in \mathcal{P}$ with join vertices $v_1, \ldots, v_m, v_{m+1} := b_1$ and with new blocks $A_m := A_m \backslash \{a_1, b_1, c_1\}$, $B_m = \{b_1, v_m\}$, and A_{m+1} equal to the sub-tournament of T induced by $\{c_1\} \cup V(T_2 \backslash a_2)$; if $T_2 = F_4$ (see Figure 8), then $T \in \mathcal{P}$ with join vertices $v_1, \ldots, v_m, v_{m+1} := b_1, v_{m+2} := z_3$ and with new blocks $A_m := A_m \backslash \{a_1, b_1, c_1\}$, $B_m = \{b_1, v_m\}$, $A_{m+1} = \{b_1, c_1\}$, $B_{m+1} = \{b_1, z_1, z_3, z_4\}$, and $A_{m+2} = \{z_2, z_3\}$.

When (v_m, b_1) and (c_1, v_m) are two arcs of T_1, we see that if $T_2 \neq F_4$ (possibly $T_2 = F_3$; see Figure 8), then $T \in \mathcal{P}$ with join vertices $v_1, \ldots, v_m, v_{m+1} := c_1, v_{m+2} := b_1$ and with new blocks $A_m := A_m \backslash \{a_1, b_1, c_1\}$, $B_m = \{c_1, v_m\}$, $A_{m+1} = \{c_1\}$, $B_{m+1} = \{b_1, c_1\}$, and $A_{m+2} = T_2 \backslash a_2$; if $T_2 = F_4$ (see Figure 8), then $T \in \mathcal{P}$ with join vertices $v_1, \ldots, v_m, v_{m+1} := c_1, v_{m+2} := b_1, v_{m+3} := z_3$ and with new blocks $A_m := A_m \backslash \{a_1, b_1, c_1\}$, $B_m = \{c_1, v_m\}$, $A_{m+1} = \{c_1\}$, $B_{m+1} = \{b_1, c_1\}$, $A_{m+2} = \{b_1\}$, $B_{m+2} = \{b_1, z_1, z_3, z_4\}$, and $A_{m+3} = \{z_2, z_3\}$.

From the induction hypothesis and the above construction, we can also see that the leftmost join vertex v_1 has an out-neighbor in the leftmost building block of A_1, and the rightmost join vertex v_k, with $k = m$, $m+1$ or $m+2$, has an in-neighbor in the rightmost building block of A_k. Therefore $T \in \mathcal{P}$. This establishes the "only if" part. ∎

3 Minimax Relation

In this section we show that every Möbius-free strong tournament other than F_1 and G_1 satisfies the minimax relation on packing and covering feedback arc sets.

Theorem 3.1. *Let $T = (V, A)$ be a Möbius-free strong tournament with $|V| \geq 3$ and $T \notin \{F_1, G_1\}$. Then T is FAS Mengerian.*

As usual, we use $\mathbb{Z}_+$ to denote the set of all nonnegative integers and use $\mathbb{Z}_+^A$ to denote the set of vectors $x = (x(a) : a \in A)$ whose coordinates belong to $\mathbb{Z}_+$. Let $w \in \mathbb{Z}_+^A$. Recall that $\mu_w(T)$ is the minimum total weight of a cycle (directed) in T. A cycle C in T is called a *minimum* cycle of (T, w) if $w(C) = \mu_w(T)$. Let u and v be two vertices of T. A *u-v path* is a path from u to v. A *u-v* path is called *minimum* with respect to w (or simply *w-minimum*) if it has the minimum total weight among all u-v paths. An FAS packing of T with respect to w is also called a *w-FAS packing*.

By Theorem 2.7, every Möbius-free strong tournament T other than F_1 and G_1 can be depicted as in Figure 7. We shall prove Theorem 3.1 by induction on the number of vertical blocks in T; the lemma below clearly yields the base statement.

Lemma 3.2. *Every tournament in $\mathcal{Q}$ (see the paragraph succeeding Theorem 2.7) is FAS Mengerian.*

Proof. Let $G = (V, A)$ be a tournament in $\mathcal{Q}$, let v be the center of G, and let $H_1, H_2, \ldots, H_k$ be the building blocks of G centered at v. We use Ω to denote the set of all subscripts i with $|H_i| = 4$ and use $\triangle_i$ to denote the triangle $H_i \backslash v$ for each $i \in \Omega$. Note that these triangles are pairwise vertex disjoint.

If v is a source or a sink of G, then the triangles $\triangle_i$ are the only cycles in G. Thus G is trivially FAS Mengerian. So we assume hereafter that v is neither a source nor a sink of G.

Let $w \in \mathbb{Z}_+^A$. Our objective is to find a w-FAS packing in G of size $r := \mu_w(G)$. For this purpose, let X (resp. Y) be the out-neighborhood (resp. in-neighborhood) of v in G, and let D be the digraph obtained from G by splitting v into a source s and a sink t, such that

- for each vertex $x \in X$, there is an arc sx in D with length $w(sx) = w(vx)$;
- for each vertex $y \in Y$, there is an arc yt in D with length $w(yt) = w(yv)$; and
- for each arc ab of G with $v \notin \{a, b\}$, there is an arc ab in D with length $w(ab)$.

Let $\mathcal{C}$ be the collection of all cycles (directed) passing through v in G, and let r' be the minimum weight of a cycle in $\mathcal{C}$. Clearly, $r' \geq r$. We call a subset of arcs in G a *$\mathcal{C}$-transversal* if it intersects each cycle in $\mathcal{C}$. We also view $\triangle_i$ for $i \in \Omega$ as a triangle in D and view each arc of D as an arc of G.

From the construction of D, we see that

(1) there is a 1-1 correspondence between cycles in $\mathcal{C}$ and s-t paths in D, and the shortest distance from s to t in D with respect to w is equal to r'.

For $i = 1, 2, \ldots, r$, let U_i be the set of vertices at distance less than i from s in D with respect to w, and let $C_i := \delta^+(U_i)$. (Possibly there are arcs entering U_i in D, yet C_i consists of arcs leaving U_i only. So C_i is an s-t cut in D.) Observe that

(2) no C_i contains two or more arcs in $\triangle_j$ for any $j \in \Omega$ and

(3) each C_i corresponds to a $\mathcal{C}$-transversal in G by (1). Furthermore, each arc a of D is contained in at most $w(a)$ of $C_1, C_2, \ldots, C_r$.

Let us construct $F_1, F_2, \ldots, F_r$ from $C_1, C_2, \ldots, C_r$ by using the following algorithm.

Initially, set $F_i := C_i$ for $1 \leq i \leq r$. While $\Omega \neq \emptyset$, do: take $j \in \Omega$, and add precisely one of the arcs $e_{j,1}, e_{j,2}, e_{j,3}$ of $\triangle_j$ to each F_i (if it contains no arc of $\triangle_j$) to form a new F_i so that each $e_{j,p}$ for $1 \leq p \leq 3$ is contained in at most $w(e_{j,p})$ of the resulting $F_1, F_2, \ldots, F_r$. Set $\Omega = \Omega - \{j\}$.

Since $\triangle_j$ is a triangle, $w(e_{j,1}) + w(e_{j,2}) + w(e_{j,3}) \geq r$. Thus the correctness of our algorithm is guaranteed by (2) and (3). Note that every cycle of G outside $\mathcal{C}$ is a $\triangle_i$ for some i. From (3), we further deduce that each F_i is an FAS of G and that each arc a of G is contained in at most $w(a)$ members of $\mathcal{F} := \{F_1, F_2, \ldots, F_r\}$. Therefore $\mathcal{F}$ is a w-FAS packing of G having size r. ∎

For convenience, we say that the w-FAS packing $\mathcal{F}$ of size r output above is obtained by first performing breadth-first search for r steps in G from v and then eliminating triangles in $G \backslash v$, and say that F_i is the *depth-i set* in $\mathcal{F}$ from v for $1 \le i \le r$. Keep in mind that breadth-first search employed in this paper always starts from a source (that is why we split v into a source and a sink as there are arcs entering and leaving it). The reader is referred to Schrijver [37] (see page 88) for more information about breadth-first search.

Let $T = (V, A)$ be as described in Theorem 2.7, and let $T^* = (V, E^*)$ be the subgraph of T arising from the vertical block A_1 by adding all arcs ab of T with $w(ab) > 0$, $a \in V(A_1)$, and $b \notin V(A_1)$. Note that $T^* = T = A_1$ if $m = 1$ and that T^* contains no arc in B_1 except possibly v_1v_2 when $|B_1| = 4$. For any collection $\mathcal{F}$ of subsets of A, we use $\mathcal{F} \cap E^*$ to denote the collection consisting of all nonempty $F \cap E^*$ for $F \in \mathcal{F}$.

To prove Theorem 3.1, we shall show the existence of a w-FAS packing in T of size $\mu_w(T)$ for all $w \in \mathbb{Z}_+^A$ by induction on the number of vertical blocks. For this purpose, reducing arc weights while preserving the minimum total weight of a cycle whenever possible, it suffices to consider the weight functions w such that each arc e with $w(e) > 0$ is contained in a minimum cycle of (T, w). To make the induction work, what we establish is the following stronger statement.

Theorem 3.3. *Let $T = (V, A)$ be a Möbius-free strong tournament with $|V| \ge 3$ and $T \notin \{F_1, G_1\}$, and let $w \in \mathbb{Z}_+^A$ such that each arc e with $w(e) > 0$ is contained in a minimum cycle of (T, w). Then T has a w-FAS packing $\mathcal{F}$ of size $\mu_w(T)$, such that $\mathcal{F} \cap E^*$ can be obtained by first performing breadth-first search for $|\mathcal{F} \cap E^*|$ steps in T^* from v_1 and then eliminating triangles in $A_1 \backslash v_1$.*

Remark. Let D be the digraph obtained from T^* by splitting v_1 into a source s and a sink t. We view each arc e of D as an arc of T and associate it with a length $w(e)$. By breadth-first search in T^* from v_1 we mean that in D from s, which proceeds as follows. For $i = 1, 2, \ldots, r := \mu_w(T)$, let U_i be the set of vertices at distance less than i from s in D with respect to w, and let $C_i := \delta^+(U_i)$. Then we can construct a w-FAS packing $\{F_1, F_2, \ldots, F_r\}$ in T^* from $\{C_1, C_2, \ldots, C_r\}$ by eliminating triangles in $A_1 \backslash v_1$, as done in the proof of Lemma 3.2. This algorithm carries over naturally to T_1^* and T_2^* involved in our proof (see (9) and (10)).

To carry out the induction step, it is natural to consider the subtournaments T_1 and T_2 of T (see the paragraph above (5)). Yet, there is no guarantee that a w-FAS packing of T_1 can be combined with a w-FAS packing of T_2 to yield a w-FAS packing of T with size r. That explains why we impose some constraint on the weight function w, refine w as w_1 and w_2 when restricted to T_1 and T_2, respectively, and introduce digraphs T_1^* and T_2^* in our proof.

Proof. By Theorem 2.7, T can be depicted as in Figure 7, with vertical blocks $A_1, A_2, \ldots, A_m$, horizontal blocks $B_1, B_2, \ldots, B_{m-1}$, and join vertices $v_1, v_2, \ldots, v_m$; subject to this, we assume that m is minimum. Then the minimum of m allows us to assume that $|A_1| \ge 2$ and $|A_m| \ge 2$ (refer to the proof of Theorem 2.7). For each vertical block A_i, let $A_{i,1}, A_{i,2}, \ldots, A_{i,n_i}$ be the building blocks of A_i, for $1 \le i \le m$.

We apply induction on m. Since each $A_i \in \mathcal{Q}$, the induction base $m = 1$ follows instantly from Lemma 3.2. So we proceed to the induction step and assume that $m \ge 2$ and that the statement holds for $m - 1$.

Let us first make some simple observations about the weight function w.

(1) For any arc uv and any path P from u to v in T, we have $w(uv) \le w(P)$.

Assume that contrary: $w(uv) > w(P)$. By hypothesis, uv is contained in a minimum cycle C of (T, w). Let D be the multiset union of P and $C[v, u]$ (that is, if an arc is contained in both P

and $C[v,u]$, then it appears twice in D). Clearly, D is an Eulerian digraph with $w(D) < w(C)$. Let C' be a directed cycle contained in D. Then $w(C') \le w(D) < w(C)$, contradicting the minimality assumption on C.

From (1) it is clear that

(2) for any minimum cycle C of (T, w) and any chord uv of C, the cycle arising from C by replacing $C[u,v]$ with uv is also minimum. So $w(uv) = w(C[u,v])$.

By Theorem 2.7, v_1 has an out-neighbor in the leftmost building block of A_1. Hence

(3) there is a path in A_1 from v_1 to each vertex in $A_1 \backslash v_1$.

(4) Let ab be an arc in T with $w(ab) > 0$, $a \in V(A_1 \backslash v_1)$, and $b \notin V(A_1 \cup B_1 \backslash v_2)$, and let P be a minimum $v_1 - a$ path in A_1 (see (3)). If $v_1 b$ is an arc of T, then $w(P) + w(ab) = w(v_1 b)$. (Possibly $b = v_2$ when $|B_1| = 4$.)

To justify this, let C be a minimum cycle of (T, w) containing ab. From the structural description of T, we see that C passes through v_1 and that $C[v_1, a]$ is fully contained in A_1. By the minimality assumptions on P and C, we obtain $w(P) = w(C[v_1, a])$. In view of (2), $w(C[v_1, b]) = w(v_1 b)$. Hence $w(P) + w(ab) = w(C[v_1, a]) + w(ab) = w(C[v_1, b]) = w(v_1 b)$, as desired.

Let $T_1 = (V_1, E_1)$ be the subtournament of T induced by all vertices in $A_1 \cup B_1$, let $T_2 = (V_2, E_2)$ be the subtournament of T induced by all vertices outside $A_1 \backslash v_1$, and let A'_2 be the subtournament of T induced by all vertices in $A_2 \cup B_1$. Then

(5) $T_1 \in \mathcal{Q}$ and T_2 can be depicted as in Figure 7, with vertical blocks $A'_2, A_3, \ldots, A_m$. For $i = 1, 2$, T_i is strongly connected, with $|V_i| \ge 3$ and $T_i \notin \{F_1, G_1\}$. So T_i is Möbius-free by Theorem 2.7.

We only check that $T_i \notin \{F_1, G_1\}$ for $i = 1, 2$, as the remaining statements hold trivially. For this purpose, observe that if $|B_1| = 2$, then v_2 is a near-sink in T_1; if $|A_1| = 2$, then the vertex in $A_1 \backslash v_1$ is a near-source in T_1; if $|A_1| \ge 3$ and $|B_1| \ge 3$, then $(A_1 \backslash v_1, B_1 \backslash v_1)$ is a nontrivial dicut in $T_1 \backslash v_1$. Moreover, if $|B_1| = 2$, then v_1 is a near-source in T_2; if $|B_1| \ge 3$, then the source of $B_1 \backslash \{v_1, v_2\}$ is a near-source in T_2. Since both F_1 and G_1 are $i2s$ and neither of them contains a near-sink or a near-source, we obtain $T_i \notin \{F_1, G_1\}$ for $i = 1, 2$, as desired.

In the remainder of our proof, we reserve u_1 for the vertex in $B_1 \backslash \{v_1, v_2\}$ if $|B_1| = 3$, and reserve u_1 and u_2 for the two vertices in $B_1 \backslash \{v_1, v_2\}$ if $|B_1| = 4$, with $u_1 u_2 \in A$. Moreover, we reserve R_1 for a minimum $v_2 - v_1$ path in B_1 with respect to w, having the fewest arcs. By (1), we obtain $R_1 = v_2 v_1$ if $|B_1| \le 3$ and $R_1 = v_2 u_1 v_1$ or $v_2 u_2 v_1$ if $|B_1| = 4$. Write $r := \mu_w(T)$. The statement below follows instantly from (2).

(6) Each arc e in B_1 with $w(e) > 0$ is contained in a cycle of T_i with weight r for $i = 1$ or 2 (but not necessarily both). Each arc e in T_i but outside B_1 with $w(e) > 0$ is contained in a cycle of T_i with weight r for $i = 1, 2$. Furthermore, if $|B_1| = 4$, then the arc $u_i v_1$ with $w(u_i v_1) > 0$ is contained in a cycle of T_1 with weight r, and the arc $v_2 u_i$ with $w(v_2 u_i) > 0$ is contained in a cycle of T_2 with weight r for $i = 1, 2$.

To justify this, let e be an arbitrary arc of T_i with $w(e) > 0$ for $i = 1$ or 2, and let C be a cycle containing e in T with weight r. If C is fully contained in T_i, we have nothing to prove. So we assume that the opposite case occurs. From the structural description of T in Theorem 2.7, we see that C passes through both v_1 and v_2 and also contains an arc ab, with $a \in A_1 \backslash v_1$ and $b \notin A_1 \cup B_1$.

Since both av_2 and $v_1 b$ are chords of C, by (2) at least one of the two cycles $C[v_2, a]av_2$ and

$v_1bC[b, v_2]$ is a cycle of weight r in T_i containing e. In particular, if e is in B_1, then $C[v_2, a]av_2$ is a cycle of weight r in T_1 containing e and $v_1bC[b, v_2]$ is a cycle of weight r in T_2 containing e. This establishes the first two statements in (6).

Finally, consider the case when $|B_1| = 4$ and $e = u_iv_1$ for $i = 1$ or 2. If C is not fully contained in T_1, then $C[v_2, v_1]$ is fully contained in B_1 and hence $C[v_2, v_1]v_1v_2$ is a cycle containing u_iv_1 in T_1 with weight r by (2). Similarly, we can prove the statement on v_2u_i. Hence (6) holds.

(7) For each vertex a in $A_1\backslash v_1$ with $w(av_2) > 0$, the path av_2R_1 is contained in a cycle of T_1 with weight r. For each vertex b outside $A_1 \cup B_1\backslash v_2$ with $w(v_1b) > 0$, the path R_1v_1b is contained in a cycle of T_2 with weight r.

We only establish the second half here, as the proof of the first half goes along the same line. By (6), arc v_1b is contained in a cycle C of T_2 with weight r. Since $\delta(B_1\backslash v_2)$ forms a dicut in $T_2\backslash v_2$, cycle C must pass through v_2. It follows that $C[v_2, v_1]$ is fully contained in B_1. Let C' be obtained from C by replacing $C[v_2, v_1]$ with R_1. Then C' is a cycle of T_2 with weight r and contains the path R_1v_1b. So (7) is justified.

(8) If R_1 is not contained in any cycle of T_1 with weight r, then $w(ab) = 0$ for any $a \in V(A_1)$ if $|B_1| = 4$ and $a \in V(A_1\backslash v_1)$ if $|B_1| \leq 3$ and $b \notin V(A_1 \cup B_1\backslash v_2)$. If R_1 is not contained in any cycle of T_2 with weight r, then $w(ab) = 0$ for any $a \in V(A_1)$ and $b \notin V(A_1 \cup B_1\backslash v_2)$ if $|B_1| = 4$ and $b \notin V(A_1 \cup B_1)$ if $|B_1| \leq 3$.

Suppose on the contrary that $w(ab) > 0$ for some $a \in V(A_1)$ if $|B_1| = 4$ or $a \in V(A_1\backslash v_1)$ if $|B_1| = 3$ and $b \notin V(A_1 \cup B_1\backslash v_2)$. Let C be a minimum cycle of (T, w) containing ab. From Theorem 2.7 we see that C passes through v_1 and v_2 and that $C[v_2, v_1]$ is fully contained in B_1. If $a \in V(A_1\backslash v_1)$, then av_2 is ab or a chord of C. By (2), we have $w(av_2) = w(C[a, v_2]) \geq w(ab) > 0$. It follows from (7) that R_1 is contained in a cycle of T_1 with weight r, a contradiction. So we assume that $a = v_1$ and $|B_1| = 4$. By (2), the cycle arising from $C[v_2, v_1]$ by adding v_1v_2 is a minimum cycle of (T, w). Therefore $v_2R_1v_1v_2$ is also a cycle of T_1 with weight r, a contradiction again. The second half of the statement can be established similarly.

For $i = 1, 2$, define $w_i \in \mathbb{Z}_+^{E_i}$ to be the weight function obtained from $w|_{E_i}$ by reducing the weights of arcs in B_1, if necessary, so that $\mu_{w_i}(T_i) = r$ and that each arc e in T_i with $w_i(e) > 0$ is contained in a minimum cycle of (T_i, w_i) (see (6)). We point out that $w_1(v_1v_2) = w_2(v_1v_2) = w(v_1v_2)$ when $|B_1| = 4$; we postpone giving its proof till this case is discussed (see (23)), as this observation has nothing to do with the case when $|B_1| \leq 3$.

Let $T_1^* = (V, E_1^*)$ be the subgraph of T obtained from $T_1 = (V_1, E_1)$ by adding all arcs ab with $w(ab) > 0$, $a \in V(A_1)$, and $b \notin V(A_1 \cup B_1)$, and define $w_1(ab) = w(ab)$ for each such arc ab. Let $T_2^* = (V_2, E_2^*)$ be the subgraph of $T_2 = (V_2, E_2)$ arising from block A_2' by adding all arcs ab with $w(ab) > 0$, $a \in V(A_2')$, and $b \notin V(A_2')$. For ease of description, we color each arc v_1b, with $w(v_1b) > 0$ and $b \notin V(A_1 \cup B_1\backslash v_2)$, by blue. (Possibly $b = v_2$ when $|B_1| = 4$). From (4) and the proof of Lemma 3.2 (recall the remark succeeding Theorem 3.3), we see that

(9) T_1^* has a w_1-FAS packing $\mathcal{F}_1$ of size r, obtained by first performing breadth-first search (with respect to the weight function w_1) for r steps from v_1 in T_1^* and then eliminating triangles in $A_1\backslash v_1$, such that each blue arc e is contained in precisely $w(e)$ members of $\mathcal{F}_1$.

Using (5) and the induction hypothesis, we deduce that

(10) T_2 has a w_2-FAS packing $\mathcal{F}_2$ of size r, such that $\mathcal{F}_2 \cap E_2^*$ can be obtained by first performing breadth-first search (with respect to the weight function w_2) for $|\mathcal{F}_2 \cap E_2^*|$ steps in

T_2^* from v_2 and then eliminating triangles in $A_2'\backslash v_2$.

We shall produce a w-FAS packing $\mathcal{F}$ of T having size r by gluing members of $\mathcal{F}_1$ together with those of $\mathcal{F}_2$, possibly with slight modification. For $i = 1, 2$, let $\mathcal{F}_i = \{F_{i,1}, F_{i,2}, \ldots, F_{i,r}\}$, where $F_{1,j}$ is the depth-j set in $\mathcal{F}_1$ from v_1, and $F_{2,j} \cap E_2^*$ is the depth-j set in $\mathcal{F}_2 \cap E_2^*$ from v_2. We color each $F_{i,j}$ containing a blue arc also by blue. Observe that no arc, except blue ones and those in B_1, is shared by members of $\mathcal{F}_1$ and members of $\mathcal{F}_2$. So, naturally, in our proof blue members of $\mathcal{F}_1$ will be glued together with blue members of $\mathcal{F}_2$. Once the members of $\mathcal{F}$ containing blue arcs are determined, the members containing arcs ab with $a \in V(A_1\backslash v_1)$ and $b \notin V(A_1 \cup B_1\backslash v_2)$ will be determined accordingly by (4).

Depending on the size of B_1, we distinguish between two cases.

Case 1. $|B_1| \le 3$.

We may assume that $|B_1| = 3$, because this situation properly contains the one when $|B_1| = 2$. Let $q := w(v_2v_1)$, $s := w(u_1v_1)$, and $t := w(v_2u_1)$. In view of (1), we have $q \le s + t$.

(11) If $s > 0$ and u_1v_1 is not contained in a cycle of T_1 having weight r with respect to the weight function w, then $q = s + t$. Furthermore, v_2v_1 is not contained in a cycle of T_1 having weight r with respect to w either.

By (6), u_1v_1 is contained in a cycle C of T_2 having weight r with respect to w. Clearly, C passes through v_2u_1. It follows instantly from (2) that $q = s + t$. Assume on the contrary that v_2v_1 is contained in a cycle Q of T_1 having weight r with respect to w. Let Q' be the cycle obtained from Q by replacing v_2v_1 with the path $v_2u_1v_1$. Then Q' has weight r and contains u_1v_1, a contradiction. So (11) is justified.

Similarly, the following statement holds.

(12) If $t > 0$ and v_2u_1 is not contained in a cycle of T_2 having weight r with respect to the weight function w, then $q = s + t$. Furthermore, v_2v_1 is not contained in a cycle of T_2 having weight r with respect to w either.

Let E_1' be the arc set obtained from E_1^* by deleting arcs in B_1, let E_2' be the arc set obtained from E_2 by deleting arcs in B_1, and let $K_{i,j}$ be the restriction of $F_{i,j}$ to E_i' for $i = 1, 2$ and $1 \le j \le r$.

(13) Let us modify $K_{i,j}$'s as follows:

- add arc v_2v_1 to $K_{1,j}$ for $r - q + 1 \le j \le r$;
- add arc u_1v_1 to $K_{1,j}$ for $r - s + 1 \le j \le r$;
- add arc v_2u_1 to $K_{1,j}$ for $r - q + 1 \le j \le r - s$;
- add arc v_2v_1 to $K_{2,j}$ for $1 \le j \le q$;
- add arc v_2u_1 to $K_{2,j}$ for $1 \le j \le t$; and
- add arc u_1v_1 to $K_{2,j}$ for $t + 1 \le j \le q$.

We use $F_{i,j}'$ to denote the resulting $K_{i,j}$.

(14) $\mathcal{F}_1' := \{F_{1,1}', F_{1,2}', \ldots, F_{1,r}'\}$ is a w-FAS packing of T_1^*, and $\mathcal{F}_2' := \{F_{2,1}', F_{2,2}', \ldots, F_{2,r}'\}$ is a w-FAS packing of T_2.

To justify this, recall that each arc $e \in F_{1,j}$ satisfies $w_1(e) > 0$ and that each arc e of T_1 with $w_1(e) > 0$ is contained in a cycle of T_1 with weight r. From (9) and breadth-first search we deduce that

- $F_{1,j}$ contains v_2v_1 iff $r - w_1(v_2v_1) + 1 \le j \le r$;
- $F_{1,j}$ contains u_1v_1 iff $r - w_1(u_1v_1) + 1 \le j \le r$; and

• $F_{1,j}$ contains v_2u_1 iff $r - w_1(v_2v_1) + 1 \leq j \leq r - w_1(u_1v_1)$.

Since $q \geq w_1(v_2v_1)$, $s \geq w_1(u_1v_1)$, and $t \geq w_1(v_2u_1)$, we deduce that if $F_{1,j}$ contains v_2v_1, then so does $F'_{1,j}$, and if $F_{1,j}$ contains u_1v_1, then so does $F'_{1,j}$. Moreover, if $F_{1,j}$ contains v_2u_1, then $F'_{1,j}$ contains v_2u_1 or u_1v_1. Note that each cycle of T_1 containing v_2u_1 must pass through u_1v_1. Since each $F_{1,j}$ is an FAS of T_1^*, so is $F'_{1,j}$. From (13) it is clear that $\mathcal{F}'_1$ is a w-FAS packing of T_1^*. Similarly, we can prove that $\mathcal{F}'_2$ is a w-FAS packing of T_2.

(15) If $F'_{2,j} \neq F_{2,j}$ for some j with $1 \leq j \leq r$, then $w(ab) = 0$ for any $a \in V(A_1)$ and $b \notin V(A_1 \cup B_1)$. In particular, there is no blue arc in T.

To justify this, note from (1), (10) and breadth-first search that $F_{2,j}$ contains v_2v_1 iff $1 \leq j \leq w_2(v_2v_1)$, $F_{2,j}$ contains v_2u_1 iff $1 \leq j \leq w_2(v_2u_1)$, and $F_{2,j}$ contains u_1v_1 iff $w_2(v_2u_1)+1 \leq j \leq w_2(v_2v_1)$. Since $F'_{2,j} \neq F_{2,j}$ for some j with $1 \leq j \leq r$, we deduce from (13) that $w_2(v_2v_1) < q$ or $w_2(v_2u_1) < t$. From (12) we further conclude that the inequality $w_2(v_2v_1) < q$ must hold. Thus (15) follows instantly from (8).

From (9), (10), (13) and (15) we see that

(16) $F'_{1,j}$ contains a blue arc iff $F'_{2,j+q}$ contains it.

Define

(17) $$F_j := \begin{cases} F'_{1,j} \cup F'_{2,j+q} & \text{if } 1 \leq j \leq r-q; \\ F'_{1,j} \cup F'_{2,j+q-r} & \text{if } r-q+1 \leq j \leq r. \end{cases}$$

(18) For F_j's defined in (17), the following statements hold:

• F_j contains v_2v_1 iff $r-q+1 \leq j \leq r$;

• F_j contains u_1v_1 iff $r-s+1 \leq j \leq r$; and

• F_j contains v_2u_1 iff $r-q+1 \leq j \leq \min\{r, r-q+t\}$ or $1 \leq j \leq \max\{0, t-q\}$.

To justify this, note from (17) that $F'_{1,j}$ is a subset of F_j for $1 \leq j \leq r$ and from (13) that

(18.1) $F'_{1,j}$ contains v_2v_1 iff $r-q+1 \leq j \leq r$, and $F'_{2,k}$ contains v_2v_1 iff $1 \leq k \leq q$;

(18.2) $F'_{1,j}$ contains u_1v_1 iff $r-s+1 \leq j \leq r$, and $F'_{2,k}$ contains u_1v_1 iff $t+1 \leq k \leq q$;

(18.3) $F'_{1,j}$ contains v_2u_1 iff $r-q+1 \leq j \leq r-s$, and $F'_{2,k}$ contains v_2u_1 iff $1 \leq k \leq t$.

First, let k be a subscript with $v_2v_1 \in F'_{2,k}$. Then $1 \leq k \leq q$ by (18.1). Let j be the subscript with $k = j+q-r$. Then $j = r-q+k$. Thus $r-q+1 \leq j \leq r$ and hence $F'_{2,k}$ is a subset of F_j by (17). Combining this with (18.1) (as $F'_{1,j} \subseteq F_j$), we see that F_j contains v_2v_1 iff $r-q+1 \leq j \leq r$.

Second, let k be a subscript with $u_1v_1 \in F'_{2,k}$. Then $t+1 \leq k \leq q$ by (18.2). Let j be the subscript with $k = j+q-r$. Then $j = r-q+k$. Thus $r-q+t+1 \leq j \leq r$. It follows from (17) that $F'_{2,k}$ is a subset of F_j. By (1), $s+t \geq q$. So $r-q+t+1 \geq r-s+1$ and hence $r-s+1 \leq j \leq r$. Combining this with (18.2) (as $F'_{1,j} \subseteq F_j$), we see that F_j contains u_1v_1 iff $r-s+1 \leq j \leq r$.

Finally, let k be a subscript with $v_2u_1 \in F'_{2,k}$. Then $1 \leq k \leq t$ by (18.3). When $1 \leq k \leq \min\{q,t\}$, let j be the subscript with $k = j+q-r$. Then $j = r-q+k$. So $r-q+1 \leq j \leq \min\{r, r-q+t\}$ and thus $F'_{2,k}$ is a subset of F_j by (17). When $\min\{q,t\}+1 \leq k \leq t$ (equivalently $q+1 \leq k < t$), let j be the subscript with $k = j+q$. Then $j = k-q$. Thus $1 \leq j \leq t-q$ and hence $F'_{2,k}$ is a subset of F_j by (17). Therefore, there exists a subscript k with $v_2u_1 \in F'_{2,k} \subseteq F_j$ iff $r-q+1 \leq j \leq \min\{r, r-q+t\}$ or $1 \leq j \leq \max\{0, t-q\}$. Combining this with (18.3) (as $F'_{1,j} \subseteq F_j$), we see that F_j contains v_2u_1 iff $r-q+1 \leq j \leq \min\{r, r-q+t\}$ or $1 \leq j \leq \max\{0, t-q\}$, because $r-s \leq \min\{r, r-q+t\}$ (recall that $s+t \geq q$ by (1)). This establishes (18).

In view of (16)-(18), we obtain

(19) each arc e of T is contained in at most $w(e)$ members of $\mathcal{F} := \{F_1, F_2, \ldots, F_r\}$.

Let us show that

(20) each F_j, with $1 \le j \le r$, is an FAS of T.

For this purpose, let C be an arbitrary cycle in T. Clearly, F_j intersects C if C is a cycle of T_1 or a cycle of T_2 by (14). So we assume that C is not fully contained in T_i for $i = 1, 2$.

Consider the subcase when u_1 is outside C. Now C contains an arc ab with $a \in V(A_1 \backslash v_1)$ and $b \notin V(A_1 \cup B_1)$. From Theorem 2.7 we deduce that C passes through v_1 and $C[v_1, a]$ is fully contained in A_1. Let C' be the cycle arising from C by replacing $C[v_1, b]$ with $v_1 b$, and let $F'_{2,k}$ be the member of $\mathcal{F}_2$ contained in F_j. Then C' is fully contained in T_2 and intersects $F'_{2,k}$. If $F'_{2,k}$ intersects $C'[b, v_1] = C[b, v_1]$, then F_j intersects C. So we assume that $F'_{2,k}$ contains $v_1 b$ and hence $w(v_1 b) > 0$, indicating that $v_1 b$ is a blue arc. By (8), $w_2(v_2 v_1) = w(v_2 v_1) = q$. Thus, by (16) and (17), $k = j + q$ and $F'_{1,j}$ contains $v_1 b$ as well. In view of (1), $w_1(C[v_1, b]) = w(C[v_1, b]) \ge w(v_1 b)$. From the construction of $\mathcal{F}_1$ using depth-first search, we see that $F'_{1,j}$ intersects $C[v_1, b]$. So F_j intersects C.

It remains to consider the subcase when C contains u_1. Assume first that $v_2 u_1 v_1$ is a segment of C. Let C' be obtained from C by replacing $v_2 u_1 v_1$ with $v_2 v_1$. As observed in the preceding paragraph, F_j intersects C'. If $v_2 v_1 \notin F_j$, then F_j intersects $C'[v_1, v_2]$ and hence C; otherwise, $v_2 v_1 \in F_j$, so $r - q + 1 \le j \le r$ by (18). Since $s + t \ge q$ by (1), we have $r - q + t \ge r - s$. Hence $r - q + 1 \le j \le \min\{r, r - q + t\}$ or $r - s + 1 \le j \le r$. It follows from (18) that F_j contains $v_2 u_1$ or $u_1 v_1$. Therefore F_j intersects C.

Next, we assume that C has a segment $a u_1 b$, where $a \in V(A_1 \backslash v_1)$ and $b \notin V(A_1 \cup B_1)$. Note that $v_2 v_1$ is contained in C and $C[v_1, a]$ is fully contained in A_1. Let C' be obtained from C by replacing $C[v_2, u_1]$ with $v_2 u_1$. Then C' is fully contained in T_2. So F_j intersects C'. If $v_2 u_1 \notin F_j$, then F_j intersects $C'[u_1, v_2]$ and hence C; otherwise, $v_2 u_1 \in F_j$, so $r - q + 1 \le j \le \min\{r, r - q + t\}$ or $1 \le j \le \max\{0, t - q\}$ by (18). If $t \le q$, then $r - q + 1 \le j \le r - q + t \le r$. Thus F_j contains $v_2 v_1$ by (18) and hence intersects C. Suppose $t > q$. Since $(r - s) + q \ge t$ by (11) or (1) (when $s > 0$ and $u_1 v_1$ is contained in a cycle Q of T_1 having weight r, consider the path $v_2 v_1 Q[v_1, u_1]$, which has weight $(r - s) + q$), we obtain $r - s \ge t - q$. Hence $r - q + 1 \le j \le r$ or $1 \le j \le r - s$. It follows from (18) that either F_j contains $v_2 v_1$ or $F'_{1,j}$ (and hence F_j) intersects $C[v_1, u_1]$ by (9). This establishes (20).

Combining (19) with (20), we conclude that $\mathcal{F}$ is a w-FAS packing of T having size r. From (9) and (15)-(17), it is clear that $\mathcal{F} \cap E^*$ is obtained by first performing breadth-first search for $|\mathcal{F} \cap E^*|$ steps in T^* from v_1 and then eliminating triangles in $A_1 \backslash v_1$.

Case 2. $|B_1| = 4$.

Observe that

(21) arc $v_1 v_2$ is contained in only three cycles, $v_1 v_2 u_1 v_1$, $v_1 v_2 u_2 v_1$, and $v_1 v_2 u_1 u_2 v_1$, of T, and $w(v_1 v_2 u_i v_1) \le w(v_1 v_2 u_1 u_2 v_1)$ for $i = 1, 2$ by (1).

(22) $w(a v_2) \le w(v_1 v_2)$ for any vertex a in $A_1 \backslash v_1$, and $w(v_1 b) \le w(v_1 v_2)$ for any vertex b outside $A_1 \cup B_1 \backslash v_2$.

We only prove the first half of this statement, as the proof of the second half does along the same line. If $w(a v_2) = 0$, then trivially $w(a v_2) \le w(v_1 v_2)$. So we assume that $w(a v_2) > 0$. By (7), the path $a v_2 R_1$ is contained in a cycle C of T_1 having weight r with respect to the weight function w. By (1), we have $w(C[v_1, v_2]) = w(v_1v_2)$. It follows that $w(av_2) \leq w(v_1v_2)$. This establishes (22).

Let $p := w(u_1u_2)$, $q := w(v_1v_2)$, $s_i := w(u_iv_1)$, and $t_i := w(v_2u_i)$ for $i = 1, 2$.

(23) $q = w_1(v_1v_2) = w_2(v_1v_2)$, $s_i = w_1(u_iv_1)$, and $t_i = w_2(v_2u_i)$ for $i = 1, 2$. Furthermore, if $p > 0$, then either $p + s_2 = s_1$ or $t_1 + p = t_2$. If $q > 0$, then $v_1v_2R_1v_1$ is a cycle having weight r with respect to the weight function w.

From (21) and (6) it follows immediately that $q = w_1(v_1v_2) = w_2(v_1v_2)$, $s_i = w_1(u_iv_1)$, and $t_i = w_2(v_2u_i)$ for $i = 1, 2$. To show the statements concerning p, let C be a cycle in T_i containing u_1u_2 and having weight r with respect to the weight function w for $i = 1$ or 2; such C exists by (6). Since u_2v_1 is the only arc leaving u_2 in T_1, and v_2u_1 is the only arc entering u_1 in T_2, cycle C contains u_2v_1 or v_2u_1. Thus $p + s_2 = s_1$ or $t_1 + p = t_2$ by (2). If $q > 0$, then v_1v_2 is contained in a cycle having weight r with respect to the weight function w. From (21) we deduce that $v_1v_2R_1v_1$ has weight r with respect to w. So (23) is established.

We proceed by considering two subcases.

Subcase 2.1. $s_i + t_i + q = r$ for $i = 1$ or 2.

From (4) and (7)-(10) we see that

(24) $F_{1,j}$ contains a blue arc v_1b iff so does $F_{2,j+r-q}$. (Hence $F_{1,j}$ is colored blue iff so is $F_{2,j+r-q}$.) Furthermore, $F_{1,j}$ contains a blue arc iff $1 \leq j \leq q$ by (22) and (23).

Define

(25) $$F_j := \begin{cases} F_{1,j} \cup F_{2,j+r-q} & \text{if } 1 \leq j \leq q; \\ F_{1,j} \cup F_{2,j-q} & \text{if } q+1 \leq j \leq r. \end{cases}$$

Thus each blue set in $\mathcal{F}_1$ is glued together with the corresponding blue set in $\mathcal{F}_2$ (see (24)), if any.

(26) For F_j's defined in (25), the following statements hold:

- F_j contains v_1v_2 iff $1 \leq j \leq q$;
- F_j contains u_iv_1 iff $r - s_i + 1 \leq j \leq r$ for $i = 1, 2$;
- F_j contains v_2u_i iff $q+1 \leq j \leq \min\{r, q+t_i\}$ or $1 \leq j \leq \max\{0, t_i - r + q\}$ for $i = 1, 2$; and
- F_j contains u_1u_2 iff $t_1 + q + 1 \leq j \leq \min\{r, t_2 + q\}$ or $1 \leq j \leq \max\{0, t_2 - r + q\}$ when $s_1 + t_1 + q = r$ and iff $r - s_1 + 1 \leq j \leq r - s_2$ when $s_2 + t_2 + q = r$.

To justify this, note from (6), (9), (10) and (23) that

(26.1) $F_{1,j}$ contains v_1v_2 iff $1 \leq j \leq q$, and $F_{2,k}$ contains v_1v_2 iff $r - q + 1 \leq k \leq r$;

(26.2) $F_{1,j}$ contains u_iv_1 iff $r - s_i + 1 \leq j \leq r$, and $F_{2,k}$ contains u_iv_1 iff $t_i + 1 \leq k \leq r - q$ for $i = 1, 2$;

(26.3) $F_{1,j}$ contains v_2u_i iff $q + 1 \leq j \leq r - s_i$, and $F_{2,k}$ contains v_2u_i iff $1 \leq k \leq t_i$ for $i = 1, 2$;

(26.4) $F_{1,j}$ contains u_1u_2 iff $r - s_1 + 1 \leq j \leq r - s_2$, and $F_{2,k}$ contains u_1u_2 iff $t_1 + 1 \leq k \leq t_2$.

First, let k be a subscript with $v_1v_2 \in F_{2,k}$. Then $r - q + 1 \leq k \leq r$ by (26.1). Let j be the subscript with $k = j + r - q$. Then $j = k - r + q$. Thus $1 \leq j \leq q$ and hence $F_{2,k}$ is a subset of F_j by (25). Combining this with (26.1) (as $F_{1,j} \subseteq F_j$), we see that F_j contains v_1v_2 iff $1 \leq j \leq q$.

Second, let k be a subscript with $u_iv_1 \in F_{2,k}$. Then $t_i + 1 \leq k \leq r - q$ by (26.2). Let j be the subscript with $k = j - q$. Then $j = k + q$. Thus $t_i + q + 1 \leq j \leq r$. Since $s_i + t_i + q \geq r$, we have $t_i + q + 1 \geq r - s_i + 1$ and hence $r - s_i + 1 \leq j \leq r$. Combining this with (26.2) (as $F_{1,j} \subseteq F_j$), we see that F_j contains u_iv_1 iff $r - s_i + 1 \leq j \leq r$.

Third, let k be a subscript with $v_2u_i \in F_{2,k}$. Then $1 \le k \le t_i$ by (26.3). When $1 \le k \le \min\{r-q, t_i\}$, let j be the subscript with $k = j-q$. Then $j = q+k$. So $q+1 \le j \le \min\{r, q+t_i\}$. Hence $F_{2,k}$ is a subset of F_j by (25). When $\min\{r-q, t_i\}+1 \le k \le t_i$ (equivalently $r-q+1 \le k \le t_i$), let j be the subscript with $k = j+r-q$. Then $j = k-r+q$. Thus $1 \le j \le t_i-r+q \le q$ and hence $F_{2,k}$ is a subset of F_j by (25). Therefore, there exists a subscript k with $v_2u_i \in F_{2,k} \subseteq F_j$ iff $q+1 \le j \le \min\{r, q+t_i\}$ or $1 \le j \le \max\{0, t_i-r+q\}$. Combining this with (26.3) (as $F_{1,j} \subseteq F_j$), we see that F_j contains v_2u_i iff $q+1 \le j \le \min\{r, q+t_i\}$ or $1 \le j \le \max\{0, t_i-r+q\}$, because $s_i+t_i+q \ge r$, which implies $r-s_i \le q+t_i$.

Finally, let k be a subscript with $u_1u_2 \in F_{2,k}$. Then $t_1+1 \le k \le t_2$ by (26.4). When $t_1+1 \le k \le \min\{r-q, t_2\}$, let j be the subscript with $k = j-q$. Then $j = k+q$. Thus $t_1+q+1 \le j \le \min\{r, t_2+q\}$. Hence $F_{2,k}$ is a subset of F_j by (25). When $\min\{r-q, t_2\}+1 \le k \le t_2$ (equivalently $r-q+1 \le k \le t_2$), let j be the subscript with $k = j+r-q$. Then $j = k-r+q$. Thus $1 \le j \le t_2-r+q \le q$ and hence $F_{2,k}$ is a subset of F_j by (25). Therefore,

(26.5) there exists a subscript k with $u_1u_2 \in F_{2,k} \subseteq F_j$ iff $t_1+q+1 \le j \le \min\{r, t_2+q\}$ or $1 \le j \le \max\{0, t_2-r+q\}$.

By the hypothesis of the present subcase, $s_i+t_i+q = r$ for $i = 1$ or 2. If $s_1+t_1+q = r$, then $r-s_1+1 = t_1+q+1$. Clearly, $r-s_2 \le \min\{r, t_2+q\}$. Combining (26.4) (as $F_{1,j} \subseteq F_j$) with (26.5), we see that F_j contains u_1u_2 iff $t_1+q+1 \le j \le \min\{r, t_2+q\}$ or $1 \le j \le \max\{0, t_2-r+q\}$. If $s_2+t_2+q = r$, then $r-s_2 = t_2+q$. Clearly, $r-s_1+1 \le t_1+q+1$. It follows from (26.4) and (26.5) that F_j contains u_1u_2 iff $r-s_1+1 \le j \le r-s_2$. Thus (26) holds.

By (1), we have $p \ge \max\{s_1-s_2, t_2-t_1\}$. In view of (24)-(26), we obtain

(27) each arc e of T is contained in at most $w(e)$ members of $\mathcal{F} := \{F_1, F_2, \ldots, F_r\}$.

Let us show that

(28) each F_j, with $1 \le j \le r$, is an FAS of T.

For this purpose, let C be an arbitrary cycle in T. Clearly, F_j intersects C if C is a cycle of T_1 or a cycle of T_2. So we assume that C is not fully contained in T_i for $i = 1, 2$.

Suppose C contains an arc ab with $a \in V(A_1 \backslash v_1)$ and $b \notin V(A_1 \cup B_1)$. From the structural description, we see that C passes through v_1 and $C[v_1, a]$ is fully contained in A_1. Let C' be the cycle arising from C by replacing $C[v_1, b]$ with v_1b, and let $F_{2,k}$ be the member of $\mathcal{F}_2$ contained in F_j. Then C' is fully contained in T_2 and intersects $F_{2,k}$. If $F_{2,k}$ intersects $C'[b, v_1] = C[b, v_1]$, then F_j intersects C. So we assume that $F_{2,k}$ contains v_1b and hence $w(v_1b) > 0$. It follows from (22) that $q \ge w(v_1b) > 0$. By (24) and (25), we get $k = j+r-q$ and $F_{1,j}$ contains the blue arc v_1b as well. By (1), we obtain $w(C[v_1, b]) \ge w(v_1b)$. From the construction of $\mathcal{F}_1$ using breadth-first search, we see that $F_{1,j}$ intersects $C[v_1, b]$. Thus F_j intersects C.

So we assume that C contains no arc ab with $a \in V(A_1 \backslash v_1)$ and $b \notin V(A_1 \cup B_1)$. Consider the situation when C contains both $v_2u_1v_1$ and au_2b as segments, where $a \in V(A_1 \backslash v_1)$ and $b \notin V(A_1 \cup B_1)$. Note that $C[v_1, a]$ is fully contained in A_1. Let C' be obtained from C by replacing $C[v_2, u_2]$ with v_2u_2. Then C' is fully contained in T_2. So F_j intersects C'. If $v_2u_2 \notin F_j$, then F_j intersects $C'[u_2, v_2]$ and hence C; otherwise, $v_2u_2 \in F_j$, so $q+1 \le j \le \min\{r, q+t_2\}$ or $1 \le j \le \max\{0, t_2-r+q\}$ by (26). If $q+1 \le j \le r$ then, by (26), F_j contains v_2u_1 or u_1v_1, because $q+t_1 \ge r-s_1$. So F_j intersects C. If $1 \le j \le t_2-r+q$ then $q+t_2 > r$ and hence $s_1+t_1+q = r$ by the hypothesis of Subcase 2.1. By (1), we have $t_1+s_1+(r-s_2) \ge t_2$ (when $s_2 > 0$, arc u_2v_1 is contained in a cycle Q of T_1 having weight r with respect to w by (6). Consider the path $v_2u_1v_1Q[v_1, u_2]$, which has weight $t_1+s_1+(r-s_2)$). It follows that

$t_2 - r + q \leq r - s_2$. Thus $1 \leq j \leq r - s_2$. So $F_{1,j}$ intersects $C[v_1, u_2]$ by (9) and hence F_j intersects C by (25).

Notice that u_1u_2 plays no role in the above proof. So the same argument (simply interchanging the subscripts 1 and 2, whenever appropriate) implies that F_j also intersects C if C contains both $v_2u_2v_1$ and au_1b as segments, where $a \in V(A_1 \backslash v_1)$ and $b \notin V(A_1 \cup B_1)$. This proves (28).

Combining (27) with (28), we conclude that $\mathcal{F}$ is a w-FAS packing of T having size r. From (9) and (25), it is clear that $\mathcal{F} \cap E^*$ is obtained by first performing breadth-first search for $|\mathcal{F} \cap E^*|$ steps in T^* from v_1 and then eliminating triangles in $A_1 \backslash v_1$.

Subcase 2.2. $s_i + t_i + q > r$ for $i = 1, 2$.

Recall that each arc e with $w(e) > 0$ is contained in a minimum cycle of (T, w). By (21), we obtain

(29) $q = 0$. So $s_i + t_i > r$ for $i = 1, 2$ and hence s_1, s_2, t_1 and t_2 are all positive.

In view of (21) and (29), R_1 is contained in no cycle of T_i having weight r with respect to w for $i = 1, 2$. It follows from (8) that

(30) $w(ab) = 0$ for any $a \in V(A_1)$ and $b \notin V(A_1 \cup B_1 \backslash v_2)$.

For $i = 1, 2$, let $T'_i = (V'_i, E'_i)$ be obtained from T_i by deleting the vertex v_{3-i}, and let $\mathcal{F}'_i = \{F'_{i,1}, F'_{i,2}, \ldots, F'_{i,r}\}$, where $F'_{i,j}$ is the restriction of $F_{i,j}$ to E'_i for $1 \leq j \leq r$. Observe that no arc is shared by a member of $\mathcal{F}'_1$ and that of $\mathcal{F}'_2$, except u_1u_2. We shall produce a w-FAS packing $\mathcal{F}$ of T having size r by gluing members of $\mathcal{F}'_1$ together with those of $\mathcal{F}'_2$, along u_1u_2 whenever possible. For this purpose, observe from (6), (9), (10), and (29) that

(31) $F'_{1,j}$ contains u_1u_2 iff $r - s_1 + 1 \leq j \leq r - s_2$, and $F'_{2,k}$ contains u_1u_2 iff $t_1 + 1 \leq k \leq t_2$;

(32) $F'_{1,j}$ contains u_iv_1 iff $r - s_i + 1 \leq j \leq r$, and no $F'_{2,k}$ contains u_iv_1 for $i = 1, 2$; and

(33) no $F'_{1,j}$ contains v_2u_i, and $F'_{2,k}$ contains v_2u_i iff $1 \leq k \leq t_i$ for $i = 1, 2$.

Let $\{g, h\}$ be a permutation of $\{1, 2\}$ with $s_g + t_g \leq s_h + t_h$. We first arrange $F'_{1,1}, F'_{1,2}, \ldots, F'_{1,r}$ on a circle O in clockwise order, and then arrange $F'_{2,1}, F'_{2,2}, \ldots, F'_{2,r}$ on O in the same order, such that members of $\mathcal{F}'_1$ alternate with those of $\mathcal{F}'_2$ in the following way:

- F'_{2,t_g+1} follows $F'_{1,r-s_g+1}$ immediately;
- F'_{2,t_g+2} follows $F'_{1,r-s_g+2}$ immediately;

  ……

- F'_{2,t_g} follows $F'_{1,r-s_g}$ immediately,

where the subscripts are taken modulo r. In particular, $F'_{i,0} = F'_{i,r}$ for $i = 1, 2$.

For $1 \leq j \leq r$, let $\pi(j)$ denote the subscript such that $F'_{2,\pi(j)}$ follows $F'_{1,j}$ immediately on O, and define $F_j = F'_{1,j} \cup F'_{2,\pi(j)}$. Observe that

(34) $\pi(j) = \begin{cases} (s_g + t_g - r) + j & \text{if } 1 \leq j \leq 2r - (s_g + t_g), \\ (s_g + t_g - 2r) + j & \text{if } 2r - (s_g + t_g) + 1 \leq j \leq r, \end{cases}$ which implies $\pi(r - s_g) = t_g$ if $s_g < r$ and $\pi(r - s_h) \leq t_h$ if $s_h < r$ (the first line of $\pi(j)$ applies now).

(35) Each F_j for $1 \leq j \leq r$ intersects each of the three paths $v_2u_1v_1$, $v_2u_2v_1$, and $v_2u_1u_2v_1$.

To justify this, imagine that circle O has r positions, $1, 2, \ldots, r$, in clockwise order, such that each position i is occupied by both $F'_{1,i}$ and $F'_{2,\pi(i)}$. By (29), we have $s_g + t_g > r$. From the arrangements of $F_{i,j}$'s on O, it follows immediately that

(35.1) circle O is covered by $F'_{1,r-s_g+1}, F'_{1,r-s_g+2}, \ldots, F'_{1,r}, F'_{2,1}, F'_{2,2}, \ldots, F'_{2,t_g}$; that is, each position of O is occupied by at least one of these sets.

(35.2) Circle O is also covered by $F'_{1,r-s_h+1}, F'_{1,r-s_h+2}, \ldots, F'_{1,r}, F'_{2,1}, F'_{2,2}, \ldots, F'_{2,t_h}$.

The statement holds trivially if $s_h = r$. So we assume that $s_h < r$. From (34) and (29) we deduce that $\pi(1) = (s_g + t_g - r) + 1 \geq 2$ and $\pi(r - s_h) \leq t_h$. Hence $\{F'_{2,\pi(1)}, F'_{2,\pi(2)}, \ldots, F'_{2,\pi(r-s_h)}\}$ $\subseteq \{F'_{2,1}, F'_{2,2}, \ldots, F'_{2,t_h}\}$, this proves (35.2).

Similarly, we can check that $\{F'_{2,\pi(1)}, F'_{2,\pi(2)}, \ldots, F'_{2,\pi(r-s_2)}\} \subseteq \{F'_{2,1}, F'_{2,2}, \ldots, F'_{2,t_1}, F'_{2,t_1+1}, F'_{2,t_1+2}, \ldots, F'_{2,t_2}\}$, where $F'_{2,t_1+1}, F'_{2,t_1+2}, \ldots, F'_{2,t_2}$ appear only when $t_1 < t_2$. Thus

(35.3) circle O is moreover covered by $F'_{1,r-s_2+1}, F'_{1,r-s_2+2}, \ldots, F'_{1,r}, F'_{2,1}, F'_{2,2}, \ldots, F'_{2,t_1}, F'_{2,t_1+1}, F'_{2,t_1+2}, \ldots, F'_{2,t_2}$.

Combining (31)-(33) and (35.1)-(35.3), we conclude that each F_j for $1 \leq j \leq r$ intersects each of the three paths $v_2u_1v_1$, $v_2u_2v_1$, and $v_2u_1u_2v_1$.

(36) u_1u_2 is contained in at most $w(u_1u_2)$ members of the family $\mathcal{F} := \{F_1, F_2, \ldots, F_r\}$.

Since $\mathcal{F}'_i$ $(i = 1, 2)$ is obtained by restricting the w_i-packing $\mathcal{F}_i$ to E'_i, the construction of $\mathcal{F}$ and (31) allow us to assume that $s_2 + 1 \leq s_1$ and $t_1 + 1 \leq t_2$. By (1) with respect to w_1 and w_2 respectively, we obtain $s_1 - s_2 \leq w_1(u_1u_2)$ and $t_2 - t_1 \leq w_2(u_1u_2)$. Hence $\max\{s_1 - s_2, t_2 - t_1\} \leq \max\{w_1(u_1u_2), w_2(u_1u_2)\} \leq w(u_1u_2)$. When $g = 1$, it is instant from the construction of $\mathcal{F}$ that exactly $\max\{s_1 - s_2, t_2 - t_1\}$ members of $\mathcal{F}$ contain u_1u_2. When $g = 2$, since $r - s_1 + 1 \geq 1$ and $t_1 \leq t_2 - 1$, it follows from (34) (the first line) that $\pi(r - s_1 + 1) \leq t_1 + 1$. Thus $\pi(r - s_1 + 1) \leq t_2 = \pi(r - s_2)$, which implies that exactly $s_1 - s_2$ members of $\mathcal{F}$ contain u_1u_2. Therefore (36) holds in either case.

(37) Each F_j, with $1 \leq j \leq r$, is an FAS of T.

To see this, let C be an arbitrary cycle in T. Clearly, F_j intersects C if C is a cycle of T'_1 or a cycle of T'_2. So we assume that C is not fully contained in T'_i for $i = 1, 2$. From the structural description of T, we deduce that C contains one of the three paths $v_2u_1v_1$, $v_2u_2v_1$, and $v_2u_1u_2v_1$ as a segment. Therefore F_j intersects C by (35), as desired.

Since no arc is shared by a member of $\mathcal{F}'_1$ and that of $\mathcal{F}'_2$, except u_1u_2, the family $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ is a w-FAS packing of T having size r by (36) and (37). From (9) and (30), it is clear that $\mathcal{F} \cap E^*$ is obtained by first performing breadth-first search for $|\mathcal{F} \cap E^*|$ steps in T^* from v_1 and then eliminating triangles in $A_1 \backslash v_1$. This completes the proof of Theorem 3.3. ∎

4 Computer-assisted Proof

In the preceding section we have established the desired minimax relation for all Möbius-free strong tournaments other than F_1 and G_1, thereby finishing the main body of the proof of Theorem 1.2. In this section we present a computer-assisted proof for G_1.

Lemma 4.1. *Tournament G_1 is FAS Mengerian.*

In Schrijver [36] there is a characterization (Corollary 22.13d) of TDI system of the form $Ax \leq b, x \geq \mathbf{0}$, where A is a nonnegative integral matrix. The same argument yields the following result.

Lemma 4.2. *Let A be a nonnegative integral matrix with no zero row, and let b be a rational vector. Then the system $Ax \geq b$, $x \geq \mathbf{0}$ is TDI iff for each $\{0,1\}$-vector y, there exists an integral vector $z \geq \mathbf{0}$ with $z^TA \leq \lceil y^TA/2 \rceil$ and $2z^Tb \geq y^Tb$.*

To prove Lemma 4.1, let A be the minimal FAS-arc incidence matrix of G_1. Clearly, G_1 is FAS Mengerian iff $Ax \geq \mathbf{1}$, $x \geq \mathbf{0}$ is a TDI system. We shall demonstrate that the dimension of A is 41×15. Since it is beyond the capacity of our computer to exhaust all possible 2^{41} cases addressed in Lemma 4.2, we have to derive a refinement of this lemma to fulfill our need.

Suppose the dimension of A in Lemma 4.2 is $m \times n$. Let $\prec$ denote the lexicographical order defined over the set of all m-dimensional $\{0,1\}$-vectors; that is, $u \prec v$ if there exists a subscript j, with $1 \leq j \leq m$, such that $u_i = v_i$ for all $1 \leq i < j$ and $u_j < v_j$.

Lemma 4.3. *Let A be a nonnegative integral matrix with no zero row. Let V and W be two sets of $\{0,1\}$-vectors such that for each $v \in V$, there exists $w \in W$ satisfying $v \prec w$, $v^T\mathbf{1} = w^T\mathbf{1}$, and $w^TA \leq v^TA$. Let U consist of all $\{0,1\}$-vectors u such that $u^T\mathbf{1}$ is odd and $u \not\geq v$ for each $v \in V$. Then the system $Ax \geq \mathbf{1}$, $x \geq \mathbf{0}$ is TDI iff for each $y \in U$, there exists an integral vector $z \geq \mathbf{0}$ with $z^TA \leq \lceil y^TA/2 \rceil$ and $2z^T\mathbf{1} \geq y^T\mathbf{1}$.*

Proof. The "only if" part follows instantly from Lemma 4.2.

To establish the "if" part, it suffices to find a desired z for every $\{0,1\}$-vector y as described in Lemma 4.2. Suppose on the contrary that such z does not exist for some y. We choose such a counterexample y with the property that

(1) $y^T\mathbf{1}$ is as small as possible, and

(2) subject to (1), the lexicographical order of y is as high as possible.

Note that $y \notin U$, and thus either $y^T\mathbf{1}$ is even or $y \geq v$ for some $v \in V$.

We first assume that $y^T\mathbf{1}$ is even. Now $y \neq \mathbf{0}$, for otherwise $z = \mathbf{0}$ would satisfy the requirements. Thus there exists a unit $\{0,1\}$-vector $e \leq y$. Since $(y-e)^T\mathbf{1} = y^T\mathbf{1} - 1$, condition (1) guarantees the existence of an integral vector $z \geq \mathbf{0}$ satisfying $z^TA \leq \lceil (y-e)^TA/2 \rceil$ and $2z^T\mathbf{1} \geq (y-e)^T\mathbf{1}$, which clearly imply $z^TA \leq \lceil y^TA/2 \rceil$ and $2z^T\mathbf{1} \geq y^T\mathbf{1}$, a contradiction.

Next, we assume that $y \geq v$ for some $v \in V$. By hypothesis, there exists $w \in W$ such that $v \prec w$, $v^T\mathbf{1} = w^T\mathbf{1}$, and $w^TA \leq v^TA$. Observe that $y - v + w$ can be expressed as $\alpha + 2\beta$ for some $\{0,1\}$-vectors α and β. We proceed by considering two subcases.

Suppose $\beta \neq \mathbf{0}$. Then $\alpha^T\mathbf{1} = y^T\mathbf{1} - v^T\mathbf{1} + w^T\mathbf{1} - 2\beta^T\mathbf{1} = y^T\mathbf{1} - 2\beta^T\mathbf{1} < y^T\mathbf{1}$. By (1), there exists an integral vector $\gamma \geq \mathbf{0}$ satisfying $\gamma^TA \leq \lceil \alpha^TA/2 \rceil$ and $2\gamma^T\mathbf{1} \geq \alpha^T\mathbf{1}$. Set $z = \gamma + \beta$. Then $z^TA = \gamma^TA + \beta^TA \leq \lceil \alpha^TA/2 \rceil + \beta^TA = \lceil (\alpha+2\beta)^TA/2 \rceil = \lceil (y-v+w)^TA/2 \rceil \leq \lceil y^TA/2 \rceil$. Similarly, $2z^T\mathbf{1} = 2\gamma^T\mathbf{1} + 2\beta^T\mathbf{1} \geq \alpha^T\mathbf{1} + 2\beta^T\mathbf{1} = (y-v+w)^T\mathbf{1} = y^T\mathbf{1}$, which is impossible as y is a counterexample.

Suppose $\beta = \mathbf{0}$. Then $\alpha^T\mathbf{1} = (y-v+w)^T\mathbf{1} = y^T\mathbf{1}$. Since $v \prec w$, we have $y \prec \alpha$, which implies, from (2), the existence of an integral vector $z \geq \mathbf{0}$ such that $z^TA \leq \lceil \alpha^TA/2 \rceil$ and $2z^T\mathbf{1} \geq \alpha^T\mathbf{1}$. Consequently, $z^TA \leq \lceil (y-v+w)^TA/2 \rceil \leq \lceil y^TA/2 \rceil$ and $2z^T\mathbf{1} \geq (y-v+w)^T\mathbf{1} = y^T\mathbf{1}$, again a contradiction. ∎

As we shall see, Lemma 4.3 can help eliminate many cases involved in our analysis.

Proof of Lemma 4.1. Tournament G_1 is as shown in Figure 5. For simplicity, we relabel each vertex v_i as i for $1 \leq i \leq 6$. Thus the vertex set of G_1 is $V_1 = \{1,2,3,4,5,6\}$ and arc set is $E_1 = \{12, 23, 34, 45, 51, 13, 35, 52, 24, 41, 16, 26, 63, 64, 65\}$ whose members are denoted by $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o$, respectively (so $a = 12$, $b = 23$, $c = 34$ and so on).

Claim 1. Let $\mathcal{F}$ be the family of all minimal feedback arc sets of G_1. Then $|\mathcal{F}| = 41$ and

$\mathcal{F} = \{ehj, afhk, dgjo, acehk, acehn, acghk, bdejl, beijl, bfikl, cehin, cgikl, cgino, degjl, dgjkl, abdfkl, acdgkl, acdgko, acdgno, acghno, adfgkl, adfgko, adghjk, aefhmn, afhmno, bceikl, bceiln, bdejmo, bdfjkl, bdfjmo, bfimno, cegijl, cegiln, cehikl, abdfkmo, abdfmno, adfgmno, adfhjmo, bceimno, befhimn, befilmn, beijmno\}$, where, for instance, ehj stands for the minimal FAS consisting of arcs e, h and j.

To justify this, we first list all subsets of E_1 in nondecreasing order of cardinality. For each term F on the list, from the first to the last, we check if $G_1 \backslash F$ is acyclic and if F contains a feedback arc set we have already found. If F is a feedback arc set and it does not contain any earlier ones, then F is a minimal feedback arc set and we put it in $\mathcal{F}$. When the process is finished, we end up with 41 minimal feedback arc sets as shown above. This step was carried out by using computer (see [40] for the source code).

Let A be the minimal FAS-arc incidence matrix of G_1, such that the ith row of A corresponds to the ith member of $\mathcal{F}$ displayed in Claim 1. We shall use Lemma 4.3 to verify that the system $Ax \geq \mathbf{1}$, $x \geq \mathbf{0}$ is TDI. To this end, let S_V and S_W be two families of 2-subsets of $\{1, 2, \ldots, 41\}$ as defined below (the subset $\{i, j\}$ is written as i-j):

$S_V = \{$2-7, 2-8, 2-13, 2-27, 2-31, 2-41, 3-4, 3-5, 3-10, 3-23, 3-33, 3-39, 4-8, 4-9, 4-13, 4-14, 4-20, 4-21, 4-22, 4-23, 4-24, 4-28, 4-29, 4-30, 4-31, 4-36, 4-37, 4-39, 4-40, 4-41, 5-8, 5-9, 5-14, 5-21, 5-22, 5-28, 5-29, 5-31, 5-37, 5-41, 6-7, 6-8, 6-9, 6-13, 6-15, 6-20, 6-21, 6-23, 6-25, 6-26, 6-27, 6-28, 6-29, 6-30, 6-31, 6-32, 6-33, 6-34, 6-35, 6-36, 6-37, 6-38, 6-39, 6-40, 6-41, 7-10, 7-12, 7-19, 7-21, 7-22, 7-24, 7-33, 7-37, 7-39, 8-10, 8-15, 8-16, 8-17, 8-18, 8-19, 8-21, 8-22, 8-23, 8-24, 8-28, 8-33, 8-35, 8-36, 8-37, 8-39, 9-19, 9-22, 9-23, 9-24, 9-31, 9-37, 9-41, 10-13, 10-14, 10-15, 10-16, 10-17, 10-18, 10-19, 10-20, 10-21, 10-22, 10-27, 10-28, 10-29, 10-31, 10-34, 10-35, 10-37, 10-41, 11-15, 11-22, 11-23, 11-24, 11-27, 11-29, 11-34, 11-35, 11-37, 11-39, 11-41, 12-13, 12-14, 12-15, 12-16, 12-22, 12-27, 12-28, 12-29, 12-33, 12-37, 12-39, 13-17, 13-19, 13-21, 13-22, 13-24, 13-25, 13-27, 13-29, 13-30, 13-33, 13-34, 13-37, 13-38, 13-39, 13-41, 14-17, 14-18, 14-19, 14-21, 14-23, 14-24, 14-25, 14-26, 14-27, 14-30, 14-31, 14-32, 14-33, 14-34, 14-35, 14-36, 14-37, 14-38, 14-39, 14-41, 15-19, 15-22, 15-24, 15-30, 15-31, 15-33, 15-37, 15-39, 15-41, 16-19, 16-22, 16-24, 16-28, 16-29, 16-30, 16-31, 16-33, 16-37, 16-39, 16-40, 16-41, 17-19, 17-22, 17-23, 17-24, 17-28, 17-29, 17-30, 17-31, 17-32, 17-33, 17-37, 17-39, 17-40, 17-41, 18-22, 18-28, 18-29, 18-30, 18-31, 18-33, 18-37, 18-39, 18-41, 19-20, 19-21, 19-22, 19-25, 19-26, 19-27, 19-28, 19-29, 19-31, 19-32, 19-33, 19-34, 19-35, 19-36, 19-37, 19-38, 19-39, 19-40, 19-41, 20-22, 20-24, 20-25, 20-26, 20-27, 20-28, 20-29, 20-30, 20-31, 20-33, 20-37, 20-38, 20-39, 20-41, 21-22, 21-23, 21-24, 21-25, 21-26, 21-27, 21-28, 21-29, 21-31, 21-33, 21-37, 21-38, 21-39, 21-40, 21-41, 22-23, 22-24, 22-25, 22-26, 22-27, 22-28, 22-29, 22-30, 22-31, 22-32, 22-33, 22-34, 22-35, 22-36, 22-37, 22-38, 22-39, 22-40, 22-41, 23-25, 23-27, 23-28, 23-29, 23-31, 23-33, 23-34, 23-37, 23-38, 23-41, 24-25, 24-26, 24-27, 24-28, 24-31, 24-33, 24-34, 24-38, 24-39, 24-40, 24-41, 25-28, 25-31, 25-35, 25-36, 25-37, 25-39, 25-41, 26-28, 26-29, 26-31, 26-33, 26-34, 26-36, 26-37, 26-39, 26-41, 27-31, 27-32, 27-33, 27-36, 27-37, 27-39, 27-40, 28-31, 28-32, 28-33, 28-34, 28-35, 28-36, 28-37, 28-38, 28-39, 28-40, 28-41, 29-31, 29-32, 29-33, 29-36, 29-38, 29-39, 29-40, 29-41, 30-31, 30-33, 30-37, 31-33, 31-34, 31-35, 31-36, 31-37, 31-38, 31-39, 31-40, 31-41, 32-33, 32-34, 32-35, 32-37, 32-39, 32-41, 33-34, 33-35, 33-36, 33-37, 33-38, 33-39, 33-40, 33-41, 34-37, 34-39, 34-40, 34-41, 35-37, 35-39, 35-41, 36-37, 36-38, 36-39, 36-41, 37-38, 37-39, 37-40, 37-41, 38-39, 39-41, 40-41$\}$ and

S_W = {1-2, 1-3, 1-6, 1-9, 1-11, 1-12, 1-14, 1-15, 1-16, 1-17, 1-18, 1-20, 1-24, 1-25, 1-26, 1-29, 1-30, 1-32, 1-34, 1-35, 1-36, 1-38, 1-40, 2-3, 2-5, 2-10, 2-11, 2-12, 2-14, 2-16, 2-17, 2-18, 2-25, 2-26, 2-29, 2-30, 2-32, 2-35, 2-36, 2-38, 2-40, 3-6, 3-7, 3-8, 3-9, 3-11, 3-15, 3-16, 3-20, 3-24, 3-25, 3-26, 3-30, 3-32, 3-34, 3-35, 3-38, 3-40, 4-11, 4-12, 5-11, 5-12, 5-30, 6-14, 6-18, 7-9, 7-11, 7-30, 8-11, 8-12, 8-30, 8-32, 8-38, 9-10, 9-16, 9-17, 9-18, 9-35, 9-36, 10-11, 10-25, 10-30, 10-40, 11-13, 11-18, 12-34, 12-35, 14-15, 15-29, 15-32, 15-38, 18-24, 18-40, 23-30, 24-29, 27-30}.

Notice that $|S_V| = 390$ and $|S_W| = 96$; these S_V and S_W will yield V and W as described in Lemma 4.3. The choices for S_V and S_W are not unique. We obtained our S_V and S_W by trial and error (see [40] for the source code). In the search process we restricted our attention to 2-subsets. It is possible to choose larger sets S_V and S_W, which would cause Γ (to be defined in Claim 3) to contain fewer stable sets.

Claim 2. Let V and W be the sets of characteristic vectors (with length 41) of members of S_V and S_W, respectively. Then V and W satisfy the conditions described in Lemma 4.3.

To justify this, for each of the 390 vectors $v \in V$ and each of the 96 vectors $w \in W$, we test if $v \prec w$ and $w^T A \leq v^T A$ hold simultaneously (note that $v^T \mathbf{1} = w^T \mathbf{1}$ is always true). Using a computer we have confirmed that, for every $v \in V$ indeed there exists $w \in W$ such that $v \prec w$ and $w^T A \leq v^T A$ are both true.

Claim 3. Let Γ be the graph with vertex set $\{1, 2, \ldots, 41\}$ such that $i, j \in \{1, 2, \ldots, 41\}$ are adjacent iff i-j is a member of S_V. Then Γ has exactly 41022 odd stable sets.

Mathematica has a function *FindClique*, which can be used to generate all 219 maximal stable sets of Γ. We also independently implemented the *Bron-Kerbosch algorithm* (see [8]) and obtained the same result. These maximal stable sets give rise to all 82044 stable sets, and exactly half of which are odd (see [40] for the source code).

Claim 4. System $Ax \geq \mathbf{1}$, $x \geq \mathbf{0}$ is TDI.

To justify this, let us choose V and W as in Claim 2. It is then clear that U (defined in Lemma 4.3) consists of exactly characteristic vectors of odd stable sets of Γ. By Claim 3, $|U| = 41022$. For each $y \in U$, we define $c = \lceil y^T A/2 \rceil$ and solve $\max\{z^T \mathbf{1}\colon\ z^T A \leq c^T,\ z \geq \mathbf{0}$ and integral$\}$ using *LinearProgramming* of Mathematica (see [40] for the source code). For each optimal solution z obtained, we verify that $2z^T \mathbf{1} \geq y^T \mathbf{1}$. We also verify that z is an integral vector satisfying $z^T A \leq c^T$. Our computational results indicate that indeed that is the case. After completing this process for all 41022 vectors in U, we conclude from Lemma 4.3 that Claim 4 is true. ∎

We can finally establish the equivalence of three statements described in Theorem 1.2, thereby obtaining a complete characterization of all FAS ideal and Mengerian touranments.

Proof of Theorem 1.2. Implication $(iii) \Rightarrow (ii)$ holds, because total-dual integrality implies primal integrality (see Edmonds-Giles theorem [19] stated in Section 1). It was proved by Lehman [29] that a clutter is ideal iff its blocker is ideal, which implies that a tournament is cycle ideal iff it is FAS ideal. Therefore the equivalence of (i) and (ii) in Theorem 1.1 yields implication $(ii) \Rightarrow (i)$. It remains to prove implication $(i) \Rightarrow (iii)$. Clearly, we may assume that T is strong. Since F_1 arises from G_1 by deleting vertex v_6 (see the labeling in Figure 5), from Lemma 4.1 we deduce that F_1 is also FAS Mengerian. So we may assume that $T \notin \{F_1, G_1\}$.

From Theorem 3.1 we thus conclude that T is FAS Mengerian. ∎

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