Supplement to Hall-type results for 3-connected projective graphs

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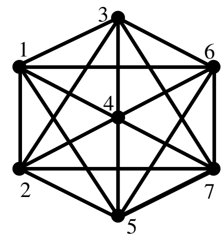
This is a supplement to a paper of Ding and Iverson [1] classifying all minimal excludable sets for the projective plane. The first section of this supplement contains details for all minor checking required in the paper. The tables in this section are of three kinds: uncontractions, undeletions, and edge-splits. In any of the tables only one graph for each isomorphism class is given. An uncontraction of G is denoted $v \to (\{u_1, u_2, \ldots, u_p\}, \{w_1, w_2, \ldots, w_q\})$, where v is the vertex in G being split, v is adjacent to u_1,u_2,$\ldots$, u_p in the uncontraction, the new vertex v' is adjacent to w_1,w_2,$\ldots$, w_p, and v' is given the label $|V(G)|+1$. An edge-split is denoted by $v \to (u_1, u_2)$ where the edge-split is created by deleting edge u_1u_2 from G and adding a new vertex with label $|V(G)|+1$ adjacent to v, u_1, and u_2. In each case, the contractions and deletions are listed that produce the desired minor.

The second section contains proofs to show that the graphs in the exception sets corresponding to the maximal splitting sets in [1] are really in the exception set. That is, that each graph contains no minor in among the graph of the corresponding excludable set.

1 Finding minors

1.1 A_2 minors

Below are the uncontractions and undeletions of A_2 and their minors needed for Lemma 3.2 of [1].



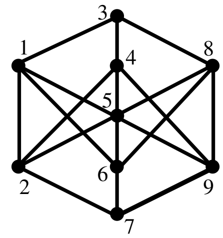
A_2

A_2 Uncontractions			
Uncontraction	Delete	Contract	Minor
$1 \to (\{4,5,6\},\{2,3\})$	$\{\{2,3\}\}$	$-$	B_7
$1 \to (\{3,5,6\},\{2,4\})$	$\{\{1,6\},\{2,4\},\{2,7\}\}$	$-$	D_3
$1 \to (\{3,4,5\},\{2,6\})$	$\{\{1,3\},\{4,5\},\{5,7\}\}$	$-$	D_3
$4 \to (\{3,5,6,7\},\{1,2\})$	$\{\{1,2\},\{1,6\},\{2,7\}\}$	$-$	D_3
$4 \to (\{2,3,5,6\},\{1,7\})$	$\{\{2,3\},\{2,5\},\{3,6\},\{5,6\}\}$	$-$	E_3
$4 \to (\{5,6,7\},\{1,2,3\})$	$\{\{1,5\},\{3,7\}\}$	$-$	C_7
$4 \to (\{3,5,7\},\{1,2,6\})$	$\{\{1,2\},\{1,6\},\{3,7\},\{5,7\}\}$	$-$	E_{18}

A_2 Undeletions			
Added Edge	Delete	Contract	Minor
$\{1,7\}$	$\{\{2,3\},\{5,6\}\}$	$-$	B_1

1.2 C_4 minors

Below are the uncontractions and undeletions of C_4 and their minors needed for Lemma 3.3 of [1].



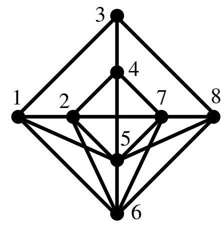
C_4

C_4 Uncontractions			
Uncontraction	Delete	Contract	Minor
$1 \to (\{5,6\},\{2,3\})$	$\{\{5,6\}\}$	$\{\{2,7\},\{3,8\}\}$	D_3
$1 \to (\{3,6\},\{2,6\})$	–	$\{\{7,9\}\}$	C_3
$5 \to (\{4,6,8,9\},\{1,2\})$	$\{\{1,2\}\}$	$\{\{3,4\},\{6,7\}\}$	D_3
$5 \to (\{2,6,8,9\},\{1,4\})$	$\{\{1,6\},\{4,9\},\{5,8\}\}$	$\{\{3,8\}\}$	F_1
$5 \to (\{2,4,6,8\},\{1,9\})$	$\{\{2,4\},\{6,8\},\{5,10\}\}$	$\{\{1,10\}\}$	F_1
$5 \to (\{6,8,9\},\{1,2,4\})$	–	$\{\{1,3\},\{7,9\}\}$	C_7
$5 \to (\{4,6,9\},\{1,2,8\})$	$\{\{1,2\},\{4,9\},\{5,10\}\}$	$\{\{2,4\}\}$	F_1
$5 \to (\{2,6,9\},\{1,4,8\})$	$\{\{1,2\},\{4,9\},\{5,10\}\}$	$\{\{2,4\}\}$	F_1

C_4 Undeletions			
Added Edge	Delete	Contract	Minor
$\{1,4\}$	–	$\{\{3,8\}\}$	B_7
$\{1,7\}$	–	$\{\{1,3\},\{7,9\}\}$	B_1
$\{1,9\}$	–	$\{\{1,3\},\{7,9\}\}$	B_1
$\{3,5\}$	$\{\{1,5\},\{4,5\},\{5,8\}\}$	–	E_{22}
$\{3,7\}$	$\{\{1,5\},\{4,5\},\{5,8\}\}$	–	E_{22}

1.3 B_7 minors

Below are the uncontractions and undeletions of B_7 and their minors needed for Lemma 3.4 of [1].



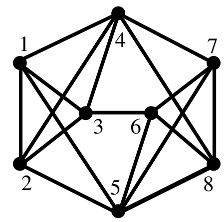
B_7

B_7 Uncontractions			
Uncontraction	Delete	Contract	Minor
$1 \to (\{5,6\},\{2,3\})$	$\{\{5,6\},\{6,7\}\}$	$\{\{3,8\}\}$	D_3
$1 \to (\{3,5\},\{2,6\})$	$\{\{2,6\}\}$	$-$	C_3
$2 \to (\{5,6,7\},\{1,4\})$	$\{\{5,6\},\{5,7\},\{6,7\}\}$	$-$	E_{22}
$2 \to (\{4,6,7\},\{1,5\})$	$\{\{1,5\},\{2,7\}\}$	$\{\{3,4\}\}$	D_3
$2 \to (\{4,5,7\},\{1,6\})$	$\{\{1,6\}\}$	$-$	C_3
$2 \to (\{4,5,6\},\{1,7\})$	$\{\{1,5\},\{2,5\}\}$	$\{\{3,4\}\}$	D_3
$2 \to (\{1,4,7\},\{5,6\})$	$\{\{2,4\},\{5,6\},\{5,7\},\{6,8\}\}$	$-$	F_1
$2 \to (\{1,4,5\},\{6,7\})$	$\{\{6,7\}\}$	$-$	C_4
$5 \to (\{4,6,7,8\},\{1,2\})$	$\{\{1,2\},\{2,7\}\}$	$\{\{3,4\}\}$	D_3
$5 \to (\{2,6,7,8\},\{1,4\})$	$\{\{1,6\},\{4,7\}\}$	$\{\{3,8\}\}$	D_3
$5 \to (\{2,4,6,8\},\{1,7\})$	$\{\{2,4\},\{6,8\}\}$	$\{\{1,9\}\}$	D_3
$5 \to (\{1,4,7,8\},\{2,6\})$	$\{\{2,4\},\{2,6\},\{5,7\},\{6,8\}\}$	$-$	F_1
$5 \to (\{6,7,8\},\{1,2,4\})$	$\{\{2,6\}\}$	$\{\{1,3\}\}$	C_7
$5 \to (\{4,6,8\},\{1,2,7\})$	$\{\{1,2\},\{2,7\},\{6,8\}\}$	$\{\{1,3\}\}$	E_{18}
$5 \to (\{2,6,7\},\{1,4,8\})$	$\{\{1,2\},\{2,7\},\{5,6\},\{7,8\}\}$	$-$	F_1

B_7 Undeletions			
Added Edge	Delete	Contract	Minor
$\{1,4\}$	$-$	$\{\{3,8\}\}$	A_2
$\{1,7\}$	$\{\{2,6\}\}$	$\{\{1,3\}\}$	B_1
$\{2,3\}$	$\{\{6,7\}\}$	$\{\{3,8\}\}$	B_1
$\{3,5\}$	$-$	$-$	$-$

1.4 C_7 minors

Below are the uncontractions, undeletions, and edge-splits of C_7 and their minors needed for Lemma 3.5 of [1].



C_7

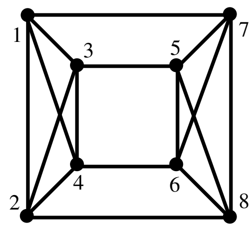
C_7 Uncontractions			
Uncontraction	Delete	Contract	Minor
$1 \to (\{4,5\},\{2,3\})$	$\{\{2,3\}\}$	$\{\{3,6\}\}$	D_3
$1 \to (\{3,5\},\{2,4\})$	$\{\{2,4\},\{5,6\},\{7,8\}\}$	–	F_1
$1 \to (\{3,4\},\{2,5\})$	$\{\{2,5\},\{3,4\}\}$	–	E_{19}
$3 \to (\{4,6\},\{1,2\})$	$\{\{1,2\}\}$	$\{\{3,6\}\}$	D_3
$3 \to (\{2,6\},\{1,4\})$	$\{\{1,4\},\{2,5\}\}$	–	E_{19}
$4 \to (\{3,7,8\},\{1,2\})$	$\{\{1,2\},\{5,6\},\{7,8\}\}$	–	F_1
$4 \to (\{2,7,8\},\{1,3\})$	$\{\{1,3\},\{2,5\}\}$	–	E_{19}
$4 \to (\{2,3,8\},\{1,7\})$	$\{\{1,5\}\}$	$\{\{1,9\}\}$	D_{17}
$4 \to (\{1,2,8\},\{3,7\})$	$\{\{1,2\}\}$	–	D_{12}
$4 \to (\{1,2,3\},\{7,8\})$	$\{\{5,6\},\{7,8\}\}$	–	E_{19}

C_7 Undeletions			
Added Edge(s)	Delete	Contract	Minor
$\{1,6\}$	–	–	–
$\{1,7\}$	$\{\{1,5\},\{4,7\}\}$	–	D_{17}
$\{3,5\}$	$\{\{5,6\},\{7,8\}\}$	–	D_3
$\{4,5\}$	–	$\{\{3,6\}\}$	B_1
$\{1,6\},\{2,6\}$	$\{\{1,5\},\{5,6\},\{7,8\}\}$		D_3
$\{1,6\},\{3,7\}$	$\{\{1,5\},\{3,6\},\{4,7\}\}$		D_{17}

C_7 Edge-Splits			
Edge Split	Delete	Contract	Minor
$3 \to \{7,8\}$	$\{\{1,2\},\{3,4\},\{4,7\},\{5,6\}\}$	–	F_1

1.5 D_{17} minors

Below are the uncontractions, undeletions, and edge-splits of D_{17} and their minors needed for Lemma 3.6 of [1].



D_{17}

D_{17} Uncontractions			
Added Edge(s)	Delete	Contract	Minor
$1 \to (\{4,7\},\{2,3\})$	$\{\{2,3\}\}$	–	E_{20}

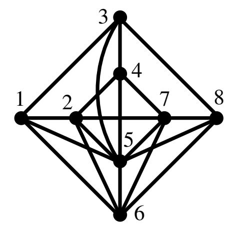
D_{17} Undeletions			
Added Edge(s)	Delete	Contract	Minor
$\{1,5\}$	–	–	–
$\{1,5\},\{1,6\}$	–	–	–
$\{1,5\},\{2,6\}$	–	–	–
$\{1,5\},\{2,7\}$	–	–	–
$\{1,5\},\{3,7\}$	–	–	–
$\{1,5\},\{1,6\},\{1,8\}$	–	–	–
$\{1,5\},\{1,6\},\{2,5\}$	–	–	–
$\{1,5\},\{1,6\},\{2,7\}$	–	–	–
$\{1,5\},\{1,6\},\{3,6\}$	$\{\{3,5\}\}$	$\{\{2,8\}\}$	B_1
$\{1,5\},\{1,6\},\{3,7\}$	–	–	–
$\{1,5\},\{1,6\},\{3,8\}$	$\{\{1,4\},\{2,3\},\{3,5\}\}$	–	D_3
$\{1,5\},\{2,6\},\{3,7\}$	$\{\{1,3\},\{2,4\},\{5,7\},\{6,8\}\}$	–	E_{18}
$\{1,5\},\{2,6\},\{3,8\}$	–	–	–
$\{1,5\},\{2,7\},\{3,8\}$	–	$\{\{4,6\}\}$	A_2
Continued on next page			

D_{17} Undeletions (continued)			
Added Edge(s)	Delete	Contract	Minor
$\{1,5\},\{1,6\},\{1,8\},\{2,5\}$	$\{\{1,4\},\{2,8\}\}$	$\{\{4,6\}\}$	B_1
$\{1,5\},\{1,6\},\{1,8\},\{2,7\}$	–	–	–
$\{1,5\},\{1,6\},\{2,5\},\{2,6\}$	–	–	–
$\{1,5\},\{1,6\},\{2,5\},\{2,7\}$	$\{\{1,4\},\{1,7\}\}$	$\{\{4,6\}\}$	B_1
$\{1,5\},\{1,6\},\{2,5\},\{3,7\}$	–	–	–
$\{1,5\},\{1,6\},\{2,5\},\{3,8\}$	$\{\{1,4\},\{2,3\},\{2,5\},\{3,5\}\}$	–	D_3
$\{1,5\},\{1,6\},\{2,5\},\{4,8\}$	$\{\{2,4\},\{4,6\}\}$	$\{\{4,8\}\}$	B_1
$\{1,5\},\{1,6\},\{2,7\},\{3,6\}$	$\{\{3,5\},\{7,8\}\}$	$\{\{2,8\}\}$	B_1
$\{1,5\},\{1,6\},\{2,7\},\{3,8\}$	$\{\{1,4\},\{2,3\},\{2,7\},\{3,5\}\}$	–	D_3
$\{1,5\},\{1,6\},\{3,6\},\{3,7\}$	$\{\{1,5\},\{1,7\}\}$	$\{\{2,8\}\}$	B_1
$\{1,5\},\{1,6\},\{3,6\},\{4,7\}$	$\{\{3,4\},\{5,7\}\}$	$\{\{2,8\}\}$	B_1
$\{1,5\},\{1,6\},\{3,7\},\{3,8\}$	$\{\{5,6\},\{6,7\}\}$	$\{\{6,8\}\}$	B_1
$\{1,5\},\{1,6\},\{3,7\},\{4,8\}$	$\{\{1,3\},\{2,4\},\{3,5\},\{4,6\}\}$	–	D_3
$\{1,5\},\{1,6\},\{3,8\},\{4,8\}$	$\{\{1,3\},\{2,4\},\{3,8\},\{4,6\}\}$	–	D_3
$\{1,5\},\{2,6\},\{3,7\},\{4,8\}$	$\{\{1,3\},\{2,4\},\{4,8\},\{5,7\},\{6,8\}\}$	–	E_{18}
$\{1,5\},\{2,6\},\{3,8\},\{4,7\}$	–	–	–
$\{1,5\},\{1,6\},\{1,8\},\{2,5\},\{2,6\}$	$\{\{1,3\},\{2,3\},\{2,8\}\}$	$\{\{3,4\}\}$	B_1
$\{1,5\},\{1,6\},\{1,8\},\{2,5\},\{2,7\}$	$\{\{1,4\},\{1,7\},\{1,8\}\}$	$\{\{4,6\}\}$	B_1
$\{1,5\},\{1,6\},\{1,8\},\{2,5\},\{3,6\}$	$\{\{1,2\},\{2,3\},\{2,8\}\}$	$\{\{2,4\}\}$	B_1
$\{1,5\},\{1,6\},\{1,8\},\{2,5\},\{3,7\}$	$\{\{1,4\},\{2,3\},\{7,8\}\}$	$\{\{4,6\}\}$	B_1
$\{1,5\},\{1,6\},\{1,8\},\{2,5\},\{3,8\}$	$\{\{1,2\},\{1,3\},\{1,4\}\}$	$\{\{4,6\}\}$	B_1
$\{1,5\},\{1,6\},\{1,8\},\{2,5\},\{4,5\}$	$\{\{1,2\},\{4,6\},\{5,8\}\}$	$\{\{2,8\}\}$	B_1
$\{1,5\},\{1,6\},\{1,8\},\{2,5\},\{4,7\}$	$\{\{1,4\},\{2,8\},\{4,6\}\}$	$\{\{4,7\}\}$	B_1
$\{1,5\},\{1,6\},\{2,5\},\{2,6\},\{3,7\}$	$\{\{1,3\},\{1,5\},\{3,5\}\}$	$\{\{3,7\}\}$	B_1
$\{1,5\},\{1,6\},\{2,5\},\{2,7\},\{3,6\}$	$\{\{1,2\},\{1,7\},\{2,3\}\}$	$\{\{2,4\}\}$	B_1
$\{1,5\},\{1,6\},\{2,5\},\{2,7\},\{3,7\}$	$\{\{1,2\},\{1,4\},\{2,3\}\}$	$\{\{4,6\}\}$	B_1
$\{1,5\},\{1,6\},\{2,5\},\{2,7\},\{3,8\}$	$\{\{1,3\},\{1,4\},\{7,8\}\}$	$\{\{4,6\}\}$	B_1
$\{1,5\},\{1,6\},\{2,5\},\{2,7\},\{4,8\}$	$\{\{1,4\},\{1,7\},\{6,8\}\}$	$\{\{4,6\}\}$	B_1
$\{1,5\},\{1,6\},\{2,5\},\{3,7\},\{4,8\}$	$\{\{2,4\},\{3,7\},\{4,6\}\}$	$\{\{4,8\}\}$	B_1
$\{1,5\},\{1,6\},\{2,5\},\{3,8\},\{4,7\}$	$\{\{1,2\},\{1,3\},\{2,3\},\{5,8\},\{6,8\}\}$	–	D_3
$\{1,5\},\{1,6\},\{2,5\},\{3,8\},\{4,8\}$	$\{\{1,2\},\{1,3\},\{4,6\}\}$	$\{\{1,4\}\}$	B_1
$\{1,5\},\{1,6\},\{2,7\},\{3,6\},\{4,5\}$	$\{\{1,2\},\{1,7\},\{2,3\}\}$	$\{\{2,4\}\}$	B_1
$\{1,5\},\{1,6\},\{2,7\},\{3,8\},\{4,8\}$	$\{\{1,2\},\{2,8\},\{3,4\},\{3,5\},\{4,6\}\}$	–	D_3
$\{1,5\},\{1,6\},\{3,6\},\{3,7\},\{4,5\}$	$\{\{1,2\},\{1,4\},\{7,8\}\}$	$\{\{2,8\}\}$	B_1

D_{17} Edge-Splits			
Edge Split	Delete	Contract	Minor
$1 \to \{5,6\}$	$\{\{7,8\}\}$	–	D_{12}

2 Proofs for exception sets

Proposition 2.1. B_7^+ *is* $(\mathcal{A}_3 - \{B_7\})$*-free.*



B_7^+

Proof. B_7^+ has 8 vertices and 19 edges. The only members of $\mathcal{A}_3 - \{B_7\}$ with 8 or fewer vertices and 19 or fewer edges are: A_2, B_1, C_7, D_3, D_{17}, E_3, and E_{18}.

If B_7^+ has A_2 as a minor, it must be obtained by contracting exactly one edge, since A_2 has 7 vertices and 18 edges. But all vertices of A_2 have degree at least 5, while B_7^+ has four degree 4 vertices, making this impossible.

If B_7^+ has B_1 as a minor, it must be obtained by contracting one edge and deleting one edge, since B_1 has 7 vertices and 17 edges. But B_1 has all vertices of degree 5 or less, and B_7 has a universal vertex of degree 7, so contracting any edge leaves the graph with a universal vertex with at least one multiple edge, which must be deleted, but this leaves a universal vertex of degree 6, so B_7 has no B_1 minor.

If B_7^+ has C_7 as a minor, it must be obtained by deleting two edges since C_7 has 8 vertices and 17 edges. But C_7 has no vertex with degree higher than 5, and B_7 has a vertex of degree 7, so both deleted edges must be adjacent to the vertex of degree 7, and since C_7 has only two vertices of degree 5, the deleted edges must both be adjacent to degree 5 vertices in B_7^+. But deleting any two such edges still leaves two degree 5 vertices adjacent to one another, while in C_7, the two degree 5 vertices are non-adjacent.

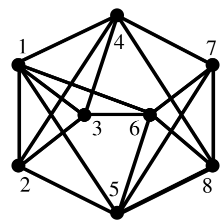
If B_7^+ has D_3 as a minor, it must be obtained by deleting three edges since D_3 has 8 vertices and 16 edges. But D_3 has exactly three vertices of degree 5 that induce a path in D_3, and no vertices of degree higher than 5. There are four vertices of degree 5 or more in B_7^+: three vertices of degree 5, and one vertex of degree 7, all of which are mutually adjacent. Thus we cannot delete any edges adjacent to two of the vertices of degree 5, but then those two vertices together with a third vertex of degree higher than 5 will always induce a triangle. Thus B_7^+ does not have D_3 as a minor.

If B_7^+ has D_{17} as a minor, it must be obtained by deleting three edges since D_{17} has 8 vertices and 16 edges. But since all vertices in D_{17} have degree 4, we must delete the three edges adjacent to the degree 7 vertex and one of the degree 5 vertices, leaving a graph that is not isomorphic to D_{17}.

If B_7^+ has E_3 as a minor, it must be obtained by deleting four edges since E_3 has 8 vertices and 15 edges. But E_3 has three mutually non-adjacent vertices of degree 5, and B_7^+ has as its only vertices of degree 5 or more three vertices of degree 5 and a vertex of degree 7, all of which are mutually adjacent, thus it is impossible to delete edges from B_7^+ to produce three mutually non-adjacent vertices of degree 5, and B_7^+ has no E_3 minor.

If B_7^+ has E_{18} as a minor, it must be obtained by deleting four edges since B_7^+ has 8 vertices and 15 edges. But E_{18} has no vertices of degree five or more, so we must delete at least three edges adjacent to vertex 5, and one edge adjacent to each of vertices 2,6, and 7. Thus we must delete two or more edges from among $\{\{2,5\},\{5,6\},\{5,7\}\}$. Now notice that 126, 345, and 678 are three edge-disjoint triangles in B_7^+ also disjoint from $\{\{2,5\},\{5,6\},\{5,7\}\}$, so since E_{18} is bipartite, we must also delete at least one edge from each of those triangles. But this is a total of more than 5 edges and E_{18} cannot be a minor of B_7^+. $\square$

Proposition 2.2. *C_7^+ is $(\mathcal{A}_3 - \{B_7\})$-free.*



C_7^+

Proof. C_7^+ has 8 vertices and 18 edges. The only members of $\mathcal{A}_3 - \{C_7\}$ with 8 or fewer vertices and 18 or fewer edges are: A_2, B_1, B_7, D_3, D_{17}, E_3, and E_{18}.

C_7^+ cannot have A_2 as a minor since A_2 has the same number of edges as C_7^+, but one fewer vertex.

If C_7^+ has B_1 as a minor, it must be obtained by contracting one edge, since B_1 has 7 vertices and 17 edges. But we must contract an edge whose ends have no common neighbors, since B_1 is simple, so the degree of the new vertex will be at least 6, since C_7+ has no vertices of degree less than 4. So C_7^+ does not have B_1 as a minor, since B_1 has no vertices of degree 6 or higher.

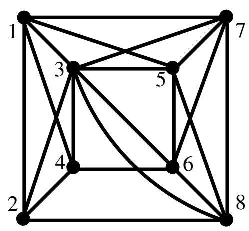
If C_7^+ has B_7 as a minor, they must be isomorphic since they have the same number of edges and vertices. But B_7 has a vertex of degree 6, while C_7^+ does not.

If C_7^+ has D_3 as a minor, it must be obtained by deleting two edges since D_3 has 8 vertices and 16 edges. C_7^+ has four vertices of degree 5, and D_3 has three vertices of degree five inducing a path. So we must delete one edge with ends at a degree 5 vertex and a degree 4 vertex in C_7^+ so that the degree 5 vertices in C_7^+ induce a path, this edge must be $\{2,5\}$,$\{3,6\}$,$\{5,7\}$,$\{5,8\}$,$\{6,7\}$, or $\{6,8\}$. The second edge must have both ends at degree 4 vertices, and thus must be $\{2,3\}$ or $\{7,8\}$. If we delete $\{2,5\}$, then we cannot delete $\{2,3\}$, because this would leave a degree 2 vertex, but deleting $\{7,8\}$ produces a graph in which the degree 3 vertices do not induce a path, while in D_3 the degree 3 vertices induce a path. Similarly, we cannot delete any edge from among $\{\{2,5\},\{3,6\},\{5,7\},\{5,8\},\{6,7\},\{6,8\}\}$ together with an edge from among $\{\{2,3\},\{7,8\}\}$ without creating a degree 2 vertex or having the degree 3 vertices not induce a path. Thus C_7^+ has no D_3 minor.

If C_7^+ has D_{17} as a minor, it must be obtained by deleting two edges since D_{17} has 8 vertices and 16 edges. But since all vertices in D_{17} have degree 4, we must delete the two non-adjacent edges with both ends at degree 5 vertices. The only two such edges are $\{1,4\}$ and $\{5,6\}$, and deleting them does not produce a graph isomorphic to D_{17}.

Now C_7^+ has four edge disjoint triangles: 124, 136, 478, and 567. So any bipartite subgraph of C_7+ has $18-4=14$ or fewer edges. E_3 and E_{18} are both bipartite graphs on 8 vertices and 15 edges, so they cannot be minors of C_7^+. $\square$

Proposition 2.3. *D_{17}^1 is $(\mathcal{A}_3 - \{C_7, D_{17}\})$-free.*



D_{17}^1

Proof. D_{17}^1 has 8 vertices and 20 edges. The only members of $\mathcal{A}_3 - \{C_7, D_{17}\}$ with 8 or fewer vertices and 20 or fewer edges are: A_2, B_1, B_7, D_3, E_3, and E_{18}.

If D_{17}^1 has A_2 as a minor, it must be obtained by contracting one edge, and deleting one edge, since A_2 has 7 vertices and 18 edges. But A_2 has all vertices of degree 5 or higher, so we must contract $\{2,4\}$, creating two parallel edges, so A_2 cannot be a minor of D_{17}^1.

If D_{17}^1 has B_1 as a minor, it must be obtained by contracting one edge and deleting two edges, since B_1 has 7 vertices and 17 edges. But B_1 has three vertices of degree 6, while D_{17}^1 has only one vertex of degree 6 or more. So contracting one edge would produce at most two vertices of degree 6 or more, and D_{17}^1 does not have B_1 as a minor.

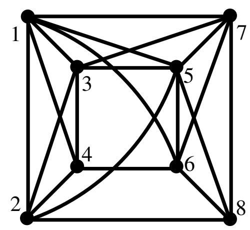
If D_{17}^1 has B_7 as a minor, it must be obtained by deleting two edges, since B_7 has 8 vertices and 18 edges. But B_7 has a vertex of degree 6, which must correspond to vertex 3 in D_{17}^1, since it is the only vertex of degree 6 or more in D_{17}^1. Also, the vertices of degree four are mutually non-adjacent in B_7, so we must delete an edge adjacent to 2 or 4 in D_{17}^1. Now since the vertex of degree 3 in B_7 is not adjacent to the vertex of degree 6, we must delete $\{2,3\}$ or $\{2,4\}$. Thus the degree 4 vertices of B_7 correspond to vertices 1 and either 2 and 6 or 4 and 8, so we must delete an edge incident to 1 and an edge incident to either 6 or 8. But we can only delete one more edge, and neither $\{1,6\}$ nor $\{1,8\}$ are edges of D_{17}^1. Thus D_{17}^1 does not have B_7 as a minor.

If D_{17}^1 has D_3 as a minor, it must be obtained by deleting four edges, since D_3 has 8 vertices and 16 edges. But in D_3 any two adjacent vertices of degree four share three neighbors, so since vertices 2 and 4 in D_{17}^1 don't share three neighbors, we must delete an edge incident with each. If we delete $\{2,4\}$, then 2 and 4 are non-adjacent degree 3 vertices, but in D_3 any non-adjacent degree 3 vertices share the same neighborhood, while 2 and 4 do not. But any adjacent degree 3 vertices in D_3 share no neighbors, while 2 and 4 share two neighbors. So D_{17}^1 cannot have D_3 as a minor.

If D_{17}^1 has E_3 as a minor, it must be obtained by deleting five edges, since E_3 has 8 vertices and 15 edges. But in E_3, the only vertices of degree less than five are non-adjacent degree 3 vertices with the same neighborhood, but vertices 2 and 4 share only two neighbors, and D_{17}^1 cannot have E_3 as a minor.

If D_{17}^1 has E_{18} as a minor, it must be obtained by deleting five edges, since E_{18} has 8 vertices and 15 edges. But in E_{18}, any adjacent degree four vertices share no neighbors, so since vertices 2 and 4 share two neighbors, we must delete an edge incident to 2 and an edge incident to 4. In E_{18}, any degree 3 vertices are non-adjacent, so we must delete $\{2,4\}$, but the degree 3 vertices in E_{18} share no neighbors, while 2 and 4 share two neighbors. Thus E_{18} is not a minor of D_{17}^1. □

Proposition 2.4. D_{17}^2 *is* $(\mathcal{A}_3 - \{C_7, D_{17}\})$*-free.*



D_{17}^2

Proof. D_{17}^2 has 8 vertices and 20 edges. The only members of $\mathcal{A}_3 - \{C_7, D_{17}\}$ with 8 or fewer vertices and 20 or fewer edges are: A_2, B_1, B_7, D_3, E_3, and E_{18}.

If D_{17}^2 has A_2 as a minor, it must be obtained by contracting one edge, and deleting one edge, since A_2 has 7 vertices and 18 edges. But A_2 has all vertices of degree 5 or higher and D_{17}^2 has two non-adjacent degree 4 vertices (4 and 8), so A_2 cannot be a minor of D_{17}^2.

If D_{17}^2 has B_1 as a minor, it must be obtained by contracting one edge and deleting two edges, since B_1 has 7 vertices and 17 edges. Note that every edge in D_{17}^2 participates in two triangles except for $\{2,8\}$ and $\{4,6\}$, so if we contract any edge other than $\{2,8\}$ or $\{4,6\}$, then we must delete two parallel edges. The resulting graph is never isomorphic to B_1. If we contract $\{2,8\}$ or $\{4,6\}$, we must delete one parallel edge, and then there are only two vertices of degree 6 or more, but B_1 has three vertices of degree 6. Thus D_{17}^2 does not have B_1 as a minor.

If D_{17}^2 has B_7 as a minor, it must be obtained by deleting two edges, since B_7 has 8 vertices and 18 edges. Neither vertex 1 nor vertex 5 can correspond to the degree 3 vertex of B_7, since this would require deleting at least three edges. If vertex v = 2,3,6, or 7 corresponds to the degree 3 vertex, then we must delete two edges adjacent to v so that its neighbors are mutually non-adjacent. But the graph induced by the vertices adjacent to 2,3,6, or 7 contains a triangle and at least one other edge, so this is not possible. Thus vertex 4 or vertex 8 corresponds to the degree 3 vertex of B_7. But the degree 3 vertex of B_7 is adjacent to all the degree 4 vertices in B_7, and vertices 4 and 8 are non-adjacent. Thus D_{17}^2 does not have B_7 as a minor.

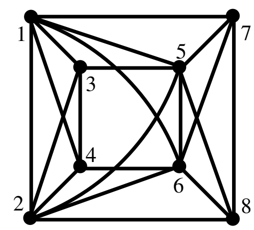
If D_{17}^2 has D_3 as a minor, it must be obtained by deleting four edges, since D_3 has 8 vertices and 16 edges. But in D_3, any vertices of degree four are adjacent, so we must delete an edge incident with vertex 4 and an edge incident vertex 8 in D_{17}^2. But any two non-adjacent degree 3 vertices in D_3 share the same neighborhood, while vertex 4 and 8 share only two neighbors. Thus D_3 is not a minor of D_{17}^2.

If D_{17}^2 has E_3 as a minor, it must be obtained by deleting five edges, since E_3 has 8 vertices and

15 edges. But in E_3, the only vertices of degree less than five are non-adjacent degree 3 vertices with the same neighborhood, but vertices 4 and 8 share only two neighbors, and D^2_{17} cannot have E_3 as a minor.

If D^2_{17} has E_{18} as a minor, it must be obtained by deleting five edges, since E_{18} has 8 vertices and 15 edges. But in E_{18}, any non-adjacent degree four vertices have the same neighborhood, so since vertices 4 and 8 share only two neighbors, we must delete an edge incident to 4 and an edge incident to 8. In E_{18}, the two degree 3 vertices share no neighbors, but it is impossible to delete an edge incident to 4 and an edge incident to 8 making them share no neighbors, since they share two neighbors in D^2_{17}. Thus E_{18} is not a minor of D^2_{17}. $\square$

Proposition 2.5. *D^3_{17} is $(\mathcal{A}_3 - \{C_7, D_{17}\})$-free.*



D^3_{17}

Proof. D^3_{17} has 8 vertices and 20 edges. The only members of $\mathcal{A}_3 - \{C_7, D_{17}\}$ with 8 or fewer vertices and 20 or fewer edges are: A_2, B_1, B_7, D_3, E_3, and E_{18}.

If D^3_{17} has A_2 as a minor, it must be obtained by contracting one edge, and deleting one edge, since A_2 has 7 vertices and 18 edges. But A_2 has all vertices of degree 5 or higher and D^3_{17} has two pairs of adjacent degree 4 vertices (3,4 and 7,8), so A_2 cannot be a minor of D^3_{17}.

If D^3_{17} has B_1 as a minor, it must be obtained by contracting one edge and deleting two edges, since B_1 has 7 vertices and 17 edges. Note that every edge in D^3_{17} participates in two triangles, so if we contract any edge, we must delete two parallel edges. The resulting graph is never isomorphic to B_1, and thus B_1 is not a minor of D^3_{17}.

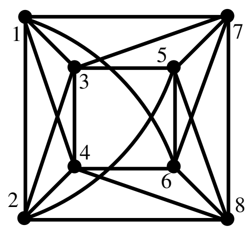
If D^3_{17} has B_7 as a minor, it must be obtained by deleting two edges, since B_7 has 8 vertices and 18 edges. But B_7 has four vertices of degree 5 or more, which must correspond to vertices 1,2,5, and 6 in D^3_{17}, so vertices 3,4,7, and 8 correspond to the vertices of degree 4 or less in B_7. But the vertices of degree four in B_7 are mutually non-adjacent, so we must delete either $\{3, 4\}$ or $\{7, 8\}$, creating two degree 3 vertices, but B_7 has only one degree 3 vertex. Thus B_7 is not a minor of D^3_{17}.

If D^3_{17} has D_3 as a minor, it must be obtained by deleting four edges, since D_3 has 8 vertices and 16 edges. But in D_3, there are two non-adjacent degree 5 vertices that share the same neighborhood, while in D^3_{17}, the only vertices of degree 5 or more are 1,2,5, and 6, and no two of these share five neighbors. Thus D_3 is not a minor of D^3_{17}.

If D^3_{17} has E_3 as a minor, it must be obtained by deleting five edges, since E_3 has 8 vertices and 15 edges. But in E_3, the only vertices of degree less than five are non-adjacent degree 3 vertices with the same neighborhood, but vertices 4 and 8 share only two neighbors, and D^3_{17} cannot have E_3 as a minor.

If D^3_{17} has E_{18} as a minor, it must be obtained by deleting five edges, since E_{18} has 8 vertices and 15 edges. But in E_{18}, any adjacent degree four vertices share no neighbors, so since vertices 3 and 4 share two neighbors, we must delete an edge incident to 3 and an edge incident to 4. In E_{18}, any degree 3 vertices are non-adjacent, so we must delete $\{3, 4\}$, but the degree 3 vertices in E_{18} share no neighbors, while 3 and 4 share two neighbors. Thus E_{18} is not a minor of D^3_{17}. □

Proposition 2.6. *D^4_{17} is $(\mathcal{A}_3 - \{C_7, D_{17}\})$-free.*



D^4_{17}

Proof. D^4_{17} has 8 vertices and 20 edges. The only members of $\mathcal{A}_3 - \{C_7, D_{17}\}$ with 8 or fewer vertices and 20 or fewer edges are: A_2, B_1, B_7, D_3, E_3, and E_{18}.

If D^4_{17} has A_2 as a minor, it must be obtained by contracting one edge, and deleting one edge, since A_2 has 7 vertices and 18 edges. But A_2 has all vertices of degree 5 or higher and D^4_{17} has all vertices of degree 5, and contracting any edge of D^4_{17} produces a parallel edge (whose deletion creates a vertex of degree 4), since D^4_{17} has fewer than 10 vertices. So A_2 cannot be a minor of D^4_{17}.

If D^4_{17} has B_1 as a minor, it must be obtained by contracting one edge and deleting two edges, since B_1 has 7 vertices and 17 edges. But B_1 has three vertices of degree 6, while D^4_{17} has no vertex of degree 6 or more. So contracting one edge would produce at most one vertex of degree 6 or more, and D^4_{17} does not have B_1 as a minor.

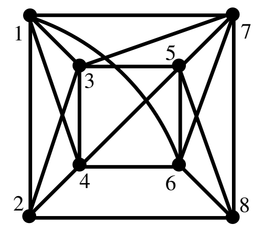
If D^4_{17} has B_7 as a minor, it must be obtained by deleting two edges, since B_7 has 8 vertices and 18 edges. But B_7 has a vertex of degree 6, while D^4_{17} has all vertices of degree 5, so B_7 is not a minor of D^4_{17}.

If D^4_{17} has D_3 as a minor, it must be obtained by deleting four edges, since D_3 has 8 vertices and 16 edges. But in D_3, there are two non-adjacent degree 5 vertices that share the same neighborhood, while in D^4_{17}, no two non-adjacent vertices share five neighbors. Thus D_3 is not a minor of D^4_{17}.

If D^4_{17} has E_3 as a minor, it must be obtained by deleting five edges, since E_3 has 8 vertices and 15 edges. But E_3 has three mutually non-adjacent degree 5 vertices, while the largest independent set in D^4_{17} has size 2, so D^4_{17} cannot have E_3 as a minor.

If D^4_{17} has E_{18} as a minor, it must be obtained by deleting five edges, since E_{18} has 8 vertices and 15 edges. But in E_{18} there are no edges between the neighbors of any vertex, so since vertex 2 is adjacent to 1,3, and 4 in D^4_{17}, we must delete $\{1, 3\}, \{1, 4\}$, and $\{3, 4\}$, creating three degree 3 vertices, yet E_{18} has only two degree 3 vertices. Thus D^4_{17} does not have E_{18} as a minor. □

Proposition 2.7. *D^5_{17} is $(\mathcal{A}_3 - \{A_2, C_7, D_{17}\})$-free.*



D^5_{17}

Proof. D^5_{17} has 8 vertices and 19 edges. The only members of $\mathcal{A}_3 - \{A_2, C_7, D_{17}\}$ with 8 or fewer vertices and 20 or fewer edges are: B_1, B_7, D_3, E_3, and E_{18}.

If D^5_{17} has B_1 as a minor, it must be obtained by contracting one edge and deleting one edge, since B_1 has 7 vertices and 17 edges. But B_1 has three vertices of degree 6, while D^5_{17} has no vertex of degree 6 or more. So contracting one edge would produce at most one vertex of degree 6 or more, and D^5_{17} does not have B_1 as a minor.

If D^5_{17} has B_7 as a minor, it must be obtained by deleting one edge, since B_7 has 8 vertices and 18 edges. But B_7 has a vertex of degree 6, while D^5_{17} has all vertices of degree 5 or less, so B_7 is not a minor of D^5_{17}.

If D^5_{17} has D_3 as a minor, it must be obtained by deleting three edges, since D_3 has 8 vertices and 16 edges. But in D_3, there are two non-adjacent degree 5 vertices that share the same neighborhood, while in D^4_{17}, no two non-adjacent degree five vertices share the same neighborhood. Thus D_3 is not a minor of D^4_{17}.

If D^5_{17} has E_3 as a minor, it must be obtained by deleting four edges, since E_3 has 8 vertices and 15 edges. But in E_3, the only vertices of degree less than five are non-adjacent degree 3 vertices with the same neighborhood, but vertex 2 and 8 share no neighbors, and D^3_{17} cannot have E_3 as a minor.

If D^5_{17} has E_{18} as a minor, it must be obtained by deleting four edges, since E_{18} has 8 vertices and 15 edges. But in E_{18} there are no edges between the neighbors of any vertex, so since vertex 2 is adjacent to 1,3, and 4 in D^5_{17}, we must delete $\{1,3\}, \{1,4\}$, and $\{3,4\}$, creating three degree 3 vertices, yet E_{18} has only two degree 3 vertices. Thus D^5_{17} does not have E_{18} as a minor. $\square$

References

[1] Guoli Ding and Perry Iverson, Hall-type results for 3-connected projective graphs, Manuscript.