Internally 4-connected projective-planar graphs

Guoli Ding and Perry Iverson
Mathematics Department, Louisiana State University, Baton Rouge, LA 70803

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Abstract

Archdeacon proved that projective-planar graphs are characterized by 35 excluded minors. Using this result we show that internally 4-connected projective-planar graphs are characterized by 23 internally 4-connected excluded minors.

1 Introduction

A classical result of Archdeacon [1, 2] states that projective-planar graphs are characterized by a set $\mathcal{A}$ of 35 excluded minors. This set consists of three disconnected graphs, three graphs of connectivity one, six graphs of connectivity two (0-, 1-, 2-sums of K_5 and $K_{3,3}$), and 23 graphs of connectivity at least three. In many applications graphs in consideration are well-connected. For this reason, it is desirable to refine Archdeacon's result for better-connected graphs.

The following is a simple fact observed by many. If a connected graph contains a 0-sum of two graphs in $\{K_5, K_{3,3}\}$ as a minor, then it contains the 1-sum of the same pair as a minor. Consequently, a connected graph is projective-planar if and only if it does not contain any connected member of $\mathcal{A}$ as a minor. More interestingly, it is confirmed by Robertson, Seymour, and Thomas (unpublished) that, for each $k \in \{2, 3\}$, a k-connected graph is projective-planar if and only if it does not contain any k-connected member of $\mathcal{A}$ as a minor.

There have been several attempts to establish similar results for internally 4-connected graphs. Maharry and Slilaty proved a result (unpublished) saying that internally 4-connected projective-planar graphs can be characterized by excluding a subset of $\mathcal{A}$ (some of which are not internally 4-connected). Thomas observed that in addition to the eleven internally 4-connected members of $\mathcal{A}$, there are at least two other minor-minimal internally 4-connected non-projective-planar graphs. Note that the property of being internally 4-connected is not a minor-closed property, so when referring to minor-minimal internally 4-connected non-projective-planar graphs, we mean those graphs for which no proper minor is both internally 4-connected and non-projective-planar. Since 3-connected projective-planar graphs are characterized by excluding the 23 3-connected members of $\mathcal{A}$, the general consensus is that internally 4-connected projective-planar graphs should be characterized by fewer internally 4-connected excluded minors. In this paper, however, we show that the total number of excluded minors is exactly 23.

Theorem 1.1. *An internally 4-connected graph is projective-planar if and only if it does not contain any of the 23 internally 4-connected graphs shown in the Appendix as a minor.*

This theorem has an interesting corollary. Let v be a cubic vertex adjacent to v_1, v_2, and v_3 in a graph G. Then a $Y\Delta$-*transformation* of G is a graph obtained by deleting v and the edges incident to v, and adding edges v_1v_2, v_1v_3, and v_2v_3. We say that H is a $Y\Delta$-*minor* of G if H is obtained from G by a series of edge deletions, edge contractions, vertex deletions, and $Y\Delta$-transformations. It is easy to verify that the class of projective-planar graphs is $Y\Delta$-minor closed. Under this relation, the number of forbidden graphs is reduced to just eight.

Corollary 1.2. *An internally 4-connected graph is projective-planar if and only if it does not contain any of* A_2, D_{17}, E_{18}, E_{22}, B_1', B_1''', D_3', *or* F_1' *as a* $Y\Delta$-*minor.*

Let $\mathcal{A}'$ consist of the twelve 3-connected members of $\mathcal{A}$ that are not internally 4-connected. These graphs are depicted in Figure 3.1. To prove Theorem 1.1, we show that if an internally 4-connected graph G contains a member of $\mathcal{A}'$ as a minor, then G contains one of the graphs in the Appendix as a minor. In the next section we explain how our approach works. Since our method is about how to fix a small separation in a general graph, its applications are not limited to problems in this paper. To illustrate our main idea, we give short proofs of the results of Robertson, Seymour, and Thomas in the 2- and 3-connected cases. In Section 3, we apply the approach outlined in Section 2 to the twelve graphs of $\mathcal{A}'$. Finally, in Section 4, we complete the proof of Theorem 1.1 and Corollary 1.2. To handle the large amount of case analysis occurred in Section 3, we use a computer to perform the routine work. Every result in this section is verified by two independent programs, so we believe that potential programming errors are eliminated. At the end of the paper, we argue that using a computer is a reasonable or even better choice for this problem. Finally, we remark that we have found 37 minor-minimal 4-connected non-projective-planar graphs and there could be even more.

2 Improving connectivity

Suppose G is non-projective-planar and it satisfies our desired connectivity. According to Archdeacon's theorem, G contains some $A \in \mathcal{A}$ as a minor. Graph A certifies the non-projectivity of G but its connectivity could be very low. Our problem is to find, based on A, a non-projective-planar minor of G that is better connected than A. In this section we illustrate how to do this. In fact, our result is independent of $\mathcal{A}$ and thus can be used to fix connectivity in a general situation.

Let $k \geq 0$ be an integer. A k-*separation* of a graph $G = (V, E)$ is a pair (G_1, G_2) of subgraphs $G_i = (V_i, E_i)$ such that (E_1, E_2) is a partition of E, $V_1 \cup V_2 = V$, and $|V_1 \cap V_2| = k < \min\{|V_1|, |V_2|\}$. Readers familiar with matroid theory will notice this is essentially a *vertical* k-*separation*. Graph G is called k-*connected* if $|V| > k$ and there is no k'-separation for any $k' < k$. In addition, G is called *internally* $(k+1)$-*connected* if G is k-connected and for every k-separation (G_1, G_2) of G it holds that $\min\{|E_1|, |E_2|\} = k$.

Let G be a minor of H and let (G_1, G_2) be a k-separation of G. If H has a k-separation (H_1, H_2) such that $E(G_i) \subseteq E(H_i)$ then we say that (G_1, G_2) *extends* to (H_1, H_2). If (G_1, G_2) does not extend to any k-separation of H, then there is a minimal graph G' such that G is a minor of G', G' is a minor of H, and (G_1, G_2) does not extend to any k-separation of G'. Clearly, we can think of G' as a result of fixing the separation (G_1, G_2) of G. According the Graph-Minor Theorem of Robertson and Seymour, there are only finitely many such graphs G' for any given G and (G_1, G_2). Therefore, we can say that every separation can be fixed in finitely many ways. In fact, using alternating walks (see Section 3.3 of [3] for its definition) one can actually construct all these graphs G'.

However, fixing k-separations may require a very long alternating walk that can add many additional edges. A drastic increase in the number of edges may make the alternating walk approach non-practical. In the following we explain how to fix a separation (G_1, G_2) of G without increasing the number of edges too much by not keeping the entire G as a minor. Instead, we will only keep G_1 and G_2. This weakened fix turns out to be the right combination: we do get a better connected graph yet we do not destroy the current graph by too much.

First, we introduce a more generalized idea of separation that will allow us to deal with multiple separations at the same time. A *k-division* of a graph $G = (V, E)$ is a triple (G_1, G_2, M), such that $G_i = (V_i, E_i)$ are subgraphs of G and M is a matching from a subset of $V_1 - V_2$ to a subset of $V_2 - V_1$, (E_1, E_2, M) is a partition of E, $V_1 \cup V_2 = V$, and $|V_1 \cap V_2| + |M| = k < \min\{|V_1|, |V_2|\}$. Note that $(G_1 \cup M_1, G_2 \cup M_2)$ is a k-separation for every partition (M_1, M_2) of M, so a k-division is in fact a collection of k-separations. On the other hand, since we allow M to be empty, every k-separation (G_1, G_2) can be considered as a special k-division $(G_1, G_2, \emptyset)$. We will not make distinction between these two in our discussions. If G is a minor of H, then we say that a k-division (G_1, G_2, M) of G *extends* to a k-separation (H_1, H_2) of H if $E(G_i) \subseteq E(H_i)$. This is equivalent to saying that $(G_1 \cup M_1, G_2 \cup M_2)$ extends to (H_1, H_2) for at least one partition (M_1, M_2) of M.

Let v be a vertex of G. The operation of *splitting* v results in a graph obtained from $G - v$ by adding two new adjacent vertices v', v'' and making each neighbor of v in G adjacent to exactly one of v', v'' such that not all such neighbors are adjacent to only one of v', v''. Note that this definition does allow v' or v'' to have degree two. A *rooted* graph (G, R) is a graph G together with a specified set R of vertices that we call *roots*. Let (G_1, G_2, M) be a k-division of G and let $V_i = V(G_i)$, $V_i' = V_i \cap V(M)$, and $X = V_1 \cap V_2$. For each $i \in \{1, 2\}$, let $\mathcal{G}_i$ consist of all rooted graphs of the following two types:

(i) (G_i, R), where $R = X \cup V_i' \cup \{v\}$ with $v \in V_i - (X \cup V_i')$;

(ii) (G_i', R), where G_i' is obtained from G_i by splitting a vertex $v \in X \cup V_i'$ and R consists of vertices in $X \cup V_i' - \{v\}$ and the two new vertices.

We point out that $|R| = k + 1$ in both cases. To avoid potential confusion in the following discussion, we assume that members of $\mathcal{G}_i$ are isomorphic copies of the above-mentioned rooted graphs. Therefore, we can say that graphs in $\mathcal{G}_1$ and $\mathcal{G}_2$ are vertex-disjoint. To make a connection with the original graphs, we assume that each root vertex x has a label $\ell(x)$ such that $\ell(x)$ is the vertex in G that corresponds to x. In case the root vertex x corresponds to a vertex obtained by splitting v then $\ell(x) = v$ (instead of v' or v'').

Example 1. Let G be the 1-sum of $K_{3,3}$ and K_5, and let (G_1, G_2) be the corresponding 1-separation. Rooted graphs in $\mathcal{G}_1$ and $\mathcal{G}_2$ are illustrated below (when two rooted graphs are isomorphic only one is shown), where square vertices are the roots and the labels are not shown.

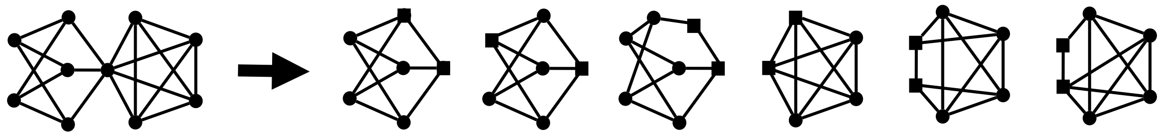


Figure 2.1: From (G_1, G_2) to rooted graphs in $\mathcal{G}_1$ and $\mathcal{G}_2$

Example 2. In the last example M is empty. Figure 2.2 below shows a 3-division of an Archdeacon graph with $M \neq \emptyset$. The only two non-isomorphic rooted graphs in $\mathcal{G}_i$ $(i = 1, 2)$ are also included.

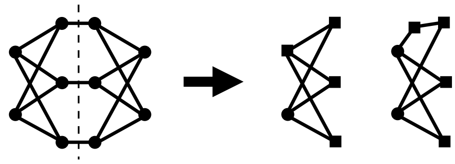


Figure 2.2: A 3-division and rooted graphs $K^1_{2,3}$, $K^2_{2,3}$

Let $\mathcal{G}$ be the set of all graphs constructed as follows: Let $(J_1, R_1) \in \mathcal{G}_1$ and $(J_2, R_2) \in \mathcal{G}_2$. Let L be a perfect matching between R_1 and R_2 and let J be the union of J_1, J_2, and L. Note that L does not necessary match vertices with the same labels under ℓ. Let L_0 be the set of edges x_1x_2 in L such that $\ell(x_1) = \ell(x_2)$. Note that this condition implies $\ell(x_1) \in X$. Then J/L_0 is a graph in $\mathcal{G}$. In case L_0 has two edges x_1x_2, y_1y_2 such that $x_1, y_1 \in R_1$, $x_2, y_2 \in R_2$, and $\ell(x_1) = \ell(x_2) = \ell(y_1) = \ell(y_2)$, then x_1 and y_1 are obtained from splitting a vertex v, and x_2, y_2 are obtained from splitting the same vertex v. In this special case, we put $J/L_0 \backslash e_1$ (instead of J/L_0) in $\mathcal{G}$ since contracting L_0 would make the two edges $e_1 = x_1y_1$, $e_2 = x_2y_2$ in parallel. Members of $\mathcal{G}$ are called *twists* of the k-division (G_1, G_2, M).

Theorem 2.1. *If G is a minor of H and (G_1, G_2, M) is a k-division of G that does not extend to a k-separation of H, then H has a twist of (G_1, G_2, M) as a minor.*

This is the result that we are going to use repeatedly to fix the connectivity of a minor. We first prove it and then show how to use it. Before we start we make a few remarks. Suppose G' is a twist of a k-division (G_1, G_2, M) of G. Then G' contains both G_1 and G_2 as minors. Moreover, G' has no k-separations that separate the two minors, which means that the given division is fixed. Furthermore, G' is only slightly bigger than G since G' may have at most $k + 2 - |M|$ extra edges. In general, however, G is no longer a minor of G'. This is the price we must pay for fixing a division with a small number of extra edges. In our applications, twists may destroy the non-projective-planar minor we started with. Fortunately, we can choose our divisions so that non-projective-planarity is maintained. This nice property makes the twist operation a very powerful tool in our proof. Note that in general a twist of a k-division of a non-projective-planar graph need not be non-projective-planar. Finally, we should clarify that although a twist can fix any given division, it may at the same time create new unwanted divisions. This could be a problem in certain applications, but it does not cause any trouble in this paper.

We will need two lemmas for proving Theorem 2.1. Let G be a graph and let A, B be subsets of $V(G)$. A path P of G is called an *A-B path* if all ends of P are in $A \cup B$ and $|V(P) \cap A| =$

$|V(P) \cap B| = 1$. A set $\mathcal{Q}$ of vertex-disjoint A-B paths *exceeds* another set $\mathcal{P}$ of vertex-disjoint A-B paths if $|\mathcal{Q}| = |\mathcal{P}| + 1$ and the set of ends of paths in $\mathcal{Q}$ is a superset of the set of ends of paths in $\mathcal{P}$. The following well-known result can be found in [3, p.63].

Lemma 2.2. *Let G be a graph, A, B be subsets of $V(G)$ with $\min\{|A|, |B|\} > k$, and $\mathcal{P}$ be a set of k vertex-disjoint A-B-paths of G. Then G has either a set of vertex-disjoint A-B-paths exceeding $\mathcal{P}$ or a k-separation (G_1, G_2) with $A \subseteq V(G_1)$ and $B \subseteq V(G_2)$.*

Let G be a graph and let A, B be subsets of $V(G)$. A subgraph G' of G is called A-B *mixed* if $V(G') \cap A \neq \emptyset \neq V(G') \cap B$. If this condition is not satisfied, then G' is called A-B *monotone*. We emphasize that a tree or a subtree must have at least one vertex. This assumption will be used implicitly several times in this section.

Lemma 2.3. *Let T be a tree and let $A, B \subseteq V(T)$. Then either there exists a vertex t such that all components of $T - t$ are A-B monotone or there is an edge e such that both components of $T \backslash e$ are A-B mixed.*

Proof. Let us assume that, for every edge e, at least one component of $T \backslash e$ is A-B monotone, for otherwise we are done. We prove the existence of vertex t for which every component of $T - v$ is A-B monotone. For any edge $e = t_1 t_2$ of T, let T_1, T_2 be the two components of $T \backslash e$ with $V(T_i) \ni t_i$. We may assume that exactly one of T_1, T_2 is A-B monotone because otherwise both t_1, t_2 could be our t. Let us direct edge e from t_i to t_j if T_i is A-B monotone. Since T is a tree, the resulting directed graph is acyclic, which implies the existence of a vertex t such that every edge incident with it is directed to it. Clearly, t is the vertex we are looking for. $\square$

Let G be a graph and let $\emptyset \neq X \subseteq V(G)$. We denote by $G[X]$ the subgraph of G induced by X.

Proof of Theorem 2.1. Since G is obtained from H by deleting vertices, deleting edges, and contracting edges, we may assume that there exist vertex-disjoint subtrees T_v $(v \in V(G))$ of H such that, if $e \in E(G)$ is incident with $u, v \in V(G)$, then, as an edge of H, e is between T_u and T_v. For each $i \in \{1, 2\}$, let $G_i = (V_i, E_i)$. Let $X = V_1 \cap V_2 = \{x_1, x_2, \ldots, x_{k_0}\}$. Let A_i be the set of vertices of T_{x_i} that are incident with edges of G_1 and let B_i be the set of vertices of T_{x_i} that are incident with edges of G_2. Suppose there is an edge e in some T_{x_i} so that both components of $T_{x_i} \backslash e$ are A_i-B_i mixed. Then contract all edges of each T_v except e, delete all other edges not in G except e, and delete remaining vertices not in G (other than the ends of e) to get a minor G'. Note that G' can be obtained from G by splitting vertex x_i. Moreover, G' is also the twist obtained by splitting x_i in both G_1 and G_2, which give rise to rooted graphs G'_1, G'_2 of type (ii), and then by identifying roots of G'_1 to roots of G'_2 with the same label and by adding the edges of M.

Thus by Lemma 2.3, we may assume there is a vertex u_i in T_{x_i} so that all components of $T_{x_i} - u_i$ are A_i-B_i monotone for each $i \in \{1, 2, \ldots, k_0\}$. It follows that T_{x_i} has two edge-disjoint subtrees T_{A_i} and T_{B_i} that contain the entire A_i and B_i, respectively. In case A_i or B_i is empty, it is clear that T_{A_i} or T_{B_i}, respectively, can be any single vertex subtree of T_{x_i}. Let us choose these two subtrees such that they are minimal and let P_i be the unique minimal path between these two subtrees in T_{x_i}. Now let $Y = V_1 \cap V(M) = \{y_{k_0+1}, y_{k_0+2}, \ldots, y_k\}$ and $Z = V_2 \cap V(M) = \{z_{k_0+1}, z_{k_0+2}, \ldots, z_k\}$. For each $i \in \{k_0 + 1, k_0 + 2, ..., k\}$, let A_i be the set of vertices in T_{y_i} incident with edges of G_1 and B_i be the set of vertices in T_{z_i} incident with edges of G_2. Then T_{y_i} and T_{z_i} have minimal subtrees

T_{A_i} and T_{B_i} containing the entire A_i and B_i, respectively. Again, if A_i or B_i is empty, T_{A_i} or T_{B_i} is a single vertex subtree of T_{y_i} or T_{z_i}. Let P_i be the unique minimal path between these two subtrees in $T_{y_i} \cup T_{z_i} + e_i$, where e_i is the edge in H corresponding to the matching edge $y_i z_i$. For each $i \in \{1, 2, \ldots, k\}$, let the ends of the path P_i be u_{i1} in T_{A_i} and u_{i2} in T_{B_i}.

Let $\mathcal{P}$ be the set of all P_i $(1 \leq i \leq k)$. Let $A = \left(\bigcup_{i=1}^{k} V(T_{A_i})\right) \cup \left(\bigcup_{v \in V_1 - (X \cup Y)} V(T_v)\right)$ and let $B = \left(\bigcup_{i=1}^{k} V(T_{B_i})\right) \cup \left(\bigcup_{v \in V_2 - (X \cup Z)} V(T_v)\right)$. Then $A, B \subseteq V(H)$ and $\mathcal{P}$ is a set of k vertex-disjoint A-B paths of H. By the definition of k-division, $V_1 - (X \cup Y) \neq \emptyset \neq V_2 - (X \cup Z)$, which implies $\min\{|A|, |B|\} > k$. Hence, by Lemma 2.2, H has either a set of vertex-disjoint A-B paths exceeding $\mathcal{P}$ or a k-separation (H_1, H_2) with $A \subseteq V(H_1)$ and $B \subseteq V(H_2)$. Note that the second alternative does not happen because otherwise $E_1 \subseteq E(H[A]) \subseteq E(H_1)$ and $E_2 \subseteq E(H[B]) \subseteq E(H_2)$, and (G_1, G_2, M) extends to (H_1, H_2).

Now we may assume that H has a set of vertex-disjoint A-B paths $\mathcal{P}' = \{P'_1, P'_2, \ldots, P'_{k+1}\}$ exceeding $\mathcal{P}$. Let $u_a \in A$ and $u_b \in B$ be the two ends of paths of $\mathcal{P}'$ that are not ends of any path of $\mathcal{P}$. We prove that H has a minor that is a twist of (G_1, G_2, M). To do so, we prove that $H[A]$ and $H[B]$ can be reduced to rooted graphs in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, and paths in $\mathcal{P}'$ provide a matching L between the two rooted graphs.

Since A and B are symmetric, it is enough for us to consider $H[A]$. Let us contract each T_v $(v \in V_1 - (X \cup Y))$ and T_{A_i}, except for T_{A_i} that contains u_a (this T_{A_i} does not exist if u_a belongs to T_v for some $v \in V_1 - (X \cup Y)$). In the exception case, let Q be the path in T_{A_i} from u_a to u_{i1}. Clearly, Q has at least one edge e since u_a is not an end of P_i. Let us contract all edges of T_{A_i} except for e. Then by deleting edges we can reduce $H[A]$ to a rooted minor $(G'_1, R_1) \in \mathcal{G}_1$, where $R_1 = \{u_a, u_{11}, u_{21}, ..., u_{k1}\}$. This is clear if u_a belongs to T_v for some $v \in V_1 - (X \cup Y)$ since we obtain a rooted graph of type (i). If u_a belongs to some T_{A_i}, from the minimality of T_{A_i} we deduce that both components of $T_{A_i} \backslash e$ contain vertices of A_i, and so we obtain a rooted graph of type (ii).

Note that paths of $\mathcal{P}'$ are between R_1 and R_2. For each path of $\mathcal{P}'$ with at least one edge we contract it to a single edge. We also contract the last edge if the path is between roots of the same label, meaning that the path is between T_{A_i} and T_{B_i} for some $i \leq k_0$. If a path of $\mathcal{P}'$ consists of a single vertex, that is, one of the x_i, then we consider the path as a result of contracting an auxiliary edge (of the matching L) between $x_i \in R_1$ and $x_i \in R_2$. Thus we have produced a minor of H that is a twist of (G_1, G_2, M) using (G'_1, R_1) and (G'_2, R_2), which proves the theorem. $\square$

Theorem 2.1 can be applied directly to determine both the 2- and 3-connected minor-minimal non-projective-planar graphs already previously determined by Robertson, Seymour and Thomas. Let $\mathcal{A}_i$ be the i-connected members of $\mathcal{A}$. We use Archdeacon's notation for the 35 graphs in $\mathcal{A}$.

Theorem 2.4. *A 2-connected graph is projective-planar if and only if it does not contain any member of $\mathcal{A}_2$ as a minor.*

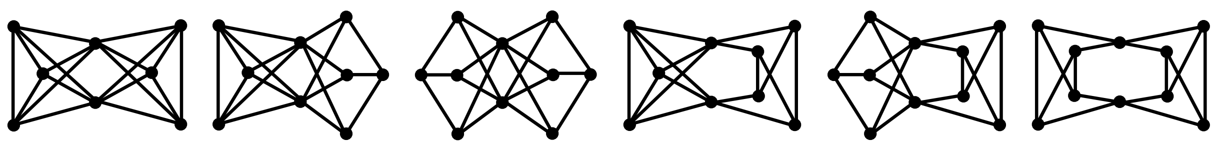


Figure 2.3: The six graphs in $\mathcal{A}$ of connectivity two: B_3, C_2, D_1, D_4, E_6, and F_6

Proof. Clearly, we only need to prove that every 2-connected non-projective-planar graph G contains a graph in $\mathcal{A}_2$ as a minor. According to our observation in the introduction we may assume that G has a minor $A \in \mathcal{A}$ that is a 1-sum of two graphs in $\{K_{3,3}, K_5\}$. By Theorem 2.1, G contains a twist J of the unique 1-separation of A as a minor. Suppose J is constructed from rooted graphs (J_1, R_1) and (J_2, R_2). Then (J_i, R_i) is one of the six graphs illustrated in Figure 2.1, which we denote by $K_{3,3}^1$, $K_{3,3}^2$, $K_{3,3}^3$, K_5^1, K_5^2, K_5^3, respectively. Note that $K_{3,3}^3$ can be contracted to $K_{3,3}^1$, K_5^3 can be contracted to K_5^1, and K_5^2 can be reduced to $K_{3,3}^1$ by deleting edges. Thus we may assume each J_i to be $K_{3,3}^1, K_{3,3}^2$, or K_5^2, which implies that there are six choices for the pair J_1, J_2. Let L be the matching that is used to construct J from J_1, J_2. Then contracting L (instead of $L_0 \subseteq L$) results in a minor J' of J and thus of G. Clearly, for the six choices of J_1, J_2, minor J' corresponds exactly to the six graphs in $\mathcal{A}$ of connectivity two, which are illustrated in Figure 2.3. $\square$

This theorem is easy to prove because of two main reasons. First, both parts of the 1-separation are highly symmetric, which reduces the number of cases. The better connected our graphs get, the less symmetric they are. Second, the entire matching L can be contracted in a twist, which also reduces the number of cases significantly. This is no longer true for higher connectivity.

Theorem 2.5. *A 3-connected graph is projective-planar if and only if it does not contain any member of $\mathcal{A}_3$ as minor.*

Proof. We need only prove that every 3-connected non-projective-planar graph contains a graph in $\mathcal{A}_3$ as a minor. By Theorem 2.4, we may assume that G has a graph $A \in \mathcal{A}_2$ as a minor, where A is one of the six graphs in $\mathcal{A}_2$ of connectivity two, which are listed in Figure 2.3. Notice that each of these graphs is a 2-sum of two graphs among $\{K_{3,3}, K_5\}$. By Theorem 2.1, G contains a twist J of the 2-separation of A as a minor where J is constructed from rooted graphs (J_1, R_2) and (J_2, R_2) that are among the graphs shown in Figure 2.4, which we call $K_{3,3}^{N1}$, $K_{3,3}^{N2}$, $K_{3,3}^{N3}$, $K_{3,3}^{E1}$, $K_{3,3}^{E2}$, K_5^1, and K_5^2, respectively. Let L be the matching used to construct J from J_1 and J_2. We prove that J contains a graph in Figure 3.1 as minor.

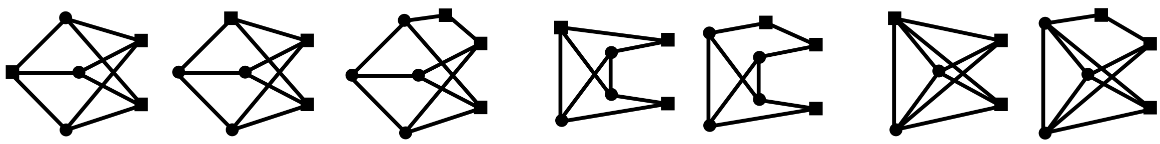


Figure 2.4: Seven possibilities for (J_i, R_i): $K_{3,3}^{N1}$, $K_{3,3}^{N2}$, $K_{3,3}^{N3}$, $K_{3,3}^{E1}$, $K_{3,3}^{E2}$, K_5^1, and K_5^2

First assume (J_1, R_1) is one of $K_{3,3}^{N1}$, $K_{3,3}^{N2}$, and $K_{3,3}^{N3}$, and contract the entire matching L to obtain J'. Since $K_{3,3}^{N3}$ can be contracted to $K_{3,3}^{N2}$, $K_{3,3}^{E2}$ can be contracted to $K_{3,3}^{E1}$, and K_5^2 can be contracted to K_5^1, we assume that (J_1, R_1) is $K_{3,3}^{N1}$ or $K_{3,3}^{N2}$ and (J_2, R_2) is one of $K_{3,3}^{N1}$, $K_{3,3}^{N2}$, $K_{3,3}^{E1}$, and K_5^1. Notice that $K_{2,3}$ rooted at the three mutually non-adjacent vertices can be obtained from $K_{3,3}^{N2}$, $K_{3,3}^{E1}$, and K_5^1 by contracting and deleting edges. Thus if (J_1, R_1) or (J_2, R_2) is $K_{3,3}^{N1}$, then J' contains $K_{3,5} = E_3 \in \mathcal{A}_3$ as a minor. Now we may assume that (J_1, R_1) is $K_{3,3}^{N2}$ and (J_2, R_2) is $K_{3,3}^{N2}$, $K_{3,3}^{E1}$, or K_5^1. If (J_2, R_2) is $K_{3,3}^{N2}$, delete an edge from it to obtain $K_{3,3}^{E1}$; if (J_2, R_2) is $K_{3,3}^{E1}$, J'

has (after deleting the edge with both ends in R_2) either $E_5 \in \mathcal{A}_3$ or $F_1 \in \mathcal{A}_3$ as a subgraph; and if (J_2, R_2) is K_5^1, J' has $D_3 \in \mathcal{A}_3$ as a subgraph.

Now (J_i, R_i) must be among $K_{3,3}^{E1}$, $K_{3,3}^{E2}$, K_5^1, and K_5^2 for each $i \in \{1, 2\}$. Suppose (J_1, R_1) is $K_{3,3}^{E2}$ or K_5^2. We contract the entire matching L to obtain J'. If (J_2, R_2) is $K_{3,3}^{E2}$ or K_5^2, contract it to $K_{3,3}^{E1}$ or K_5^1, respectively. In case (J_1, R_1) is $K_{3,3}^{E2}$, if (J_2, R_2) is $K_{3,3}^{E1}$, J' has F_1 as a minor, and if (J_2, R_2) is K_5^1, J' has D_3 as a minor. So (J_1, R_1) is K_5^2. If (J_2, R_2) is K_5^1, J' has $C_7 \in \mathcal{A}_3$ as a subgraph (by deleting edges with both ends in R_2). So (J_2, R_2) is $K_{3,3}^{E1}$. If the degree-two root of R_1 is contracted to the degree-three root of R_2, then J' has F_1 as a minor. Else, J' has D_3 as a minor (by contracting K_5^2 to K_5^1).

So (J_i, R_i) is either $K_{3,3}^{E1}$ or K_5^1 for each $i \in \{1, 2\}$. In this case, we may no longer contract the entire matching L since this may result in a projective-planar graph. Let $\{v_1, v_2\}$ be the 2-cut of A and let x, y be the third vertex of R_1, R_2, respectively. Suppose both (J_1, R_1) and (J_2, R_2) are K_5^1. If $xy \notin L$, then J/L is isomorphic to B_1 (after deleting a parallel edge); if $xy \in L$, then contracting the other two edges of L leads to a C_7 minor. Thus we assume that (J_2, R_2) is $K_{3,3}^{E1}$. By contracting the two edges of L that are not incident with x, and reducing (J_2, R_2) to $K_{2,3}$ rooted at the three mutually non-adjacent vertices, it is clear that either D_3 or F_1 is a minor. □

It may be of use to notice that in the previous theorem we actually show that a 3-connected graph with a minor in $\mathcal{A}_2 - \mathcal{A}_3$ must have a minor in $\{B_1, C_7, D_3, E_3, E_5, F_1\} \subseteq \mathcal{A}_3$. We also point out that none of these six graphs is internally 4-connected.

3 Twists of graphs in $\mathcal{A}_3$

In this section we apply Theorem 2.1 to the twelve graphs in $\mathcal{A}_3$ that are not internally 4-connected. These twelve are B_1, C_7, D_3, D_9, D_{12}, E_3, E_5, E_{11}, E_{19}, E_{27}, F_1, and G_1 shown in Figure 3.1.

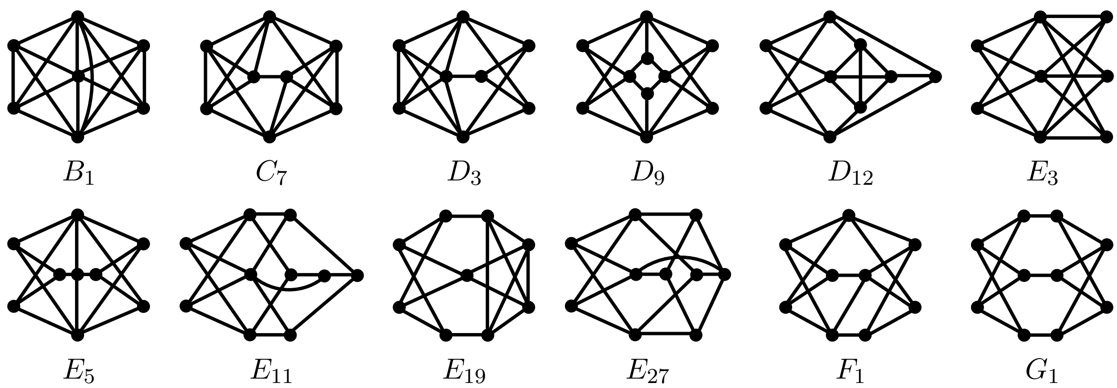


Figure 3.1: Graphs in $\mathcal{A}_3$ that are not internally 4-connected

From the proof of Theorem 2.4 and Theorem 2.5 we have seen how the twist operation works. Proof in this section will go through exactly the same process. However, the amount of case checking increases significantly. For each of the twelve graphs, there are hundreds of possible twists, which makes a proof by hand very tedious. Therefore, we choose to use a computer to perform the routine work. Our proof is verified by two independent computer programs to decrease the chance of programming errors. We use the computer program in two ways. First, to generate a list of all possible twists of a given 3-division. Second, to verify that each twist has a desired minor. In the following proof, we will only present a summary of the computation. The edge lists of the intermediate graphs are available as online material, which could help the reader to verify the details.

The following twelve lemmas deal with the twelve graphs in Figure 3.1, and the lemmas are listed according to the order that the twelve graphs are listed. Throughout this section we will indicate a 3-division (G_1, G_2, M) as a figure with a dashed line through the vertices of $V(G_1) \cap V(G_2)$ and edges of M, where edges of G_1 are left of the dashed line, and edges of G_2 are right of the dashed line. Note that some output graphs in these lemmas are not internally 4-connected, which means that there are dependencies among the non-internally 4-connected members of $\mathcal{A}_3$. We will handle these dependencies in Section 4.

Lemma 3.1. *Any internally 4-connected graph with B_1 as a minor has a minor among: B_1', B_1'', B_1''', and D_3.*

Proof. Consider the 3-separation of B_1 shown in Figure 3.2. There are 146 twists of this separation, and 11 of these have none of the other 146 as a minor. Among these 11, one is B_1^a, the second graph shown in Figure 3.2, and each of the other graphs has B_1', B_1'', B_1''', or D_3 as a minor. The 3-separation of B_1^a shown has 329 twists, and 21 of these have none of the other 329 as a minor. Each of those 21 graphs has B_1', B_1'', B_1''', or D_3 as a minor. □

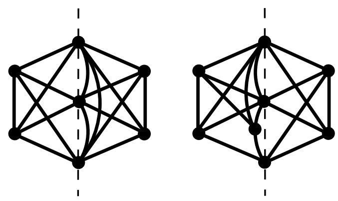


Figure 3.2: A 3-separation of B_1 and B_1^a

Lemma 3.2. *Any internally 4-connected graph with C_7 as a minor has a minor among: D_3, D_{12}, D_{17}, and F_1.*

Proof. There are 206 twists of the 3-division of C_7 shown in Figure 3.3, and 14 of these have none of the other 206 as a minor. Each of those 14 graphs has D_3, D_{12}, D_{17}, or F_1 as a minor. □

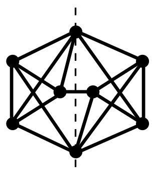


Figure 3.3: A 3-division of C_7

Lemma 3.3. *Any internally 4-connected graph with D_3 as a minor has a minor among: D_3', D_3'', E_{20}, and F_1.*

Proof. D_3 has a natural 3-division in which M consists of the center horizontal edge. If we start with this 3-division, we will have to perform the twist operation at least five times. However, the following alternative allows us to complete the proof by performing the twist operation only four times. There are 116 twists of the 3-separation of D_3 shown in Figure 3.4. Only 10 of these have none of the other 116 as a minor. Among these 10, two are D_3^a and D_3^b, and each of the other has D_3', D_3'', E_{20}, or F_1 as a minor. There are 409 twists of the 3-separation of D_3^a shown in the figure. Only 25 of these have none of the other 409 as a minor. Among these 25, one is D_3^{aa} and each of the other has D_3', D_3'', or F_1 as a minor. There are 480 twists of the 3-separation of D_3^{aa} shown in the figure. 79 of these have none of the other 480 as a minor. Each of these 79 has D_3', D_3'', or F_1 as a minor. There are 269 twists of the 3-separation of D_3^b shown in the figure. Only 13 of these have none of the other 269 as a minor. Each of these 13 has D_3', D_3'', or F_1 as a minor. □

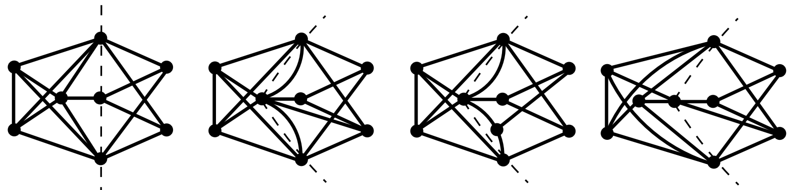


Figure 3.4: A 3-separation of D_3, D_3^a, D_3^b, and D_3^{aa}

Lemma 3.4. *Any internally 4-connected graph with D_9 as a minor has a minor among: E_{11}, E_{22}, and E_{27}.*

Proof. D_9 has two equivalent 3-separations. There are 232 graphs that are twists of either of those separations, and only 16 of these have none of the other 232 as a minor. Each of those 16 graphs has E_{11}, E_{22}, or E_{27} as a minor. □

Lemma 3.5. *Any internally 4-connected graph with D_{12} as a minor has a minor among: D_{17}, E_{20}, E_{22}, and F_1'.*

Proof. D_{12} has only one 3-separation. There are 226 graphs that are twists of that separation, and only 14 of these have none of the other 226 as a minor. Each of those 14 graphs has D_{17}, E_{20}, E_{22}, or F_1' as a minor. □

Lemma 3.6. *Any internally 4-connected graph with E_3 as a minor has a minor among: D_3', D_3'', E_3', E_3'', E_5, E_{18}, and F_1.*

Proof. There are 43 twists of the 3-separation of E_3 shown in Figure 3.5. Only 4 of these have none of the other 43 as a minor. Two of these 4 are E_3^a and E_3^b, and the other two have E_5 or F_1 as a minor. There are 45 twists of the 3-separation of E_3^a shown. Only 4 of these have none of the other 45 as a minor. One of these 4 is E_3^{aa} and the other three have E_3^b, E_5, or F_1 as a minor. There are 90 twists of the 3-separation of E_3^{aa} shown. Only 8 of these have none of the other 90 as a minor. Each of these 8 has D_3', E_3', E_{18}, or F_1 as a minor. There are 57 twists of the 3-division of E_3^b shown. Only 4 of these have none of the other 57 as a minor. Two of these 4 are E_3^{ba} and E_3^{bb}, and the other two have E_5 or F_1 as a minor. There are 303 twists of the 3-separation of E_3^{ba} shown. Only 17 of these have none of the other 303 as a minor. Each of these 17 has D_3', D_3'', E_3'',

E_5, E_{18}, or F_1 as a minor. There are 251 twists of the 3-separation of E_3^{bb} shown. Only 12 of these have none of the other 251 as a minor. Each of these 12 has D_3'', E_5, E_{18}, or F_1 as a minor. □

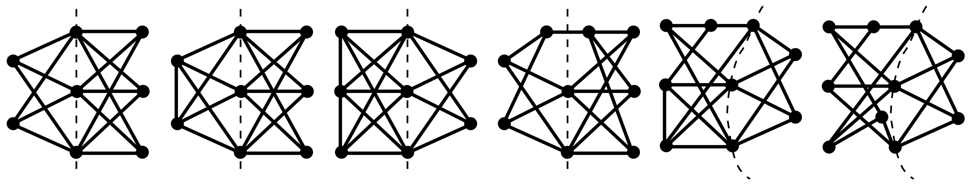


Figure 3.5: A 3-division of E_3, E_3^a, E_3^{aa}, E_3^b, E_3^{ba}, and E_3^{bb}

Lemma 3.7. *Any internally 4-connected graph with E_5 as a minor has a minor among: D_3, E_3'', E_5', E_5'', E_{18}, and F_1.*

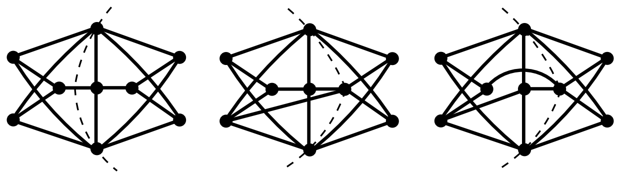


Figure 3.6: A 3-division of E_5, E_5^a, and E_5^b

Proof. There are 143 twists of the 3-division of E_5 shown in Figure 3.6. Only 10 of these have none of the other 143 as a minor. Among these 10, two are E_5^a and E_5^b and each of the others has E_5', E_5'', or F_1 as a minor. There are 198 twists of the 3-separation of E_5^a shown in the figure. Only 14 of these have none of the other 198 as a minor. Each of these 14 has D_3, E_5', E_{18}, or F_1 as a minor. Note that E_5^b is isomorphic to E_3^{ba} shown in Figure 3.5. We saw in Lemma 3.6 that the twists of the 3-separation shown each have D_3, E_3'', E_5', E_{18}, or F_1 as a minor. □

Lemma 3.8. *Any internally 4-connected graph with E_{11} as a minor has a minor among: E_{20}, E_{22}, F_1', and F_4.*

Proof. E_{11} has only one 3-separation. There are 265 twists of that separation, and only 16 of these have none of the other 265 as a minor. Each of those 16 has E_{20}, E_{22}, F_1', or F_4 as a minor. □

Lemma 3.9. *Any internally 4-connected graph with E_{19} as a minor has a minor among: E_{20}, E_{27}, and F_1.*

Proof. There are 55 twists of the 3-division of E_{19} shown in Figure 3.7, and 7 of these have none of the other 55 as a minor. Each of those 7 graphs has E_{20}, E_{27}, or F_1 as a minor. □

Lemma 3.10. *Any internally 4-connected graph with E_{27} as a minor has a minor among: E_{20}, E_{22}, F_1', and F_4.*

Proof. E_{27} has only one 3-separation. There are 216 twists of that separation, and only 15 of these have none of the other 216 as a minor. Each of those 15 has E_{20}, E_{22}, F_1', or F_4 as a minor. □

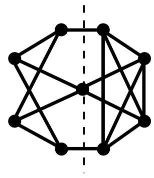


Figure 3.7: A 3-division of E_{19}

Lemma 3.11. *Any internally 4-connected graph with F_1 as a minor has a minor among: E_{27}, F_1', F_1'', F_4, and G_1.*

Proof. There are 127 twists of the 3-division of F_1 shown in Figure 3.8, and 8 of these have none of the other 127 as a minor. Four of these 8 are F_1^a, F_1^b, F_1^c, and F_1^d, and the other four have E_{27}, F_1', F_1'', or F_4 as a minor. There are 163 twists of the 3-division of F_1^a shown, and 8 of these have none of the other 163 as a minor. Each of those 8 has F_1' or F_4 as a minor. There are 175 twists of the 3-separation of F_1^b shown, and 9 of these have none of the other 175 as a minor. Each of those 9 has F_1' or F_4 as a minor. There are 110 twists of the 3-division of F_1^c shown, and 8 of these have none of the other 110 as a minor. Each of those 8 has F_1', F_1'', or F_4 as a minor. There are 98 twists of the 3-division of F_1^d shown, and 11 of these have none of the other 98 as a minor. Each of those 11 has E_{27}, F_4, or G_1 as a minor. $\square$

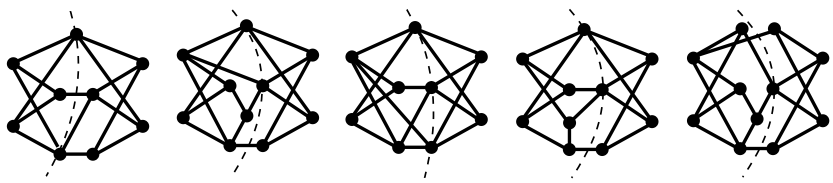


Figure 3.8: A 3-division of F_1, F_1^a, F_1^b, F_1^c, and F_1^d

Lemma 3.12. *Any internally 4-connected graph with G_1 as a minor has a minor among: F_4 and G_1'.*

Proof. There are 7 twists of the 3-division of G_1 shown in Figure 2.2, and only 2 of these have none of the other 7 as a minor. Those two are isomorphic to F_4 and G_1', respectively. $\square$

It is worth mentioning that the proof of Lemma 3.12 can also be easily completed without using a computer, which we explain here. Let J be a twist of the 3-division of G_1 shown in Figure 2.2, and let J be constructed from matching L and two rooted graphs, which are $K_{2,3}^1$ or $K_{2,3}^2$ illustrated in Figure 2.2. By contracting $K_{2,3}^2$ to $K_{2,3}^1$ we may assume that both rooted graphs are $K_{2,3}^1$. Up to symmetry, there are exactly two ways to put $K_{2,3}^1$, $K_{2,3}^1$, and L together, and the two resulting graphs are isomorphic to F_4 and G_1', respectively.

This proof raises a natural question: can proofs in this section be simplified into computer-free proofs? In the above proof, $K_{2,3}^2$ is always contracted to $K_{2,3}^1$, which simplifies the proof. The same idea was also used in the proof of Theorem 2.5, where we contracted $K_{3,3}^{E2}$ and K_5^2 to $K_{3,3}^{E1}$ and K_5^1, respectively, several times. However, we also saw in that proof that there are cases when such a contraction is not allowed. What this means is that the rooted graphs could be simplified in some cases, but they cannot be simplified in general. We also point out that, as illustrated in the proof of Theorem 2.5, matching L can be contracted in many cases, but it cannot be contracted in general. Therefore, the twist operation cannot be further simplified in general.

There is certainly a chance that a proof with fewer cases could be extracted from the current proof since certain cases could be combined together. However, a price we have to pay is to end up with a complicated proof, because we have to make fine distinctions between the cases in order to put similar cases together. In other words, we have to lose the simplicity of our current proof. On the other hand, in terms of computing time on a computer, the improvement would be negligible since both proofs will be considered short.

In proving the twelve lemmas of this section, we performed the twist operation 26 times and generated 4759 twists, among which 360 are minor-minimal. Then we verified that these minimal twists converge to 87 desired minors (some minors appeared multiple times). If we still follow the same main steps, a simplified proof would still be a list of verifications of hundreds of cases, since very likely most of the minimal twists would still be there. Such a proof might be checkable by hand, but, since it consists of mainly routine work, the proof would be boring and going through the proof would be a torture to a reader. Furthermore, checking hundreds of cases by hand is potentially less reliable than doing it with a computer. From this point of view, using a computer is not only a reasonable choice, but a better choice for our problem.

4 Proof of main results

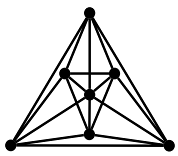
Let $\mathcal{A}_4'$ denote the set of 23 graphs in the Appendix.

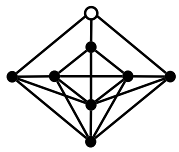
Proof of Theorem 1.1. Each graph in $\mathcal{A}_4'$ is non-projective-planar since it contains a graph in $\mathcal{A}_3$ as a minor. Now, let G be an internally 4-connected non-projective-planar graph. By Theorem 2.5, G contains a graph in $\mathcal{A}_3$ as a minor. We order the twelve members of $\mathcal{A}_3 - \mathcal{A}_4'$ as follows: B_1, C_7, E_3, E_5, D_3, D_9, D_{12}, E_{11}, E_{19}, F_1, E_{27}, G_1. Let us denote this sequence by $Z_1, Z_2, ..., Z_{12}$. Then the twelve lemmas of the last section can be expressed uniformly as: for $i = 1, 2, ..., 12$, any internally 4-connected graph with Z_i as a minor contains either some Z_j $(j > i)$ or some graph in $\mathcal{A}_4'$ as a minor. Consequently, G must contain a member of $\mathcal{A}_4'$ as a minor, which proves the theorem. $\square$

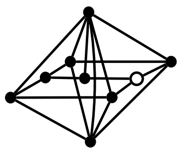
Proof of Corollary 1.2. Let G be an internally 4-connected graph. If G contains one of the eight YΔ-minors, then G is non-projective-planar since the eight graphs are non-projective-planar and the class of projective graphs is closed under YΔ-minors. Conversely, if G is non-projective-planar then by Theorem 1.1, G contains a graph in $\mathcal{A}_4'$ as a minor. Let us write $A \to B$ if B is a YΔ-transformation of A. In the Appendix, if a graph has a cubic vertex represented by an open circle, it is easy to see that performing a YΔ-transformation at that vertex results in another graph in $\mathcal{A}_4'$, which leads to the following YΔ relationships: $E_2 \to D_2 \to C_3 \to B_7 \to A_2$, $C_4 \to B_7$, $G_1' \to E_{20} \to D_{17}$, $F_4 \to E_{20}$, $E_5'' \to E_5' \to D_3'$, $D_3'' \to D_3'$, $E_3'' \to E_3' \to B_1'$, $B_1'' \to B_1'$, and $F_1'' \to F_1'$. Therefore, G has one of the eight graphs as a YΔ-minor. $\square$

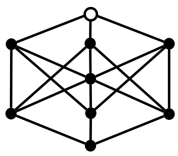
Appendix. The 23 minor-minimal internally 4-connected non-projective-planar graphs

The first eleven graphs are internally 4-connected members of $\mathcal{A}$, where we keep Archdeacon's original notation. The last twelve graphs are new, where notation Z', Z'', and Z''' indicate that these graphs contain $Z \in \mathcal{A}_3$ as a minor. We point out that, in all cases, Z is the only graph in $\mathcal{A}_3$ that is a minor of any of Z', Z'', and Z'''. Furthermore, Z', Z'', and Z''' have the same number of edges for a given Z, and thus no graph in this list contains another graph in this list as a minor. If a vertex is represented by an open circle, it means that a YΔ-transformation at that vertex results in another graph on this list.


A_2


B_7


C_3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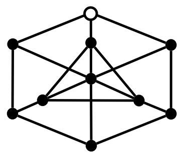
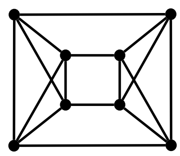
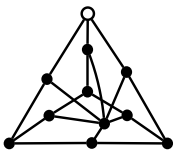
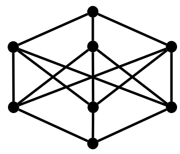
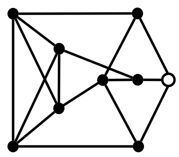

D_2


D_{17}


E_2


E_{18}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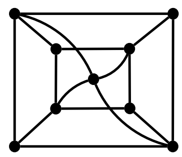
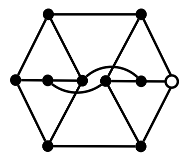
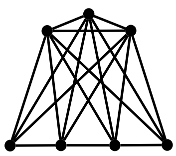
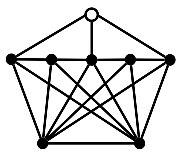

E_{22}


F_4


B_1'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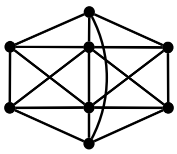
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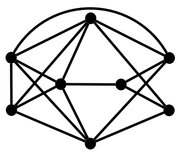
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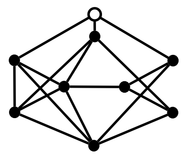
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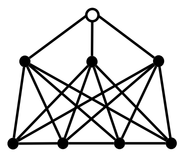
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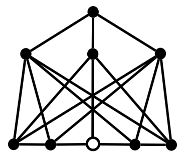
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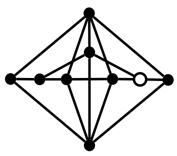
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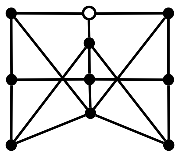
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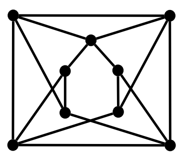
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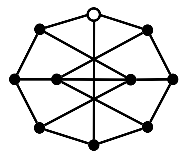
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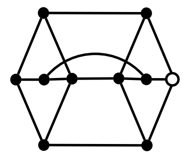
G_1'

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