Hall-type results for 3-connected projective graphs

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Abstract

Projective planar graphs can be characterized by a set $\mathcal{A}$ of 35 excluded minors. However, these 35 are not equally important. A set $\mathcal{E}$ of 3-connected members of $\mathcal{A}$ is *excludable* if there are only finitely many 3-connected non-projective planar graphs that do not contain any graph in $\mathcal{E}$ as a minor. In this paper we show that there are precisely two minimal excludable sets, which have sizes 19 and 20, respectively.

1 Introduction

Archdeacon [1] proved that projective planar graphs can be characterized by a set $\mathcal{A}$ of 35 excluded minors. This set consists of three graphs of connectivity zero, three graphs of connectivity one, six graphs of connectivity two (0-, 1-, 2-sums of graphs in $\{K_5, K_{3,3}\}$), and 23 graphs of connectivity at least three. Even though every graph in $\mathcal{A}$ is necessary for characterizing projective planarity, these 35 graphs are not equally important. For instance, it is easy to see that if a connected graph contains a 0-sum of two graphs in $\{K_5, K_{3,3}\}$ as a minor, then it contains the 1-sum of the same pair as a minor. Therefore, a connected graph is projective planar if and only if it does not contain any connected member of $\mathcal{A}$ as a minor. Robertson, Seymour, and Thomas proved (unpublished) similar results for 2- and 3-connected graphs. The 3-connected version of this theorem is stated below, where $\mathcal{A}_3$ is the set of all 23 3-connected members of $\mathcal{A}$. For the convenience of the reader, we include a drawing of these 23 graphs in Appendix A. We refer the reader to [2] for a short proof of this result.

Theorem 1.1. *A 3-connected graph is projective planar if and only if it does not contain any member of $\mathcal{A}_3$ as minor.*

The goal of this paper is to further refine Theorem 1.1. For any graph H, a graph G is called H*-free* if no minor of G is isomorphic to H. For any set $\mathcal{H}$ of graphs, G is $\mathcal{H}$*-free* if G is H-free for all $H \in \mathcal{H}$. Recall Hall's refinement [4] of Kuratowski theorem, which says that a 3-connected graph G is planar if and only if G is $K_{3,3}$-free, except for $G = K_5$. We will prove a theorem of the same type. Let us call $\mathcal{E} \subseteq \mathcal{A}_3$ *excludable* if there are only finitely many 3-connected $\mathcal{E}$-free graphs that are not projective planar. In other words, with finitely many exceptions, a 3-connected graph is projective planar if and only if it is $\mathcal{E}$-free. If $\mathcal{E}$ is excludable, we call $\mathcal{A}_3 - \mathcal{E}$ a *splitting set* since its members behave like splitters, and we call the set of finitely many 3-connected non-projective planar $\mathcal{E}$-free graphs the corresponding *exception set.*

We will make use of Archdeacon's notation for the 23 graphs in $\mathcal{A}_3$, calling them A_2, B_1, B_7, C_3, C_4, C_7, D_2, D_3, D_9, D_{12}, D_{17}, E_2, E_3, E_5, E_{11}, E_{18}, E_{19}, E_{20}, E_{22}, E_{27}, F_1, F_4, and G_1 (see Appendix A). The main result of this paper is:

Theorem 1.2. *There are precisely two maximal splitting sets,* $\{A_2, C_4, C_7, D_{17}\}$ *and* $\{B_7, C_7, D_{17}\}$.

Equivalently, $\mathcal{A}_3 - \{A_2, C_4, C_7, D_{17}\}$ and $\mathcal{A}_3 - \{B_7, C_7, D_{17}\}$ are the only two minimal excludable sets, which means that there are exactly two "unimprovable" Hall-type theorems for projective planar graphs. We can also explicitly determine all the exception graphs.

Proposition 1.3. *For* $\mathcal{A}_3 - \{A_2, C_4, C_7, D_{17}\}$*, the exception set has exactly 20 graphs:* A_2*,* C_4*,* C_7*,* C_7^+ *(pictured in Figure 3.4), and the sixteen subgraphs of the five graphs in Figure 3.5 that have* D_{17} *as a subgraph. For* $\mathcal{A}_3 - \{B_7, C_7, D_{17}\}$*, the exception set has exactly 19 graphs:* B_7*,* B_7^+ *(pictured in Figure 3.3),* C_7*,* C_7^+*, and the fifteen subgraphs of the first four graphs in Figure 3.5 that have* D_{17} *as a subgraph.*

This proposition indicates that every exception graph has at most 8 vertices and at most 20 edges. Therefore, the assertion in Theorem 1.2 that $\{A_2, C_4, C_7, D_{17}\}$ and $\{B_7, C_7, D_{17}\}$ are splitting sets can be rephrased as: *the following are equivalent for any 3-connected graph* G *with at least 9 vertices or at least 21 edges:*

(i) G *is projective planar;*

(ii) G *is* $\mathcal{E}$*-free for* $\mathcal{E} = \mathcal{A}_3 - \{A_2, C_4, C_7, D_{17}\}$*;*

(iii) G *is* $\mathcal{E}$*-free for* $\mathcal{E} = \mathcal{A}_3 - \{B_7, C_7, D_{17}\}$.

In Section 2 we discuss several graph operations and related results that are needed for the proof of Theorem 1.2. In Section 3, we first demonstrate that any graph $A \in \mathcal{A}_3 - \{A_2, B_7, C_4, C_7, D_{17}\}$ is in no splitting set by providing an infinite family of 3-connected non-projective planar $(\mathcal{A}_3 - \{A\})$-free graphs, and we then prove several Lemmas that require careful analysis of A_2, B_7, C_4, C_7, and D_{17}. In Section 4, we use these results to complete the proof of Theorem 1.2.

2 Graph operations

All graphs considered in this paper are simple. We say a graph G is an *uncontraction* of a graph H if $G/e = H$, where both ends of e have degree at least three in G. Equivalently, G is obtained from H by deleting a vertex v of degree at least four, then adding two adjacent vertices v_1 and v_2, and finally making every neighbor of v adjacent to precisely one of v_1, v_2 such that, for $i = 1, 2$, at least two of these neighbors are adjacent to v_i. We will denote such an uncontraction by $v \to (\{u_1, u_2, \ldots, u_p\}, \{w_1, w_2, \ldots, w_q\})$, where $u_1, u_2, \ldots, u_p, w_1, w_2, \ldots, w_q$ are all the vertices adjacent to v in H and $u_1, u_2, \ldots, u_p$ are adjacent to v_1 and $w_1, w_2, \ldots, w_q$ are adjacent to v_2 in G. Throughout the paper, if the vertices of H are labeled, we will give v_1 the label of v and v_2 the label $|V(H)|+1$ in such an uncontraction. A graph G is an *undeletion* of H if G is obtained from H by adding an edge between two non-adjacent vertices. It is easy to see that both uncontraction and undeletion preserve the 3-connectivity of a graph. The following is a corollary of Seymour's well-known splitter theorem for matroids [6], proved independently by Negami in [5]. We will make use of this theorem several times throughout the paper.

Theorem 2.1. *Let H be a 3-connected minor of a 3-connected graph G such that H is not a wheel. Then a graph isomorphic to G can be obtained from H by repeatedly applying the operations of uncontraction and undeletion.*

A graph G is an *edge-split* of a graph H if G is obtained from H by deleting an edge uv and adding a new vertex adjacent to u,v, and w, where w is different from u and v. We denote such an edge-splitting by $w \to uv$. We call G a *triad-addition* of H if G is obtained from H by adding a vertex u incident to exactly three distinct vertices v_1, v_2, v_3 of H. We denote such a triad-addition by $v_1v_2v_3$. It is easy to see that both edge-splitting and triad-addition preserve the 3-connectivity of a graph. The following is a modification of Theorem 2.1 that will make the analysis in Section 3 much simpler than using Theorem 2.1 alone.

Lemma 2.2. *Suppose a 3-connected graph G has a 3-connected minor H with $|V(G)| > |V(H)|$. Let F be a set of edges e in the complement of H such that $H+e$ is not a minor of G. Then G contains one of the following as a minor:*

- *An uncontraction of H.*
- *An edge-split $u \to v_1v_2$ of H where, for $i=1,2$, either $uv_i \in E(H)$ with $d_H(v_i)=3$ or $uv_i \notin E(H) \cup F$.*
- *A triad-addition $v_1v_2v_3$ of H where $v_1v_2, v_1v_3, v_2v_3 \notin E(H) \cup F$.*

Proof. Suppose no uncontraction of H is a minor of G. Then G has a subdivision H' of H as a subgraph. We first prove that G contains either an edge-split or a triad-addition of H as a minor. Suppose in H' an edge $e = v_1v_2$ of H is replaced by a path P_e on at least three vertices. Since G is 3-connected, $G \backslash \{v_1, v_2\}$ has a path P between $V(P_e) - \{v_1, v_2\}$ and $x \in V(H') - V(P_e)$. Then $H' \cup P$ can be contracted to an edge-split $u \to v_1v_2$ of H, where x is contracted to u. So we may assume $H' = H$. Since $|V(G)| > |V(H)|$, G has a vertex x not in H. By the 3-connectedness of G, there are three paths P_1, P_2, P_3 from x to $V(H)$ that are vertex-disjoint except for at x. Then $H \cup P_1 \cup P_2 \cup P_3$ can be contracted to a triad-addition of H.

If G has an edge-split $u \to v_1v_2$ with new vertex w as a minor where $uv_1 \in F$, then contract wv_1 to find $H + uv_1$ as a minor of G, a contradiction. If $uv_1 \in E(H)$ and $d_H(v_1) \geq 4$, then delete uv_1 from the edge-split to find an uncontraction of H as a minor of G, again a contradiction. So the edge-split satisfies the requirements. If G has a triad-addition $v_1v_2v_3$ with new vertex u as a minor with $v_1v_2 \in E(H)$, then delete v_1v_2 to get an edge-split of H, which reduces to the last case. If $v_1v_2 \in S$, then delete uv_3 and contract uv_1 to find $H + v_1v_2$ as a minor of G, a contradiction. So the triad-addition satisfies the requirement. □

A *k-separation* of a graph G is a pair of subgraphs $G_i = (V_i, E_i)$ $(i = 1, 2)$ of G such that $E_1 \cup E_2 = E(G)$, $E_1 \cap E_2 = \emptyset$, $V_1 \cup V_2 = V(G)$, $V_1 - V_2 \neq \emptyset \neq V_2 - V_1$, and $|V_1 \cap V_2| = k$. If $k = 3$, we define G_i^{Δ} $(i = 1, 2)$ to be the graph obtained from G_i by adding all missing edges among vertices of $V_1 \cap V_2$, and we define G_i^{Y} $(i = 1, 2)$ to be the graph obtained from G_i by adding an extra vertex with an edge to each vertex of $V_1 \cap V_2$. If G has a vertex v of degree three then there is a unique 3-separation (G_1, G_2) with $G_1 = G - v$. We will use the notation $(G - v)^{\Delta}$ without referencing the 3-separation. Note that this is the simplification of a $Y\Delta$-transformation.

Lemma 2.3. *Let H be a 3-connected minor of a 3-connected graph G with 3-separation (G_1, G_2). Let $U = V(G_1) \cap V(G_2)$. Then either H is a minor of G_1^{Δ}, H is a minor of G_2^{Δ}, or H has a 3-separation (H_1, H_2) with $V(H_1) \cap V(H_2) = U$ and H_i is a minor of G_i $(i = 1, 2)$.*

Proof. We may assume the minor is produced by only contracting and deleting edges (without deleting vertices) since G is connected. So $V(G)$ has a partition $(X_v : v \in V(H))$ such that each $G_v = G[X_v]$ is connected and H is obtained from G by contracting every $E(G_v)$ and then deleting edges. If there exists $i \in \{1, 2\}$ such that no X_v is a subset of $V(G_i) - U$ then every X_v that meets $V(G_i)$ also meets U, which implies that H is a minor of G_j^{Δ} $(j \neq i)$. So for $i = 1, 2$, there exists $X_{v_i} \subseteq V(G_i) - U$. Since H is 3-connected, it has three internally vertex-disjoint v_1v_2-paths, which forces the three vertices in U to be contained in three distinct sets in $(X_v : v \in V(H))$. For $i = 1, 2$, let H_i be obtained from G_i by deleting and contracting edges that were deleted and contracted when producing H. Then (H_1, H_2) is a 3-separation of H with $V(H_1) \cap V(H_2) = U$, as required. □

Lemma 2.4. *Let G be a 3-connected graph with a 3-separation (G_1, G_2) where G_2^{Y} is planar. Then G is projective planar if and only if G_1^{Y} is projective planar.*

Proof. Let $U = V(G_1) \cap V(G_2) = \{u_1, u_2, u_3\}$ and let v_i $(i = 1, 2)$ be the unique vertex in $V(G_i^{Y}) - V(G_i)$. If G is projective planar, then since G is 3-connected, G_1^{Y} is a minor of G and thus G_1^{Y} is projective planar. Conversely, if G_1^{Y} is projective planar, it has an embedding in the projective plane with edges u_1v_1, u_2v_1, u_3v_1 embedded in a disc D_1 such that D_1 is disjoint from the rest of the embedding. Also since G_2^{Y} is planar, G_2^{Y} has an embedding in the sphere with edges u_1v_2, u_2v_2, u_3v_2 embedded in a disc D_2 such that D_2 is disjoint from the rest of the embedding. Let disc D_2' be the complement of D_2 in the sphere. Then replacing D_1 with D_2' and identifying the corresponding vertices in U results in a projective embedding of G. □

Let $\mathcal{C}$ be a set of graphs. We call a graph $G \in \mathcal{C}$ a *splitter* in $\mathcal{C}$ if there are only finitely many $(\mathcal{C} - \{G\})$-free 3-connected graphs with G as a minor. Note that this is a generalization of Seymour's definition [6] of a splitter, which requires G to be the only $(\mathcal{C} - \{G\})$-free 3-connected graph with G as a minor. Now we can prove a result that helps us to show nearly all graphs in $\mathcal{A}_3$ are not splitters in $\mathcal{A}_3$.

Lemma 2.5. *Let G be a 3-connected graph with a vertex v of degree three. If both G and $(G - v)^{\Delta}$ are $\mathcal{C}$-free, where $\mathcal{C} \subseteq \mathcal{A}_3$, then for any minor J of G, J is not a splitter in $\mathcal{C} \cup \{J\}$.*

Proof. Let $H_1 = G - v$ and let $U \subseteq V(H_1)$ be the set of the three neighbors of v. Let $\mathcal{F}$ be the set of 3-connected graphs $F = H_1 \cup H$ such that (H_1, H) is a 3-separation of F, $V(H_1) \cap V(H) = U$, and H^Y is planar. Clearly, every graph in $\mathcal{F}$ contains $H_1^Y \cong G$, and thus J, as a minor. We prove that all graphs in $\mathcal{F}$ are $\mathcal{C}$-free, which shows that J is not a splitter in $\mathcal{C} \cup \{J\}$.

Suppose some $H_1 \cup H \in \mathcal{F}$ contains some $A \in \mathcal{C}$ as a minor. Then by Lemma 2.3 either A is a minor of H_1^Δ, A is a minor of H^Δ, or A has a 3-separation (A_1, A_2) with $V(A_1) \cap V(A_2) = U$, where A_1 is a minor of H_1 and A_2 is a minor of H. It is not possible for A to be a minor of H_1^Δ since $A \in \mathcal{C}$ and H_1^Δ is $\mathcal{C}$-free. If A is a minor of H^Δ, we deduce from the planarity of H^Y the planarity of H^Δ and thus also of A, which is not possible since $A \in \mathcal{C} \subseteq \mathcal{A}_3$. Therefore, A has a 3-separation (A_1, A_2) with A_2^Y planar. Since A is non-projective planar, by Lemma 2.4, A_1^Y is non-projective planar. But A_1^Y is a minor of A and all proper minors of A are projective planar, so $A = A_1^Y$. It follows that A is a minor of $H_1^Y = G$, which is not possible since $A \in \mathcal{C}$ and G is $\mathcal{C}$-free. This contradiction completes the proof of the lemma. □

3 Case analysis

We begin this section with a Lemma showing that nearly all graphs in $\mathcal{A}_3$ are not splitters in $\mathcal{A}_3$.

Lemma 3.1. *Any graph $A \in \mathcal{A}_3 - \{A_2, B_7, C_4, C_7, D_{17}\}$ is not a splitter in $\mathcal{A}_3$. Also, B_7 is not a splitter in $\mathcal{A}_3 - \{A_2\}$ and C_4 is not a splitter in $\mathcal{A}_3 - \{B_7\}$.*

Proof. Each graph $A \in \mathcal{A}_3 - \{A_2, B_1, B_7, C_4, C_7, D_{17}\}$ is obviously $(\mathcal{A}_3 - \{A\})$-free, and for every such A, there is a vertex v of degree 3 shown as the square vertex in Appendix A such that $(A - v)^\Delta$ is projective planar and thus $(\mathcal{A}_3 - \{A\})$-free. Projective embeddings for each of these graphs can be found in Appendix B. So each such A cannot be a splitter in $\mathcal{A}_3$ by Lemma 2.5.

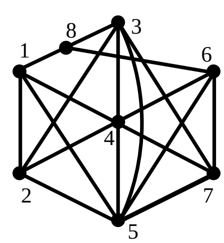


Figure 3.1: An uncontraction B_1^+ of B_1

To see that B_1 is not a splitter in $\mathcal{A}_3$, we consider the uncontraction B_1^+ of B_1 shown in Figure 3.1. By Lemma 2.5 we only need to show that B_1^+ and $(B_1^+ - 8)^\Delta$ are $(\mathcal{A}_3 - \{B_1\})$-free. First, $(B_1^+ - 8)^\Delta$ has seven vertices and the only graph in $(\mathcal{A}_3 - \{B_1\})$ with seven or fewer vertices is A_2, which has seven vertices yet a bigger minimum degree. It follows that $(B_1^+ - 8)^\Delta$ does not contain A_2 and so $(B_1^+ - 8)^\Delta$ is $(\mathcal{A}_3 - \{B_1\})$-free. Similarly, notice that B_1^+ has eight vertices and eighteen edges, and the only graphs in $(\mathcal{A}_3 - \{B_1\})$ with eight or fewer vertices are $A_2(7, 18)$, $B_7(8, 18)$, $C_7(8, 17)$, $D_3(8, 16)$, $D_{17}(8, 16)$, $E_3(8, 15)$, $E_{18}(8, 15)$, where the two numbers indicate the number of vertices and edges of the graph. Clearly B_1^+ does not contain A_2 as a minor since they have the same number of edges yet different number of vertices. The remaining graphs have the same number of vertices as B_1^+ and would have to be spanning subgraphs. By checking their degree sequences it is easy to see B_1^+ is not isomorphic to B_7, and thus does not contain B_7 as a minor since they have the same number of edges. Neither C_7 nor D_{17} can be a spanning subgraph of B_1^+ since they have no vertices of degree three, but B_1^+ does. D_3 has three vertices of degree five, and two other adjacent vertices that are adjacent to each of the vertices of degree five, but B_1^+ has only three vertices of degree at least five, and no two of the remaining vertices are adjacent to each other and also adjacent to these three vertices, so D_3 cannot be a spanning subgraph of B_1^+. There are four edge-disjoint triangles in B_1^+: 125, 234, 357, and 467, which implies that any bipartite subgraph of B_1^+ can have at most $18-4=14$ edges. Since E_3 and E_{18} are bipartite with 15 edges, none of them is a subgraph of B_1^+. Thus B_1^+ is $(\mathcal{A}_3 - \{B_1\})$-free, which completes the proof that B_1 is not a splitter in $\mathcal{A}_3$.

Since $(B_7 - v)^\Delta$ is isomorphic to A_2, where v is the unique cubic vertex of B_7, both B_7 and $(B_7 - v)^\Delta$ are $(\mathcal{A}_3 - \{A_2, B_7\})$-free and thus B_7 is not a splitter in $\mathcal{A}_3 - \{A_2\}$. Similarly, since $(C_4 - v)^\Delta$ is isomorphic to B_7, where v is any cubic vertex of C_4, both C_4 and $(C_4 - v)^\Delta$ are $(\mathcal{A}_3 - \{B_7, C_4\})$-free, so C_4 is not a splitter in $\mathcal{A}_3 - \{B_7\}$. □

The remainder of this section will consist of five Lemmas classifying the finitely many graphs with A_2, B_7, C_4, C_7, or D_{17} as a minor that do not have minors among most of $\mathcal{A}_3$. This will allow us to classify all graphs in splitting sets and the corresponding exception sets.

Lemma 3.2. *The only 3-connected $(\mathcal{A}_3 - \{A_2\})$-free graph with A_2 as a minor is A_2.*

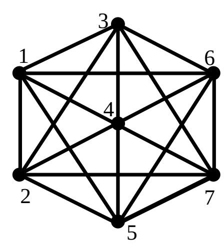


Figure 3.2: A labeling of A_2

Proof. By Theorem 2.1 we only need to verify that every undeletion and every uncontraction of A_2 contains a member of $\mathcal{A}_3 - \{A_2\}$ as a minor. We will use a labeling of the vertices of A_2 shown in Figure 3.2. Observe that

(*) *the complement of A_2 consists of three non-incident edges and an isolated vertex.*

Using (*) we see that, up to isomorphism, there is only one undeletion of A_2, obtained by adding edge $\{1,7\}$ which contains B_1 as a minor by deleting edges $\{2,3\}$ and $\{5,6\}$. It is also not difficult to see from (*) that there are seven uncontractions of A_2 up to isomorphism. They are listed below along with their minors:

A_2 Uncontractions			
Uncontraction	Delete	Contract	Minor
$1 \to (\{4,5,6\},\{2,3\})$	$\{\{2,3\}\}$	–	B_7
$1 \to (\{3,5,6\},\{2,4\})$	$\{\{1,6\},\{2,4\},\{2,7\}\}$	–	D_3
$1 \to (\{3,4,5\},\{2,6\})$	$\{\{1,3\},\{4,5\},\{5,7\}\}$	–	D_3
$4 \to (\{3,5,6,7\},\{1,2\})$	$\{\{1,2\},\{1,6\},\{2,7\}\}$	–	D_3
$4 \to (\{2,3,5,6\},\{1,7\})$	$\{\{2,3\},\{2,5\},\{3,6\},\{5,6\}\}$	–	E_3
$4 \to (\{5,6,7\},\{1,2,3\})$	$\{\{1,5\},\{3,7\}\}$	–	C_7
$4 \to (\{3,5,7\},\{1,2,6\})$	$\{\{1,2\},\{1,6\},\{3,7\},\{5,7\}\}$	–	E_{18}

So all undeletions and uncontractions of A_2 have a member of $(\mathcal{A}_3 - \{A_2\})$ as a minor and A_2 is the only 3-connected $(\mathcal{A}_3 - \{A_2\})$-free graph. □

Our proofs for the next four lemmas are similar to that of the last one. We first generate all expansions of a graph according to Theorem 2.1 or Lemma 2.2, and then we check if the generated graphs contain any required minors. For some graphs we have to iterate this process several times. In proving the next four lemmas, we generate about 90 non-isomorphic graphs, and we find required minors for each of them. This task is straightforward but very tedious. To save space, we choose not to include the lengthy cases analysis here. In particular, details like the chart in the proof of Lemma 3.2 are omitted. These details are included in a supplement [3] we prepared for those who wish to get into it. It is worth mentioning that proofs in this section concerning graph expansion and minor checking are also verified by a computer. In fact, the two authors wrote two independent programs and they derived the same conclusions.

Lemma 3.3. *The only 3-connected $(\mathcal{A}_3 - \{A_2, C_4\})$-free graph with C_4 as a minor is C_4.*

Proof. There are eight uncontractions of C_4 up to isomorphism. Each uncontraction has C_3, C_7, D_3 or F_1 as a minor. There are five undeletions of C_4 up to isomorphism. Each undeletion has B_1, B_7, or E_{22} as a minor. So by Theorem 2.1, C_4 is the only 3-connected $(\mathcal{A}_3 - \{A_2, C_4\})$-free graph with C_4 as a minor. □

Lemma 3.4. *The only 3-connected $(\mathcal{A}_3 - \{B_7\})$-free graphs with B_7 as a minor are among B_7 and B_7^+ as shown in Figure 3.3.*

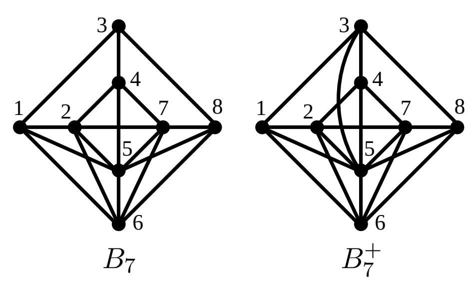


Figure 3.3: B_7 and an undeletion of B_7

Proof. Let G be a 3-connected $(\mathcal{A}_3 - \{B_7\})$-free graph with B_7 as a minor. Up to isomorphism, B_7 has four undeletions. One of these four is B_7^+ (which can be obtained only by adding the edge $\{3, 5\}$, as shown by its degree sequence), and the others have A_2 or B_1 as a minor. So clearly, if $|V(G)| \leq |V(B_7)|$ then G is isomorphic to B_7 or B_7^+. Now assume $|V(G)| > |V(B_7)|$, and note that adding any edge other than $\{3, 5\}$ to B_7 gives a minor among A_2 or B_1, so by Lemma 2.2 with $F = E(\overline{B_7}) - \{\{3, 5\}\}$, G must have a minor among the uncontractions, certain edge-splits, and certain triad-additions of B_7. There are fifteen uncontractions of B_7 up to isomorphism, each of which has C_3, C_4, C_7, D_3, E_{18}, E_{22}, or F_1 as a minor. No edge-splits of B_7 as required by Lemma 2.2 exist, since there is only one vertex of degree three in B_7 and it is incident to $\{3, 5\}$, the only edge not in $E(B_7) \cup F$. Finally, no triad-addition of B_7 as required by Lemma 2.2 exists, since $\{3, 5\}$ is the only edge not in $E(B_7) \cup F$, and no G with $|V(G)| > |V(B_7)|$ exists. Thus the only possible $\mathcal{A}_3 - \{B_7\}$-free graphs with B_7 as a minor are B_7 and B_7^+. □

Lemma 3.5. *The only 3-connected $(\mathcal{A}_3 - \{A_2, B_7, C_4, C_7\})$-free graphs with C_7 as a minor are among C_7 and C_7^+ as shown in Figure 3.4.*

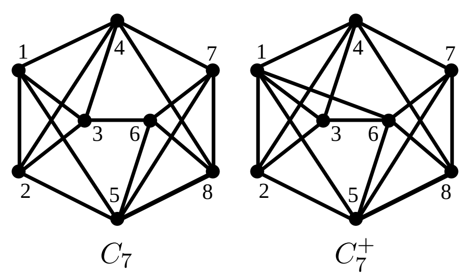


Figure 3.4: C_7 and an undeletion of C_7

Proof. Let G be a 3-connected $(\mathcal{A}_3 - \{A_2, B_7, C_4, C_7\})$-free graph with C_7 as a minor. There are four undeletions of C_7 up to isomorphism, one of which is C_7^+ (which can be obtained only by adding one of: $\{1, 6\}$, $\{2, 6\}$, $\{3, 7\}$, and $\{3, 8\}$), and the others have B_1, D_3 or D_{17} as a minor. Now assume $|V(G)| > |V(C_7)|$, and note that adding any edge other than $\{1, 6\}$, $\{2, 6\}$, $\{3, 7\}$, or $\{3, 8\}$ to C_7 gives a minor among B_1, D_3 or D_{17}, so by Lemma 2.2 with $F = E(\overline{C_7}) - \{\{1, 6\}, \{2, 6\}, \{3, 7\}, \{3, 8\}\}$, G must have a minor among the uncontractions, certain edge-splits, and certain triad-additions of C_7. There are ten uncontractions of C_7 up to isomorphism, each of which has D_3, D_{12}, D_{17}, E_{19}, or F_1 as a minor. There are no vertices of degree three in C_7, so the only possible edge-splits of C_7 as required by Lemma 2.2 are $3 \to \{7, 8\}$ and $6 \to \{1, 2\}$. These are isomorphic and have F_1 as a minor. Finally, no triad-additions of C_7 as required by Lemma 2.2 exist, since the only edges not in $E(C_7) \cup F$ are $\{1, 7\}$, $\{1, 8\}$, $\{2, 7\}$, and $\{2, 8\}$, and there is no triangle among them. So we must have $|V(G)| \leq |V(B_7)|$. Now there are two possible ways to add two edges from among $\{1, 6\}$, $\{2, 6\}$, $\{3, 7\}$, and $\{3, 8\}$ to C_7 up to isomorphism. They have either D_3 or D_{17} as a minor. Thus the only possible $(\mathcal{A}_3 - \{A_2, B_7, C_4, C_7\})$-free graphs with C_7 as a minor are C_7 and C_7^+. □

Lemma 3.6. *The only 3-connected $(\mathcal{A}_3 - \{B_7, C_7, D_{17}\})$-free graphs containing D_{17} as a minor are among the fifteen subgraphs of the first four graphs in Figure 3.5 that also have D_{17} as a subgraph.*

The only 3-connected ($\mathcal{A}_3 - \{A_2, C_4, C_7, D_{17}\}$)-free graphs with D_{17} as a minor are among those fifteen graphs together with the fifth graph in Figure 3.5.

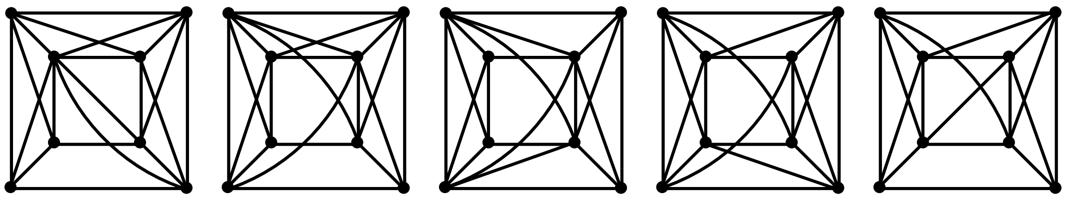


Figure 3.5: Five undeletions of D_{17}

Proof. Let G be a 3-connected ($\mathcal{A}_3 - \{A_2, B_7, C_4, C_7, D_{17}\}$)-free graph with D_{17} as a minor. Now assume $|V(G)| > |V(D_{17})|$, and by Lemma 2.2 with $F = \emptyset$, G must have a minor among the uncontractions, certain edge-splits, and certain triad-additions of D_{17}. There is one uncontraction of D_{17} up to isomorphism and it has E_{20} as a minor. Furthermore, D_{17} has no vertices of degree three, so any edge-split of D_{17} as required by Lemma 2.2 must be of the form $u \to v_1v_2$ where $uv_1, uv_2 \in E(\overline{D_{17}})$ and $v_1v_2 \in E(D_{17})$. There is only one such edge-split up to isomorphism, and it has D_{12} as a minor. Finally, no triad-additions of D_{17} as required by Lemma 2.2 exist, since this would require there to be a triangle among the edges of $\overline{D_{17}}$, and largest independent set of vertices in D_{17} has size two. So G does not contain any such required triad-addition as a minor, and we have $|V(G)| \leq |V(D_{17})|$.

Now all eighteen graphs consisting of D_{17} with five extra edges have either B_1 or D_3 as a minor. All twelve graphs consisting of D_{17} with four extra edges that are not one of the first four graphs in Figure 3.5 have minors among B_1, D_3, and E_{18}. There are only four graphs consisting of D_{17} together with three extra edges that are not subgraphs of one of the first four graphs in Figure 3.5. One of these is the fifth graph in Figure 3.5, and the other three have minors among B_1, D_3, and E_{18}. All graphs consisting of D_{17} together with one or two extra edges are subgraphs of one of the first four graphs in Figure 3.5. So the only possible 3-connected ($\mathcal{A}_3 - \{A_2, B_7, C_4, C_7, D_{17}\}$)-free graphs with D_{17} as a minor are subgraphs of the first four graphs of Figure 3.5 with D_{17} as a subgraph or the fifth graph in that Figure. The fifth graph in that Figure has A_2 as a minor, so the Lemma follows immediately. □

4 Final proofs

Now we are ready to prove Theorem 1.2 and Proposition 1.3. By Lemma 3.1, the only graphs in any splitting set are A_2, B_7, C_4, C_7, and D_{17}, and the pairs $\{A_2, B_7\}$ and $\{B_7, C_4\}$ are in no splitting set. Thus the maximal possible splitting sets are $\{A_2, C_4, C_7, D_{17}\}$ and $\{B_7, C_7, D_{17}\}$. We claim both are splitting sets.

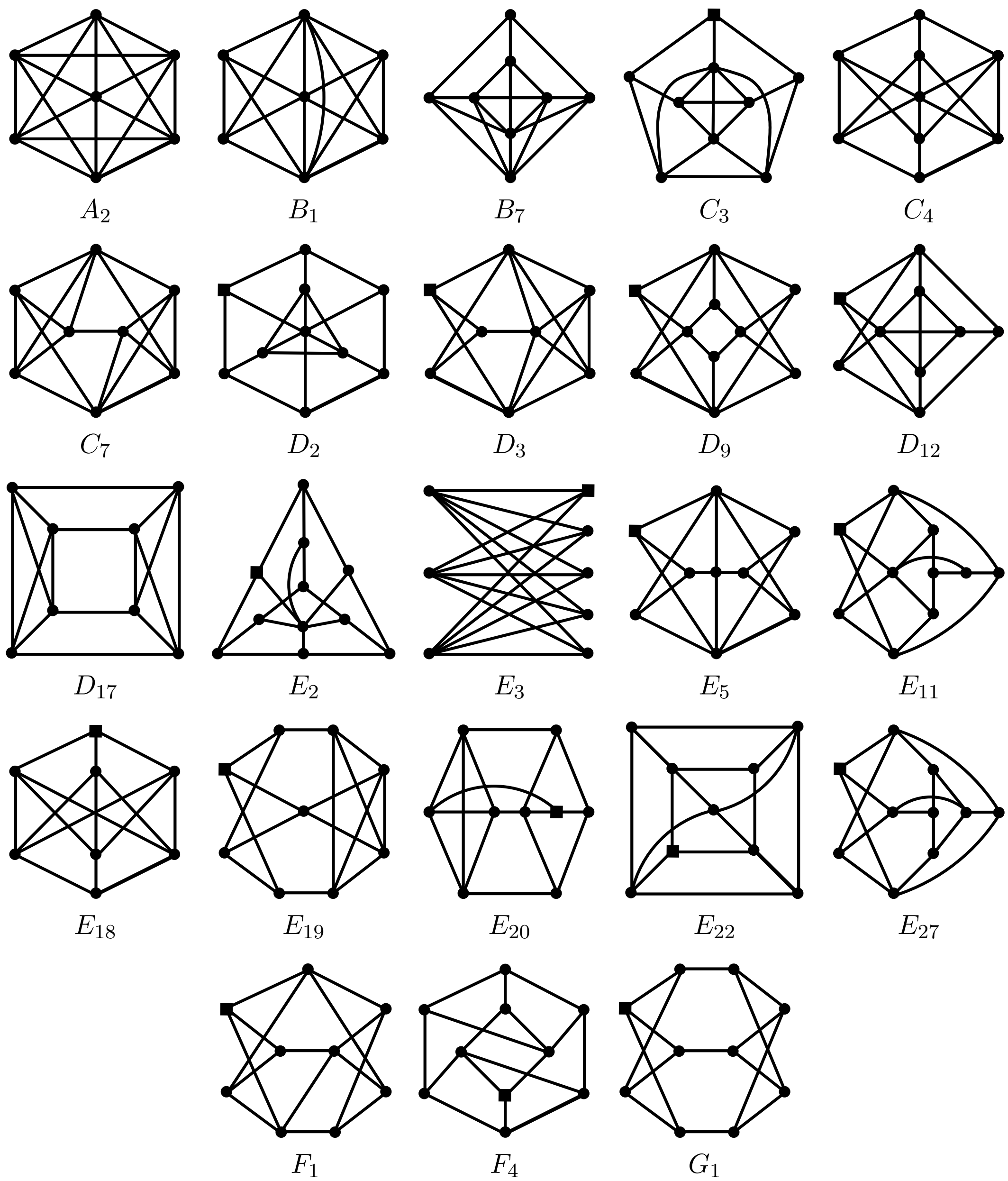
By Lemma 3.6, the only possible ($\mathcal{A}_3 - \{A_2, C_4, C_7, D_{17}\}$)-free graphs with D_{17} as a minor are subgraphs of the five graphs in Figure 3.5, so the only remaining ($\mathcal{A}_3 - \{A_2, C_4, C_7, D_{17}\}$)-

free graphs are also D_{17}-free, and are thus $(\mathcal{A}_3 - \{A_2, C_4, C_7\})$-free. By Lemma 3.5, the only possible $(\mathcal{A}_3 - \{A_2, C_4, C_7\})$-free graphs with C_7 as a minor are C_7 and C_7^+, so the only remaining $(\mathcal{A}_3 - \{A_2, C_4, C_7\})$-free graphs are $(\mathcal{A}_3 - \{A_2, C_4\})$-free. By Lemma 3.3, the only $(\mathcal{A}_3 - \{A_2, C_4\})$-free graph with C_4 as a minor is C_4, and we only need to consider $(\mathcal{A}_3 - \{A_2\})$-free graphs. But by Lemma 3.2, the only $(\mathcal{A}_3 - \{A_2\})$-free graph with A_2 as a minor is A_2. So $\{A_2, C_4, C_7, D_{17}\}$ is a splitting set whose corresponding exception set is a subset of $\{A_2, C_4, C_7, C_7^+\}$ together with the sixteen subgraphs of the five graphs in Figure 3.5 with D_{17} as a subgraph.

Similarly, Lemma 3.6, Lemma 3.5, and Lemma 3.4 tell us that $\{B_7, C_7, D_{17}\}$ is a splitting set whose corresponding exception set is a subset of $\{B_7, B_7^+, C_7, C_7^+\}$ together with the fifteen subgraphs of the first four graphs in Figure 3.5 with D_{17} as a subgraph. Therefore, the proof of Theorem 1.2 is complete.

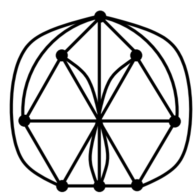
To prove Proposition 1.3, we need to show that B_7^+ is $(\mathcal{A}_3 - \{B_7\})$-free, C_7^+ is $(\mathcal{A}_3 - \{C_7\})$-free, the first four graphs in Figure 3.5 are $(\mathcal{A}_3 - \{C_7, D_{17}\})$-free, and the fifth graph in Figure 3.5 is $(\mathcal{A}_3 - \{A_2, C_7, D_{17}\})$-free. These can be proved with a tedious analysis similar to the proof in Lemma 3.1 for B_1^+. To save space, we choose not to include the proofs here and, instead, we put them in the supplement [3]. We point out that these minor testing problems can be very easily verified by a computer. In conclusion, the exception sets are exactly those given, which proves Proposition 1.3.

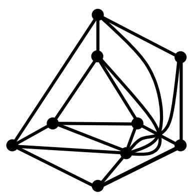
A The 23 graphs in $\mathcal{A}_3$

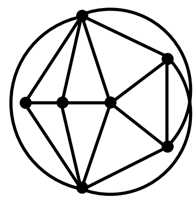


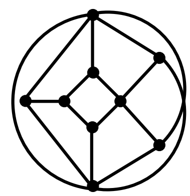
B Projective Embeddings

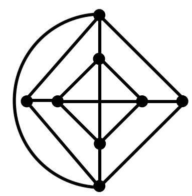
For each graph of $A \in \mathcal{A}$, let v be the square vertex shown in Appendix A. Below are several drawings of $(A-v)^{\Delta}$ needed for the proof of Lemma 3.1. Each drawing can be interpreted as a projective embedding by adding a crosscap at the crossing.

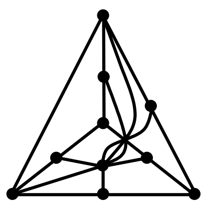

$(C_3 - v)^{\Delta}$

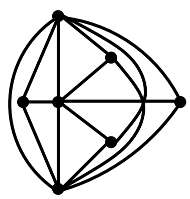

$(D_2 - v)^{\Delta}$

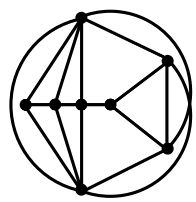

$(D_3 - v)^{\Delta}$

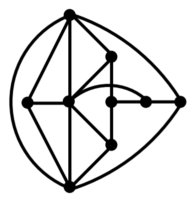

$(D_9 - v)^{\Delta}$

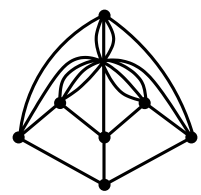

$(D_{12} - v)^{\Delta}$

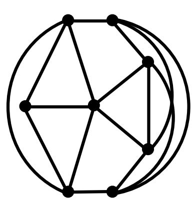

$(E_2 - v)^{\Delta}$

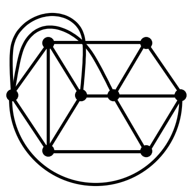

$(E_3 - v)^{\Delta}$

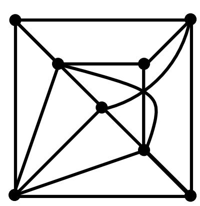

$(E_5 - v)^{\Delta}$

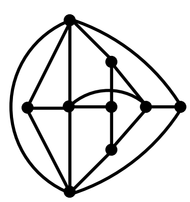

$(E_{11} - v)^{\Delta}$

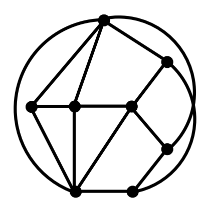

$(E_{18} - v)^{\Delta}$

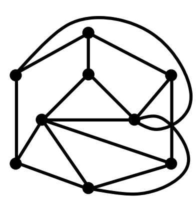

$(E_{19} - v)^{\Delta}$

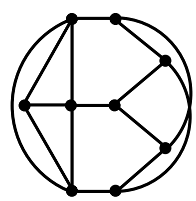

$(E_{20} - v)^{\Delta}$


$(E_{22} - v)^{\Delta}$


$(E_{27} - v)^{\Delta}$


$(F_1 - v)^{\Delta}$


$(F_4 - v)^{\Delta}$


$(G_1 - v)^{\Delta}$

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