When is the Matching Polytope Box-totally Dual Integral?

Guoli Ding[a*] Lei Tan[b] Wenan Zang[b†]

[a] Mathematics Department, Louisiana State University
Baton Rouge, LA 70803, USA

[b] Department of Mathematics, The University of Hong Kong
Hong Kong, China

Abstract

Let $G = (V, E)$ be a graph. The matching polytope of G, denoted by $P(G)$, is the convex hull of the incidence vectors of all matchings in G. As proved by Edmonds in 1965, $P(G)$ is determined by the following linear system $\pi(G)$:

- $x(e) \geq 0$ for each $e \in E$;
- $x(\delta(v)) \leq 1$ for each $v \in V$;
- $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ for each $U \subseteq V$ with $|U|$ odd.

In 1978, Cunningham and Marsh strengthened this theorem by showing that $\pi(G)$ is always totally dual integral. In 1984, Edmonds and Giles initiated the study of graphs G for which $\pi(G)$ is box-totally dual integral. In this paper we present a structural characterization of all such graphs, and develop a general and powerful method for establishing box-total dual integrality.

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1 Introduction

Let $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ be a rational linear system and let P denote the polyhedron $\{\boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$. We call P *integral* if it is the convex hull of all integral vectors contained in P. As shown by Edmonds and Giles [10], P is integral if and only if the maximum in the LP-duality equation

$$\max\{\boldsymbol{w}^T\boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{y}^T\boldsymbol{b} : \boldsymbol{y}^TA \geq \boldsymbol{w}^T,\ \boldsymbol{y} \geq \boldsymbol{0}\}$$

has an integral optimal solution, for every integral vector $\boldsymbol{w}$ for which the optimum is finite. If, instead, the minimum in the equation enjoys this property, then the system $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is called *totally dual integral* (TDI). Furthermore, the system is called *box-totally dual integral* (box-TDI) if $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is TDI for all rational vectors $\boldsymbol{l}$ and $\boldsymbol{u}$; in the literature there is an equivalent definition of box-TDI systems, where the coordinates of $\boldsymbol{u}$ are also allowed to be $+\infty$ (see Schrijver [17], page 318). It is well known that many combinatorial optimization problems can be naturally formulated as integer programs of the form $\max\{\boldsymbol{w}^T\boldsymbol{x} : \boldsymbol{x} \in P,\ \text{integral}\}$; if P is integral, then such a problem reduces to its LP-relaxation, thereby is solvable in polynomial time. Edmonds and Giles [10] proved that total dual integrality implies primal integrality: if $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is TDI and $\boldsymbol{b}$ is integer-valued, then P is integral. So the model of TDI systems plays a crucial role in combinatorial optimization; in particular, it serves as a general framework for establishing various min-max theorems. The importance of box-TDI systems can be seen from the fact that box constraints arise frequently in practice and that box-total dual integrality strengthens total dual integrality. Therefore, these three integrality properties have been subjects of extensive research; they are also the major concern of polyhedral combinatorics (see Schrijver [17, 18] for comprehensive and in-depth accounts). Since it is NP-hard in general to recognize linear systems with such integrality properties [14, 7], we restrict our attention to Edmonds' system for defining the matching polytope in this paper.

Let $G = (V, E)$ be a graph. The *matching polytope* of G, denoted by $P(G)$, is the convex hull of the incidence vectors of all matchings in G. For each $v \in V$, we use $\delta(v)$ to denote the set of all edges incident with v in G, and use $d(v)$ (or $d_G(v)$ under some circumstances) to denote the degree of v. For each $U \subseteq V$, we use $G[U]$ to denote the subgraph of G induced by U, and use $E[U]$ to denote the edge set of $G[U]$. Consider the linear system $\pi(G)$ consisting of the following inequalities:

(i) $x(e) \geq 0$ for each $e \in E$;

(ii) $x(\delta(v)) \leq 1$ for each $v \in V$;

(iii) $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ for each $U \subseteq V$ with $|U|$ odd,

where and throughout $x(F) = \sum_{e \in F} x(e)$ for any $F \subseteq E$. From a theorem of Birkhoff [1], it follows that $P(G)$ is determined by the inequalities (i) and (ii) if and only if G is bipartite. For a general graph G, Edmonds [9] showed that adding (iii) is enough to give a description of $P(G)$.

Theorem 1.1. (Edmonds [9]) *For any graph $G = (V, E)$, the matching polytope $P(G)$ is determined by $\pi(G)$.*

As remarked by Schrijver [18], the matching polytope forms the first class of polytopes whose characterization does not simply follow just from total unimodularity, and its description was a breakthrough in polyhedral combinatorics.

Pulleyblank and Edmonds [15] characterized which of the inequalities in $\pi(G)$ give a facet of the matching polytope. Define

- $I(G) = \{v \in V : d(v) \geq 3$, or $d(v) = 2$ and v is contained in no triangle, or $d(v) = 1$ and the neighbor of v also has degree $1\}$,
- $\mathcal{T}(G) = \{U \subseteq V : |U| \geq 3$, $G[U]$ is factor-critical and 2-connected$\}$.

Recall that a graph H is *factor-critical* if $H \backslash v$ has a perfect matching for each vertex v of H (see Lovász and Plummer [13]).

Theorem 1.2. (Pulleyblank & Edmonds [15]) *For any graph $G = (V, E)$, each inequality in $\pi(G)$ is a nonnegative integer combination of the following inequalities:*

(i) $x(e) \geq 0$ *for each $e \in E$;*

(ii) $x(\delta(v)) \leq 1$ *for each $v \in I(G)$;*

(iii) $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ *for each $U \subseteq \mathcal{T}(G)$.*

So they also determine the matching polytope $P(G)$.

Let $\sigma(G)$ be the system consisting of all the inequalities exhibited in Theorem 1.2. We call $\sigma(G)$ the *restricted Edmonds system* for defining $P(G)$.

Cunningham and Marsh [6] strengthened Edmonds' matching polytope theorem (that is, Theorem 1.1) by showing that $\pi(G)$ is actually TDI, which yields a min-max relation for the maximum weight of a matching in G (see Theorem 25.2 in Schrijver [18]).

Theorem 1.3. (Cunningham & Marsh [6]) *For any graph $G = (V, E)$, the Edmonds system $\pi(G)$ is TDI.*

Motivated by Theorems 1.1 and 1.3, Edmonds and Giles [11] initiated the study of graphs G for which $\pi(G)$ is box-TDI, and discovered the following counterexmaple. The purpose of this paper is to give a structural characterization of all such graphs.

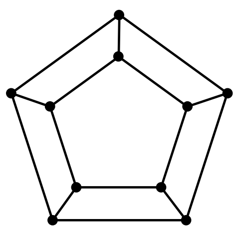
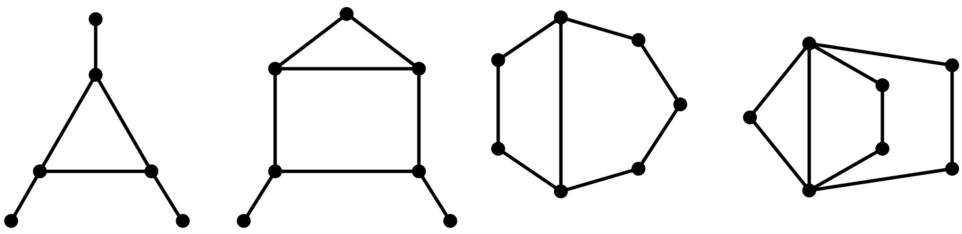


Figure 1: A graph G for which $\pi(G)$ is not box-TDI

We define one term before presenting the main theorem. A graph K is called *a fully odd subdivision* of a graph H if K is obtained from H by subdividing each edge of H into a path of odd length (possibly the length is one).

Theorem 1.4. *Let $G = (V, E)$ be a graph. Then the Edmonds system $\pi(G)$ is box-TDI if and only if G contains no fully odd subdivision of $F_1, F_2, F_3,$ or F_4 (see Figure 2) as a subgraph.*



F_1 F_2 F_3 F_4

Figure 2: Forbidden subgraphs

A polyhedron is called *box-TDI* if it can be defined by a box-TDI system. Cook [5] observed that box-total dual integrality essentially is a property of polyhedra (see Theorem 22.8 in Schrijver [17]).

Theorem 1.5. (Cook [5]) *Let Q be a box-TDI polyhedron and let $C\boldsymbol{x} \le \boldsymbol{d}$ be an arbitrary TDI system such that $Q = \{\boldsymbol{x} : C\boldsymbol{x} \le \boldsymbol{d}\}$. Then $C\boldsymbol{x} \le \boldsymbol{d}$ is box-TDI.*

So Theorem 1.4 actually tells us when the matching polytope is box-TDI. To establish the "if" part of this theorem, we need a structural description of all graphs under consideration. Due to the strict parity restriction, fully odd subdivisions are much more difficult to manipulate than subdivisions, minors, and odd minors (see, for instance, [2, 12, 16]); this drawback makes our description rather delicate and sophisticated. The other difficulty with the proof lies in the lack of a proper tool for establishing box-total dual integrality. To the best of our knowledge, there are only two general-purpose methods presently available, which are described below.

Theorem 1.6. (Cook [5]) *A rational system $A\boldsymbol{x} \le \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$, with $\boldsymbol{x} \in \mathbb{R}^n$, is box-TDI if and only if it is TDI and for any rational vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^T$, there exists an integral vector $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)^T$ such that $\lfloor c_i \rfloor \le \tilde{c}_i \le \lceil c_i \rceil$, for all $1 \le i \le n$, and such that every optimal solution of $\max\{\boldsymbol{c}^T\boldsymbol{x} : A\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}\}$ is also an optimal solution of $\max\{\tilde{\boldsymbol{c}}^T\boldsymbol{x} : A\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}\}$.*

Nevertheless, this necessary and sufficient condition is very difficult to verify in practice. In [18], Schrijver proved the following theorem (see Theorem 5.35), which implies that a number of classical min-max theorems can be further strengthened with box-TDI properties.

Theorem 1.7. (Schrijver [18]) *Let $A\boldsymbol{x} \le \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$ be a rational system. Suppose that for any rational vector $\boldsymbol{c}$, the program $\max\{\boldsymbol{c}^T\boldsymbol{x} : A\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}\}$ has (if finite) an optimal dual solution $\boldsymbol{y}$ such that the rows of A corresponding to positive components of $\boldsymbol{y}$ form a totally unimodular submatrix of A. Then $A\boldsymbol{x} \le \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$ is box-TDI.*

Since the aforementioned Edmonds system does not necessarily meet the total unimodularity requirement, Schrijver's theorem can hardly be applied in our proof directly. In this paper we shall develop a general and powerful method for establishing box-total dual integrality; our proof of Theorem 1.4 will rely heavily on this new approach.

Let us introduce some notations and terminology before proceeding. As usual, let $\mathbb{Q}$ and $\mathbb{Z}$ denote the sets of rationals and integers, respectively, and let $\mathbb{Q}_+$ and $\mathbb{Z}_+$ denote the sets of nonnegative numbers in the corresponding sets. Set $\mathbb{Z}/k = \{x/k : x \in \mathbb{Z}\}$ for each integer $k \geq 2$. For any set Ω of numbers and any finite set K, we use Ω^K to denote the set of vectors $\boldsymbol{x} = (x(k) : k \in K)$ whose coordinates are members of Ω. For each $J \subseteq K$, the $|J|$-dimensional vector $\boldsymbol{x}|_J = (x(j) : j \in J)$ stands for the projection of $\boldsymbol{x}$ to Ω^J.

Throughout this paper, a *collection* is a synonym of a multiset in which elements may occur more than once, while elements of a *set* or a *subset* (of a collection) are all distinct. So if $X = \{x_1, x_2, \ldots, x_m\}$ is a collection, then possibly $x_i = x_j$ for some distinct i, j. The *size* $|X|$ of X is defined to be m. If $Y = \{y_1, y_2, \ldots, y_n\}$ is also a collection, then the *union* $X \cup Y$ of X and Y is the collection $\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$. Thus the size of $X \cup Y$ is $|X| + |Y|$, which is different from what happens to the union of two sets. Similarly, we can define $X \cap Y$ and $X - Y$ of these two collections.

Let $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ be a rational system, where $A = [a_{ij}]_{m \times n}$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)^T$. We call A *integral* if all a_{ij} are integers (not necessarily nonnegative). Let R be the set of indices of all rows of A, and let S be the set of indices of all columns of A. For any collection Λ of elements of R and any element s of S, set $b(\Lambda) = \sum_{r \in \Lambda} b_r$ and $d_\Lambda(s) = \sum_{r \in \Lambda} a_{rs}$. Notice that if r appears k times in Λ, then b_r is counted k times in $b(\Lambda)$, and a_{rs} is counted k times in $d_\Lambda(s)$. An *equitable subpartition* of Λ consists of two collections Λ_1 and Λ_2 of elements of R (which are not necessarily in Λ) such that

(i) $b(\Lambda_1) + b(\Lambda_2) \leq b(\Lambda)$;

(ii) $d_{\Lambda_1 \cup \Lambda_2}(s) \geq d_\Lambda(s)$ for all $s \in S$; and

(iii) $\min\{d_{\Lambda_1}(s), d_{\Lambda_2}(s)\} \geq \lfloor d_\Lambda(s)/2 \rfloor$ for all $s \in S$.

We call the system $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ *equitably subpartitionable*, abbreviated ESP, if every collection Λ of elements of R admits an equitable subpartition. We refer to the above (i), (ii), and (iii) as *ESP property.*

Theorem 1.8. *Every ESP system $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$, with A integral, is box-TDI.*

We point out that the ESP property was first introduced by Ding and Zang [8] for linear systems of the form $A\boldsymbol{x} \geq \boldsymbol{1}$, $\boldsymbol{x} \geq \boldsymbol{0}$, where A is a 0-1 matrix and $\boldsymbol{1}$ is an all-one vector, which has proved to be very effective in dealing with various packing and covering problems (see [8, 4]). The property defined above is clearly a natural extension of the original definition in the most general setting. Although recognizing box-TDI systems is an optimization problem, as we shall see, our approach based on the ESP property is of transparent combinatorial nature and hence is fairly easy to work with. Recently we have successfully characterized several important classes of box-perfect graphs (see, for instances, [3, 5]) using this approach; one of our theorems asserts that every parity graph is box-perfect, which confirms a conjecture made by Cameron and Edmonds [3] in 1982. We strongly believe that the ESP property is exactly the tool one needs for the study of box-perfect graphs, and shall further explore its connection with other optimization problems.

The remainder of this paper is organized as follows. In Section 2, we show that the ESP property implies box-total dual integrality, thereby proving Theorem 1.8. In Section 3, we demonstrate that every fully odd subdivision of F_1, F_2, F_3, and F_4 is an obstruction to box-total dual integrality, which establishes the "only if" part of Theorem 1.4. In Section 4, we present a structural description of all internally 2-connected graphs with no fully odd subdivision of F_1, F_2, F_3, or F_4. In Section 5, we show that the restricted Edmonds system specified in Theorem 1.2 is ESP for all graphs considered in the preceding section. In Section 6, we derive the "if" part of Theorem 1.4 (thus finish the proof) based on two summing operations.

2 ESP Property

The purpose of this section is to prove Theorem 1.8, which asserts that the ESP property is sufficient for a linear system to be box-TDI. With a slight abuse of notation, we write $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ for both the linear program $\min\{\boldsymbol{\alpha}^T\boldsymbol{b} - \boldsymbol{\beta}^T\boldsymbol{l} + \boldsymbol{\gamma}^T\boldsymbol{u} : \boldsymbol{\alpha}^T A - \boldsymbol{\beta}^T + \boldsymbol{\gamma}^T \geq \boldsymbol{w}^T,\ \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \geq \boldsymbol{0}\}$ and its optimal value. When integrality is imposed on its solutions, we write $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$ for both the corresponding integer program and its optimal value. Similarly, we can define $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z}/2)$. Recall the notations introduced in the preceding section: R is the set of indices of all rows of A, and S is the set of indices of all columns of A. Suppose $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ is an optimal solution to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$. Let Λ be the collection of elements in R, such that each $r \in R$ appears precisely $\alpha^*(r)$ times in Λ; we call Λ the row-index collection of A corresponding to $\boldsymbol{\alpha}^*$.

We propose to establish the following statement, which clearly implies Theorem 1.8.

Theorem 2.1. *Let $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ be a rational system, with A integral. Suppose that for any $\boldsymbol{l}, \boldsymbol{u} \in \mathbb{Q}^S$ and $\boldsymbol{w} \in \mathbb{Z}^S$ with finite $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, there exists an optimal solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$, such that the row-index collection of A corresponding to $\boldsymbol{\alpha}^*$ admits an equitable subpartition. Then $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is box-TDI.*

The proof given below is an adaption of that of Theorem 1.2 in [4]. For completeness and ease of reference, we include all details here.

Schrijver and Seymour [17] established that a rational system $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is TDI if and only if $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z}/2) = \mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$ for any integral vector $\boldsymbol{w}$ for which $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ is finite (see Theorem 22.13 in Schrijver [17]), which amounts to saying that $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z}) = 2 \cdot \mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$. By definition, the LHS is bounded above by the RHS. So we get the following necessary and sufficient condition for total dual integrality.

Lemma 2.2. *The rational system $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is TDI if and only if*

$$\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z}) \geq 2 \cdot \mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$$

for any integral vector $\boldsymbol{w}$ for which $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ is finite.

Proof of Theorem 2.1. By Lemma 2.2, it suffices to show that for any $\boldsymbol{l} \in \mathbb{Q}^S$, $\boldsymbol{u} \in (\mathbb{Q} \cup \{+\infty\})^S$, and $\boldsymbol{w} \in \mathbb{Z}^S$ with finite $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, we have $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z}) \geq 2\cdot\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$. According to the hypothesis, there exists an optimal solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$, such that the row-index collection Λ of A corresponding to $\boldsymbol{\alpha}^*$ admits an equitable subpartition (Λ_1, Λ_2). Our objective is to find a feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$ based on both $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ and (Λ_1, Λ_2), with $\boldsymbol{\alpha}^T\boldsymbol{b} - \boldsymbol{\beta}^T\boldsymbol{l} + \boldsymbol{\gamma}^T\boldsymbol{u} \leq [(\boldsymbol{\alpha}^*)^T\boldsymbol{b} - (\boldsymbol{\beta}^*)^T\boldsymbol{l} + (\boldsymbol{\gamma}^*)^T\boldsymbol{u}]/2$

Let us make some observations about $\boldsymbol{\beta}^*$ and $\boldsymbol{\gamma}^*$. For convenience, we may assume that

(1) $\beta^*(s)\gamma^*(s) = 0$ for all $s \in S$.

Otherwise, $\beta^*(s) \neq 0 \neq \gamma^*(s)$ for some column index $s \in S$. Set $\delta = \text{Min}\{\beta^*(s), \gamma^*(s)\}$. Clearly $\delta > 0$. Let $\boldsymbol{\beta}'$ be the vector obtained from $\boldsymbol{\beta}^*$ by replacing $\beta^*(s)$ with $\beta^*(s) - \delta$, and let $\boldsymbol{\gamma}'$ be the vector obtained from $\boldsymbol{\gamma}^*$ by replacing $\gamma^*(s)$ with $\gamma^*(s) - \delta$. Observe that $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ is a feasible solution to $\text{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$, and that $(\boldsymbol{\alpha}^*)^T\boldsymbol{b} - (\boldsymbol{\beta}')^T\boldsymbol{l} + (\boldsymbol{\gamma}')^T\boldsymbol{u} = (\boldsymbol{\alpha}^*)^T\boldsymbol{b} - (\boldsymbol{\beta}^*)^T\boldsymbol{l} + (\boldsymbol{\gamma}^*)^T\boldsymbol{u} - (u(v) - l(v))\delta \leq (\boldsymbol{\alpha}^*)^T\boldsymbol{b} - (\boldsymbol{\beta}^*)^T\boldsymbol{l} + (\boldsymbol{\gamma}^*)^T\boldsymbol{u}$. So $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ is also an optimal solution to $\text{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$. Hence (1) holds because otherwise we can replace $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ with $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ and repeat this process.

(2) $\beta^*(s) = 0$ for all $s \in S$ with $l(s) < 0$.

Otherwise, $\beta^*(s) > 0$ for some $s \in S$ with $l(s) < 0$. Let $\boldsymbol{\beta}'$ be the vector obtained from $\boldsymbol{\beta}^*$ by replacing $\beta^*(s)$ with zero. Clearly, $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}^*)$ is a feasible solution to $\text{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$, whose objective value is smaller than that of $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$; this contradiction justifies (2).

The inequality contained in $(\boldsymbol{\alpha}^*)^T A - (\boldsymbol{\beta}^*)^T + (\boldsymbol{\gamma}^*)^T \geq 2\boldsymbol{w}^T$ corresponding to a column index s reads $d_\Lambda(s) - \beta^*(s) + \gamma^*(s) \geq 2w(s)$, which can be strengthened as follows.

(3) $d_\Lambda(s) - \beta^*(s) + \gamma^*(s) = 2w(s)$ for all $s \in S$ with $\beta^*(s) + \gamma^*(s) > 0$.

Assume the contrary: $d_\Lambda(s) - \beta^*(s) + \gamma^*(s) > 2w(s)$ for some $s \in S$ with $\beta^*(s) + \gamma^*(s) > 0$. Set $\delta = d_\Lambda(s) - \beta^*(s) + \gamma^*(s) - 2w(s)$. By assumption, $\delta > 0$. If $\beta^*(s) > 0$, then $\gamma^*(s) = 0$ and $l(s) \geq 0$ by (1) and (2); in this case, let $\boldsymbol{\beta}'$ be the vector obtained from $\boldsymbol{\beta}^*$ by replacing $\beta^*(s)$ with $\beta^*(s) + \delta$ and let $\boldsymbol{\gamma}' = \boldsymbol{\gamma}^*$. If $\gamma^*(s) > 0$, then $\beta^*(s) = 0$ by (1); in this case, let $\boldsymbol{\gamma}'$ be the vector obtained from $\boldsymbol{\gamma}^*$ by replacing $\gamma^*(s)$ with $\max\{0, \gamma^*(s) - \delta\}$ and let $\boldsymbol{\beta}' = \boldsymbol{\beta}^*$. Observe that $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ is a feasible solution to $\text{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$, and that $(\boldsymbol{\alpha}^*)^T\boldsymbol{b} - (\boldsymbol{\beta}^*)^T\boldsymbol{l} + (\boldsymbol{\gamma}^*)^T\boldsymbol{u} \geq (\boldsymbol{\alpha}^*)^T\boldsymbol{b} - (\boldsymbol{\beta}')^T\boldsymbol{l} + (\boldsymbol{\gamma}')^T\boldsymbol{u}$. Hence $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ is also an optimal solution to $\text{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$. Let us replace $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ with $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ and repeat this process until we get stuck. Clearly, the resulting solution satisfies (1), (2) and (3) simultaneously.

For $i = 1, 2$, define a vector $\boldsymbol{\alpha}_i \in \mathbb{Z}_+^R$, such that $\boldsymbol{\alpha}_i(r)$ is precisely the multiplicity of row index r in Λ_i for all $r \in R$. By (i) of the ESP property, $b(\Lambda_1) + b(\Lambda_2) \leq b(\Lambda)$. So

(4) $\boldsymbol{\alpha}_1^T\boldsymbol{b} + \boldsymbol{\alpha}_2^T\boldsymbol{b} \leq (\boldsymbol{\alpha}^*)^T\boldsymbol{b}$.

Consider an arbitrary column index $s \in S$. Suppose $d_{\Lambda_p}(s) \geq d_{\Lambda_q}(s)$, where $\{p, q\} = \{1, 2\}$. Since A is integral, $d_\Lambda(s)$ and $d_{\Lambda_i}(s)$ for $i = p, q$ are all integers. By (ii) and (iii) of the ESP property, we have

(5) $d_{\Lambda_p}(s) \geq \lceil d_\Lambda(s)/2 \rceil$ and $d_{\Lambda_q}(s) \geq \lfloor d_\Lambda(s)/2 \rfloor$.

Set

- $\beta_p(s) = \lceil \beta^*(s)/2 \rceil$ and $\gamma_p(s) = \lfloor \gamma^*(s)/2 \rfloor$, and
- $\beta_q(s) = \lfloor \beta^*(s)/2 \rfloor$ and $\gamma_q(s) = \lceil \gamma^*(s)/2 \rceil$.

Then

(6) $\beta_p(s) + \beta_q(s) = \beta^*(s)$ and $\gamma_p(s) + \gamma_q(s) = \gamma^*(s)$.

Let us show that

(7) $d_{\Lambda_i}(s) - \beta_i(s) + \gamma_i(s) \geq w(s)$ for $i = 1, 2$.

We distinguish between two cases according to the parity of $d_\Lambda(s)$. If $d_\Lambda(s)$ is even, then both $\beta^*(s)$ and $\gamma^*(s)$ are even by (1) and (3). Thus $d_{\Lambda_i}(s) - \beta_i(s) + \gamma_i(s) \geq (d_\Lambda(s) - \beta^*(s) + \gamma^*(s))/2 \geq w(s)$ for $i = 1, 2$ by (5). It remains to consider the case when $d_\Lambda(s)$ is odd. If $\beta^*(s) = \gamma^*(s) = 0$, then, by (5) for $i = 1, 2$, we have $d_{\Lambda_i}(s) - \beta_i(s) + \gamma_i(s) = d_{\Lambda_i}(s) \geq (d_\Lambda(s) - 1)/2 \geq (2w(s) - 1)/2 = w(s) - \frac{1}{2}$. Thus $d_{\Lambda_i}(s) - \beta_i(s) + \gamma_i(s) \geq w(s)$ for $i = 1, 2$ as the left-hand side is an integer. So we assume that $\beta^*(s)+\gamma^*(s) > 0$. It follows from (3) that $d_\Lambda(s)-\beta^*(s)+\gamma^*(s) = 2w(s)$. Since $d_\Lambda(s)$ is odd, so is $-\beta^*(s)+\gamma^*(s)$. Moreover, $\beta^*(s)\gamma^*(s) = 0$ by (1). From the definition, we see that $-\beta_p(s)+\gamma_p(s) = (-\beta^*(s)+\gamma^*(s)-1)/2$ and $-\beta_q(s)+\gamma_q(s) = (-\beta^*(s)+\gamma^*(s)+1)/2$. Combining them with (5), we conclude that $d_{\Lambda_i}(s) - \beta_i(s) + \gamma_i(s) \geq (d_\Lambda(s) - \beta^*(s) + \gamma^*(s))/2 = w(s)$ for $i = p, q$, which establishes (7).

For $i = 1, 2$, set $\boldsymbol{\beta}_i = (\beta_i(s) : s \in S)$ and $\boldsymbol{\gamma}_i = (\gamma_i(s) : s \in S)$. By (7), we have $\boldsymbol{\alpha}_i^T A - \boldsymbol{\beta}_i + \boldsymbol{\gamma}_i \geq \boldsymbol{w}^T$, and thus $(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \boldsymbol{\gamma}_i)$ is a feasible solution to $\text{Min}(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$. From (6), it follows that $\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2 = \boldsymbol{\beta}^*$ and $\boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2 = \boldsymbol{\gamma}^*$. Hence $\sum_{i=1}^2 (-\boldsymbol{\beta}_i^T \boldsymbol{l} + \boldsymbol{\gamma}_i^T \boldsymbol{u}) = -(\boldsymbol{\beta}^*)^T \boldsymbol{l} + (\boldsymbol{\gamma}^*)^T \boldsymbol{u}$. Using (4), we obtain $\boldsymbol{\alpha}_i^T \boldsymbol{b} - \boldsymbol{\beta}_i^T \boldsymbol{l} + \boldsymbol{\gamma}_i^T \boldsymbol{u} \leq [(\boldsymbol{\alpha}^*)^T \boldsymbol{b} - (\boldsymbol{\beta}^*)^T \boldsymbol{l} + (\boldsymbol{\gamma}^*)^T \boldsymbol{u}]/2$ for $i = 1$ or 2; the corresponding $(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \boldsymbol{\gamma}_i)$ is a solution to $\text{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$ as desired. ∎

3 Forbidden Structures

Let $G = (V, E)$ be a graph. By Theorems 1.2 and 1.3, the restricted Edmonds system $\sigma(G)$ is also TDI (see (41) on page 322 in Schrijver [17]). Thus the following statement follows instantly from Theorem 1.5.

Lemma 3.1. *The system $\sigma(G)$ is box-TDI if and only if $\pi(G)$ is box-TDI.*

By definition, $\sigma(G)$ is box-TDI if and only if, for any $\boldsymbol{l} \in \mathbb{Q}^E$, $\boldsymbol{u} \in (\mathbb{Q} \cup \{+\infty\})^E$, and $\boldsymbol{w} \in \mathbb{Z}^E$, the minimum in the LP-duality equation

$$
\begin{array}{rlll}
 & \text{Maximize} & \sum_{e \in E} w(e)x(e) & \\
 & \text{Subject to} & \sum_{e \in \delta(v)} x(e) \leq 1 & \text{for each } v \in I(G) \\
 & & \sum_{e \in E[U]} x(e) \leq \lfloor \frac{1}{2}|U| \rfloor & \text{for each } U \subseteq \mathcal{T}(G) \\
 & & l(e) \leq x(e) \leq u(e) & \text{for each } e \in E \\
 & & x(e) \geq 0 & \text{for each } e \in E \\
= & \text{Minimize} & \multicolumn{2}{l}{\sum_{v \in I(G)} \alpha(v) + \sum_{U \in \mathcal{T}(G)} \lfloor \frac{1}{2}|U| \rfloor \alpha(U) - \sum_{e \in E} l(e)\beta(e) + \sum_{e \in E} u(e)\gamma(e)} \\
 & \text{Subject to} & \sum_{e \in \delta(v)} \alpha(v) + \sum_{e \in E[U]} \alpha(U) - \beta(e) + \gamma(e) \geq w(e) & \text{for each } e \in E \\
 & & \alpha(u) \geq 0 & \text{for each } u \in I(G) \cup \mathcal{T}(G) \\
 & & \beta(e),\ \gamma(e) \geq 0 & \text{for each } e \in E
\end{array}
$$

has an integral optimal solution, provided the optimum is finite. These two problems are referred to as $G-$Max and $G-$Min, respectively.

In this section we aim to prove the following theorem, which establishes the "only if" part of Theorem 1.4.

Theorem 3.2. *Let $G = (V, E)$ be a graph containing a fully odd subdivision of some F_i (see Figure 2), with $1 \leq i \leq 4$, as a subgraph. Then $\pi(G)$ (equivalently $\sigma(G)$) is not box-TDI.*

We break the proof into a few lemmas.

Lemma 3.3. *The system $\sigma(F_i)$ is not box-TDI for $1 \leq i \leq 4$.*

Proof. Let $F_i = (V_i, E_i)$ and set $R_i = I(F_i) \cup \mathcal{T}(F_i)$ for $1 \leq i \leq 4$. To establish the statement, we need to find $\boldsymbol{l} \in \mathbb{Q}^{E_i}$, $\boldsymbol{u} \in \mathbb{Q}^{E_i}$, and $\boldsymbol{w} \in \mathbb{Z}^{E_i}$ such that F_i-Min has no integral

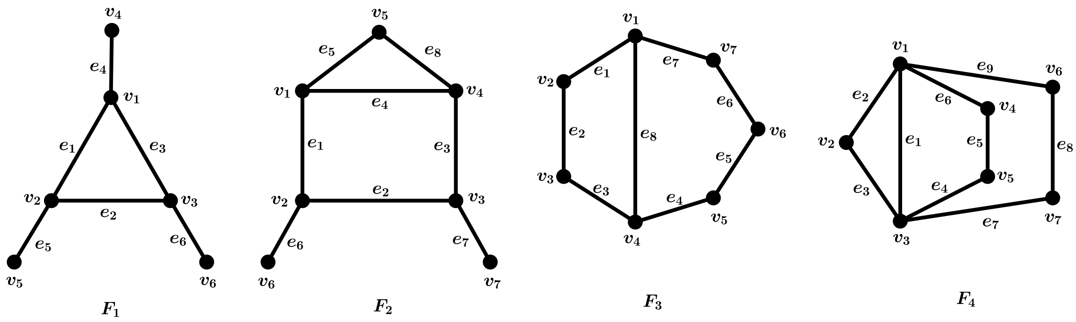


Figure 3: A labeling of forbidden subgraphs

optimal solution for each i. For this purpose, we label each F_i as depicted in Figure 3, and distinguish among four cases.

Case 1. $i = 1$. Set $w(e) = 1$, $l(e) = 0$, and $u(e) = 1/2$ for each $e \in E_1$. Define $\boldsymbol{x} \in \mathbb{Q}^{E_1}$, $\boldsymbol{\alpha} \in \mathbb{Q}^{R_1}$, $\boldsymbol{\beta} \in \mathbb{Q}^{E_1}$, and $\boldsymbol{\gamma} \in \mathbb{Q}^{E_1}$ as follows:

- $x(e) = 1/4$ if $e \in \{e_1, e_2, e_3\}$ and 1/2 otherwise;
- $\alpha(u) = 1/2$ if $u \in \{v_1, v_2, v_3\}$ and 0 otherwise;
- $\beta(e) = 0$ for each $e \in E_1$; and
- $\gamma(e) = 1/2$ if $e \in \{e_4, e_5, e_6\}$ and 0 otherwise.

It is straightforward to verify that $\boldsymbol{x}$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ are feasible solutions to F_1-Max and F_1-Min, respectively, and have the same objective value of 9/4. By the LP-duality theorem, $\boldsymbol{x}$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ are optimal solutions to F_1-Max and F_1-Min, respectively, with optimal value $z^* = 9/4$. Since $\{l(e), u(e)\} \subseteq \mathbb{Z}/2$ for all $e \in E_1$ while $z^* \notin \mathbb{Z}/2$, it follows that F_1-Min has no integral optimal solution.

Case 2. $i = 2$. Set $w(e) = 1$ if $e \in E_2 \backslash e_4$ and $w(e_4) = 2$, and set $l(e) = 0$ and $u(e) = 1/2$ for each $e \in E_2$. Define $\boldsymbol{x} \in \mathbb{Q}^{E_2}$, $\boldsymbol{\alpha} \in \mathbb{Q}^{R_2}$, $\boldsymbol{\beta} \in \mathbb{Q}^{E_2}$, and $\boldsymbol{\gamma} \in \mathbb{Q}^{E_2}$ as follows:

- $x(e) = 1/4$ if $e \in \{e_1, e_2, e_3, e_5, e_8\}$ and 1/2 otherwise;
- $\alpha(u) = 1/2$ if $u \in \{v_1, v_2, v_3, v_4, \{v_1, v_4, v_5\}\}$ and 0 otherwise;
- $\beta(e) = 0$ for each $e \in E_2$; and
- $\gamma(e) = 1/2$ if $e \in \{e_4, e_6, e_7\}$ and 0 otherwise.

It is easy to see that $\boldsymbol{x}$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ are feasible solutions to F_2-Max and F_2-Min, respectively, and have the same objective value of 13/4. Similar to Case 1, we can thus deduce that F_2-Min has no integral optimal solution.

Case 3. $i = 3$. Set $w(e) = 1$ if $e \in \{e_1, e_2, e_3, e_5\}$ and 2 otherwise, set $l(e) = 0$ for each $e \in E_3$, and set $u(e) = 1$ if $e \in \{e_5, e_7, e_8\}$ and 1/2 otherwise. Define $\boldsymbol{x} \in \mathbb{Q}^{E_3}$, $\boldsymbol{\alpha} \in \mathbb{Q}^{R_3}$, $\boldsymbol{\beta} \in \mathbb{Q}^{E_3}$, and $\boldsymbol{\gamma} \in \mathbb{Q}^{E_3}$ as follows:

- $x(e) = 1/4$ if $e \in \{e_1, e_3, e_5, e_8\}$ and 1/2 otherwise;
- $\alpha(u) = 1/2$ if $u \in \{v_1, v_4, v_7, \{v_1, v_4, v_5, v_6, v_7\}, V_3\}$ and 0 otherwise;
- $\beta(e) = 0$ for each $e \in E_2$; and
- $\gamma(e) = 1/2$ if $e \in \{e_2, e_4, e_6\}$ and 0 otherwise.

It is routine to check that $\boldsymbol{x}$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ are feasible solutions to F_3-Max and F_3-Min, respectively, and have the same objective value of 19/4. Similar to Case 1, we can thus imply that F_3-Min has no integral optimal solution.

Case 4. $i = 4$. Set $w(e) = 1$ if $e \in \{e_5, e_7, e_8, e_9\}$ and 2 otherwise, set $l(e) = 0$ for each $e \in E_4$, and set $u(e) = 2/3$ if $e \in \{e_5, e_8\}$ and 1/3 otherwise. Define $\boldsymbol{x} \in \mathbb{Q}^{E_4}$, $\boldsymbol{\alpha} \in \mathbb{Q}^{R_4}$, $\boldsymbol{\beta} \in \mathbb{Q}^{E_4}$, and $\boldsymbol{\gamma} \in \mathbb{Q}^{E_4}$ as follows:

- $x(e) = 1/6$ if $e \in \{e_1, e_7, e_9\}$, $x(e_5) = 1/2$, $x(e_8) = 2/3$, and 1/3 otherwise;
- $\alpha(u) = 1/2$ if $u \in \{v_1, v_3, \{v_1, v_2, v_3, v_4, v_5\}, V_4\}$ and 0 otherwise;
- $\beta(e) = 0$ for each $e \in E_2$; and
- $\gamma(e) = 1/2$ if $e \in \{e_2, e_3, e_4, e_6, e_8\}$ and 0 otherwise.

It is not difficult to verify that $\boldsymbol{x}$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ are feasible solutions to F_4-Max and F_4-Min, respectively, and have the same objective value of 9/2. Similar to Case 1, we can thus conclude that F_4-Min has no integral optimal solution. ∎

The following simple observation can be found in Schrijver [17], on page 323.

Lemma 3.4. *Let C' be obtained from a matrix C by deleting a column. If $C\boldsymbol{x} \le \boldsymbol{d}$, $\boldsymbol{x} \ge \boldsymbol{0}$ is box-TDI, then so is $C'\boldsymbol{x} \le \boldsymbol{d}$, $\boldsymbol{x} \ge \boldsymbol{0}$.*

The following lemma essentially states that each face of a TDI system is TDI again (see Theorem 22.2 in Schrijver [17]).

Lemma 3.5. *Let $C\boldsymbol{x} \le \boldsymbol{d}$ be a TDI system and let $\boldsymbol{\alpha}^T\boldsymbol{x} \le \beta$ be one of its inequalities. Then the system $C\boldsymbol{x} \le \boldsymbol{d}$, $-\boldsymbol{\alpha}^T\boldsymbol{x} \le -\beta$ is also TDI.*

The lemma below follows immediately from the definition of box-TDI systems.

Lemma 3.6. *Suppose $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are two rational vector with $\boldsymbol{\alpha}_1 \le \boldsymbol{\alpha}_2$, and β_1 and β_2 are two rational numbers with $\beta_1 \ge \beta_2$. Then the system $C\boldsymbol{x} \le \boldsymbol{d}$, $\boldsymbol{\alpha}_1^T\boldsymbol{x} \le \beta_1$, $\boldsymbol{\alpha}_2^T\boldsymbol{x} \le \beta_2$, $\boldsymbol{x} \ge \boldsymbol{0}$ is box-TDI if and only if $C\boldsymbol{x} \le \boldsymbol{d}$, $\boldsymbol{\alpha}_2^T\boldsymbol{x} \le \beta_2$, $\boldsymbol{x} \ge \boldsymbol{0}$ is box-TDI.*

Lemma 3.7. *Let $A\boldsymbol{x} \le \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$ and $A'\boldsymbol{x}' \le \boldsymbol{b}'$, $\boldsymbol{x}' \ge \boldsymbol{0}$ be two rational systems such that*

$$A = \begin{bmatrix} \boldsymbol{a}_1^T & 1 \\ \boldsymbol{a}_2^T & 1 \\ A_1 & \boldsymbol{0} \\ A_2 & \boldsymbol{1} \end{bmatrix}, \quad A' = \begin{bmatrix} \boldsymbol{0}^T & 1 & 1 & 0 \\ \boldsymbol{0}^T & 0 & 1 & 1 \\ \boldsymbol{0}^T & 0 & -1 & -1 \\ \boldsymbol{a}_1^T & 1 & 0 & 0 \\ \boldsymbol{a}_2^T & 0 & 0 & 1 \\ A_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ A_2 & \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{1} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ \boldsymbol{b_1} \\ \boldsymbol{b_2} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{b}' = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ \boldsymbol{b}_1 \\ \boldsymbol{b}_2 + \boldsymbol{1} \end{bmatrix},$$

where $\boldsymbol{a}_2 \ge \boldsymbol{0}$. If $A'\boldsymbol{x}' \le \boldsymbol{b}'$, $\boldsymbol{x}' \ge \boldsymbol{0}$ is box-TDI, then so is $A\boldsymbol{x} \le \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$.

Proof. Let the rows and columns of A be indexed by disjoint sets R and S, respectively. We partition R into $\{r_1, r_2\} \cup R_1 \cup R_2$, where r_i is the index of row i and R_i is the set of the indices of all rows corresponding to A_i, for $i = 1, 2$, and partition S into $T \cup \{q_1\}$, where q_1 is the index of the last column. Next, let the rows of A' be indexed by the set $R' = \{p_1, p_2, p_3, r_1, r_2\} \cup R_1 \cup R_2$ and let the columns of A' be indexed by the set $S' = T \cup \{q_1, q_2, q_3\}$, where p_i is the index of row i and q_i is the index of the i^{th} column succeeding T, for $i = 1, 2, 3$. Thus q_3 is the index of the last column of A'.

We aim to show that the system $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is TDI for all $\boldsymbol{l} \in \mathbb{Q}^S$ and $\boldsymbol{u} \in \mathbb{Q}^S$. To this end, let $\boldsymbol{w}$ be an arbitrary vector in $\mathbb{Z}^S$ such that the optimal value of the following LP-duality equation

$$\max\left\{\boldsymbol{w}^T\boldsymbol{x} \,\middle|\, \begin{bmatrix} A \\ I \\ -I \end{bmatrix} \boldsymbol{x} \leq \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{u} \\ -\boldsymbol{l} \end{bmatrix}, \boldsymbol{x} \geq \boldsymbol{0}\right\} = \min\left\{\boldsymbol{y}^T \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{u} \\ -\boldsymbol{l} \end{bmatrix} \,\middle|\, \boldsymbol{y}^T \begin{bmatrix} A \\ I \\ -I \end{bmatrix} \geq \boldsymbol{w}^T, \boldsymbol{y} \geq \boldsymbol{0}\right\} \tag{3.1}$$

is finite.

To verify that the minimum in (3.1) has an integral optimal solution, we define $\boldsymbol{u}' \in (\mathbb{Q} \cup \{+\infty\})^{S'}$, $\boldsymbol{l}' \in \mathbb{Q}^{S'}$, and $\boldsymbol{w}' \in \mathbb{Z}^{S'}$, such that

(1) $\boldsymbol{l}'|_S = \boldsymbol{l}$, $\boldsymbol{u}'|_S = \boldsymbol{u}$, $\boldsymbol{w}'|_S = \boldsymbol{w}$, $(l'_{q_2}, u'_{q_2}) = (\max\{0, 1 - u_{q_1}\}, +\infty)$, $(l'_{q_3}, u'_{q_3}) = (l'_{q_1}, u'_{q_1})$, and $w'_{q_2} = w'_{q_3} = 0$,

and consider the primal-dual pair

$$\max\left\{\boldsymbol{w}'^T\boldsymbol{x}' \,\middle|\, \begin{bmatrix} A' \\ I \\ -I \end{bmatrix} \boldsymbol{x}' \leq \begin{bmatrix} \boldsymbol{b}' \\ \boldsymbol{u}' \\ -\boldsymbol{l}' \end{bmatrix}, \boldsymbol{x}' \geq \boldsymbol{0}\right\} = \min\left\{\boldsymbol{y}'^T \begin{bmatrix} \boldsymbol{b}' \\ \boldsymbol{u}' \\ -\boldsymbol{l}' \end{bmatrix} \,\middle|\, \boldsymbol{y}'^T \begin{bmatrix} A' \\ I \\ -I \end{bmatrix} \geq \boldsymbol{w}'^T, \boldsymbol{y}' \geq \boldsymbol{0}\right\}. \tag{3.2}$$

In what follows, we refer to the four problems in (3.1) and (3.2) as (3.1)-Max, (3.1)-Min, (3.2)-Max and (3.2)-Min, respectively. We first claim that

(2) The two problems (3.1)-Max and (3.2)-Max have the same optimal value.

To justify this, let $\boldsymbol{x}$ be an arbitrary feasible solution to (3.1)-Max, and let $\boldsymbol{x}' \in \mathbb{R}^{S'}$ be defined by $\boldsymbol{x}'|_S = \boldsymbol{x}$, $x'_{q_2} = 1 - x_{q_1}$, and $x'_{q_3} = x_{q_1}$. It is easy to see that $\boldsymbol{x}'$ is a feasible solution to (3.2)-Max. By (1), we have $(\boldsymbol{w}')^T\boldsymbol{x}' = \boldsymbol{w}^T\boldsymbol{x}$.

Conversely, for any feasible solution $\boldsymbol{x}'$ to (3.2)-Max, set $\boldsymbol{x} = \boldsymbol{x}'|_S$. From the first three inequalities contained in $A'\boldsymbol{x}' \leq \boldsymbol{b}'$, we deduce that $x'_{q_1} + x'_{q_2} \leq 1$ and $x'_{q_2} + x'_{q_3} = 1$. So $x'_{q_1} \leq x'_{q_3}$. It follows that $A\boldsymbol{x} \leq \boldsymbol{b}$ and hence $\boldsymbol{x}$ is a feasible solution to (3.1)-Max. Clearly, $\boldsymbol{w}^T\boldsymbol{x} = (\boldsymbol{w}')^T\boldsymbol{x}'$. Combining the above two observations, we establish (2).

Since $A'\boldsymbol{x}' \leq \boldsymbol{b}'$, $\boldsymbol{x}' \geq \boldsymbol{0}$ is a box-TDI system, the definition and (2) guarantee the existence of an integral optimal solution $\bar{\boldsymbol{y}}'$ to (3.2)-Min. Let

- $\bar{y}'_t$ be the coordinate of $\bar{\boldsymbol{y}}'$ corresponding to constraint t contained in $A'\boldsymbol{x}' \leq \boldsymbol{b}'$ for each $t \in R'$,
- $\bar{y}'_t$ be the coordinate of $\bar{\boldsymbol{y}}'$ corresponding to the constraint $x'_t \leq u'_t$ for each $t \in S'$, and
- $\bar{y}'_{-t}$ be the coordinate of $\bar{\boldsymbol{y}}'$ corresponding to the constraint $-x'_t \leq -l'_t$ for each $t \in S'$.

Observe that neither the box constraint $x'_t \leq u'_t$ when $u'_t = +\infty$ nor $-x'_t \leq -l'_t$ when $l'_t = 0$ appears in (3.2)-Max, so

(3) $\bar{y}'_{q_2} = 0$. Moreover, $\bar{y}'_{q_1} = \bar{y}'_{q_3} = 0$ if $u'_{q_1} = u_{q_1} = +\infty$, and $\bar{y}'_{-q_2} = 0$ if $u_{q_1} \geq 1$ (as $l'_{q_2} = 0$).

Consider the constraints corresponding to the last three columns of A' in (3.2)-Min, which respectively, read

(4) $\bar{y}'_{p_1} + \bar{y}'_{r_1} + \sum_{t \in R_2} \bar{y}'_t + \bar{y}'_{q_1} - \bar{y}'_{-q_1} \geq w'_{q_1}$,

(5) $\bar{y}'_{p_1} + \bar{y}'_{p_2} - \bar{y}'_{p_3} + \sum_{t \in R_2} \bar{y}'_t - \bar{y}'_{-q_2} \geq 0$ (see (3)), and

(6) $\bar{y}'_{p_2} - \bar{y}'_{p_3} + \bar{y}'_{r_2} + \sum_{t\in R_2} \bar{y}'_t + \bar{y}'_{q_3} - \bar{y}'_{-q_3} \geq 0$.

We may assume that (5) holds with equality; that is,

(7) $\bar{y}'_{p_1} + \bar{y}'_{p_2} - \bar{y}'_{p_3} + \sum_{t\in R_2} \bar{y}'_t - \bar{y}'_{-q_2} = 0$.

Suppose the contrary: $\bar{y}'_{p_1} + \bar{y}'_{p_2} - \bar{y}'_{p_3} + \sum_{t\in R_2} \bar{y}'_t - \bar{y}'_{-q_2}$, denoted by δ, is nonzero. By (1) and (3.2), we have $\delta > 0$. Let $\boldsymbol{y}'$ be obtained from $\bar{\boldsymbol{y}}'$ by replacing $\bar{y}'_{r_2}$ with $\bar{y}'_{r_2} + \delta$ and replacing $\bar{y}'_{p_3}$ with $\bar{y}'_{p_3} + \delta$. Since $\boldsymbol{a}_2 \geq \boldsymbol{0}$, $\boldsymbol{y}'$ is a feasible solution to (3.2)-Min. Since $y'_{r_2} - y'_{p_3} = \bar{y}'_{r_2} - \bar{y}'_{p_3}$, from the definition of $\boldsymbol{b}'$ we see that $\boldsymbol{y}'$ has the same objective value as $\bar{\boldsymbol{y}}'$. Hence $\boldsymbol{y}'$ is also an optimal solution to (3.2)-Min. Therefore (7) follows, otherwise we replace $\bar{\boldsymbol{y}}'$ with $\boldsymbol{y}'$.

Let us proceed with the construction of an integral optimal solution $\bar{\boldsymbol{y}}$ to (3.1)-Min. Set

- $\bar{y}_t = \bar{y}'_t$ for $t \in R \cup T$,
- $\bar{y}_{-t} = \bar{y}'_{-t}$ for $t \in T$,
- $\bar{y}_{q_1} = \bar{y}'_{q_1} + \bar{y}'_{q_3} + \bar{y}'_{-q_2}$, and
- $\bar{y}_{-q_1} = \bar{y}'_{-q_1} + \bar{y}'_{-q_3}$.

By (3), we have $\bar{y}_{q_1} = 0$ if $u_{q_1} = +\infty$. So $\bar{\boldsymbol{y}}$ is well defined. We propose to prove that

(8) $\bar{\boldsymbol{y}}$ is a feasible solution to (3.1)-Min.

For this purpose, it suffices to show that $\bar{\boldsymbol{y}}$ satisfies the constraint corresponding to column q_1 in (3.1)-Min, because $\boldsymbol{w}|_T = \boldsymbol{w}'|_T$. Note that $\bar{y}_{r_1} + \bar{y}_{r_2} + \sum_{t\in R_2} \bar{y}_t + \bar{y}_{q_1} - \bar{y}_{-q_1} = \bar{y}'_{r_1} + \bar{y}'_{r_2} + \sum_{t\in R_2} \bar{y}'_r + (\bar{y}'_{q_1} + \bar{y}'_{q_3} + \bar{y}'_{-q_2}) - (\bar{y}'_{-q_1} + \bar{y}'_{-q_3})$ =LHS of (4)+LHS of (6)−LHS of (5)≥RHS of (4)$= w'_{q_1} = w_{q_1}$, where the last inequality follows from (7). Thus (8) is established.

(9) $(\boldsymbol{b}^T, \boldsymbol{u}^T, -\boldsymbol{l}^T)\bar{\boldsymbol{y}} = ((\boldsymbol{b}')^T, (\boldsymbol{u}')^T, -(\boldsymbol{l}')^T)\bar{\boldsymbol{y}}'$.

To justify this, set $\bar{\boldsymbol{y}}_{R_i} = (\bar{y}_t : t \in R_i)$ for $i = 1, 2$, and set $\bar{\boldsymbol{y}}_K = (\bar{y}_t : t \in K)$ and $\bar{\boldsymbol{y}}_{-K} = (\bar{y}_{-t} : t \in K)$ for any $K \subseteq S$. Similarly, we can define $\bar{\boldsymbol{y}}'_{R_i}$, $\bar{\boldsymbol{y}}'_K$ and $\bar{\boldsymbol{y}}'_{-K}$ for any $K \subseteq S'$. By direct computation, we obtain

$$
\begin{aligned}
& (\boldsymbol{b}^T, \boldsymbol{u}^T, -\boldsymbol{l}^T)\bar{\boldsymbol{y}} \\
= \;& \bar{y}_{r_1} + \bar{y}_{r_2} + \boldsymbol{b}_1^T \bar{\boldsymbol{y}}_{R_1} + \boldsymbol{b}_2^T \bar{\boldsymbol{y}}_{R_2} + \boldsymbol{u}_T^T \bar{\boldsymbol{y}}_T - \boldsymbol{l}_T^T \bar{\boldsymbol{y}}_{-T} + u_{q_1}\bar{y}_{q_1} - l_{q_1}\bar{y}_{-q_1} \\
= \;& \bar{y}_{r_1} + \bar{y}_{r_2} + \boldsymbol{b}_1^T \bar{\boldsymbol{y}}_{R_1} + \boldsymbol{b}_2^T \bar{\boldsymbol{y}}_{R_2} + \boldsymbol{u}_T^T \bar{\boldsymbol{y}}_T - \boldsymbol{l}_T^T \bar{\boldsymbol{y}}_{-T} + u_{q_1}\bar{y}_{q_1} - l_{q_1}\bar{y}_{-q_1} + \text{LHS of (7)} \\
= \;& \bar{y}'_{r_1} + \bar{y}'_{r_2} + \boldsymbol{b}_1^T \bar{\boldsymbol{y}}'_{R_1} + \boldsymbol{b}_2^T \bar{\boldsymbol{y}}'_{R_2} + \boldsymbol{u}_T^T \bar{\boldsymbol{y}}'_T - \boldsymbol{l}_T^T \bar{\boldsymbol{y}}'_{-T} + u_{q_1}(\bar{y}'_{q_1} + \bar{y}'_{q_3} + \bar{y}'_{-q_2}) - l_{q_1}(\bar{y}'_{-q_1} + \bar{y}'_{-q_3}) \\
& + \bar{y}'_{p_1} + \bar{y}'_{p_2} - \bar{y}'_{p_3} + \sum_{t\in R_2} \bar{y}'_t - \bar{y}'_{-q_2} \\
= \;& \bar{y}'_{p_1} + \bar{y}'_{p_2} - \bar{y}'_{p_3} + \bar{y}'_{r_1} + \bar{y}'_{r_2} + \boldsymbol{b}_1^T \bar{\boldsymbol{y}}'_{R_1} + (\boldsymbol{b}_2^T \bar{\boldsymbol{y}}'_{R_2} + \sum_{t\in R_2} \bar{y}'_t) + [\boldsymbol{u}_T^T \bar{\boldsymbol{y}}'_T + u_{q_1}(\bar{y}'_{q_1} + \bar{y}'_{q_3})] \\
& - [\boldsymbol{l}_T^T \bar{\boldsymbol{y}}'_{-T} + l_{q_1}(\bar{y}'_{-q_1} + \bar{y}'_{-q_3}) + (1 - u_{q_1})\bar{y}'_{-q_2}] \\
= \;& \bar{y}'_{p_1} + \bar{y}'_{p_2} - \bar{y}'_{p_3} + \bar{y}'_{r_1} + \bar{y}'_{r_2} + \boldsymbol{b}_1^T \bar{\boldsymbol{y}}'_{R_1} + (\boldsymbol{b}_2 + \boldsymbol{1})^T \bar{\boldsymbol{y}}'_{R_2} + (\boldsymbol{u}')^T \bar{\boldsymbol{y}}'_{S'} - (\boldsymbol{l}')^T \bar{\boldsymbol{y}}'_{-S'} \\
= \;& ((\boldsymbol{b}')^T, (\boldsymbol{u}')^T, -(\boldsymbol{l}')^T)\bar{\boldsymbol{y}}'.
\end{aligned}
$$

In view of (3), the same statement holds as well if $u_{q_1} = +\infty$ or $u_{q_1} \geq 1$. So (9) is true.

Combining (2), (8) and (9), we conclude that $\bar{\boldsymbol{y}}$ is an integral optimal solution to (3.1)-Min. This proves the lemma. ∎

Lemma 3.8. *Let H be a subgraph of a graph G. If $\pi(G)$ is box-TDI, then so is $\pi(H)$.*

Proof. Let $A\boldsymbol{x} \le \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$ stand for the system $\pi(G)$. Clearly, $\pi(H)$ arises from $\pi(G)$ by deleting the columns of A corresponding to edges outside H and then deleting some resulting redundant inequalities as described in Lemma 3.6. It follows immediately from Lemma 3.4 and Lemma 3.6 that $\pi(H)$ is box-TDI as well, completing the proof. ∎

Lemma 3.9. *Let G be obtained from a graph H by subdividing one edge into a path of length three. If $\sigma(G)$ is box-TDI, then so is $\sigma(H)$.*

Proof. By hypothesis, G arises from H by subdividing one edge $f = r_1r_2$ into a path $P = r_1p_1p_2r_2$, where $d_H(r_1) \le d_H(r_2)$. Let q_1, q_2, q_3 denote the three edges r_1p_1, p_1p_2, p_2r_2 on P, respectively. Note that if $E[U]$, with $U \in \mathcal{T}(G)$, contains one of q_1, q_2, q_3, then it contains all of them. Let $\sigma'(G)$ be obtained from $\sigma(G)$ by adding two inequalities $-x(q_2) - x(q_3) \le -1$ and $x(q_1) + x(q_2) + x(q_3) \le 2$ (redundant). Let $\sigma''(G) = \sigma'(G)$ if $d_G(r_1) \ge 2$ and let $\sigma''(G)$ be obtained from $\sigma'(G)$ by adding one more redundant inequality $x(q_1) \le 1$. Let $A'\boldsymbol{x}' \le \boldsymbol{b}'$, $\boldsymbol{x}' \ge \boldsymbol{0}$ be the linear system corresponding to $\sigma''(G)$. Clearly, we can write A' and $\boldsymbol{b}'$ as specified in Lemma 3.7, where the last three columns of A' correspond to q_1, q_2, q_3, respectively, the first two rows of A' correspond to p_1, p_2, respectively, the third row corresponds to inequality $-x(q_2) - x(q_3) \le -1$, the fourth and fifth rows correspond to r_1, r_2, respectively, and the rows intersecting A_2 correspond to those $U \in \mathcal{T}(G)$ such that $E[U]$ contains all of q_1, q_2, q_3, if any, and the inequality $x(q_1) + x(q_2) + x(q_3) \le 2$. Let $A\boldsymbol{x} \le \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$ be as described in Lemma 3.7, such that the first two rows of A correspond to vertices r_1 and r_2, respectively, and the last column corresponds to edge r_1r_2, and let $C\boldsymbol{x} \le \boldsymbol{d}$, $\boldsymbol{x} \ge \boldsymbol{0}$ stand for $\sigma(H)$. Clearly, $C\boldsymbol{x} \le \boldsymbol{d}$ is a subsystem of $A\boldsymbol{x} \le \boldsymbol{b}$. If $d_H(r_1) = 1$, then the inequality $x(f) \le 1$ is contained in $A\boldsymbol{x} \le \boldsymbol{b}$ but not in $C\boldsymbol{x} \le \boldsymbol{d}$ (recall the definition of $\sigma(H)$ in Theorem 1.2). Moreover, if H contains a triangle $r_1r_2r_3$ such that $d_H(r_i) = 2$ for some $1 \le i \le 3$, then $x(\delta(r_i)) \le 1$ is included in the system $A\boldsymbol{x} \le \boldsymbol{b}$ but not in $C\boldsymbol{x} \le \boldsymbol{d}$. Nevertheless, such an inequality is implied by the constraint $x(E[U]) \le 1$, with $U = \{r_1, r_2, r_3\}$, which appears in both $A\boldsymbol{x} \le \boldsymbol{b}$ and $C\boldsymbol{x} \le \boldsymbol{d}$. Thus $C\boldsymbol{x} \le \boldsymbol{d}$ can be obtained from $A\boldsymbol{x} \le \boldsymbol{b}$ by possibly deleting the redundant inequality $x(f) \le 1$ and those created by some degree-2 vertices contained in triangles in H. Since $\sigma(G)$ is box-TDI, so are $\sigma'(G)$ and $\sigma''(G)$ by Lemma 3.5 and Theorem 1.5. Hence $A\boldsymbol{x} \le \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$ is also box-TDI by Lemma 3.7. From Lemma 3.6 we deduce that $\sigma(H)$ is box-TDI as well. ∎

Proof of Theorem 3.2 . Let H be a fully odd subdivision of F_i contained in G for some $1 \le i \le 4$. By Lemma 3.9, $\sigma(H)$ and hence $\pi(H)$ by Lemma 3.1 is not a box-TDI system. It follows immediately from Lemma 3.8 that $\pi(G)$ is not box-TDI either, completing the proof. ∎

4 Structural Description

A graph G is called *good* if it contains no fully odd subdivision of F_1, F_2, F_3, or F_4 (see Figure 2) as a subgraph. To establish the "if" part of Theorem 1.4, we need a structural description of good graphs. As stated in Section 1, due to the strict parity restriction, it is very difficult to thoroughly use fully odd subdivisions in our investigation. To overcome this difficulty, we shall view G as a signed graph (with all edges odd initially), and obtain a smaller and smaller signed graph from G by repeatedly using B-reductions (see Subsection 4.4), in which each odd/even edge is the place holder for a certain bipartite subgraph of G. We call a resulting signed graph *irreducible* if no more B-reductions can be applied to it. Depending on the presence or absence of the so-called D-subgraphs, we shall be able to determine all irreducible graphs (see Lemmas 4.10 and 4.13). The original graph G can finally be retrieved from such graphs by using B-extensions (see Subsections 4.4 and 4.6), which are reverse operations of the above-mentioned B-reductions. To guarantee the validity of these reduction and extension processes, we shall prove some technical lemmas in Sections 4.1-4.3 – they carry over naturally to signed graphs with all edges odd!

4.1 Preliminaries

We digress to introduce some other notations and terminology before proceeding. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex and edge sets of G, respectively. For any $X \subseteq V(G) \cup E(G)$, we use $G\backslash X$ to denote the graph arising from G by deleting all members of X, and set $G\backslash x = G\backslash X$ if $X = \{x\}$. For any two nonadjacent vertices u and v of G, we use $G + uv$ to denote the graph obtained from G by adding an edge uv. For any subgraph K of G, a *K-bridge* of G is a subgraph B of G induced by *either* (i) an edge in $E(G)\backslash E(K)$ with both ends in $V(K)$ *or* (ii) the edges in a component Ω of $G\backslash V(K)$ together with edges of G between Ω and K. We call B *nontrivial* if it satisfies (ii). The vertices in $V(B) \cap V(K)$ are called *feet* of B. Throughout, by a path we mean a *simple* one, which contains no repeated vertices. A path with ends x and y is called an *xy-path.* A path is called *odd* if it is of odd length and *even* otherwise. For any two vertices u, v on a path P, we use $P[u, v]$ to denote the subpath of P connecting u and v, and define $P(u, v] = P[u, v]\backslash u$, $P[u, v) = P[u, v]\backslash v$, and $P(u, v) = P[u, v]\backslash\{u, v\}$. For any vertex u on a cycle C, we use u^- (resp. u^+) to denote the vertex preceding (resp. succeeding) u on C in the clockwise direction. For any two vertices u and v on C, we use $C[u, v]$ to denote the segment of C from u to v in the clockwise direction.

A graph G is called *internally 2-connected* (i-2-c) if it is connected and, for any $v \in V$, if $G\backslash v$ is disconnected, then it has precisely two components with one of them being an isolated vertex, and called *fully subdivided* if it is connected and bipartite, with bipartition (X, Y), such that all vertices in Y have degree at most two (we call X and Y the *color 1 class* and *color 2 class* of G, respectively). For convenience, a single vertex is also viewed as a fully subdivided graph, which has only color 1 class. Notice that if a fully subdivided graph G contains no pendant edges, then it arises from a connected graph H by subdividing each edge exactly once.

In our structural description of good graphs, the most complicated one arises from a ladder-like structure by replacing each edge with a fully subdivided graph. The precise definition is given below.

Let C be a cycle with two distinguished edges u_1u_2 and v_1v_2 (not necessarily disjoint) such that u_1, v_1, v_2, u_2 occur on C in clockwise cyclic order, and let H be obtained from C by adding chords between $C[u_1, v_1]$ and $C[v_2, u_2]$, such that each vertex on C is incident with at least one chord and such that if two chords x_1y_1 and x_2y_2 cross, then $\{x_1, y_1, x_2, y_2\}$ induces a 4-cycle. (Possibly a chord is parallel to u_1u_2 or v_1v_2.) We call H a *ladder* with *top* u_1u_2, *bottom* v_1v_2, and *outer cycle* C. Let G be obtained from H by

- replacing each chord e with a complete bipartite graph $L_e = K_{2,n}$ for some $n \geq 1$, in which one color class consists of the two ends of e only; and

- replacing each edge f in $C\backslash\{u_1v_1, u_2v_2\}$ with a fully subdivided graph L_f, in which both ends of f belong to the color 1 class, where $L_f = K_{2,t}$ for some $t \geq 1$ if f is contained in a 4-cycle induced by two crossing chords.

We call G a *plump ladder* generated from H. Now we are ready to present the structural description.

Theorem 4.1. *Let $G = (V, E)$ be an i-2-c nonbipartite graph. Then G is good iff it is a subgraph of one of the nine graphs depicted in Figure 4, where G_9 is an arbitrary plump ladder, and the words "odd" and "any" indicate the parities of the corresponding paths.*

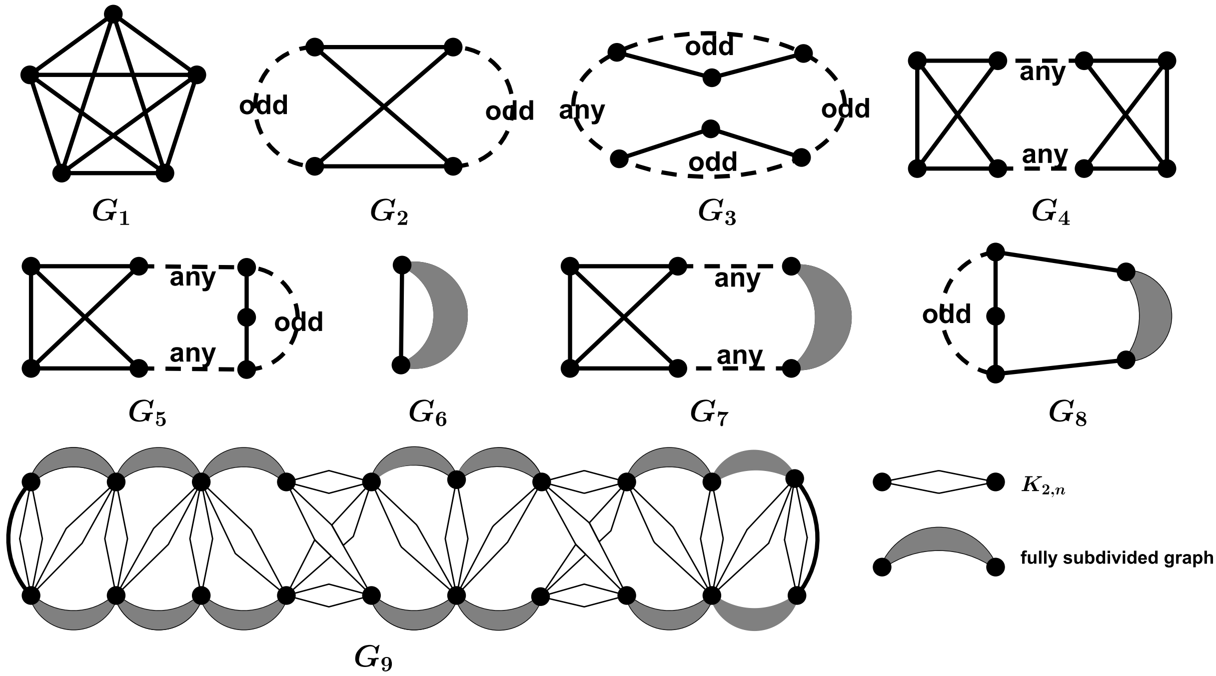


Figure 4: Primitive graphs

The remainder of this section is devoted to a proof of this theorem, in which we shall repeatedly apply the following simple lemmas.

Lemma 4.2. *Let $G = (V, E)$ be an i-2-c graph, let $U \subseteq V$ with $|U| \geq 2$, and let $v \in V\backslash U$. If each vertex in $U \cup \{v\}$ has degree at least two in G, then there exist two paths from v to U that have only v in common.*

Proof. Suppose the contrary. Then G has a vertex w separating v from U, with $w \neq v$, by Menger's theorem. Let Ω_1 be the component of $G\backslash w$ that contains v, and let Ω_2 be the component of $G\backslash w$ that contains a vertex x in $U\backslash w$. Since both v and x have degree at least two in G, each of Ω_1 and Ω_2 contains at least two vertices, contradicting the hypothesis that G is i-2-c. ∎

Lemma 4.3. *Let x, y be two distinct vertices in a connected graph G. Then G contains an xy-path P together with an edge uv, with $u \in V(P)$ while $v \notin V(P)$, unless the entire G is an xy-path.* ∎

Lemma 4.4. *Let $H = (X, Y; E)$ be a connected bipartite graph and let $G = H + x_1x_2$, with $\{x_1, x_2\} \subseteq X$. Suppose G is i-2-c and $d_G(y_0) \geq 3$ for some $y_0 \in Y$. Then the following statements hold:*

(i) If $d_G(x_0) \geq 3$ for some $x_0 \in X$, then G has a cycle C that contains the edge x_1x_2 and contains some $x \in X$ and $y \in Y$, with $d_G(z) \geq 3$ for $z = x, y$;

(ii) If $d_H(x_1) \geq 2$, then H contains an x_1x_2-path P and two disjoint edges x_1y_1 and y_2x_3, with $y_2 \in V(P) \cap Y$ while $\{y_1, x_3\} \cap V(P) = \emptyset$.

Proof. (i) By Lemma 4.2 with $U = \{x_1, x_2\}$ and $v = y_0$, there exists a cycle C in G that contains both edge x_1x_2 and vertex y_0. We may assume that C contains no vertex $x \in X$ with $d_G(x) \geq 3$, otherwise we are done. Thus $x_0 \notin V(C)$. By Lemma 4.2 with $U = V(C)$ and $v = x_0$, there exists two paths P_1 and P_2 from x_0 to C that have only x_0 in common. For $i = 1, 2$, let q_i be the end of P_i in C. Then $d_G(q_i) \geq 3$, so $q_i \in Y$. Let Q denote the subpath of $C\backslash x_1x_2$ between q_1 and q_2, and let C' be the cycle obtained from $C \cup P_1 \cup P_2$ by deleting all vertices on $Q(q_1, q_2)$. Then C' is as desired.

(ii) By (i), G has a cycle that contains the edge x_1x_2 and some $y \in Y$ with $d_G(y) \geq 3$; let C be such a shortest cycle. Observe that C is an induced cycle in G, for otherwise it would have a chord ab. Thus $d_G(z) \geq 3$ for $z = a, b$. Let Q be the subpath of $C\backslash x_1x_2$ between a and b, and let C' be obtained from C by deleting all vertices on $Q(a, b)$ and adding edge ab. Then the existence of C' contradicts the choice of C. It follows that G contains two edges x_1y_1 and yx_3 such that $\{y_1, x_3\} \cap V(C) = \emptyset$. Set $P = C\backslash x_1x_2$ and $y_2 = y$; we are done. ∎

Lemma 4.5. *Let $H = (X, Y; E)$ be a connected bipartite graph and let $G = H + x_1y_1$, with $x_1 \in X$ and $y_1 \in Y$. Suppose G is i-2-c. Then at least one of the following statements holds:*

(i) H contains an x_1y_1-path P and an x_2y_2-path Q, such that $V(P) \cap V(Q) = \{x_2, y_2\}$ and that both $P[x_2, y_2]$ and Q are of odd length. (Possibly $x_1 = x_2$ or $y_1 = y_2$.)

(ii) H contains an x_1y_1-path P and two disjoint edges y_2x_3 and x_2y_3, with $\{x_2, y_2\} \subseteq V(P)$ while $\{x_3, y_3\} \cap V(P) = \emptyset$ and with y_2 on $P[x_1, x_2]$, such that $P[x_1, y_2]$, $P[y_2, x_2]$, and $P[x_2, y_1]$ are all of odd length.

(iii) H contains an edge x_2y_2 such that $H\backslash x_2y_2$ has precisely two components $H_1 = (X_1, Y_1; E_1)$ and $H_2 = (X_2, Y_2; E_2)$, with $\{x_1, x_2\} \subseteq X_1$ and $\{y_1, y_2\} \subseteq Y_2$, and that $d_H(v) \leq 2$ for each $v \in Y_1 \cup X_2$. (Possibly $x_1 = x_2$ or $y_1 = y_2$.)

Proof. Assume on the contrary that none of (i)-(iii) holds for H and, subject to this, $|V(H)|$ is minimum. Let A be the set of all pendant vertices of H outside $\{x_1, y_1\}$. Then $H\backslash A$ is not 2-connected, for otherwise, there would be two internally disjoint x_1y_1-paths in H, which satisfy (i), contradicting our assumption. Since G is i-2-c, $H\backslash A$ contains a block chain $B_1, B_2, \ldots, B_t$ connecting x_1 and y_1, with $t \geq 2$, $x_1 \in V(B_1)$ and $y_1 \in V(B_t)$. Let z_i be the common vertex of B_i and B_{i+1} for $1 \leq i \leq t-1$, and set $z_0 = x_1$ and $z_t = y_1$.

(1) For each nontrivial block B_i, the vertices z_i and z_{i+1} belong to the same color class of B_i.

Otherwise, there would be two internal disjoint z_iz_{i+1}-paths R_1 and R_2 in B_i. Let S_1 (resp. S_2) be a z_0z_i-path (resp. $z_{i+1}z_t$-path) in H. Let $P = S_1 \cup R_1 \cup S_2$ and $Q = R_2$. Then they satisfy (i), contradicting our assumption.

(2) Both B_1 and B_t are trivial blocks.

Suppose the contrary: B_1, say, is nontrivial. Let B_1' be obtained from B_1 by adding all pendant edges with one end in B_1 and the other end in A, and let (X_1', Y_1') be the bipartition of B_1', with $\{z_0, z_1\} \subseteq X_1'$ (see (1)). Note that both z_0 and z_1 have degree at least two in B_1'. If some vertex in Y_1' has degree at least three in B_1', then Lemma 4.3 guarantees the existence of a z_0z_1-path R and two disjoint edges z_1a_1 and a_2b_2, with $a_2 \in V(P) \cap Y_1'$ while $\{a_1, b_2\} \cap V(R) = \emptyset$. Let S be a z_1z_t-path in H. Then $R \cup S$, z_1a_1 and a_2b_2 satisfy (ii), contradicting our assumption. It follows that each vertex in Y_1' has degree at most two in B_1'. Let H' be obtained from H be deleting all vertices in $B_1' \backslash z_1$. With $\{z_1, y_1\}$ in place of $\{x_1, y_1\}$, we see that neither (i) nor (ii) holds H' (otherwise, the corresponding statement holds for H). Thus H' has the property exhibited in (iii), and hence $H = H' \cup B_1'$ is also as described in (iii). This contradiction yields (2).

By (2), we have $B_1 = z_0z_1$ and $B_t = z_{t-1}z_t$. If z_1 or z_{t-1} has degree two in H, say the former, letting H' be obtained from $H \backslash \{z_0, z_1\}$ by deleting vertices in A which are adjacent to z_0 or z_1, then at least one of (i), (ii) and (iii) holds for H', with $\{z_2, y_1\}$ in place of $\{x_1, y_1\}$. Clearly, the corresponding statement holds for H. This contradiction implies that both z_1 and z_{t-1} has degree at least three in H. Let R be a shortest z_1z_{t-1}-path in $H \backslash \{x_1, y_1\}$. Then $H \backslash \{x_1, y_1\}$ contains edges z_1z_1' and $z_{t-1}z_{t-1}'$, with $\{z_1', z_{t-1}'\} \cap V(R) = \emptyset$. Note that $z_1' \neq z_{t-1}'$ because they belong to different color classes of H. Since z_1z_1', $z_{t-1}z_{t-1}'$ and the path $x_1z_1Rz_{t-1}y_1$ satisfy (ii), we reach a contradiction to the assumption again. ∎

Lemma 4.6. *Let G be obtained from two disjoint paths $P = p_0p_1 \ldots p_m$ and $Q = q_0q_1 \ldots q_n$ by adding three edges p_0q_0, p_0q_1, p_1q_0 and adding a $p_{m-1}q_{n-1}$-path R of odd length, whose internal vertices are all outside $P \cup Q$, where $m \geq 2$ and $n \geq 2$. Then G contains a fully odd subdivision of F_1 if $m + n$ is even and a fully odd subdivision of F_2 otherwise.*

Proof. We first consider the case when $m + n$ is even. Set $K = G \backslash \{p_0q_0, p_0q_1\}$ if m is odd and $K = G \backslash \{p_1q_0, q_0q_1\}$ otherwise. Then K is a fully odd subdivision of F_1.

It remains to consider the case when $m + n$ is odd. By symmetry, we may assume that m is odd and n is even. Consequently, $G \backslash p_0q_1$ is a fully odd subdivision of F_2. ∎

Lemma 4.7. *Let G_1 (resp. G_2) be obtained from a cycle C by adding two paths P_1, P_2 and a pendant edge u_3v_4 (resp. by adding three paths P_1, P_2, P_3), as shown in Figure 5, where the parity of each u_iv_i-path is indicated by even or odd, and possibly $v_i = u_{i+1}$ for $1 \leq i \leq 3$ (with $u_4 = u_1$). Suppose $C[v_j, u_{j+1}]$ in G_1 is of odd length for at least one j with $1 \leq j \leq 3$. Then both G_1 and G_2 contain a fully odd subdivision of F_1 or F_2.*

Proof. In both G_1 and G_2, let u_iu_i' and v_iv_i' be the edges incident with u_i and v_i on P_i, respectively, for each i.

To prove the statement for G_1, we first consider the case when C is an odd cycle. If one of $C[v_2, u_3]$ and $C[v_3, u_1]$ is odd, say the former, then either $C \cup \{v_2v_2', u_3v_4, u_1u_1'\}$ or $C \cup \{v_2v_2', u_3v_4, v_1v_1'\}$ is a fully odd subdivision of F_1. So we assume that both $C[v_2, u_3]$ and $C[v_3, u_1]$ are of even length, and hence $C[v_1, u_2]$ is of odd length by hypothesis. Thus $C \cup \{v_1v_1', u_2u_2', u_3v_4\}$ is a fully odd subdivision of F_1. It remains to consider the case when C is an even cycle. Observe that at least one of $C[v_2, u_3]$ and $C[v_3, u_1]$ is of odd length, for otherwise the parity of C implies that $C[v_1, u_2]$ is also of even length, contradicting the hypothesis. By symmetry, we may assume

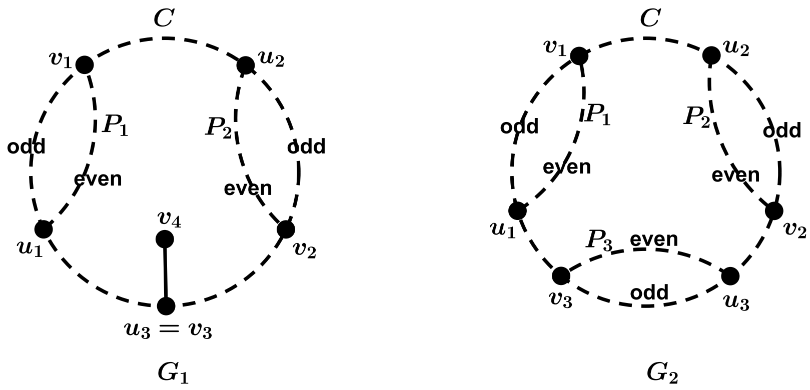


Figure 5: Two configurations with F_1 or F_2

that $C[v_2, u_3]$ is of odd length. Then either $C \cup P_2 \cup \{v_1v_1', u_3v_4\}$ or $C \cup P_2 \cup \{u_1u_1', u_3v_4\}$ is a fully odd subdivision of F_2.

Let us proceed to prove the statement for G_2. If $C[v_j, u_{j+1}]$ is of odd length for at least one j with $1 \le j \le 3$, say $C[v_2, u_3]$, then $C \cup P_1 \cup P_2 \cup \{u_3u_3'\}$ contains a fully odd subdivision of F_1 or F_2 by the statement for G_1. So we assume that $C[v_j, u_{j+1}]$ is of even length for all j with $1 \le j \le 3$. Thus C is an odd cycle. It follows that $C \cup \{u_1u_1', u_2u_2', u_3u_3'\}$ is a fully odd subdivision of F_1. ∎

Lemma 4.8. *Let G be obtained from a connected bipartite graph $H = (X, Y; E)$ by adding two x_1x_2-paths P_1 and P_2 of odd length, with $\{x_1, x_2\} \subseteq X$, such that H, $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$ are pairwise disjoint. If G is i-2-c and good, then $X = \{x_1, x_2\}$.*

Proof. By symmetry, we may assume that $|V(P_1)| \le |V(P_2)|$. So P_2 has length at least three. Observe that H contains no x_1x_2-path of length at least four, for otherwise, the union of such a path and $P_1 \cup P_2$ would yield a fully odd subdivision of F_3 in G, a contradiction. We claim that H contains no vertex in $X \backslash \{x_1, x_2\}$ with degree at least two. Suppose the contrary: $d_H(x_3) \ge 2$ for some x_3 in $X \backslash \{x_1, x_2\}$. Since G is i-2-c, Lemma 4.2 guarantees the existence of two paths Q_1 and Q_2 from x_3 to $\{x_1, x_2\}$ in H that have only x_3 in common. Thus $Q_1 \cup Q_2$ would be a x_1x_2-path with length at least four in H, contradicting our previous observation. So the claim is justified.

Suppose x_3 is a vertex in $X \backslash \{x_1, x_2\}$. Then x_3 has only one neighbor y in H by the above claim. Since G is i-2-c, from the claim we further deduce that y has no neighbor outside $\{x_1, x_2, x_3\}$. If y is adjacent to both x_1 and x_2, letting x_ix_i' be the edge on P_2 incident with x_i for $i = 1, 2$, then $P_1 \cup \{x_1y, x_2y, x_3y, x_1x_1', x_2x_2'\}$ would yield a fully odd subdivision of F_1 in G, a contradiction. So y is adjacent to precisely one of x_1 and x_2, say the former. Thus $G \backslash x_1$ has at least two components with two or more vertices, which contradicts the hypothesis that G is i-2-c. ∎

4.2 Nearly Bipartite Graphs

A graph G is called *nearly bipartite* if G is nonbipartite but $G\backslash e$ is bipartite for some edge e of G. In this subsection we determine nearly bipartite good graphs.

Lemma 4.9. *Let $H = (X, Y; E)$ be a connected bipartite graph and let $G = H + x_1x_2$, with $\{x_1, x_2\} \subseteq X$. If G is i-2-c and good, then G is one of the six graphs depicted in Figure 6, where $\alpha \in \{\text{odd}, \text{even}\}$.*

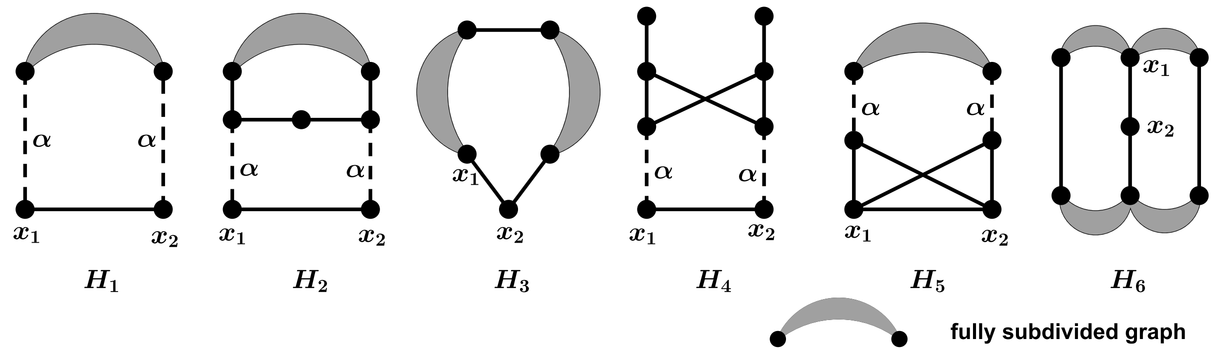


Figure 6: Nearly bipartite good graphs

Proof. Suppose $G \neq H_1$ in Figure 6. Then G contains a vertex in X and a vertex in Y, both with degree at least three. So G has a cycle C containing x_1x_2 such that at least one C-bridge has a foot in X and at least one C-bridge has a foot in Y by Lemma 4.4(i). Since H is bipartite, C is an odd cycle.

In what follows, all bridges are C-bridges unless otherwise stated. For any vertex a on C, we use $\bar{N}_C(a)$ to denote the set of all neighbors of a outside C. We proceed by considering two cases.

Case 1. Each bridge has its feet only in X or only in Y.

From the hypothesis of this case, it is clear that

(1) all bridges are nontrivial.

We say that a bridge is of *type* X (resp. *type* Y) if it has feet only in X (resp. Y), and that a type-X bridge B_1 and a type-Y bridge B_2 *cross* if there exist four vertices u_1, v_1, u_2, v_2 which occur on C in clockwise cyclic order, such that u_1, u_2 are two feet of B_1 and v_1, v_2 are two feet of B_2. Observe that

(2) no type-X bridge crosses with a type-Y bridge.

Assume the contrary: some type-X bridge B_1 and type-Y bridge B_2 cross. Let $\{u_1, u_2, v_1, v_2\}$ be as specified in the above definition. Let P_1 be a u_1u_2-path in B_1 and let P_2 be a v_1v_2-path in B_2. By (1), each of P_1 and P_2 has length at least two. Renaming the subscripts if necessary, we may assume that x_1x_2 is contained in $C[v_2, u_1]$. Then the graph obtained from $C \cup P_1 \cup P_2$ by deleting all vertices on $C(u_2, v_2)$ would be a fully odd subdivision of F_3, contradicting the hypothesis that G is good.

By symmetry and (2), one of the following five subcases occurs, where $\{A, B\} = \{X, Y\}$.

Subcase 1.1. There exist four vertices u_1, u_2, v_1, v_2, such that $x_1, v_1, u_1, u_2, v_2, x_2$ occur on C in clockwise cyclic order and that u_1, u_2 are two feet of a type-A block B_1 and v_1, v_2 are two feet of a type-B block B_2. (Possibly $x_i = v_i$ for $i = 1$ or 2.)

In this subcase, note that

(3) no type-B block has a foot outside $\{v_1, v_2\}$.

Assume the contrary: B_3 is a type-B block with a foot $v_3 \neq v_i$ for $i = 1, 2$. Then v_3 is on $C[u_1, x_2]$ or on $C[x_1, u_2]$, say the former. Let u_1u_1' be an edge in B_1, v_1v_1' an edge in B_2, and v_3v_3' an edge in B_3. Then $C \cup \{u_1u_1', v_1v_1', v_3v_3'\}$ would be an odd subdivision of F_1 in G; this contradiction justifies (3).

(4) $|\bar{N}_C(v_1) \cup \bar{N}_C(v_2)| = 1$.

Otherwise, there exist two distinct vertices v_1' and v_2' outside C, such that both v_1v_1' and v_2v_2' are edges of G. Let u_1u_1' be an edge in B_1. Then $C \cup \{u_1u_1', v_1v_1', v_2v_2'\}$ would be an odd subdivision of F_1 in G, a contradiction.

Let v_0 be the only vertex in $\bar{N}_C(v_1) \cup \bar{N}_C(v_2)$. Then

(5) B_2 is the only type-B bridge in G, which is the path $R = v_1v_0v_2$.

From (3) and (4), it follows instantly that B_2 is the only type-B bridge in G and $R = v_1v_0v_2$ is a path in B_2. If B_2 contains an edge v_0v_3 with $v_3 \notin \{v_1, v_2\}$, then $C[v_2, v_1] \cup R \cup \{v_1v_1^+, v_2^-v_2, v_0v_3\}$ would be an odd subdivision of F_1 in G, a contradiction.

The same argument implies that

(6) no type-A bridge has a foot on $C[v_2, v_1]$.

Let u_3, u_4 be two vertices on C, such that v_1, u_3, u_4, v_2 occur on C in clockwise cyclic order, each of u_3 and u_4 is a foot of some type-A bridge, and no vertex in $C(v_1, u_3) \cup C(u_4, v_2)$ is a foot of a type-A bridge. Then

(7) $d_G(u_j) \geq 3$ for $j = 3, 4$.

Set $K = C(u_4, u_3) \cup R$ and $L = G \backslash V(K)$. By (5) and (6), the only edges between K and L are $u_3^-u_3$ and $u_4u_4^+$. It follows from (7) that

(8) $d_L(u_j) \geq 2$ for $j = 3, 4$.

As H is a bipartite graph, so is L. Let (S, T) be the bipartition of L, with $\{u_3, u_4\} \subseteq S$. If $d_L(t) \geq 3$ for some $t \in T$, then Lemma 4.4(ii) guarantees the existence of a u_3u_4-path P and two disjoint edges u_4w_1 and w_2u_5 in L, with $w_2 \in V(P) \cap T$ while $\{w_1, u_5\} \cap V(P) = \emptyset$. Thus $C[u_4, u_3] \cup P \cup \{v_2v_0, u_4w_1, w_2u_5\}$ would be a fully odd subdivision of F_1 in G. This contradiction implies that L is a fully subdivided graph in which both u_3 and u_4 belong to color 1 class. Hence $G = H_2$ in Figure 6, because $L \cup C(v_1, u_3] \cup C(u_4, v_2)$ is also fully subdivided.

Subcase 1.2. There exist four vertices u_1, u_2, v_1, v_2, such that $x_1, u_1, u_2, v_1, v_2, x_2$ occur on C in clockwise cyclic order and that u_1, u_2 are two feet of a type-A bridge B_1 and v_1, v_2 are two feet of a type-B bridge B_2. (Possibly $x_1 = u_1$ or $x_2 = v_2$.)

In this subcase, note that no type-B bridge B_3 has a foot v_3 on $C(u_1, u_2)$, for otherwise, let u_2u_2' be an edge in B_1, v_1v_1' an edge in B_2, and v_3v_3' an edge in B_3. Then $C \cup \{v_3v_3', u_2u_2', v_1v_1'\}$ would be a fully odd subdivision of F_1, a contradiction. The same argument implies the existence of four vertices u_3, u_4, v_3, v_4, such that $x_1, u_3, u_4, v_3, v_4, x_2$ occur on C in clockwise cyclic order, no type-A (resp. type-B) bridge has a root outside $C[u_3, u_4]$ (resp. $C[v_3, v_4]$), and each of u_3 and u_4 (resp. v_3 and v_4) is a foot of some type-A (resp. type-B) bridge.

Let K denote the union of $C[u_3, u_4]$ and all type-A bridges, and L denote the union of $C[v_3, v_4]$ and all type-B bridges. Since H is bipartite, so are K and L. Using the same argument as employed in the paragraph right above the description of the present subcase, with an edge v_3v_3' in a type-B bridge in place of v_2v_0 over there, we deduce that K is a fully subdivided graph in which both u_3 and u_4 belong to color 1 class. Similarly, we can prove that L is a fully subdivided graph in which both v_3 and v_4 belong to color 1 class. Renaming the subscripts of x_1 and x_2 if necessary, we may assume that $A = X$ and $B = Y$. It follows that $G = H_3$ in Figure 6, because both $K \cup C[x_1, u_4]$ and $L \cup C[v_3, x_2)$ are fully subdivided as well.

Subcase 1.3. There exist three vertices u_1, u_2, v, such that x_1, u_1, v, u_2, x_2 occur on C in clockwise cyclic order and that u_1, u_2 are two feet of a type-A bridge B_1 and v is the only foot of a type-B bridge B_2. (Possibly $x_i = u_i$ for $i = 1$ or 2.)

In this subcase, we may assume that each type-B bridge has only one foot, otherwise one of the previous two subcases occurs. Since G is i-2-c, we further obtain

(9) each type-B bridge is an edge.

Using the same argument as employed in Subcase 1.1, we deduce that

(10) B_1 is only type-A bridge in G, which is either a path $R = u_1u_0u_2$ or a star R^* arising from R by adding an edge u_0u_3. Moreover, no type-B bridge has a foot on $C[u_2, u_1]$. (Note that if $B_1 \neq R$, then $B_1 = R^*$ because G is i-2-c.)

When $B_1 = R$, let K be the union of $C(u_1, u_2)$ and all type-B bridges. Then K is a fully subdivided graph in which both u_1^+ and u_2^- belong to color 1 class. So $G = H_2$ in Figure 5 by (9) and (10). When $B_1 = R^*$, the length of $C[u_1, u_2]$ is two, for otherwise, $C[u_2, u_1] \cup R \cup \{u_0u_3, u_1u_1^+, u_2^-u_2\}$ would be a fully odd subdivision of F_1, a contradiction. Since G is i-2-c, B_2 is the only type B-bridge having v as the root. Thus $G = H_4$ in Figure 6.

Subcase 1.4. There exist three vertices u_1, u_2, v, such that x_1, u_1, u_2, v, x_2 occur on C in clockwise cyclic order and that u_1, u_2 are two feet of a type-A bridge B_1 and v is the only foot of a type-B bridge B_2. (Possibly $x_1 = u_1$ or $x_2 = v$.)

Similar to Subcase 1.3, we may assume that each type-B bridge is an edge. The remainder of the proof goes along the same line as that in Subcase 1.2. The same argument implies the existence of four vertices u_3, u_4, v_3, v_4, such that $x_1, u_3, u_4, v_3, v_4, x_2$ occur on C in clockwise cyclic order, no type-A (resp. type-B) bridge has a foot outside $C[u_3, u_4]$ (resp. $C[v_3, v_4]$), and each of u_3 and u_4 (resp. v_3 and v_4) is a foot of some type-A (resp. type-B) bridge. Let K denote the union of $C[u_3, u_4]$ and all type-A bridges, and L denote the union of $C[v_3, v_4]$ and all type-B bridges. Since G contains no fully odd subdivision of F_1, from Lemma 4.4(ii) we deduce that K is a fully subdivided graph in which both u_3 and u_4 belong to color 1 class. (The details can be found in the paragraph right above Subcase 1.2.) Clearly, L is a fully subdivided graph in which both both v_3 and v_4 belong to color 1 class. Renaming the subscripts of x_1 and x_2 if necessary, we see that $G = H_3$ in Figure 6.

Subcase 1.5. There exist two vertices u, v, such that x_1, u, v, x_2 occur on C in clockwise cyclic order and that u is the only foot of a type-A bridge B_1 and v is the only foot of a type-B bridge B_2. (Possibly $x_1 = u$ or $x_2 = v$.)

In this subcase, we may assume that each bridge has only one foot in C, otherwise one of the previous subcases occurs. It follows that each bridge is an edge because G is i-2-c. Using the same argument as employed in Subcase 1.2, we obtain four vertices u_3, u_4, v_3, v_4, such that $x_1, u_3, u_4, v_3, v_4, x_2$ occur on C in clockwise cyclic order, no type-A (resp. type-B) bridge has a foot outside $C[u_3, u_4]$ (resp. $C[v_3, v_4]$), and each of u_3 and u_4 (resp. v_3 and v_4) is a foot of some type-A (resp. type-B) bridge. Renaming the subscripts of x_1 and x_2 if necessary, it is easy to see that $G = H_3$ in Figure 6.

Therefore, if $G \neq H_1$ and Case 1 occurs, then G is H_i for some $2 \leq i \leq 4$

Case 2. Some bridge has feet in both X and Y.

In this case, we may assume that

(11) the length of C is at least five.

Suppose the contrary: C is a triangle $x_1x_2y_1$ (as C is an odd cycle). By hypothesis, some bridge B has feet y_1 and x_i for $i = 1$ or 2, say the former. Let C' be obtained from the path $x_1x_2y_1$ by adding an x_1y_1-path in B. Since C' contains the edge x_1x_2 and H is bipartite, this new cycle C' is again odd and of length at least five. Note that C' has a bridge, x_1y_1, with feet in both X and Y. So (11) holds, otherwise we replace C by C'.

Let B be an arbitrary bridge with a foot $x_3 \in X$ and a foot $y_1 \in Y$. Let us show that

(12) If x_1, x_3, y_1, x_2 occur on C in clockwise cyclic order, then $x_3 = x_1$ and $y_1 = x_2^-$. If x_1, y_1, x_3, x_2 occur on C in clockwise cyclic order, then $x_3 = x_2$ and $y_1 = x_1^+$. (So B has precisely two feet in C.)

To justify this, we only consider the situation when x_1, x_3, y_1, x_2 occur on C in clockwise cyclic order, as the other situation is simply a mirror image. If B has a foot $x \in X$ on $C(x_1, x_2]$, then B contains a path P connecting x and y_1. Since H is bipartite, the length of P is odd. Thus $C \cup P$ would be a fully odd subdivision of F_3 by (11), a contradiction. So $x_3 = x_1$. The same argument implies that B has no foot $y \in Y$ on $C(x_1, x_2^-)$ and hence $y_1 = x_2^-$. This proves (12).

Symmetry allows us to assume hereafter that some bridge B has feet x_1 and $y_1 = x_2^-$. Observe that x_2 has no neighbor z outside C, for otherwise, let P be an x_1y_1-path in B. Then the union of the odd cycle $Py_1x_2x_1$ and $\{x_1x_1^+, y_1^-y_1, x_2z\}$ would be a fully odd subdivision of F_1, a contradiction. So

(13) every bridge having x_2 as a foot is the edge $x_1^+x_2$ (see (12)).

Similarly, we can prove that

(14) if $x_1^+x_2$ is an edge of G, then every bridge having x_1 as a foot is the edge x_1y_1.

Let us distinguish between two subcases.

Subcase 2.1. $x_1^+x_2$ is an edge of G.

In this subcase, let u and v be two vertices on $C[x_1^+, y_1]$, such that $\{u, v\}$ is a subset of X or of Y, $d_G(a) \geq 3$ for $a = u$ or v (say the former), and $d_G(b) = 2$ for any vertex b in $C(x_1^+, u) \cup C(v, y_1)$, if any. Let K be the union of $C[u, v]$ and all bridges with a foot in $C[u, v]$, and let (S, T) be the bipartition of K, with $\{u, v\} \subseteq S$. If $d_K(t) \geq 3$ for some $t \in T$, then Lemma 4.4(ii) guarantees the existence of a uv-path P and two disjoint edges uw_1 and w_2w_3 in K, with $w_2 \in V(P) \cap T$ while $\{w_1, w_3\} \cap V(P) = \emptyset$. Set $L = C[v, u] \cup P \cup \{x_1^+x_2, uw_1, w_2w_3\}$ if $C[x_1^+, u]$ is of odd length and $L = C[v, u] \cup P \cup \{x_1y_1, uw_1, w_2w_3\}$ otherwise. Then L is a fully odd subdivision of F_2. This contradiction implies that K is a fully subdivided graph in which both u and v belong to color 1 class. Thus $G = H_5$ in Figure 6 by (13) and (14).

Subcase 2.2. $x_1^+x_2$ is not an edge of G.

In this subcase, $d_G(x_2) = 2$ by (13). Let G_1 be the graph arising from G by deleting all vertices in $B \backslash \{x_1, y_1\}$, and let G_2 be obtained from B by adding the path $x_1x_2y_1$. Then G_1 and G_2 have only path $x_1x_2y_1$ in common. For $i = 1, 2$, let C_i be an induced cycle containing the path $x_1x_2y_1$ in G_i. Note that C_i is an odd cycle.

(15) Every C_i-bridge in G_i has its feet only in X or only in Y for $i = 1, 2$.

Suppose the contrary: some C_i-bridge K in G_i has a foot $x \in X$ and a foot $y \in Y$. Let P be an xy-path in K. Notice that P is of odd length. If $\{x, y\} = \{x_1, y_1\}$, then $C_1 \cup C_2 \cup P$ is a full odd subdivision of F_4. If $\{x, y\} \neq \{x_1, y_1\}$, then $C_i \cup P$ is a fully odd subdivision of F_3. So we reach a contradiction in either situation.

It follows from (15) and the structural description in Case 1 that

(16) G_i is isomorphic to H_i in Figure 5 for some $1 \leq i \leq 4$.

We may assume that

(17) the fully subdivided graph involved in H_2 is not a path, otherwise such an H_2 can be drawn as an H_1.

Let us now prove that

(18) Neither G_1 nor G_2 is H_2.

Suppose the contrary: G_1 is H_2, say. Let K denote the fully subdivided graph involved in H_2 (see Figure 5). Let $P_1 = P_1[x_1, u_1]$ and $P_2 = P_2[x_2, u_2]$ be the two paths marked with α in H_2 in Figure 5, and let u_0 be the common neighbor of u_1 and u_2, which is of degree two. By (17) and Lemma 4.3, we can find an edge ab in K and a u_1u_2-path Q, such that $Q(u_1, u_2)$ is fully contained in K, $a \in V(Q)$ while $b \notin V(Q)$, and both $Q[u_1, a]$ and $Q[a, u_2]$ have odd length. Let x_1x_1' and y_1y_1' be two disjoint edges in G_2, with $x_2 \notin \{x_1', y_1'\}$. Set $L = P_1 \cup P_2 \cup Q \cup \{ab, u_1u_0, x_1x_1', x_1x_2\}$ if $\alpha =$odd and $L = P_1 \cup P_2 \cup Q \cup \{ab, u_2u_0, y_1y_1', x_1x_2\}$ otherwise. Then L is a fully odd subdivision of F_1. Thus (18) follows.

The same argument implies that

(19) Neither G_1 nor G_2 is H_4.

From (16), (18) and (19), we see that G_i is isomorphic to either H_1 or H_3 in Figure 5 for $i = 1, 2$. Let $G_1 = H_p$ and $G_2 = H_q$, where both p and q belong to $\{1, 3\}$. It is a routine matter to check for all possible combinations of p and q, the resulting graph G can always be drawn as an H_6 in Figure 6.

Therefore, if $G \neq H_1$ and Case 2 occurs, then G is either H_5 or H_6 in Figure 5.

Combining the observations in both Case 1 and Case 2, we conclude that G is one of the six graphs as depicted in Figure 6. ∎

4.3 D-subgraphs

A *diamond* is obtained from K_4 (the complete graph with four vertices) by deleting an edge. A diamond K with vertices s, t, u, v in a graph $G = (V, E)$ is called a *D-subgraph* of G if $uv \notin E$, $d_G(s) = d_G(t) = 3$, and $G\backslash\{s, t\}$ is connected. In this subsection we determine good graphs with D-subgraphs.

Lemma 4.10. *Let $G = (V, E)$ be an i-2-c and good graph with a D-subgraph. Then G is one of the three graphs depicted in Figure 7, where odd and any stand for the parities of the corresponding paths.*

Proof. By hypothesis, G contains a diamond K with vertices s, t, u, v such that $uv \notin E$, $d_G(s) = d_G(t) = 3$, and $G\backslash\{s, t\}$ is connected. Depending on the structure of $G\backslash\{s, t\}$, we distinguish among three cases.

Case 1. $G\backslash\{s, t\}$ is bipartite, in which u and v are in the same color class.

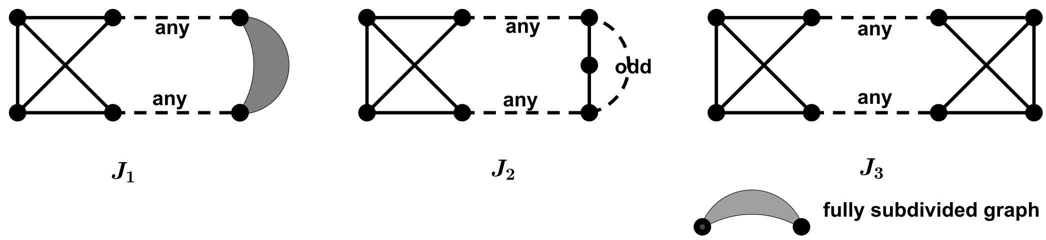


Figure 7: Good graphs with D-subgraphs

In this case, set $H = G\backslash\{s,t\}$ and $G' = H + uv$. From Lemma 4.9 with $(x_1, x_2) = (u, v)$, we see that G' is H_i in Figure 5 for some $1 \le i \le 6$. So G is obtained from H_i by replacing the edge x_1x_2 with the diamond K.

Subcase 1.1. $i = 1$. In this subcase, clearly $G = J_1$ in Figure 7.

Subcase 1.2. $i = 2$. In this subcase, we may assume that the fully subdivided graph L involved in H_2 in Figure 5 is not a path, otherwise H_2 can be drawn as H_1, so the current subcase is the same as the previous one. Let $P_1 = P_1[x_1, y_1]$ and $P_2 = P_2[x_2, y_2]$ be the two paths marked with α in H_2 in Figure 5, and let y_0 be the common neighbor of y_1 and y_2, which is of degree two. By Lemma 4.3, we can find an edge ab in L and a y_1y_2-path Q, such that $Q(y_1, y_2)$ is fully contained in L, $a \in V(Q)$ while $b \notin V(Q)$, and both $Q[y_1, a]$ and $Q[a, y_2]$ have odd length. By Lemma 4.6, $K \cup P_1 \cup P_2 \cup Q \cup \{y_1y_0, ab\}$ contains a fully odd subdivision of F_2 in G, a contradiction.

Subcase 1.3. $i = 3$. In this subcase, we may assume that neither of the fully subdivided graphs L_1 (with $x_1 \in V(L_1)$) and L_2 involved in H_3 in Figure 5 is a path, otherwise Subcase 1.1 occurs. By Lemma 4.3, we can find an edge ab in L_1, an edge cd in L_2, and an x_1x_2-path Q in $H_3\backslash x_1x_2$, such that $\{a, c\} \in V(Q)$ while $\{b, d\} \cap V(Q) = \emptyset$, and $Q[x_1, a]$ is of even length while $Q[c, x_2]$ is of odd length. Note that Q is of even length. By Lemma 4.6, $K \cup Q \cup \{ab, cd\}$ contains a fully odd subdivision of F_2 in G, a contradiction.

Subcase 1.4. $i = 4$ and 5. In this subcase, Lemma 4.6 guarantees the existence of a fully odd subdivision of F_2 in G, a contradiction.

Subcase 1.5. $i = 6$. In this subcase, let $x_1x_2y_1$ be the path of length two contained in H_6. From its structure, we see that H_6 contains two odd cycles C_1 and C_2 which have only the path $x_1x_2y_1$ in common. Let L be obtained from $C_1 \cup C_2$ by replacing the edge x_1x_2 with the path $ustv$ in K. Then L is a fully odd subdivision of F_3 in G, a contradiction again.

Therefore, if Case 1 occurs, then $G = J_1$ in Figure 7.

Case 2. $G\backslash\{s,t\}$ is bipartite, in which u and v are in different color classes.

In this case, set $G' = G\backslash t$ and $H = G'\backslash us$. From Lemma 4.9 with $(x_1, x_2) = (u, s)$, we see that G' is H_i in Figure 5 for some $1 \le i \le 6$ and $i \ne 5$ (because $d_{H_5}(x_2) = 3$ while $d_{G'}(s) = 2$). So G is obtained from H_i by adding vertex t and three edges tx_1, tx_2, ty_1, where y_1 is the only neighbor of x_2 other than x_1 in H_i. Note that y_1 corresponds to v in K.

Subcase 2.1. $i = 1$. In this subcase, clearly $G = J_1$ in Figure 7.

Subcase 2.2. $i = 2$. In this subcase, once again we may assume that the fully subdivided graph L involved in H_2 in Figure 5 is not a path. Let $P_1 = P_1[x_1, z_1]$ and $P_2 = P_2[x_2, z_2]$ be the two paths marked with α in H_2 in Figure 5, and let z_0 be the common neighbor of z_1 and z_2, which is of degree two. By Lemma 4.3, we can find an edge ab in L and a z_1z_2-path Q, such that $Q(z_1, z_2)$ is fully contained in L, $a \in V(Q)$ while $b \notin V(Q)$, and both $Q[z_1, a]$ and $Q[a, z_2]$ have odd length. By Lemma 4.6, $K \cup P_1 \cup P_2 \cup Q \cup \{z_1z_0, ab\}$ contains a fully odd subdivision of F_1 in G, a contradiction.

Subcase 2.3. $i = 3$. In this subcase, once again we may assume that neither of the fully subdivided graphs L_1 (with $x_1 \in V(L_1)$) and L_2 involved in H_3 in Figure 5 is a path. By Lemma 4.3, we can find an edge ab in L_1, an edge cd in L_2, and an x_1x_2-path Q in $H_3 \backslash x_1x_2$, such that $\{a, c\} \in V(Q)$ while $\{b, d\} \cap V(Q) = \emptyset$, and $Q[x_1, a]$ is of even length while $Q[c, x_2]$ is of odd length. Note that Q is of even length. By Lemma 4.6, $K \cup Q \cup \{ab, cd\}$ contains a fully odd subdivision of F_1 in G, a contradiction.

Subcase 2.4. $i = 4$. In this subcase, Lemma 4.6 guarantees the existence of a fully odd subdivision of F_1 in G, a contradiction.

Subcase 2.5. $i = 6$. In this subcase, let $x_1x_2y_1$ be the path of length two contained in H_6. From its structure, we see that H_6 contains two odd cycles C_1 and C_2 which have only the path $x_1x_2y_1$ in common. Since G is simple, at least one of C_1 and C_2 has length at least five, say C_1. Let e, f be the two edges incident with x_1, y_1 in $C_1 \backslash x_2$, respectively. Then $C_2 \cup \{e, f, x_2t\}$ would be a fully odd subdivision of F_1 in G, a contradiction again.

Therefore, if Case 2 occurs, then $G = J_1$ in Figure 7 as well.

Case 3. $G \backslash \{s, t\}$ is nonbipartite.

By hypothesis of the present case, $G \backslash \{s, t\}$ contains an *induced* odd cycle C. By Lemma 4.2, G contains two disjoint paths from s to C which have only s in common, and these two paths yield two *induced* disjoint paths $P_1 = P_1[u, x]$ and $P_2 = P_2[v, y]$, where x, y are two vertices on C. Let Q_1 (resp. Q_2) be the xy-segment of C with odd (resp. even) length. Observe that the length of Q_2 is two, for otherwise, let R be an xy-path of odd length contained in $K \cup P_1 \cup P_2$. Then $R \cup C$ would be a fully odd subdivision of F_3, a contradiction. We reserve the symbol z for the internal vertex of Q_2 hereafter, and consider two subcases.

Subcase 3.1. $d_G(z) = 2$.

Let $L = K \cup P_1 \cup P_2 \cup C$. In view of the degrees of s, t and z, no edge of G outside L is incident with any one in $\{s, t, z\}$. Recall that P_1 and P_2 are induced paths in G, and C is an induced cycle. If $G \neq L$, then G contains an edge e outside L, which is between P_1 and P_2, or has precisely one end in $P_1 \cup P_2 \cup Q_1$ (and the other end outside L), or is between Q_1 and $P_1 \cup P_2$; in each situation, it is a routine matter to check, using Lemma 4.6, that $L \cup \{e\}$ contains a fully odd subdivision of F_1 or F_2. This contradiction implies that $G = L$ and hence $G = J_2$ in Figure 7.

Subcase 3.2. $d_G(z) \geq 3$.

In this subcase, let x' be the vertex adjacent to x on the path suP_1, and let y' be the vertex adjacent to y on the path tvP_2. Let us show that

(1) $d_G(z) = 3$ and $N_G(z)$, the neighborhood of z in G, is either $\{x, y, x'\}$ or $\{x, y, y'\}$.

To justify this, note that z has no neighbor w outside $\{x, x', y, y'\}$, for otherwise, $C \cup \{zw, xx', yy'\}$ would be a fully odd subdivision of F_1, a contradiction. Assume on the contrary that z is adjacent to both x' and y'. Then $x' \neq s$ because $N_G(s) = \{t, u, v\}$. Let a be the vertex adjacent to x on Q_1, and let b be the neighbor of x' on suP_1 other than x. Then the union of the triangle $xx'z$ and $\{xa, x'b, zy'\}$ would be a fully odd subdivision of F_1 in G. This contradiction yields (1).

Symmetry allows us to assume that

(2) $N_G(z) = \{x, y, x'\}$.

Notice that

(3) $Q_1 = xy$.

Otherwise, let a and b be as defined in the proof of (1). Then $a \neq y$. So the union of the triangle $xx'z$ and $\{xa, x'b, zy\}$ would be a fully odd subdivision of F_1 in G. This contradiction justifies (3).

With x in place of z, the same argument implies that

(4) $N_G(x) = \{x', y, z\}$.

Let $L = K \cup P_1 \cup P_2 \cup \{x'z\}$. In view of (2), (4) and the degrees of s and t, no edge of G outside L is incident with any one in $\{s, t, x, z\}$. Recall that P_1 and P_2 are induced paths in G. If $G \neq L$, then G contains an edge e outside L, which either is between $P_1[u, x']$ and P_2 or has precisely one end in $P_1[u, x'] \cup P_2$; in either situation $L \cup \{e\}$ contains a fully odd subdivision of F_1 or F_2 by Lemma 4.6. This contradiction implies that $G = L$ and hence $G = J_3$ in Figure 7.

Therefore, if Case 3 occurs, then $G = J_2$ or J_3 in Figure 7.

Combining the above three cases, we conclude that G is one of the three graphs depicted in Figure 7. ∎

4.4 Reductions and Extensions

A *signed graph* is a triple $G = (V, E, \Sigma)$, where (V, E) is an undirected graph and $\Sigma \subseteq E$. Throughout this section,

$$G \text{ may have parallel edges, but neither } \Sigma \text{ nor } E\backslash\Sigma \text{ contains multiple members.} \tag{4.1}$$

An edge e of G is called *odd* if $e \in \Sigma$ and *even* otherwise. The *realization* G^* of G is the ordinary graph arising from G by subdividing each even edge exactly once. A path or a cycle in G is called *odd* (resp. *even*) if it contains an odd (resp. even) number of odd edges. Naturally, G is called *bipartite* if it contains no odd cycles. It is easy to see that G is bipartite if and only if G^* is bipartite if and only if V can be partitioned into (X, Y) such that edges between X and Y are precisely odd edges of G (as usual, X and Y are called two *color classes* of G). We also call G *good* if G^* is good, and call G *i-2-c* if G^* is i-2-c.

In this subsection we explore properties of signed graphs that are determined by their realizations. So we may simply think of a signed graph G as a (compact) representation of G^*.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two subgraphs of a signed graph $G = (V, E, \Sigma)$, such that

- E_1 and E_2 form a partition of E;
- $V_1 \cap V_2 = \{x, y\}$; and
- both G_1 and G_2 are connected, G_2 is bipartite, and $|E_2| \geq 2$.

We define *bipartite reduction* (or simply *B-reduction*) as follows. When x, y are in different color classes of G_2 and xy is not an odd edge in G_1, the operation of reducing G to $G_1' = G_1 + xy$, where xy is defined to be odd in G_1', is called a *B_1-reduction*; when x, y are in the same color class of G_2 and xy is not an even edge in G_1, the operation of reducing G to $G_1' = G_1 + xy$, where xy is defined to be even in G_1', is called a *B_2-reduction.* Correspondingly, we say that G is a *B_i-extension* of G_1' by using xy for $i = 1$ or 2, and call both B_1- and B_2-extensions *B-extensions.*

Notice that (4.1) is preserved on G_1' under either reduction operation. So a reduction of a signed graph results in a signed graph again. Let us make some other trivial observations about signed graphs, which will be used implicitly in our discussion.

- A reduction of a nonbipartite signed graph is again nonbipartite;
- A reduction of an i-2-c signed graph is again i-2-c;
- A reduction of a good signed graph is again good; and
- A reduction of a signed graph has fewer edges than the original graph.

The following simple observation reveals that the B-extensions enjoy some transitivity property.

Lemma 4.11. *If G' is a B-extension of G'' obtained by replacing an edge e with a bipartite graph H_e, and G is a B-extension of G' using an edge in H_e, then G is also a B-extension of G'' using e.* ∎

A diamond K with vertices s, t, u, v in a signed graph $G = (V, E, \Sigma)$ is called a *D-subgraph* of G if all five edges of K are odd, $uv \notin \Sigma$, $d_G(s) = d_G(t) = 3$, and $G \backslash \{s, t\}$ is connected. For simplicity, G is called *D-free* if it contains no D-subgraph.

Lemma 4.12. *Let $G = (V, E, \Sigma)$ be an i-2-c and good signed graph, and let $G' = (V', E', \Sigma')$ be obtained from G by a series of B-reductions. Suppose G is D-free, while G' contains a D-subgraph K with vertices s, t, u, v such that $uv \notin \Sigma'$. Then the following statements hold:*

(i) There exists an i-2-c and good signed graph G'' which also contains K as a D-subgraph, such that G is obtained from G'' by performing precisely one B-extension using edge st;

(ii) If all edges of G are odd, then G is the graph shown in Figure 8.

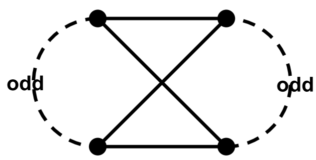


Figure 8: Good graphs containing D-subgraphs in reductions

Proof. Since $L = G' \backslash \{s, t\}$ is connected and $uv \notin \Sigma'$, there exists a uv-path Q of length at least two in the realization L^* of L. Let uu' and vv' be the two edges of Q incident with u and v, respectively. Our proof is based on the following two observations about B-reductions.

(1) If the edge su in K is created to replace a bipartite graph H_{su} in a B-reduction, then H_{su} consists of precisely two edges incident with u, including su;

To justify this, note that H_{su} contains no edge sw with $w \neq u$, for otherwise, the union of the triangle stv and three edges sw, tu, vv' would yield a fully odd subdivision of F_1 in G^*, a contradiction. So u is cut vertex of H_{su}. As G' is also i-2-c, H_{su} contains precisely two edges, including su.

(2) If the edge st in K is created to replace a bipartite graph H_{st} in a B-reduction, then H_{st} is an odd st-path of length at least three, each uv-path in L is odd, and none of four edges in $K\backslash st$ arises from B-reductions.

Assume the contrary: H_{st} is not an odd st-path. Then, by Lemma 4.3, H_{st} contains an odd st-path P (as $st \in \Sigma'$) and an edge ab with $a \in V(P)$ while $b \notin V(P)$. By symmetry, we may assume that $P[a,t]$ is odd. Thus the union of the odd cycle $sPtvs$ and three edges ab, tu, vv' would yield a fully odd subdivision of F_1 in G^*. This contradiction implies that H_{st} is an odd path with at least three edges.

If L contains an even uv-path R, then $R \cup \{su, sv, tv\} \cup H_{st}$ would yield a fully odd subdivision of F_3 in G^*, a contradiction again. In particular, it follows that the path Q is of odd length.

If one of the remaining four edges in K, say su (by symmetry), is created to replace a bipartite graph H_{su} in a B-reduction. By (1), H_{su} contains precisely two edges us and ux. Let ss' be the edge on H_{st} incident with s. Then $Q \cup usv \cup \{ux, ss', vt\}$ would yield a fully odd subdivision of F_1 in G^*. This contradiction establishes (2).

Now we are ready to present a proof of (i) and (ii). Since G contains no D-subgraph and all edges in $K\backslash st$ are symmetric, from (1) we deduce that st in K is created to replace a bipartite graph H_{st} in a B-reduction. Hence none of the four edges in $K\backslash st$ arises from B-reductions by (2). In view of Lemma 4.11, we may assume the existence of a subset Ω of E, such that G is obtained from G' by performing B-extensions using all edges e in Ω, with each e replaced by a bipartite graph H_e. Let G'' be obtained from G' by replacing each $e \in \Omega\backslash st$ with H_e. Clearly, G is a B-extension of G'' using st. As G'' is an i-2-c and good signed graph and contains K as a D-subgraph, (i) is established.

Without loss of generality, we assume that $G' = G''$. Recall (2), H_{st} is an odd st-path of length at least three. Imitating the proof of this statement, we deduce that L is an odd uv-path as well. Therefore, G is as depicted in Figure 8. This proves (ii). ∎

4.5 Irreducible Graphs

Let $G = (V, E, \Sigma)$ be an i-2-c good signed graph with all edges odd. By virtue of Lemma 4.10, we may assume that G is D-free, otherwise a structural description of G is already available. If G can be reduced by a series of B-reductions to a signed graph G' that contains a D-subgraph, then G is as depicted in Figure 8 by Lemma 4.12. We may further assume that G can be reduced by a series of B-reductions to an i-2-c, good, D-free signed graph, to which no B-reduction is applicable. This class of signed graphs is exactly the subject of our study in this subsection. For convenience, we call a signed graph *irreducible* if it is i-2-c, good, D-free, nonbipartite, and admits no B-reductions.

Lemma 4.13. *The list of all irreducible signed graphs is as given in Figure 9, where T_{10} is an arbitrary ladder in which only the top and bottom are odd edges.*

Let us exhibit some properties satisfied by an irreducible signed graph $G = (V, E, \Sigma)$ and analyze a few cases before presenting a proof of this lemma. Note that if $|V| = 2$, then G is T_1 in Figure 9 (for G is nonbipartite), which is called a 2-*gon*.

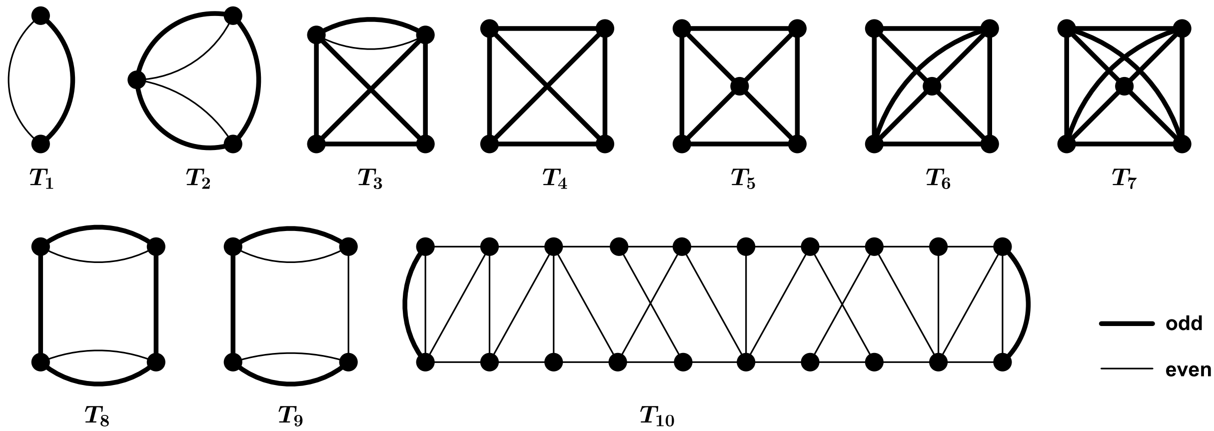


Figure 9: Irreducible signed graphs

Lemma 4.14. *Let $G = (V, E, \Sigma)$ be an irreducible signed graph with $|V| \geq 3$. Then the following statements hold:*

(i) $d_G(v) \geq 3$ for all $v \in V$, so both G and G^* are 2-connected; and

(ii) $|\Sigma| \geq 2$.

Proof. (i) Let us first show that $d_G(v) \neq 1$ for any $v \in V$. Otherwise, let u be neighbor of v, let uw be an edge in G with $w \neq v$, and let H be the bipartite graph consisting of uv and uw only. Then we can perform a B-reduction on G by replacing H with uw (having the same parity as before), which contradicts the hypothesis that G is irreducible.

Let us turn to proving that $d_G(v) \neq 2$ for any $v \in V$. Otherwise, let u and w be the neighbors of v, and let H be the graph consisting of edges uv, uw, and the edge vw with the same parity as the path $R = uvw$, if any. (Possibly G contains both odd and even vw.) Then H is a bipartite graph, and hence we can perform a B-reduction on G by replacing H with an edge vw (having the same parity as R), a contradiction again.

Combining the above two observations, we see that $d_G(v) \geq 3$ for all $v \in V$. Since G is i-2-c, it follows instantly that G and G^* are both 2-connected.

(ii) Suppose on the contrary that $|\Sigma| = 1$. Let $\Sigma = \{uv\}$ and $H = G \backslash uv$. Then we can perform a B-reduction on G by replacing H with an even edge uv, contradicting the hypothesis that G is irreducible. ∎

Lemma 4.15. *Let $G = (V, E, \Sigma)$ be an irreducible signed graph that contains a triangle with three odd edges. Then G is T_i in Figure 9 for some i with $2 \leq i \leq 7$.*

Proof. We shall first give a structural description of G^* (the realization of G), and then transform it into information about G. Depending on presence or absence of K_4 (the complete graph with four vertices) in G^*, we consider two cases.

Case 1. G^* contains a K_4.

In this case, let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of a K_4 in G^*. Observe that

(1) G^* contains no two edges u_iu_5 and u_ju_6 with $1 \leq i \neq j \leq 4$ and $\{u_5, u_6\} \cap U = \emptyset$.

Suppose the contrary. Symmetry allows us to assume that $i = 1$ and $j = 2$. Then the union of the triangle $u_1u_2u_3$ and three edges u_1u_5, u_2u_6, u_3u_4 would be an F_1 in G^*. This contradiction justifies (1).

Throughout $\bar{N}_A(v)$ stands for the set of all neighbors of a vertex v outside a vertex subset A in G^*.

(2) $|\bar{N}_U(u_i)| \leq 1$ for all $1 \leq i \leq 4$.

Otherwise, $|\bar{N}_U(u_1)| \geq 2$, say. Then $\bar{N}_U(u_i) = \emptyset$ for $i = 2, 3, 4$ by (1). Thus u_1 is a cut vertex of G^*, a contradiction.

From (1) and (2), we deduce that G^* contains at most one vertex outside U. If U is the whole vertex set of G^*, then G^* and hence $G = K_4$, which is exactly T_4 in Figure 9. It remains to consider the situation when G^* contains a fifth vertex u_5. By Lemma 4.14(i), G^* is 2-connected, so u_5 has at least two neighbors in U, which implies that G is T_3, T_6 or T_7 in Figure 9.

Case 2. G^* contains no K_4.

In this case, let $A = \{u_1, u_2, u_3\}$ be the vertex set of a triangle in G^* (see the hypothesis). By Lemma 4.14(i), we have $d_G(u_i) \geq 3$, so $\bar{N}_A(u_i) \neq \emptyset$ for $i = 1, 2, 3$. If these three sets are pairwise disjoint, then G^* would contain an F_1, a contradiction. Thus, by symmetry, we may assume that u_1 and u_2 have a common neighbor $u_4 \neq u_3$. So $U = \{u_1, u_2, u_3, u_4\}$ induces a diamond K in G^* by the hypothesis of the present case. Notice that

(3) if $\bar{N}_U(u_i) \neq \emptyset \neq \bar{N}_U(u_j)$ for $i = 1$ or 2 and $j = 3$ or 4, then $|\bar{N}_U(u_i) \cup \bar{N}_U(u_j)| = 1$.

Suppose the contrary: $|\bar{N}_U(u_i) \cup \bar{N}_U(u_j)| \geq 2$, say $i = 1$ and $j = 3$. Then G^* contains two edges u_1u_1' and u_3u_3' with $\{u_1', u_3'\} \cap U = \emptyset$. Thus the union of the triangle $u_1u_2u_3$ and three edges u_1u_1', u_2u_4, u_3u_3' would be an F_1 in G^*. This contradiction establishes (3).

If both u_3 and u_4 have degree two in G^*, then there would be two even edges between u_1 and u_2 in G, contradicting (4.1). So we may assume, by symmetry, that

(4) $\bar{N}_U(u_3) \neq \emptyset$.

(5) $\bar{N}_U(u_i) \neq \emptyset$ for $i = 1$ or 2.

Assume the contrary: $\bar{N}_U(u_i) = \emptyset$ for $i = 1, 2$. If $\bar{N}_U(u_4) \neq \emptyset$, then $G \backslash \{u_1, u_2\}$ is connected (for otherwise each of u_3 and u_4 would be a cut vertex of G^*). Thus K is a D-subgraph of G, contradicting the hypothesis that G is irreducible. If $\bar{N}_U(u_4) = \emptyset$, then u_3 would be a cut vertex of G^*. This contradiction proves (5).

By (5) and symmetry, we may assume that u_5 is a vertex in $\bar{N}_U(u_1)$. By (3) and (4), u_3 is adjacent to u_5 in G^*. Furthermore, u_5 is the only vertex outside U that is adjacent to U. As $G^* \backslash u_5$ is connected, the whole vertex set of G^* is $\{u_1, u_2, \ldots, u_5\}$. From the hypothesis of the present case, we see that u_5 is nonadjacent to u_2. Thus G is either T_2 or T_5 in Figure 9, depending on whether u_5 is adjacent to u_4. ∎

Lemma 4.16. *Let $G = (V, E, \Sigma)$ be an irreducible signed graph such that $G \backslash \{u_1v_1, u_2v_2\}$ is disconnected for some two odd edges u_1v_1 and u_2v_2. If one component of $G \backslash \{u_1v_1, u_2v_2\}$ contains a 2-gon on $\{u_1, u_2\}$, then G is T_8 in Figure 9.*

Proof. By Lemma 4.14(i), G is 2-connected. So u_1v_1 and u_2v_2 are disjoint, and $G \backslash \{u_1v_1, u_2v_2\}$ has precisely two components G_1 and G_2, with $\{u_1, u_2\} \subseteq V(G_1)$. According to the hypothesis, $\{u_1, u_2\}$ induces a 2-gon in G_1; let $P = u_1u_0u_2$ be the path corresponding to the even u_1u_2 in G_1^*. We claim that

(1) G_2^* contains a v_1v_2-path of odd length.

Suppose the contrary: there is no v_1v_2-path of odd length in G_2^*. Since G^* is 2-connected, G_2^* contains no odd cycle by Menger's theorem, and hence is a bipartite graph in which v_1 and v_2 belong to the same color class. By Lemma 4.14(i), $d_G(v_i) \geq 3$ for $i = 1, 2$, so G_2 contains at least two edges. Hence we can perform a B_2-reduction on G by replacing the whole G_2 with an even edge v_1v_2, contradicting the hypothesis that G is irreducible.

(2) Let Q be a v_1v_2-path of odd length in G_2^*, and let $\bar{N}_Q(v)$ be the set of all neighbors of a vertex v in G_2^* outside Q. If $\bar{N}_Q(v_i) \neq \emptyset$ for $i = 1, 2$, then $|\bar{N}_Q(v_1) \cup \bar{N}_Q(v_2)| = 1$.

Otherwise, let v_1v_1' and v_2v_2' be two disjoint edges in G_2^*, with $\{v_1', v_2'\} \cap V(Q) = \emptyset$. Then $P \cup Q \cup v_1u_1u_2v_2 \cup \{v_1v_1', v_2v_2'\}$ would be a fully odd subdivision of F_2 in G^*, a contradiction.

(3) G_2^* contains an edge v_1v_2.

To justify this, let Q be a v_1v_2-path of odd length in G_2^* (see (1)); subject to this, Q is as short as possible. Suppose for a contradiction that the length of Q is at least three. Write $Q = a_0a_1a_2 \ldots a_t$, where t is odd and at least three.

(4) If G_2^* contains an edge a_ia_j, with $j \geq i + 2$, then $j = i + 2$

Suppose the contrary: $j - i \geq 3$. Let Q' be the path arising from Q by replacing $Q[a_i, a_j]$ with edge a_ia_j. If $j - i$ is odd, then Q' is also a v_1v_2-path of odd length, which is shorter than Q. Thus the existence of Q' contradicts the choice of Q. So we assume that $j - i$ is even. Consequently, $j - i \geq 4$. Since Q is of odd length, either $Q[a_0, a_i]$ or $Q[a_j, a_t]$ is of odd length, say the former. Thus the union of the odd cycle $Q' \cup v_2u_2u_1v_1$ and three edges $a_ia_{i+1}, a_{j-1}a_j, u_0u_2$ would be a fully odd subdivision of F_1, a contradiction.

(5) If G_2^* contains two edges a_ia_{i+2} and a_ja_{j+2}, with $j \geq i + 1$, then $j = i + 1$.

Otherwise, let Q' be the path arising from Q by replacing $Q[a_i, a_{i+2}]$ with edge a_ia_{i+2} and replacing $Q[a_j, a_{j+2}]$ with edge a_ja_{j+2}. Then Q' is also a v_1v_2-path of odd length, which is shorter than Q, a contradiction.

(6) If G_2^* contains an edge a_ia_{i+2}, then a_{i+1} has no neighbor in G_2^* outside Q.

Otherwise, let a_{i+1}' be such a neighbor of a_{i+1}. Then the triangle $a_ia_{i+1}a_{i+2}$ together with three edges $a_{i-1}a_i, a_{i+1}a_{i+1}', a_{i+2}a_{i+3}$, with $a_{-1} = u_1$ and $a_{t+1} = u_2$, would be an F_1 in G_2^*. This contradiction justifies (6).

If G_2^* contains only one edge of the form a_ia_{i+2}, then a_{i+1} has degree two in G_2^* by (4) and (6), contradicting Lemma 4.14(i). If G_2^* contains two edges of the form a_ia_{i+2} and $a_{i+1}a_{i+3}$, then both a_{i+1} and a_{i+2} have degree three in G^* by (4), (5) and (6). Thus the diamond on $\{a_i, a_{i+1}, a_{i+2}, a_{i+3}\}$ would be a D-subgraph of G, contradicting the hypothesis that G is irreducible. From (4) and (5), we thus deduce that Q is an induced path in G^*. So, by Lemma 4.14(i), each vertex a_i has at least one neighbor in G_2^* outside Q. In view of (2), there exists a vertex b in G_2^* such that $\bar{N}_Q(a_0) \cup \bar{N}_Q(a_t) = \{b\}$. The same proof of (2) yields $\bar{N}_Q(a_0) \cup \bar{N}_Q(a_1) = \{b\}$. Thus the triangle a_0a_1b together with three edges a_0u_1, a_1a_2, ba_t would be an F_1 in G^*. This contradiction completes the proof of (3).

Let Q stand for the odd edge v_1v_2. By (2), we have $\bar{N}_Q(v_1) \cup \bar{N}_Q(v_2) = \{v_3\}$ for some vertex v_3 in G_2^*. Since $G_2^* \backslash v_3$ is connected, $\{v_1, v_2, v_3\}$ is the whole vertex set of G_2^*, and hence G_2 is a 2-gon on $\{v_1, v_2\}$, which in turn implies that G_1 is also a 2-gon on $\{u_1, u_2\}$ by symmetry. Therefore G is T_8 in Figure 9. ∎

Lemma 4.17. *Let $G = (V, E, \Sigma)$ be an irreducible signed graph that contains an odd cycle with at least three odd edges. Suppose G contains no triangle with three odd edges and contains no* *cut with two odd edges as described in Lemma 4.16. Then G is T_9 in Figure 9.*

Proof. An odd cycle in G is called a *long cycle* if it contains at least three odd edges. In our proof we reserve the symbol C for a long cycle in G such that $|V(C)|$ is minimum and, subject to this, $|E(C) \cap \Sigma|$ is maximum. As usual, an edge outside C is called a *chord* of C if it has two ends on C. Each component of $C \backslash \Sigma$ is called a *gap* of C. Note that if a gap contains at least two vertices, then it consists of even edges only. For convenience, a chord e of C is also called a *chord of a gap* R if e is between two vertices of R.

(1) Each chord of a gap is an odd edge.

Assume the contrary: some chord uv of a gap R is even. Let C' be obtained from C by replacing $R[u, v]$ with this chord uv. Then C' is an odd cycle and contains all odd edges in C. Since C' is shorter than C, the existence of C' contradicts the choice of C.

(2) Each gap has at most one chord.

Suppose for a contradiction that some gap R has two chords u_1v_1 and u_2v_2. By (1), both u_1v_1 and u_2v_2 are odd edges. If $R[u_2, v_2] \subset R[u_1, v_1]$, then $R[u_1, v_1]$ corresponds to a path of length at least four in G^*, and hence $C \cup \{u_1v_1\}$ would yield a fully odd subdivision of F_3 in G^*, a contradiction. So, renaming subscripts of vertices if necessary, we assume that both v_1 and u_2 are on $R[u_1, v_2]$. Let C' be the cycle obtained from C by replacing $R[u_1, v_2]$ with the path $u_1v_1R[v_1, u_2]u_2v_2$. Then C' is an odd cycle and contains two more odd edges than C. Since C' is not longer than C, the existence of C' contradicts the choice of C.

An edge e outside C is called a *leaving edge* of a gap R if e has precisely one end in R.

(3) Each gap has at least one leaving edge.

To justify this, let R be an arbitrary gap, and let u and v be its two ends such that u^-u and vv^+ are two odd edges on C. By Lemma 4.14(i), $d_G(x) \geq 3$ for all vertices x. So the statement holds trivially if $u = v$. It remains to consider the case when $u \neq v$. Suppose on the contrary that R has no leaving edge. In view of (2) and the degrees of vertices on R, we deduce that $\{u, v\}$ induces a 2-gon in G and $\{u^-u, vv^+\}$ is a cut as described in Lemma 4.16, contradicting the hypothesis of our lemma.

A path P is called *C-external* if all internal vertices of P are outside C.

(4) Let P be a C-external uv-path between two different gaps. If $C[u, v]$ is even and corresponds to a path in G^* of length at least four, then P is even.

Otherwise, $C \cup P$ would correspond to a fully odd subdivision of F_3 in G^*. This contradiction justifies (4).

(5) Each chord between two different gaps is an even edge.

To justify this, let uv be such a chord. Renaming the vertices if necessary, we may assume that $C[u, v]$ is even. Since $C[u, v] \cup \{uv\}$ is not a triangle with three odd edges by hypothesis, $C[u, v]$ corresponds to a path of length at least four in G^*. Thus (5) follows instantly from (4).

(6) Let P_1, P_2 be two disjoint even C-external paths. If u_i, v_i are the ends of P_i for $i = 1, 2$ such that u_1, u_2, v_1, v_2 occur on C in clockwise cyclic order, then precisely one of $C[u_1, u_2]$, $C[u_2, v_1]$, $C[v_1, v_2]$, and $C[v_2, u_1]$ is odd.

Suppose the contrary: at least two of $C[u_1, u_2]$, $C[u_2, v_1]$, $C[v_1, v_2]$, and $C[v_2, u_1]$ are odd, so exactly three of them are odd as C is odd. By symmetry, we may assume that $C[u_1, u_2]$, $C[u_2, v_1]$, and $C[v_1, v_2]$ are odd. Let $Q_1 = C[v_2, u_1] \cup P_1$ and $Q_2 = P_2 \cup C[u_2, v_1]$. Then Q_1

corresponds to an even path of length at least four in G^*, and Q_2 corresponds to an odd path. Thus $Q_1 \cup Q_2 \cup C[v_1, v_2]$ would correspond to a fully odd subdivision of F_3 in G^*, a contradiction.

The following statements (7)-(9) are concerned with three leaving edges $e_i = u_i v_i$ for $i = 1, 2, 3$ of three different gaps of C, such that u_1, u_2, u_3 occur on C in clockwise cyclic order and that $C[u_1, u_2]$, $C[u_2, u_3]$ and $C[u_3, u_1]$ are all odd.

(7) At most one of e_1, e_2, e_3 is even, and at least two of them have vertices in common (possible are identical).

Assume that contrary: at least two of e_1, e_2, e_3 are even, or they are pairwise disjoint. For $i = 1, 2, 3$, let $u_i w_i v_i$ be the path corresponding to e_i in G^* if e_i is even, and let $w_i = v_i$ if e_i is odd. Note that e_i is even if $v_i \in V(C)$ by (5). It is then a routine matter to check that the union of C^* (realization of C) and three edges $u_i w_i$ for $i = 1, 2, 3$ would yield a fully odd subdivision of F_1 in G^*, no matter what the locations of the vertices v_i are. This contradiction justifies (7).

(8) The vertices v_1, v_2, v_3 are not all identical.

Otherwise, $v_1 = v_2 = v_3$. By (5) and (7), this vertex is outside C. Observe that at least one of e_1, e_2, e_3 is even, for otherwise, let C'_i be the cycle $C[u_i, u_{i+1}] \cup u_i v_1 u_{i+1}$ for $i = 1, 2, 3$, where $u_4 = u_1$ if $i = 3$. From the choice of C, we see that C'_i is not shorter than C. So C is a triangle with three odd edges, which contradicts the hypothesis of the present lemma. It follows from (7) that precisely one of e_1, e_2, e_3 is even, say e_3. Since $u_1 v_1 u_2$ is not a triangle with three odd edges, $C[u_3, u_2]$ corresponds to a path of length at least four. From (4) with $(u, v) = (u_3, u_2)$, we conclude that $P = u_3 v_1 u_2$ is even, a contradiction.

(9) If $v_1 = v_3$, then it is outside C. Furthermore, both e_1 and e_3 are odd edges.

Suppose the contrary: v_1 is on C. Then both e_1 and e_3 are even by (5), contradicting (7). In view of (8), we have $v_2 \neq v_1$. Also, e_2 is even if $v_2 \in V(C)$. Thus, if one of e_1 and e_3 is even, then $C \cup \{e_1, e_2, e_3\}$ corresponds to a subgraph of G^* which contains a fully odd subdivision of F_1. This contradiction implies that both e_1 and e_3 are odd.

Let $R_1, R_2, \ldots, R_\kappa$ be all the gaps of C that occur on C in clockwise cyclic order, and let $e_i = u_i v_i$ be a leaving edge of R_i, with $u_i \in V(R_i)$ for $1 \leq i \leq \kappa$.

(10) If $\kappa \geq 5$ and $e_2 \notin \{e_1, e_3\}$, then v_2 is outside C and $v_2 = v_i$ for $i = 1$ or 3. Furthermore, both e_2 and e_i are odd edges.

Suppose the contrary: $v_2 \neq v_i$ for $i = 1, 3$. From (7) and (9), we deduce that $v_1 = v_3$ and is outside C. Furthermore, both e_1 and e_3 are odd. Applying (7), (8) and (9) to edges e_2, e_3, e_4, we see that either $e_2 = e_4$ or $v_2 = v_4 \neq v_1$, and v_2 is outside C. Moreover, both e_2 and e_4 are odd. Let $P_1 = u_1 v_1 u_3$ and let $P_2 = e_2$ if $e_2 = e_4$ and $P_2 = u_2 v_2 u_4$ otherwise. Then the existence of these two paths contradicts (6). Thus $v_2 = v_i$ for $i = 1$ or 3, which is outside C by (5) and (7).

(11) $\kappa = 3$.

Suppose on the contrary that $\kappa \neq 3$. Then $\kappa \geq 5$ because it equals the total number of odd edges on C. By (10) and symmetry, we may assume that G contains a $u_1 u_2$-path P_1, which is either $e_1 = e_2$ or $u_1 v_1 u_2$. Using the edges e_2, e_3, e_4 and (10), we see that G also contains a $u_3 u_4$-path P_2, which is either $e_3 = e_4$ or $u_3 v_3 u_4$. By (8), P_1, P_2 and e_5 are pairwise disjoint. It thus follows from Lemma 4.7 that G^* contains a fully odd subdivision of F_1 or F_2. This contradiction yields (11).

Symmetry and (11) allow us to assume hereafter that $e_1 = e_3$ or $v_1 = v_3$. Let $u_4 u_5$ be the odd edge contained in $C[u_3, u_1]$ such that u_3, u_4, u_5, u_1 occur on C in clockwise cyclic order. We claim that

(12) $u_3 = u_4$, $u_5 = u_1$, and $e_1 = e_3$.

Assume that contrary: $u_3 \neq u_4$, say. If R_3 has a chord e_4 incident with u_4, then e_4 is odd by (1). Thus $C \cup \{e_4, e_1, e_2\}$ would yield a fully odd subdivision of F_2 in G^*. This contradiction implies that u_4 is not adjacent to any vertex on R_3 except u_4^-. Next, we show that R_3 has no leaving edge incident with u_4. Otherwise, let $e_5 = u_4v_4$ be such a leaving edge. Using the edges e_1, e_2, e_5 and using (5), (7) and (10), we conclude that either $v_4 = v_1$ or v_2 and e_5 is odd, or $e_2 = e_5$ and is even. Observe that if $v_4 = v_1$, then $u_1v_1u_4C[u_4, u_1]u_1$ would be an odd cycle that contradicts the choice of C. In the remaining two cases, the existence of the two paths with edge sets $\{e_1, e_3\}$ and $\{e_2, e_5\}$, respectively, would contradict (6). Combining the above two observations, we conclude that $d_G(u_4) = 2$, contradicting Lemma 4.14(i). Hence $u_3 = u_4$ and $u_5 = u_1$. Since G contains no triangle with three odd edges, we further have $e_1 = e_3$. So (12) is justified.

Let u_6u_7 and u_8u_9 be two odd edges in $C \backslash u_4u_5$, such that $u_4, u_5, \ldots, u_9$ occur on C in clockwise cyclic order.

(13) R_1 or R_3 has at least one leaving edge outside $\{e_1, u_4u_5\}$.

Otherwise, neither R_1 nor R_3 has a leaving edge outside $\{e_1, u_4u_5\}$. By Lemma 4.14(i), $d_G(u_i) \geq 3$ for $i = 6, 9$, so $u_1 = u_6$ and $u_3 = u_9$. Consequently, $\{u_6u_7, u_8u_9\}$ is an edge cut as described in Lemma 4.16, contradicting the hypothesis of the present lemma.

Let e_6 be an arbitrary leaving edge of R_1 or R_3 outside $\{e_1, u_4u_5\}$, having at least one end u_{10} in $R_1 \cup R_3$. With $\{e_1, e_2, e_6\}$ in place of $\{e_1, e_2, e_3\}$, from (7) we see that e_6 and e_2 have vertex in common. From (12) we can further deduce that $e_6 = e_2$. Furthermore, $(u_{10}, u_2) = (u_6, u_7)$ if u_{10} is on R_1, and $(u_{10}, u_2) = (u_9, u_8)$ otherwise. It follows that e_2 is the unique leaving edge of R_1 and R_3 outside $\{e_1, u_4u_5\}$. Next, R_2 has no leaving edge f other than e_2, for otherwise $C \cup \{e_1, e_2, f\}$ would yield a fully odd subdivision of F_1 or F_2 by Lemma 4.7, a contradiction. Finally, since $d_G(u_i) \geq 3$ for $6 \leq i \leq 9$, we have $u_7 = u_8$ and $u_3 = u_9$ if u_{10} is on R_1 and $u_1 = u_6$ otherwise. Combining the above observations, we conclude that G is T_9 in Figure 9. ∎

Proof of Lemma 4.13. In view of Lemmas 4.15–4.17, we may assume that

(1) each odd cycle in G contains precisely one odd edge.

By Lemma 4.14(i), G is 2-connected. Since G is nonbipartite, it has an odd cycle C, with odd edge $e_1 = u_1u_2$. As $|\Sigma| \geq 2$ by Lemma 4.14(ii), there exists an odd edge $e_2 = v_1v_2$ outside C in G. By Menger's theorem, G contains two disjoint paths Q_1, Q_2 from v_1, v_2 to two distinct vertices w_1, w_2 on C, respectively, where u_1, w_1, w_2, u_2 occur on C in clockwise cyclic order, and w_i, v_i are the two ends of Q_i for $i = 1, 2$. Set $P_1 = C[u_1, w_1]$, $P_2 = C[w_2, u_2]$, and $K = C \cup Q_1 \cup Q_2 \cup \{e_2\}$. From (1) it is easy to see that

(2) e_1, e_2 are the only odd edges in K.

Consequently, $C[w_1, w_2] = w_1w_2$, for otherwise, K would correspond to a fully odd subdivision of F_3 in G^*, a contradiction. For convenience, we assume that

(3) each of P_1, P_2, Q_1, Q_2 is an induced path in G.

We claim that

(4) e_1, e_2 are the only odd edges in G.

Suppose the contrary: G contains a third odd edge e_3. Then Menger's theorem guarantees the existence of a path R traversing e_3 with both ends x, y in K. Using (1) and (2), it is a routine matter to check that e_1, e_2, e_3 are the only odd edges in $K \cup R$. Now let us proceed by considering all possible locations of x and y. If $\{x, y\} = \{w_1, w_2\}$, then $K \cup R$ would yield a fully odd subdivision of F_4 in G^*. So $\{x, y\} \neq \{w_1, w_2\}$. If $\{x, y\} \subseteq V(P_i \cup Q_i)$ for $i = 1$ or 2, or $x \in V(P_1)\backslash w_1$ and $y \in V(Q_2)\backslash w_2$, or $x \in V(P_2)\backslash w_2$ and $y \in V(Q_1)\backslash w_1$, then we can easily find a cycle with precisely three odd edges, contradicting (1). If R is between P_1 and P_2 or between Q_1 and Q_2, say the former, then $C \cup R$ would yield a fully odd subdivision of F_3 in G^*. Thus we can reach a contradiction in each case.

(5) $P_i \cup Q_i$ is an induced path in G for $i = 1, 2$.

Suppose the contrary: some edge f is a bridge of $P_1 \cup Q_1$, say. From (3) we see that one end a of f is on $P_1\backslash w_1$ and the other end b on $Q_1\backslash w_1$. Thus the graph obtained from $K \cup \{f\}$ by deleting all vertices on $Q_1(w_1, b)$ would correspond to a fully odd subdivision of F_3 in G^*. This contradiction establishes (5).

Let H be the union of the cycle $C' = K\backslash w_1w_2$ and all its chords. Then $G = H$, because any bridge of H would cause a B_2-reduction in G by (4) and (5), contradicting the hypothesis that G is irreducible.

(6) If x_1y_1 and x_2y_2 are two disjoint chords of C' such that u_1, x_1, x_2, y_1, y_2 occur on C' in clockwise cyclic order, then x_1x_2 and y_1y_2 are two edges of C'.

Assume the contrary: x_1x_2 or y_1y_2 is not an edge of C', say the former. Then $C'[y_2, y_1] \cup \{x_1y_1, x_2y_2\}$ would yield a fully odd subdivision of F_3 in G^*. This contradiction yields (6).

By Lemma 4.14(i), $d_G(v) \geq 3$ for all vertices v of G. So each vertex is incident with at least one chord of C'. Combining this with the above observations, we conclude that G is a ladder with only top e_1 and bottom e_2 odd. So G is F_{10} in Figure 9. ∎

4.6 B-extensions

In the previous subsections we have observed that the property of being good is preserved under B-reductions, and have determined all irreducible signed graphs. Although this property is not maintained under B-extensions, let us proceed to show that the original graph can still be "deciphered" from irreducible signed graphs by using such reverse operations of B-reductions and meanwhile avoiding occurrence of forbidden structures.

Throughout this subsection, $G = (V, E, \Sigma)$ is an i-2-c good signed graph with all edges odd, and $Ir(G)$ is the set of all irreducible graphs arising from G. Moreover, G_i for $1 \leq i \leq 9$ are all as depicted in Figure 4, and T_j for $1 \leq i \leq 10$ are all as shown in Figure 9.

Lemma 4.18. *If $T_1 \in Ir(G)$, then G is a subgraph of one of G_2 and $G_6 - G_9$.*

Proof. Let $\{v_1, v_2\}$ be the vertex set of T_1 and let e (resp. f) denote the even (resp. odd) edge of T_1. We may assume that f is created in T_1 to replace a connected bipartite subgraph of G in a B-reduction, for otherwise, f is an edge of G and $G\backslash f$ is bipartite. So G is nearly bipartite, and hence is one of the six graphs depicted in Figure 6 by Lemma 4.9, which are subgraphs of G_2 and $G_6 - G_9$, respectively.

Let L (resp. R) be the connected bipartite subgraph of G replaced by e (resp. f) in a B-reduction. Let L' be obtained from L by adding an edge v_1v_2 and let R' be obtained from R

by adding a path $v_1v_3v_2$, where v_3 is a new vertex outside R. Since G is i-2-c and good, so are L' and R'. Since $L'\backslash v_1v_2$ and $R'\backslash v_1v_3$ are bipartite graphs, both L' and R' are nearly bipartite. By Lemma 4.9, L' is one of H_i for $1 \le i \le 6$ in Figure 6, and R' is one of H_j for $1 \le j \le 6$ and $j \ne 5$ (as $d_{H_5}(x_2) = 3$ while $d_{R'}(v_3) = 2$). Let x_3 be the neighbor of x_2 in H_j corresponding to v_2 in R'; keep in mind that x_3 in on a path marked by α in Figure 6 when $j = 1, 2, 4$. Let L_i be obtained from H_i (potential L') by deleting x_1x_2, and let R_j be obtained from H_j (potential R') by deleting x_2. For convenience, we relabel (x_1, x_2) as (v_1, v_2) in L_i, and relabel (x_1, x_3) as (v_1, v_2) in R_j. Observe that

(1) $L \ne L_5$.

Assume on the contrary that $L = L_5$. By Lemma 4.3, either R contains a v_1v_2-path P together with an edge u_1u_2, with $u_1 \in V(P)$ while $u_2 \notin V(P)$ or R is a path of odd length at least three. In the former case, symmetry allows us to assume that $P[v_1, u_1]$ is odd. Let w_i be the vertex above v_i in H_5 (see Figure 6) for $i = 1, 2$. Then the cycle $w_2v_1Pv_2w_2$ together with v_1w_1, u_1u_2 and an edge incident with w_2 outside $\{w_2v_1, w_2v_2\}$ would be a fully odd subdivision of F_1 in G. In the latter case, let Q be a w_1w_2-path in $L_5\backslash\{v_1, v_2\}$. Then the three paths $w_1Qw_2v_2$, w_1v_2, and $w_1v_1Rv_2$ would be a fully odd subdivision of F_3. So we reach a contradiction in either case.

(2) If $L = L_4$, then G is a subgraph of G_2.

To justify this, note that R is a v_1v_2-path, for otherwise, Lemma 4.3 guarantees the existence of a v_1v_2-path P together with an edge u_1u_2 in R, with $u_1 \in V(P)$ while $u_2 \notin V(P)$. By symmetry, we may assume that $P[v_1, u_1]$ is odd. For $i = 1, 2$, let a_i be the pendant vertex right above v_i in Figure 6, let Q_i be the a_iv_i-path corresponding to the straight line segment linking a_i and v_i, and let b_i, c_i be the two vertices succeeding a_i on Q_i. If α =odd (see Figure 6), then the cycle $b_1Q_1[b_1, v_1]v_1Pv_2Q_2[v_2, c_2]c_2b_1$ together with edges u_1u_2, a_1b_1, b_2c_2 would yield a fully odd subdivision of F_1. If α =even (see Figure 6), then the cycle $c_1Q_1[c_1, v_1]v_1Pv_2Q_2[v_2, b_2]b_2c_1$ together with edges u_1u_2, b_1c_1, a_2b_2 would yield a fully odd subdivision of F_1. So we reach a contradiction in either case. As R is a path, it is clear that G is a subgraph of G_2, as desired.

The same argument yields the following statement.

(3) If $R = R_4$, then G is a subgraph of G_2.

(4) If $L = L_2$ and L cannot be drawn as L_1, then G is G_8.

To justify this, let P_i be the path starting with v_i and marked by α in L_2, let u_i be the end of P_i other than v_i for $i = 1, 2$, and let u_3 be the common neighbor of u_1 and u_2 (see Figure 6). Since L_2 cannot be drawn as L_1, the fully subdivided subgraph in L_2 is not a path. So, by Lemma 4.3, there exists a u_1u_2-path Q in $L_2\backslash u_3$ and an edge w_1w_2, with $w_1 \in V(Q)$ while $w_2 \notin V(Q)$, such that $Q[u_1, w_1]$ is of odd length. We claim that R is a path, for otherwise, Lemma 4.3 guarantees the existence of a v_1v_2-path S together with an edge z_1z_2, with $z_1 \in V(S)$ while $z_2 \notin V(S)$. By symmetry, we may assume that $S[v_1, z_1]$ is odd. Then the cycle $P_1 \cup P_2 \cup Q \cup S$ together with edges w_1w_2, z_1z_2 and one of u_2u_3 and u_1u_3 (depending on whether α =odd; see Figure 6) would yield a fully odd subdivision of F_1. This contradiction justifies our claim. It follows instantly that G is G_8.

Similarly, we can establish the following statement.

(5) If $R = R_2$ and R cannot be drawn as R_1, then G is G_8.

(6) If $L = L_6$, then G is a subgraph of a plump ladder G_9.

To justify this, let v_0 be the neighbor of v_2 other than v_1, let P_1, P_2 be two v_0v_1-paths of odd length in $L_6\backslash v_2$, and let J be the bipartite subgraph of G induced by $V(R) \cup \{v_0\}$. By Lemma 4.8, one color class of J is $\{v_0, v_1\}$. So G is a subgraph of a plump ladder G_9, as desired.

Using the same argument, we get the following statement.

(7) If $R = R_6$, then G is a subgraph of a plump ladder G_9.

(8) If $L = L_1$ and $R = R_1$, then clearly G is G_6 or a subgraph of G_8.

(9) If $L = L_1$ and $R = R_3$, then G is G_6 or a subgraph of G_8.

To justify this, let C be a shortest cycle in G containing v_1 and v_2 and intersecting both $L_1\backslash\{v_1, v_2\}$ and $R_3\backslash\{v_1, v_2\}$, let $a_1 = v_1$ and $a_4 = v_2$, and let $a_1, a_2, \dots, a_6$ be six vertices occur on C in clockwise cyclic order, where $C[a_4, a_5]$ and $C[a_6, a_1]$ are the two paths marked by α in L_1 (see Figure 6), and a_2a_3 is the edge connecting two fully subdivided subgraphs in R_3. Let B_i stand for the fully subdivided subgraph containing both a_{2i-1} and a_{2i} in G for $i = 1, 2, 3$. If α =even or if one of B_1, B_2, B_3 is an $a_{2i-1}a_{2i}$-path, then clearly G is G_6 or a subgraph of G_8. Otherwise, each B_i contains an edge b_ib_i' such that b_i is on $C[a_{2i-1}, a_{2i}]$ and that $C[a_{2i-1}, b_i]$ is of even length, because a_{2i-1} and a_{2i} are both contained in the color 1 class of B_i. Thus $C \cup \{b_1b_1', b_2b_2', b_3b_3'\}$ would be a fully odd subdivision of F_1. This contradiction establishes (9).

The same argument yields the following two statements.

(10) If $L = L_3$ and $R = R_1$, then clearly G is G_6 or a subgraph of G_8.

(11) If $L = L_3$ and $R = R_3$, then clearly G is G_6 or a subgraph of G_8.

Combining the above observations, we see that G is a subgraph of one of G_2, G_6, G_8 and G_9, as desired. ∎

Lemma 4.19. *If $T_i \in Ir(G)$ for $i = 5, 6$ or 7, then $G = T_i$ and hence is a subgraph of G_1.*

Proof. Let $v_1, v_2, \dots, v_5$ for all the vertices of T_i. We propose to show that no edge e in T_i is created to replace a connected bipartite subgraph H_e of G in a B-reduction. Assume the contrary: some edge $e = v_sv_t$ of T_i is a counterexample. Let P be a shortest v_sv_t-path in H_e. Note that P is of odd length. So either P has length at least three or $H_e\backslash e$ contains an edge f incident with e. Let K be obtained from T_i by replacing e with P or with $\{e, f\}$. It is then a routine matter to check that K and hence G contains a fully odd subdivision of F_1. This contradiction establishes the desired statement. Hence $G = T_i$, as desired. ∎

Lemma 4.20. *If $T_i \in Ir(G)$ for $i = 2, 8$ or 9, then G is G_3.*

Proof. Label the vertices of T_i as v_1, v_2, v_3, v_4, with $v_4 = v_1$ in T_2, so that $C = v_1v_2v_3v_4v_1$ is a cycle in T_i, and that each of $\{v_1, v_2\}$ and $\{v_3, v_4\}$ induces a 2-gon in T_i. For convenience, we assume that v_1, v_2, v_3, v_4 occur on C in clockwise cyclic order, and that both the odd v_1v_2 and odd v_3v_4 are contained in C. Let $H_e = (X_e, Y_e; E_e)$ be the connected bipartite subgraph of G replaced by an edge e in T_i in a B-reduction, where the two ends of e are contained in X_e if e is even. Let us show that

(1) for even $e = a_1a_2 \in \{v_1v_2, v_3v_4\}$ in T_i, the entire H_e is a a_1a_2-path of length two. Moreover, for $e = v_2v_3$ in T_9, the entire H_e is a v_2v_3-path.

To justify this, let P_1 and P_2 be two disjoint odd a_1a_2-paths in T_i. By Lemma 4.8 with $H = H_e$, we have

(2) $X_e = \{a_1, a_2\}$.

Let $C' = C$ if $i = 2$ or 8, and let C' be obtained from C by replacing v_2v_3 with a shortest v_2v_3-path in $H_{v_2v_3}$ if $i = 9$. By (2), $H_{v_{2j-1}v_{2j}}$ contains a $v_{2j-1}v_{2j}$-path Q_j of length two for

$j = 1, 2$. Set $K = C' \cup Q_1 \cup Q_2$. Suppose on the contrary that (1) is false. Then G has an edge f with one end in $\{v_1, v_4\}$ or on $C'[v_2, v_3]$ and the other end outside K (see (2)). From Lemma 4.7 we deduce that $K \cup \{f\}$ contains a fully odd subdivision of F_1 or F_2. This contradiction establishes (1).

(3) For each odd edge $e = b_1b_2$ in T_i, the entire H_e is a b_1b_2-path.

Otherwise, Lemma 4.3 guarantees the existence of a b_1b_2-path R and an edge c_1c_2 in H_e, with $c_1 \in V(R)$ while $c_2 \notin V(R)$. Let Q_j be a $v_{2j-1}v_{2j}$-path in $H_{v_{2j-1}v_{2j}}$, and let K be obtained from T_i by replacing even $v_{2j-1}v_{2j}$ with Q_j for $j = 1, 2$ and replacing e with $R \cup \{c_1c_2\}$. It is then a routine matter to check that K contains a fully odd subdivision of F_1 or F_2. So (3) holds.

Combining (1) and (3), we conclude that G is G_3. ∎

Lemma 4.21. *If $T_3 \in Ir(G)$, then G is G_5.*

Proof. Let v_1, v_2, v_3, v_4 be four vertices of T_3 such that $\{v_1, v_2\}$ induces a 2-gon, and let $H_e = (X_e, Y_e; E_e)$ be the connected bipartite subgraph of G replaced by an edge e in T_i in a B-reduction, where the two ends of e are contained in X_e if e is even. We propose to show that

(1) for the even $e = v_1v_2$ in T_3, the entire H_e is a v_1v_2-path of length two.

To justify this, let P_1 and P_2 be two disjoint odd v_1v_2-paths in T_3. By Lemma 4.8 with $H = H_e$, we have $X_e = \{v_1, v_2\}$. Let Q be a v_1v_2-path of length two. If $H_e \neq Q$, then $H_e \backslash e$ contains an edge v_iv_5 for $i = 1$ or 2. Let K be obtained from T_3 by replacing the even v_1v_2 with $Q \cup \{v_iv_5\}$. Then K and hence G contains a fully odd subdivision of F_1. This contradiction establishes (1).

(2) For the odd $e = v_1v_2$ in T_3, the entire H_e is a v_1v_2-path.

Otherwise, Lemma 4.3 guarantees the existence of a v_1v_2-path Q and an edge u_1u_2 in H_e, with $u_1 \in V(Q)$ while $u_2 \notin V(R)$. Let R be a v_1v_2-path in H_f, where f is the even v_1v_2, and let K be obtained from T_i by replacing even v_1v_2 with R and replacing e with $Q \cup \{u_1u_2\}$. It is then a routine matter to check that K and hence G contains a fully odd subdivision of F_1. So (2) follows.

(3) No odd edge $e \neq v_1v_2$ in T_3 is created to replace a connected bipartite subgraph H_e of G in a B-reduction.

Otherwise, imitating the proof of Lemma 4.19, we can easily find a fully odd subdivision of F_1 in G.

Combining (1)-(3), we see that G is G_5. ∎

Lemma 4.22. *If $T_4 \in Ir(G)$, then G is G_2 or G_7.*

Proof. Let v_1, v_2, v_3, v_4 be the vertices of T_4, and let H_e be the connected bipartite subgraph of G replaced by an edge e of T_4 in a B-reduction. We propose to show that

(1) One end v_i of e has precisely one neighbor v_i' in H_e, such that $H_e \backslash v_i$ is a fully subdivided graph in which both v_i' and v_j, the other end of e, belong to the color 1 class.

To justify this, note that $H_e + v_iv_j$ is i-2-c, so at least one of (i), (ii) and (iii) in Lemma 4.5 holds with $H = H_e$ and $(x_1, y_1) = (v_i, v_j)$. It is easy to see that if (i) or (ii) is true, then G would contain a fully odd subdivision of F_3 or F_1. So (iii) of Lemma 4.5 occurs; that is, H contains an edge x_2y_2 such that $H_e \backslash x_2y_2$ has precisely two components $H_1 = (X_1, Y_1; E_1)$ and $H_2 = (X_2, Y_2; E_2)$, with $\{x_1, x_2\} \subseteq X_1$ and $\{y_1, y_2\} \subseteq Y_2$, and that $d_H(v) \leq 2$ for any $v \in Y_1 \cup X_2$. (Possibly $x_1 = x_2$ or $y_1 = y_2$.) Let P be a shortest x_1y_1-path in H. Then P

traverses x_1, x_2, y_2, y_1 in order. We claim that $H_1 = P[x_1, x_2]$ or $H_2 = P[y_2, y_1]$. Otherwise, H_1 contains an edge z_1z_1' with z_1 on $P[x_1, x_2]$ while z_1' outside $P[x_1, x_2]$, and H_2 contains an edge z_2z_2' with z_2 on $P[y_2, y_1]$ while z_2' outside $P[y_2, y_1]$. Observe that $z_1 \in X_1$ and $z_2 \in Y_2$. Let K be obtained from T_4 by replacing e with $P \cup \{z_1z_1', z_2z_2'\}$. It is easy to see that K and hence G contains a fully odd subdivision of F_1. This contradiction proves our claim, which immediately yields (1).

(2) We may assume that for any three vertices v_i, v_j, v_k of T_4, at least one of the edges v_iv_j and v_jv_k is not created in T_4 to replace a connected bipartite subgraph of G in a B-reduction.

Suppose the contrary: v_iv_j (resp. v_jv_k) is created in T_4 to replace a connected bipartite subgraph $H_{v_iv_j}$ (resp. $H_{v_jv_k}$) of G in a B-reduction. Observe that

(3) if K is a complete graph with vertex set $U = \{u_1, u_2, u_3, u_3\}$, then each H described below contains a fully odd subdivision of F_1.

- H arises from K by adding two disjoint edges u_1u_5 and u_2u_6, with $\{u_5, u_6\} \cap U = \emptyset$;
- H arises from K by adding one edge u_1u_5, with $u_5 \notin U$, and subdividing u_2u_3 into a path of length at least two; and
- H arises from K by subdividing each of u_1u_2 and u_1u_3 into a path of length at least two.

From (3) it is easy to see that at least one of $H_{v_iv_j}$ and $H_{v_jv_k}$, say the former, consists of two edges incident with v_j only; let v_jv_5 be the edge other than v_iv_j. Let $H'_{v_jv_k}$ be obtained from $H_{v_jv_k}$ by adding the edge v_jv_5. We may thus assume that v_jv_k is created in T_4 to replace the connected bipartite subgraph $H'_{v_jv_k}$ of G in a B-reduction, while v_iv_j is not created in T_4 in any B-reduction. So (2) follows.

(4) If both v_1v_2 and v_3v_4 are created in T_4 to replace connected bipartite subgraph $H_{v_1v_2}$ and $H_{v_3v_4}$ of G, respectively, in B-reductions, then $H_{v_{2i-1}v_{2i}}$ is a $v_{2i-1}v_{2i}$-path for $i = 1, 2$.

Suppose the contrary: $H_{v_1v_2}$, say, is not a v_1v_2-path. Then there exist a v_1v_2-path P and an edge a_1a_2 in $H_{v_1v_2}$, with $a_1 \in V(P)$ while $a_2 \notin V(P)$. By Lemma 4.3, $H_{v_3v_4}$ contains either a v_3v_4-path Q of length at least three or two edges v_3v_4 and v_jv_5 for $j = 3$ or 4. Let S be obtained from T_4 by replacing v_1v_2 with $P \cup \{a_1a_2\}$ and replacing v_3v_4 with Q or with $\{v_3v_4, v_jv_5\}$. It is then a routine matter to check that S and hence G contains a fully odd subdivision of F_1. This contradiction implies (4).

From (1) we deduce that if precisely one edge of T_4 is created to replace a connected bipartite subgraph of G in a B-reduction, then G is G_7. In view of (4), if two disjoint edges of T_4 are created to replace connected bipartite subgraphs of G in B-reductions, then G is G_2. By (2), the present lemma is thus established. ∎

Lemma 4.23. *If $T_{10} \in Ir(G)$, then G is a subgraph of a plump ladder G_9.*

Proof. By Lemma 4.13, T_{10} is a ladder in which only the top u_1u_2 and bottom v_1v_2 are odd edges. Let C be the outer cycle of T_{10}. Renaming the subscripts of vertices, we assume that u_1, v_1, v_2, u_2 occur on C in clockwise cyclic order. By definition, each even edge $e = x_1x_2$ in T_{10} is created to replace a connected bipartite subgraph H_e of G in a B-reduction; let (X_e, Y_e) be the bipartition of H_e, such that $\{x_1, x_2\} \subseteq X_e$. For an odd edge e, we also use H_e to denote the corresponding bipartite subgraph of G involved in a B-reduction, if any. We propose to show that

(1) if $e = x_1x_2$ is a chord of C, then $X_e = \{x_1, x_2\}$.

To justify this, let C^* be the cycle corresponding to C in G^*, and let $P_1 = C^*[x_1, x_2]$ and $P_2 = C^*[x_2, x_1]$. Then both P_1 and P_2 are of odd length. So (1) follows instantly from Lemma 4.8.

(2) If $e = x_1x_2$ is in $C\backslash\{u_1u_2, v_1v_2\}$, then $d_{H_e}(y) \le 2$ for all $y \in Y_e$.

Suppose the contrary: $d_{H_e}(y) \ge 3$ for some $y \in Y_e$. Since G is i-2-c, Lemma 4.2 guarantees the existence of two paths Q_1 and Q_2 from y to $\{x_1, x_2\}$ in H_e that have only y in common. Clearly, we may further assume that both Q_1 and Q_2 are induced. Thus y has a third neighbor y' outside $Q_1 \cup Q_2$ in H_e. By Lemma 4.14(i), both x_1 and x_2 have degree at least three in T_{10}. So C has a chord r_i incident with x_i for $i = 1, 2$. Let x_1x_1' be an edge in H_{r_1}, let R be a path connecting the two ends of r_2 in H_{r_2}, and let C' be obtained from C by replacing edge x_1x_2 with path $Q_1 \cup Q_2$. Then $C' \cup R \cup \{x_1x_1', yy'\}$ would yield a fully odd subdivision of F_2 in G^*. This contradiction establishes (2).

(3) If $e = x_1x_2$ in $C\backslash\{u_1u_2, v_1v_2\}$ is contained in a 4-cycle induced by two crossing chords of C, then $X_e = \{x_1, x_2\}$.

To justify this, let x_1y_1 and x_2y_2 be two crossing chords of C, and let C' be the cycle obtained from C replacing edges x_1x_2, y_1y_2 with x_1y_1, x_2y_2. Then x_1x_2 becomes a chord of C'. Using (1), with C' in place of C, we deduce that $X_e = \{x_1, x_2\}$.

(4) If $e \in \{u_1u_2, v_1v_2\}$, then H_e is as described in Lemma 4.5(iii), with $H = H_e$ and x_1, y_1 being the ends of e.

To justify this, we only need to consider the case when $e = u_1u_2$ by symmetry. Thus $x_1 = u_1$ and $y_1 = u_2$. Symmetry also allows us to assume that $C[u_1, v_1]$ contains at least one edges. By Lemma 4.14(i), both u_1 and v_1 have degree at least three in T_{10}. So C contains two chords $r_1 = u_1u_3$ and $r_2 = v_1v_3$. Let $R_1 = u_1u_4u_3$ by a path in H_{r_1} and let $R_2 = v_1v_4v_3$ be a path in H_{r_2} (see (1)). Since $H + u_1u_2$ is i-2-c, at least one of (i), (ii) and (iii) in Lemma 4.5 holds.

If (i) is true, then H contains an u_1u_2-path P and an x_2y_2-path Q, such that $V(P) \cap V(Q) = \{x_2, y_2\}$ and that both $P[x_2, y_2]$ and Q are of odd length. Let S be obtained from $C\backslash u_1u_2$ by replacing $C[v_1, v_3]$ with R_2. Then $S \cup P \cup Q$ would yield a fully odd subdivision of F_3 in G^*, a contradiction.

If (ii) is true, then H contains an u_1u_2-path P and two disjoint edges y_2x_3 and x_2y_3, with $\{x_2, y_2\} \in V(P)$ while $\{x_3, y_3\} \cap V(P) = \emptyset$ and with y_2 on $P[u_1, x_2]$, such that $P[u_1, y_2]$, $P[y_2, x_2]$, and $P[x_2, u_2]$ are all of odd length. Let C'' be obtained from C by replacing $C[u_2, u_1]$ with P and replacing $C[v_1, v_3]$ with R_2. Then $C'' \cup \{u_1u_4, y_2x_3, x_2y_3\}$ would yield a fully odd subdivision of F_1 in G^*, a contradiction again.

So neither (i) nor (ii) of Lemma 4.5 occurs, and hence (4) follows.

By (4) and Lemma 4.5(iii), if $H_{a_1a_2}$, with $a_1a_2 \in \{u_1u_2, v_1v_2\}$, exists, then $H_{a_1a_2}$ contains an edge $a_1'a_2'$ such that $H_{a_1a_2}\backslash a_1'a_2'$ has precisely two components $H_{a_1a_1'} = (X_1, Y_1; E_1)$ and $H_{a_2a_2'} = (X_2, Y_2; E_2)$, with $\{a_1, a_1'\} \subseteq X_1$ and $\{a_2, a_2'\} \subseteq X_2$, and that $d_H(v) \le 2$ for any $v \in Y_1 \cup Y_2$. Let K be obtained from T_{10} by first replacing each edge e with H_e as specified in (1)-(4) and then adding a bipartite graph $L_f = K_{2,n}$ for some $n \ge 1$, in which one color class consists of the two ends of f only, for each f in $\{u_1u_2', u_1'u_2', v_1v_2', v_1'v_2, \}$, if any. Clearly, G is a subgraph of K, and K is a subgraph of a plump ladder G_9. So the desired statement holds. ∎

We are now ready to finish the structural description of good graphs.

Proof of Theorem 4.1. It is routine to check that none of $G_1, G_2, \ldots, G_9$ depicted in Figure 4 contains a fully odd subdivision of F_1, F_2, F_3 or F_4 as a subgraph. So if G is a subgraph of one of these nine graphs, then G is good.

Conversely, let G be an i-2-c, good and nonbipartite graph; we view it as a signed graph with all edges odd. By Lemma 4.13, $\{T_1, T_2, \ldots, T_{10}\}$ in Figure 9 is the set of all possible irreducible signed graphs arising from G. The lemmas proved in this subsection assert that G is a subgraph of one of $G_1, G_2, \ldots, G_9$ depicted in Figure 4, no matter what the irreducible signed graphs T_i arising from G are, completing the proof. ∎

5 Primitive Graphs

By Theorem 4.1, every i-2-c good graph is bipartite or is a subgraph of one of the nine graphs $G_1, G_2, \ldots, G_9$ (see Figure 4). The purpose of this section is to show that the restricted Edmonds system $\sigma(G)$ is ESP when G is bipartite or G_i for $1 \le i \le 9$, thereby establishing the "only if" part of Theorem 1.4 when G is i-2-c.

To facilitate better understanding of an ESP system $\sigma(G)$, we give an intuitive interpretation of this concept using graph-theoretic language. Recall the notations $I(G)$ and $\mathcal{T}(G)$ introduced right above Theorem 1.2. For each $v \in I(G)$, we call $\delta(v)$ the *star* centered at v and define its *rank* $\rho(\delta(v))$ to be 1. For each $U \subseteq \mathcal{T}(G)$, we call $E[U]$ the *odd set* generated by U and define its *rank* $\rho(E[U])$ to be $(|U|-1)/2$. For each collection Λ of stars and odd sets of G, let $\rho(\Lambda) = \sum_{K \in \Lambda} \rho(K)$ and let $d_\Lambda(e)$ denote the number of members of Λ that contain an edge e. For each star or odd set K in G, let $m_\Lambda(K)$ stand for the multiplicity of K in Λ. Observe that $\rho(K)$ is counted $m_\Lambda(K)$ times in $\rho(\Lambda)$, and K is counted $m_\Lambda(K)$ times in $d_\Lambda(e)$ if $e \in K$. An *equitable subpartition* of Λ consists of two collections Λ_1 and Λ_2 of stars and odd sets (which are not necessarily in Λ) such that

(i) $\rho(\Lambda_1) + \rho(\Lambda_2) \le \rho(\Lambda)$;

(ii) $d_{\Lambda_1 \cup \Lambda_2}(e) \ge d_\Lambda(e)$ for all $e \in E$; and

(iii) $\min\{d_{\Lambda_1}(e), d_{\Lambda_2}(e)\} \ge \lfloor d_\Lambda(e)/2 \rfloor$ for all $e \in E$.

We call G *equitably subpartitionable*, abbreviated ESP, if every collection Λ of stars and odd sets of G admits an equitable subpartition. We refer to the above (i), (ii), and (iii) as *ESP property.*

The following statement follows instantly from definitions.

Lemma 5.1. *A graph G is ESP if and only if $\sigma(G)$ is ESP.*

Let $G = (V, E)$ be a graph, and let Λ_1 and Λ_2 be two collections of stars and odd sets in G. We say that Λ_1 *dominates* Λ_2 if $\rho(\Lambda_1) \le \rho(\Lambda_2)$ while $d_{\Lambda_1}(e) \ge d_{\Lambda_2}(e)$ for all $e \in E$. Suppose G is not ESP. We reserve the symbol Δ for a collection of stars and odd sets of G such that

(5a) Δ admits no equitable subpartition;

(5b) subject to (5a), $\rho(\Delta)$ is minimized;

(5c) subject to (5a-b), $f(\Delta) = \sum_{e \in E} d_\Delta(e)$ is maximized;

(5d) subject to (5a-c), $g(\Delta)$, the number of odd sets in Δ, is minimized.

Lemma 5.2. *The collection Δ has the following properties:*

(i) If Ω dominates Δ, then $m_\Omega(K) = 1$ for all $K \in \Omega$.

(ii) If Ω dominates Δ, then $\rho(\Omega) = \rho(\Delta)$, $f(\Omega) = f(\Delta)$, and $g(\Omega) \geq g(\Delta)$.

(iii) If $\delta(v) \in \Delta$ and no odd set in Δ contains any edge in $\delta(v)$, then v has two distinct neighbors u_1, u_2 such that $\delta(u_i) \in \Delta$ for $i = 1, 2$.

Proof. (i) Assume the contrary: a star or an odd set K appears at least twice in Ω. Let $\Omega' = \Omega - \{K, K\}$. As $\rho(\Omega') < \rho(\Omega) \leq \rho(\Delta)$, from (5b) we deduce that Ω' admits an equitable subpartition (Ω'_1, Ω'_2). Set $\Omega_i = \Omega'_i \cup \{K\}$. It is a routine matter to check that (Ω_1, Ω_2) is an equitable subpartition of Ω and hence of Λ, a contradiction.

(ii) Since Ω dominates Δ, by definition $\rho(\Omega) \leq \rho(\Delta)$ and $f(\Omega) \geq f(\Delta)$. If one of the inequalities $\rho(\Omega) < \rho(\Delta)$, $f(\Omega) > f(\Delta)$, and $g(\Omega) < g(\Delta)$ holds, then (5a-d) would guarantee the existence of an equitable subpartition (Ω_1, Ω_2) of Ω, which is also an equitable subpartition of Δ, a contradiction.

(iii) Assume the contrary: there is at most one neighbor u of v such that $\delta(u) \in \Delta$. Let $\Delta' = \Delta - \{\delta(v)\}$. Then $\rho(\Delta') < \rho(\Delta)$. So Δ' admits an equitable subpartition (Δ'_1, Δ'_2) by (5b). Renaming subscripts if necessary, we assume that $d_{\Delta'_1}(uv) \leq d_{\Delta'_2}(uv)$ if u exists. Set $\Delta_1 = \Delta'_1 \cup \{\delta(v)\}$ and $\Delta_2 = \Delta'_2$. It is straightforward to verify that (Δ_1, Δ_2) is an equitable subpartition of Δ, a contradiction. ∎

Lemma 5.3. *Let v be a vertex of G with $d_G(v) = 2$. If $d_\Delta(e)$ is odd for an edge $e \in \delta(v)$, then $\delta(v) \notin \Delta$.*

Proof. Assume on the contrary that $\delta(v) \in \Delta$. Let $\Delta' = \Delta - \{\delta(v)\}$. Then $\rho(\Delta') < \rho(\Delta)$ and Δ' admits an equitable subpartition (Δ'_1, Δ'_2) by (5b). Let f be the edge incident with v other than e. Renaming subscripts if necessary, we assume that $d_{\Delta'_1}(f) \leq d_{\Delta'_2}(f)$. Set $\Delta_1 = \Delta'_1 \cup \{\delta(v)\}$ and $\Delta_2 = \Delta'_2$. Clearly, (Δ_1, Δ_2) is an equitable subpartition of Δ, a contradiction. ∎

For convenience, we introduce some notations which will be used throughout this section. For each $U \subseteq V$, define $\delta(U) = \{\delta(v) : v \in U\}$. For each path P in G, define $\delta(P) = \delta(V(P))$.

Lemma 5.4. *Let $E[S]$ and $E[T]$ be two distinct odd sets in G, such that $G[A] \backslash B$ is a path for each permutation A, B of S, T with $A \backslash B \neq \emptyset$. Then the following statements hold:*

(i) If $|S \cap T|$ is even and $S \backslash T \neq \emptyset \neq T \backslash S$, then $\{E[S], E[T]\} \nsubseteq \Delta$.

(ii) If $S \subseteq T$ and $d_G(v) = 2$ and $\delta(v) \in \Delta$ for all $v \in T \backslash S$, then $E[S] \notin \Delta$.

Proof. (i) Assume the contrary: $\{E[S], E[T]\} \subseteq \Delta$. Let P (resp. Q) denote the path $G[T] \backslash S$ (resp. $G[S] \backslash T$). Then both P and Q are of even length. Let (U_1, U_2) (resp. (U_3, U_4)) be the bipartition of P (resp. Q) with $|U_1| > |U_2|$ (resp. $|U_3| > |U_4|$), and let Δ' be the collection obtain from Δ by deleting $\{E[S], E[T]\}$ and adding $\delta(S \cap T) \cup \delta(U_2 \cup U_4)$. Then $\rho(\Delta) = \rho(\Delta')$ and $d_\Delta(e) \leq d_{\Delta'}(e)$ for all $e \in E$. So Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

(ii) Assume the contrary: $E[S] \in \Delta$. Let P denote the path $G[T] \backslash S$. Then P is of odd length. Let (U_1, U_2) be the bipartition of P, and let Δ' be the collection obtain from Δ by deleting $\{E[S], \delta(U_2)\}$ and adding $E[T]$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii). ∎

Lemma 5.5. *Let $H = (X, Y; E)$ be a bipartite graph, let a and b be two distinct vertices in X, and let Ω be a set of stars in H such that each ab-path contains a vertex v with $\delta(v) \notin \Omega$. Then Ω can be partitioned into Ω_1, Ω_2 such that (Ω_1, Ω_2) is an equitable subpartition (and hence called equitable partition) of Ω and that $|\Omega_i \cap \{\delta(a), \delta(b)\}| \leq 1$ for $i = 1, 2$.*

Proof. Let us first consider the case when $\delta(a)$ or $\delta(b)$ is outside Ω. Set $\Omega_1 = \delta(X) \cap \Omega$ and $\Omega_2 = \delta(Y) \cap \Omega$. Clearly, (Ω_1, Ω_2) is as desired. It remains to consider the case when $\{\delta(a), \delta(b)\} \subseteq \Omega$. Let Z be the set of all vertices v with $\delta(v) \notin \Omega$. By hypothesis, a and b are in different components of $H \backslash Z$. Let $H_1 = (X_1, Y_1; E_1)$ be the component of $H \backslash Z$ containing a and let $H_2 = (X_2, Y_2; E_2)$ be the union of the remaining components of $H \backslash Z$, with $a \in X_1$ and $b \in X_2$. Set $\Omega_1 = \delta(X_1 \cup Y_2)$ and $\Omega_2 = \delta(X_2 \cup Y_1)$. Obviously, $\{\Omega_1, \Omega_2\}$ is a partition of Ω with the desired properties. ∎

Let us proceed to show that the ESP property is satisfied by all bipartite graphs and all G_i's.

Lemma 5.6. *Every bipartite graph is ESP.*

Proof. Suppose on the contrary some bipartite graph $G_0 = (V_1, V_2; E)$ is not ESP. Let Δ be a collection of stars and odd sets in G_0 as specified by (5a-d) (with G_0 in place of G). By Lemma 5.2(i), we have $m_\Delta(K) = 1$ for all $K \in \Delta$. Observe that Δ contains no odd set, for otherwise, let $S = E[U]$ be such a set. Renaming subscripts if necessary, we may assume that $|U \cap V_1| < |U \cap V_2|$. Let Δ' be obtained from Δ by replacing S with $\delta(U \cap V_1)$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii). Set $\Delta_i = \delta(V_i) \cap \Delta$ for $i = 1, 2$. Clearly, (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). ∎

Lemma 5.7. *The graph $G_1 = (V_1, E_1)$ (see Figure 10) is ESP.*

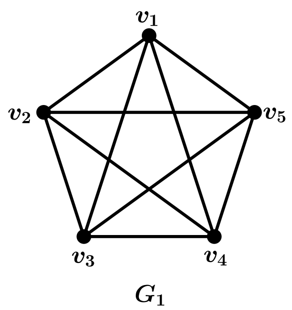


Figure 10: The primitive graph G_1

Proof. Suppose on the contrary that G_1 is not ESP. Let Δ be a collection of stars and odd sets in G_1 as specified by (5a-d) (with G_1 in place of G). By Lemma 5.2(i), we have $m_\Delta(K) = 1$ for all $K \in \Delta$. Let us make some observations about Δ.

(1) The total number of stars contained in Δ, denoted by $h(\Delta)$, is at most 2.

Otherwise, symmetry allows us to assume that $\delta(v_i) \in \Delta$ for $i = 1, 2, 3$. Let $U = \{v_1, v_2, v_3\}$ and let Δ' be the collection obtained from Δ by replacing $\delta(U)$ with $\{E[U], E_1\}$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

(2) If $E(U_i) \in \Delta$ with $|U_i| = 3$ for $i = 1, 2$, then $|U_1 \cap U_2| = 2$.

Assume on the contrary that $|U_1 \cap U_2| = 1$. Let $\Lambda' = (\Lambda - \{E[U_1], E[U_2]\}) \cup \{E_1\}$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$; this contradiction to Lemma 5.2(ii) establishes (2).

(3) Δ contains at least one odd set.

Otherwise, we may assume that Δ consists of stars only and $\delta(v_1) \in \Delta$. From (1), we see that $(\{\delta(v_1)\}, \Delta - \{\delta(v_1)\})$ is an equitable subpartition of Δ.

(4) Δ contains precisely one odd set $E[U]$ with $|U| = 3$.

Assume the contrary. If Δ contains no odd set $E[U]$ with $|U| = 3$, then E_1 is the only odd set in Δ by (3). Hence $(\{E_1\}, \Delta - \{E_1\})$ is an equitable subpartition of Δ by (1), a contradiction. So Δ contains at least two odd sets $E[U_1]$ and $E[U_2]$, with $|U_i| = 3$ for $i = 1, 2$. By symmetry and (2), we may assume that $U_1 \cap U_2 = \{v_1, v_2\}$. Let Δ' be obtained from Δ by replacing $\{E[U_1], E[U_2]\}$ with $\{\delta(v_1), \delta(v_2)\}$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

In view of (4), we reserve $E[U]$ for the only odd set in Δ with $|U| = 3$ hereafter.

(5) $v \in U$ if $\delta(v) \in \Delta$.

Otherwise, $v \notin U$. Let $\Delta' = (\Delta - \{E[U], \delta(v)\}) \cup \{E_1\}$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

(6) $E_1 \in \Delta$.

Otherwise, $(\{E[U]\}, \Delta - \{E[U]\})$ would be an equitable subpartition of Δ by (1) and (5); this contradiction implies (6).

Combining (4) and (6), we see that Δ contains precisely two odd sets $E[U]$ and E_1. If $h(\Delta) \leq 1$, then $(\{E_1\}, \Delta - \{E_1\})$ is an equitable subpartition of Δ, a contradiction. Hence, by (1), we have $h(\Delta) = 2$. By symmetry, we may assume that $\{\delta(v_1), \delta(v_2)\} \subseteq \Delta$. By (5), we further obtain $\{v_1, v_2\} \subseteq U$. Let $\Delta_1 = \{E[U], E_1\}$ and $\Delta_2 = \{\delta(v_1), \delta(v_2)\}$. Clearly, (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Therefore G_1 is ESP. ∎

Lemma 5.8. *The graph $G_2 = (V_2, E_2)$ (see Figure 11) is ESP.*

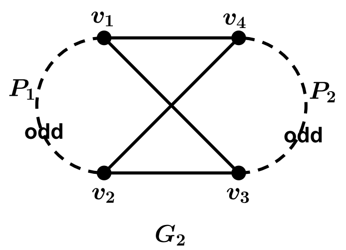


Figure 11: The primitive graph G_2

Proof. Suppose on the contrary that G_2 is not ESP. Let Δ be a collection of stars and odd sets in G_2 as specified by (5a-d) (with G_2 in place of G). By Lemma 5.2(i), we have $m_\Delta(K) = 1$ for all $K \in \Delta$. By Lemma 3.1, Lemma 3.8 and Lemma 5.7, G_2 is not a subgraph of G_1. So

(1) $P_1 \cup P_2$ contains at least two vertices outside $X = \{v_1, v_2, v_3, v_4\}$.

Repeated application of Lemma 5.2(iii) yields

(2) for $i = 1$ and 2, if $\delta(v) \in \Delta$ for some $v \in V(P_i)\backslash X$, then $\delta(P_i) \subseteq \Delta$.

Let $U_1 = \{v_4\} \cup V(P_1)$, $U_2 = \{v_3\} \cup V(P_1)$, $U_3 = \{v_1\} \cup V(P_2)$, and $U_4 = \{v_2\} \cup V(P_2)$, and let $S_i = E[U_i]$ for $1 \leq i \leq 4$. Since both P_1 and P_2 are odd, each S_i is an odd set in G_2. Furthermore, G_2 contains no other odd sets. Using Lemma 5.4(i), we obtain

(3) Δ contains at most one odd set.

(4) Δ contains no odd set.

Otherwise, by (3) and symmetry, we may assume that $S_1 \in \Delta$. Let (U_1, U_2) be the bipartition of P_2 with $v_3 \in U_1$. Set $\Delta_1 = \{S_1\} \cup \{\delta(v) \in \Delta : v \in U_1\}$ and $\Delta_2 = \Delta - \Delta_1$. Using (2) it is routine to check that (Δ_1, Δ_2) is an equitable subpartition of Δ; this contradiction justifies (4).

In view of (4), each member of Δ is a star. If $|P_i| > 1$ for $i = 1, 2$ and $\delta(v) \notin \Delta$ for all $v \in V_2\backslash X$, then $\Delta = \delta(X)$ by Lemma 5.2(iii). Let $\Delta_1 = \{\delta(v_1), \delta(v_2)\}$ and $\Delta_2 = \{\delta(v_3), \delta(v_4)\}$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, a contradiction. So

(5) for $i = 1$ or 2, either $|P_i| = 1$ or $\delta(v) \in \Delta$ for some $v \in V(P_i)\backslash X$ and hence $\delta(P_i) \subseteq \Delta$ by (2).

(6) For $i = 1$ or 2, there holds $\delta(P_i) \subseteq \Delta$.

Assume the contrary. By (5) and (1), we may assume that $|P_1| = 1$ and $|P_2| \geq 2$. Furthermore, $\delta(v) \notin \Delta$ for all $v \in V(P_2)\backslash X$. Since $\delta(v_i) \in \Delta$ for some $1 \leq i \leq 4$, repeated application of Lemma 5.2(iii) yields $\delta(v_j) \in \Delta$ for $j = 1, 2$. Thus $\delta(P_1) \subseteq \Delta$ and hence (6) is justified.

By (6) and symmetry, we may assume that $\delta(P_1) \subseteq \Delta$. It follows from Lemma 5.2(iii) that at least one of $\delta(v_3)$ and $\delta(v_4)$, say the former, belongs to Δ. Let (U_1, U_2) be the bipartition of P_2 with $v_3 \in U_1$. Set $\Delta_1 = \{S_1\} \cup \{\delta(v) \in \Delta : v \in U_1\}$ and $\Delta_2 = \{S_2\} \cup \{\delta(v) \in \Delta : v \in U_2\}$. It is easy to see that (Δ_1, Δ_2) is an equitable subpartition of Δ, a contradiction. Therefore G_2 is ESP. ∎

Lemma 5.9. *The graph $G_3 = (V_3, E_3)$ (see Figure 12) is ESP.*

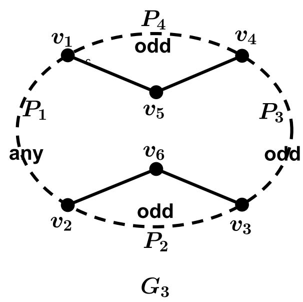


Figure 12: The primitive graph G_3

Proof. Suppose on the contrary that G_3 is not ESP. Let Δ be a collection of stars and odd sets in G_3 as specified by (5a-d) (with G_3 in place of G). By Lemma 5.2(i), we have $m_\Delta(K) = 1$ for all $K \in \Delta$. Let $U_1 = \{v_5\} \cup V(P_4)$ and $U_2 = \{v_6\} \cup V(P_2)$. Then $S_i = E[U_i]$ is an odd set in G_3 for $i = 1, 2$. Throughout the proof, we reserve

- $\mathcal{O}$ for the family consisting of all odd sets in Δ;
- X for $\{v_1, v_2, v_3, v_4\}$;

• (A_1, A_2) (resp. (A_3, A_4)) for the bipartition of P_1 (resp. P_3) with $v_2 \in A_1$ (resp. $v_3 \in A_3$);
• (B_1, B_2) (resp. (B_3, B_4)) for the bipartition of P_2 (resp. P_4) with $v_2 \in B_1$ (resp. $v_1 \in B_3$).

We break the proof into a series of simple observations. Repeated application of Lemma 5.2(iii) yields

(1) for $1 \le i \le 4$, if no odd set in Δ contains P_i and $\delta(v) \in \Delta$ for some $v \in V(P_i)\backslash X$, then $\delta(P_i) \subseteq \Delta$.

(2) If $\delta(P_2) \subseteq \Delta$, then $\delta(v_6) \notin \Delta$. Also, if $\delta(P_4) \subseteq \Delta$, then $\delta(v_5) \notin \Delta$.

Suppose the contrary: $\delta(P_2) \cup \{\delta(v_6)\} \subseteq \Delta$. Let Δ' be obtained from Δ by replacing $\delta(B_2\backslash v_3) \cup \{\delta(v_6)\}$ with S_1. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii). By symmetry, the second half also holds.

(3) $\mathcal{O} \neq \emptyset$.

Assume the contrary: $\mathcal{O} = \emptyset$. Let $Y = \{v \in V_3 : \delta(v) \in \Delta\}$ and let H be the subgraph of G_3 induced by Y. By (1) and (2), the maximum degree of H is at most two. Furthermore, H is an odd cycle, for otherwise H would be a bipartite graph. Let (Y_1, Y_2) be a bipartition of H and let $\Delta_i = \delta(Y_i)$ for $i = 1, 2$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Observe that at least one of v_5 and v_6 is outside H, for otherwise, let Δ' be obtained from Δ by replacing $\{\delta(v_5), \delta(v_6)\} \cup \delta(A_2) \cup \delta(A_4\backslash v_4)$ with $E(H)$ (an odd set by (2)). Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii). If neither v_5 nor v_6 is contained in H, set $\Delta_1 = E_3$ (the odd set induced by V_3) and $\Delta_2 = E(H)$; if exactly one of v_5 and v_6, say the latter, is contained in H, set $\Delta_1 = \delta(A_1 \cup A_3) \cup \{S_1\}$ and $\Delta_2 = E(H)$. It is easy to see that (Δ_1, Δ_2) is an equitable subpartition of Δ in either case, contradicting (5a).

Depending on the parity of P_1, we consider two cases.

Case 1. P_1 is of odd length.

Let $U_3 = \{v_5\} \cup V(P_1 \cup P_2 \cup P_3)$, $U_4 = \{v_6\} \cup V(P_1 \cup P_3 \cup P_4)$, $U_5 = V_3\backslash v_6$, and $U_6 = V_3\backslash v_5$. Then $S_i = E[U_i]$ is an odd set in G_3 for $3 \le i \le 6$. Let us make some observations about $\mathcal{O}$.

(4) $|\mathcal{O}| \ge 2$.

Assume the contrary. Then $|\mathcal{O}| = 1$ by (3). Let $\mathcal{O} = \{S_i\}$. Symmetry allows us to distinguish among the following subcases.

• $i = 1$. In this subcase, if $\delta(v_6) \in \Delta$, then $\delta(v) \notin \Delta$ for all $v \in V(P_2)\backslash X$ by (1) and (2). Repeated applications of Lemma 5.2(iii) also yields $\delta(P_1) \cup \delta(P_3) \subseteq \Delta$. Set $\Delta_1 = \{S_1\} \cup \delta(A_1 \cup A_3)$, and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). So we assume that $\delta(v_6) \notin \Delta$. Observe that $\delta(v) \in \Delta$ for some v on $P_1 \cup P_2 \cup P_3\backslash\{v_1, v_4\}$, for otherwise $(S_1, \Delta - \{S_1\})$ would be an equitable subpartition of Δ, contradicting (5a). From Lemma 5.2(iii), we further deduce that $\delta(v) \in \Delta$ for all v on $P_1 \cup P_2 \cup P_3$. Since $\{S_1, \delta(v_1)\} \subseteq \Delta$, by Lemma 5.3, we have $\delta(v_5) \notin \Delta$. Let $\Delta_1 = \{S_5\}$ and $\Delta_2 = \{S_2\} \cup \delta(A_2 \cup A_4) \cup (\delta(P_4) \cap \Delta)$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $i = 3$ or 5. In this subcase, observe that if $i = 3$ (that is, $\mathcal{O} = \{S_3\}$), then $\delta(v) \notin \Delta$ for some and hence for all $v \in V(P_4)\backslash X$ by Lemma 5.4(ii) and by (1). Let $\Delta_1 = \{S_i, \delta(v_6)\} \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ; this contradiction to (5a) proves (4).

(5) If $\{S_i, S_j\} \subseteq \mathcal{O}$ with $1 \le i < j \le 6$, then $\{i, j\}$ is one of the following five pairs:

$$\{1, 2\},\ \{1, 5\},\ \{2, 6\},\ \{3, 5\},\ \{4, 6\}.$$

To justify this, note that

• $\{i,j\} \notin \{\{1,4\},\{2,3\},\{5,6\}\}$ by Lemma 5.4(i).

• $\{i,j\} \neq \{3,4\}$. Otherwise, let Δ' be obtained from Δ by replacing $\{S_3, S_4\}$ with $\delta(B_1 \cup B_3 \backslash X) \cup \delta(P_1) \cup \delta(P_3)$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

• $\{i,j\} \notin \{\{1,3\},\{2,4\}\}$. Otherwise, by symmetry we may assume that $\{i,j\} = \{1,3\}$. Let $\Delta' = (\Delta - \{S_1, S_3\}) \cup \{S_5, \delta(v_5)\}$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

• $\{i,j\} \notin \{\{3,6\},\{4,5\}\}$. Otherwise, by symmetry we may assume that $\{i,j\} = \{3,6\}$. Let Δ' be obtained from Δ by replacing $\{S_3, S_6\}$ with $\delta(B_4 \backslash v_4) \cup \delta(P_1 \cup P_2 \cup P_3)$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

• $\{i,j\} \notin \{\{1,6\},\{2,5\}\}$. Otherwise, by symmetry we may assume that $\{i,j\} = \{1,6\}$. Let Δ' be obtained from Δ by replacing $\{S_1, S_6\}$ with $\{S_2\} \cup \delta(P_4 \backslash X) \cup \delta(A_2 \cup A_4)$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

Combining the above observations, we see that (5) holds.

(6) $\mathcal{O}$ is $\{S_1, S_2\}$, $\{S_1, S_5\}$, $\{S_2, S_6\}$, $\{S_3, S_5\}$, or $\{S_4, S_6\}$.

To justify this, let H be the graph with vertex set $\{S_1, S_2, \ldots, S_6\}$ and with five edges $\{S_i, S_j\}$ as described in (5). Since H contains no triangle, $|\mathcal{O}| < 3$ and hence $|\mathcal{O}| = 2$ by (4). Thus the statement follows instantly.

(7) If $\mathcal{O} = \{S_i, S_5\}$ for $i = 1$ or 3, then $\delta(v_5) \notin \Delta$. Otherwise, let Δ' be obtained from Δ by replacing $\{S_5, \delta(v_5)\}$ with $\{S_1, S_3\}$. Then Δ' dominates Δ and $m_{\Delta'}(S_i) \geq 2$, contradicting Lemma 5.2(i).

By (6) and symmetry, we only need to consider the following three subcases.

• $\mathcal{O} = \{S_1, S_2\}$. In this subcase, let $\Delta_1 = \{S_1\} \cup ((\delta(A_1 \cup A_3) \cup \delta(P_2)) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\mathcal{O} = \{S_1, S_5\}$. In this subcase, $\delta(v_5) \notin \Delta$ by (7). Notice that if $\delta(v) \in \Delta$ for some $v \in V(P_4) \backslash X$, then $\delta(P_4) \subseteq \Delta$ by Lemma 5.3. Set $\Delta_1 = \{S_1, S_5, \delta(v_6)\} \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(P_4) \subseteq \Delta$, and set $\Delta_1 = \{S_5, \delta(v_6)\} \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$ otherwise. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\mathcal{O} = \{S_3, S_5\}$. In this subcase, $\delta(v_5) \notin \Delta$ by (7). Moreover, $\delta(v_6) \notin \Delta$, for otherwise, let Δ' be obtained from Δ by replacing $\{S_5, \delta(v_6)\}$ with $\{S_6, \delta(v_5)\}$. Then Δ' satisfies (5a-d) and contains $\{S_3, S_6\}$, contradicting (5). Notice that if $\delta(v) \in \Delta$ for some $v \in V(P_i) \backslash X$, then $\delta(P_i) \subseteq \Delta$ for $i = 1, 2, 3$ by Lemma 5.3. Let $\Delta_1 = \{S_3, S_5\}$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(P_1 \cup P_2 \cup P_3) \subseteq \Delta$, let $\Delta_1 = \{S_5\} \cup (\delta(A_i \cup B_2) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \in \Delta$ for all $v \in V(P_j) \backslash X$, where $\{i,j\} = \{1,3\}$, and let $\Delta_1 = \{S_5\} \cup (\delta(A_1 \cup A_3) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \in \Delta$ for all $v \in V(P_2) \backslash X$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining the above three subcases, we conclude that G_3 is ESP if Case 1 occurs.

Case 2. P_1 is of even length.

Let $U_7 = V_3 \backslash \{v_5, v_6\}$, $U_8 = \{v_5, v_6\} \cup V(P_1 \cup P_3)$, $U_9 = V_3$, let $S_i = E[U_i]$ for $i = 7, 8, 9$, and let $S_{10} = E_3 \backslash E(P_2)$, $S_{11} = E_3 \backslash E(P_4)$. Then S_i is an odd set in G_3 for $7 \leq i \leq 11$. Let us make some observations about $\mathcal{O}$.

(8) If $S_1 \in \mathcal{O}$, then $\delta(v) \notin \Delta$ for some $v \in \{v_6\} \cup V(P_1 \cup P_3)$. Otherwise, let Δ' be obtained from Δ by replacing $\{S_1\} \cup \{\delta(v_6)\} \cup \delta(A_2 \cup A_3 \backslash v_3)$ with S_{10}. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

(9) If $\{S_1, S_9\}$ or $\{S_8, S_{10}\}$ or $\{S_9, S_{11}\} \subseteq \mathcal{O}$, then $\delta(v_5) \notin \Delta$. Moreover, if $\{S_2, S_9\}$ or $\{S_8, S_{11}\}$ or $\{S_9, S_{10}\} \subseteq \mathcal{O}$, then $\delta(v_6) \notin \Delta$.

Suppose $\{S_1, S_9\} \subseteq \mathcal{O}$ while $\delta(v_5) \in \Delta$. Let $\Delta' = (\Delta - \{S_9, \delta(v_5)\}) \cup \{S_1, S_{11}\}$. Then Δ' dominates Δ and $m_{\Delta'}(S_1) \geq 2$, contradicting Lemma 5.2(i). Similarly, we can prove the statement for the other cases.

(10) $|\mathcal{O}| \geq 2$.

Assume the contrary. Then $|\mathcal{O}| = 1$ by (3). Let $\mathcal{O} = \{S_i\}$. Symmetry allows us to distinguish among the following subcases.

• $i = 1$. In this subcase, at least one of $\delta(v_2)$ and $\delta(v_3)$ belongs to Δ, for otherwise $\delta(v) \notin \Delta$ for any $v \in V(P_1 \cup P_2 \cup P_3) \backslash X$ by (1). Thus $(\{S_1\}, \Delta - \{S_1\})$ would be an equitable subpartition of Δ, contradicting (5a). Moreover, $\delta(v_6) \in \Delta$, for otherwise $\delta(v) \in \Delta$ for all $v \in V(P_1 \cup P_2 \cup P_3)$ by Lemma 5.2(iii). Let Δ' be obtained from Δ by replacing $\{S_1\} \cup \delta(A_2 \cup B_2 \cup A_3)$ with S_9. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii). So $\delta(v) \notin \Delta$ for all $v \in V(P_2) \backslash X$ (which is nonempty) by (1) and (2), which implies from (1) that $\delta(P_1) \cup \delta(P_3) \subseteq \Delta$, contradicting (8).

• $7 \leq i \leq 11$. In this subcase, observe that $\delta(v) \notin \Delta$ for each v on $P_2 \cup P_4$ not covered by S_i, if any, using Lemma 5.4(ii). Let $\Delta_1 = \{S_7\} \cup (\{\delta(v_5), \delta(v_6)\} \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$ if $i = 7$, and let $\Delta_1 = \{S_i\}$ and $\Delta_2 = \Delta - \Delta_1$ otherwise. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). This proves (10).

(11) If $\{S_i, S_j\} \subseteq \mathcal{O}$ with $i \neq j$, then $7 \notin \{i, j\}$ and $\{i, j\} \notin \{\{1, 8\}, \{2, 8\}, \{1, 11\}, \{2, 10\}\}$.

To justify this, note that

• $\{i, j\} \notin \{\{1, 7\}, \{2, 7\}\}$ by Lemma 5.4(i).

• $\{i, j\} \neq \{7, j\}$ for $8 \leq j \leq 11$. Otherwise, if $\{i, j\} = \{7, 8\}$, letting Δ' be obtained from Δ by replacing $\{S_7, S_8\}$ with $\delta(P_1) \cup \delta(P_3) \cup \delta(B_1 \cup B_3 \backslash X)$, then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii). Similarly, we can prove that $\{i, j\} \neq \{7, j\}$ for $9 \leq j \leq 11$.

• $\{i, j\} \notin \{\{1, 8\}, \{2, 8\}, \{1, 11\}, \{2, 10\}\}$. Otherwise, if $\{i, j\} = \{1, 8\}$, letting Δ' be obtained from Δ by replacing $\{S_1, S_8\}$ with $\{S_{10}, \delta(v_5)\}$, then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii). Similarly, we can prove that $\{i, j\} \notin \{\{2, 8\}, \{1, 11\}, \{2, 10\}\}$.

Combining the above observations, we see that (11) holds.

(12) If $\{S_i, S_j, S_k\} \subseteq \mathcal{O}$ with i, j, k distinct, then $\{i, j, k\} \nsubseteq \{8, 9, 10, 11\}$.

Suppose the contrary. Consider the case when $\{i, j, k\} = \{8, 9, 10\}$. Let $\Delta' = (\Delta - \{S_8, S_9\}) - \{S_{10}, S_{11}\}$. Then Δ' dominates Δ and $m_{\Delta'}(S_{10}) \geq 2$, contradicting Lemma 5.2(i). Similarly, we can prove the statement for other cases.

(13) $|\mathcal{O}| \geq 3$.

Assume the contrary. Then $|\mathcal{O}| = 2$ by (10). Let $\mathcal{O} = \{S_i, S_j\}$. In view of (11), we distinguish among the following subcases.

• $\{i, j\} = \{1, 2\}$. In this subcase, $\delta(P_t) \subseteq \Delta$ for $t = 1, 3$ if $\delta(v) \in \Delta$ for some $v \in V(P_t) \backslash X$ by (1). Observe that $\delta(v) \notin \Delta$ for some $v \in V(P_1 \cup P_3) \backslash X$, for otherwise, let Δ' be obtained from Δ by replacing $\{S_1, S_2\} \cup \delta(A_2 \cup A_4 \backslash v_4)$ with S_9. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii). Let $\Delta_1 = \{S_1, S_2\} \cup (\delta(A_2) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$ for all $v \in V(P_3) \backslash X$, and let $\Delta_1 = \{S_1\} \cup ((\delta(P_2) \cup \delta(A_3)) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1) \backslash X$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} \in \{\{1, 9\}, \{2, 9\}\}$. By symmetry, we may assume that $\{i, j\} = \{1, 9\}$. In this subcase, $\delta(v_5) \notin \Delta$ by (9). Observe that $\delta(P_4) \subseteq \Delta$ if $\delta(v) \in \Delta$ for some $v \in V(P_4) \backslash X$ by Lemma 5.3. Let $\Delta_1 = \mathcal{O}$ if $\delta(P_4) \subseteq \Delta$ and $\Delta_1 = S_9$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} \in \{\{1, 10\}, \{2, 11\}\}$. By symmetry, we may assume that $\{i, j\} = \{1, 10\}$. In this subcase, we can similarly obtain an equitable subpartition of Δ as in the preceding subgraph.

• $\{i, j\} \in \{\{8, 9\}, \{10, 11\}\}$. By symmetry, we may assume that $\{i, j\} = \{8, 9\}$. In this subcase, set $\Delta' = (\Delta - \{S_8, S_9\}) \cup \{S_{10}, S_{11}\}$. Clearly, Δ' satisfies (5a-d). Observe that $\delta(P_i) \subseteq \Delta$ if $\delta(v) \in \Delta$ for some $v \in V(P_i)\backslash X$ for $i = 1, 3$ by Lemma 5.3. Let $\Delta_1 = \mathcal{O}$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(P_1) \cup \delta(P_3) \subseteq \Delta$, let $\Delta_1 = (\{\delta(v_5), S_{10}\} \cup \delta(P_2) \cup \delta(A_3)) \cap \Delta'$ and $\Delta_2 = \Delta' - \Delta_1$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1)\backslash X$, and let $\Delta_1 = (\{\delta(v_5), \delta(v_6), S_9\} \cup \delta(A_2)) \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} \in \{\{8, 10\}, \{8, 11\}\}$. By symmetry, we may assume that $\{i, j\} = \{8, 10\}$. In this subcase, $\delta(v_5) \notin \Delta$ by (9) and $\delta(v) \notin \Delta$ for all $v \in V(P_2)\backslash X$ by Lemma 5.4(ii) and (1). Observe that $\delta(P_t) \subseteq \Delta$ for $t = 1, 3$ if $\delta(v) \in \Delta$ for some $v \in V(P_t)\backslash X$ by Lemma 5.3. If $\delta(v_6) \notin \Delta$, letting $\Delta_1 = \{S_{10}\} \cup (\delta(A_2 \cup A_3) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$, then (Δ_1, Δ_2) is an equitable subpartition of Δ, this contradiction to (5a) implies that $\delta(v_6) \in \Delta$. Let $\Delta_1 = \mathcal{O}$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(P_1) \cup \delta(P_3) \subseteq \Delta$, let $\Delta_1 = (\{\delta(v_2), S_{10}\} \cup \delta(A_3)) \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1)\backslash X$, and let $\Delta_1 = (\{\delta(v_6), S_{10}\} \cup \delta(A_2)) \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} \in \{\{9, 10\}, \{9, 11\}\}$. By symmetry, we may assume that $\{i, j\} = \{9, 10\}$. In this subcase, $\delta(v_6) \notin \Delta$ by (9). Observe that $\delta(v_5) \in \Delta$, for otherwise, let $\Delta_1 = \{S_9\} \cup (\delta(A_2 \cup A_4 \cup B_4) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, a contradiction. Let $\Delta_1 = \mathcal{O}$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(P_1) \cup \delta(P_3) \subseteq \Delta$, let $\Delta_1 = (\{\delta(v_1), S_9\} \cup \delta(A_4)) \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1)\backslash X$, and let $\Delta_1 = (\{\delta(v_5), S_9\} \cup \delta(A_2)) \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining the above observations, we see that (13) holds.

Recall that $\{S_8, S_9, S_{10}, S_{11}\} \not\subseteq \mathcal{O}$ (see (12)). We may further assume that

(14) $\{S_{10}, S_{11}\} \not\subseteq \mathcal{O}$. Otherwise, let Δ' be obtained from Δ by replacing $\{S_{10}, S_{11}\}$ with $\{S_8, S_9\}$. Then Δ' dominates Δ. Since every equitable subpartition of Δ' is one for Δ, we may consider Δ' instead of Δ.

(15) $\mathcal{O}$ is $\{S_1, S_2, S_9\}$, $\{S_1, S_9, S_{10}\}$, or $\{S_2, S_9, S_{11}\}$.

To justify this, let H be the graph with vertex set $\{S_1, S_2, S_7, S_8, \ldots, S_{11}\}$ and with all edges $\{S_i, S_j\}$ as described in (11) and (15). Note that H contains precisely ten edges, in which v_7 is an isolated vertex. Since H contains no K_4, we have $|\mathcal{O}| < 4$ and hence $|\mathcal{O}| = 3$ by (13). The triangles in H are $\{S_1, S_2, S_9\}$, $\{S_1, S_9, S_{10}\}$, $\{S_2, S_9, S_{11}\}$, $\{S_8, S_9, S_{10}\}$, and $\{S_8, S_9, S_{11}\}$. In view of (12), we obtain (15).

By (15) and symmetry, we only need to consider the following two subcases.

• $\mathcal{O} = \{S_1, S_2, S_9\}$. In this subcase, $\{\delta(v_5), \delta(v_6)\} \cap \Delta = \emptyset$ by (9). Observe that $\delta(P_i) \subseteq \Delta$ for $i = 2, 4$ if $\delta(v) \in \Delta$ for some $v \in V(P_i)\backslash X$ by Lemma 5.3. Let $\Delta_1 = \mathcal{O}$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(P_2) \cup \delta(P_4) \subseteq \Delta$, let $\Delta_1 = \{S_9\}$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$ for all $v \in V(P_2 \cup P_4)\backslash X$, and let $\Delta_1 = \{S_i, S_9\}$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$ for some $v \in V(P_{2i})\backslash X$ and $\delta(P_j) \subseteq \Delta$, where $\{i, j\} \in \{\{1, 4\}, \{2, 2\}\}$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\mathcal{O} = \{S_1, S_9, S_{10}\}$. In this subcase, $\{\delta(v_5), \delta(v_6)\} \cap \Delta = \emptyset$ by (9). Observe that $\delta(P_i) \subseteq \Delta$ for $i = 1, 3$ if $\delta(v) \in \Delta$ for some $v \in V(P_i)\backslash X$ by Lemma 5.3. Let $\Delta_1 = \{S_9, S_{10}\}$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(P_1) \cup \delta(P_3) \subseteq \Delta$, let $\Delta_1 = \{S_1, S_{10}\} \cup (\delta(P_2) \cup \delta(A_3)) \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1)\backslash X$, and let $\Delta_1 = \{S_1, S_9\} \cup (\delta(A_2) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$ if $\delta(v) \notin \Delta$

for some $v \in V(P_3)\backslash X$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining the above two subcases, we conclude that G_3 is also ESP if Case 2 occurs. This completes the proof of the present lemma. ∎

Lemma 5.10. *The graph $G_4 = (V_4, E_4)$ (see Figure 13) is ESP.*

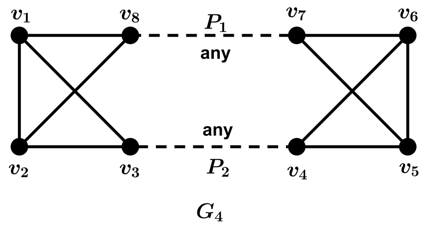


Figure 13: The primitive graph G_4

Proof. Suppose on the contrary that G_4 is not ESP. Let Δ be a collection of stars and odd sets in G_4 as specifies by (5a-d) (with G_4 in place of G). By Lemma 5.2(i), we have $m_\Delta(K) = 1$ for all $K \in \Delta$. Let $U_1 = \{v_1, v_2, v_8\}$, $U_2 = \{v_1, v_2, v_3\}$, $U_3 = \{v_5, v_6, v_7\}$, and $U_4 = \{v_4, v_5, v_6\}$. Then $S_i = E[U_i]$ is an odd set in G_4 for $i = 1, 2, 3, 4$. Throughout this proof, we reserve

- $\mathcal{O}$ for the family consisting of all odd sets in Δ;
- X for $\{v_1, v_2, v_5, v_6\}$;
- Y for $\{v_3, v_4, v_7, v_8\}$; and
- (A_1, A_2) (resp. (B_1, B_2)) for the bipartition of P_1 (resp. P_2) with $v_8 \in A_1$ (resp. $v_3 \in B_1$).

Repeated application of Lemma 5.2(iii) yields

(1) for $i = 1, 2$, if no odd set in Δ contains P_i and $\delta(v) \in \Delta$ for some $v \in V(P_i)\backslash Y$, then $\delta(P_i) \subseteq \Delta$.

(2) $|\delta(X) \cap \Delta| \geq 2$ if $\mathcal{O} = \emptyset$ (by Lemma 5.2(iii) and (1)).

(3) $\{S_i, \delta(v_1), \delta(v_2)\} \not\subseteq \Delta$ and $\{S_j, \delta(v_5), \delta(v_6)\} \not\subseteq \Delta$, for $i = 1, 2$ and $j = 3, 4$.

Suppose the contrary: $\{\delta(v_1), \delta(v_2), S_1\} \subseteq \Delta$. Let $\Delta' = (\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\}$. Then Δ' dominates Δ and $m_{\Delta'}(S_1) \geq 2$, contradicting Lemma 5.2(i). The statement for other cases can be justified similarly.

Depending on the parities of P_1 and P_2, we consider two cases.

Case 1. P_1 and P_2 have the same parity.

Let $U_5 = V_4\backslash v_6$, $U_6 = V_4\backslash v_5$, $U_7 = V_4\backslash v_1$ and $U_8 = V_4\backslash v_2$. Then $S_i = E[U_i]$ is an odd set in G_4 for $i = 5, 6, 7, 8$. Let us make some observations about $\mathcal{O}$.

(4) If $\mathcal{O} = \emptyset$, then $\delta(v) \notin \Delta$ for some $v \in V(P_1 \cup P_2)$.

Otherwise, $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$. If $\{\delta(v_1), \delta(v_2)\}$ or $\{\delta(v_5), \delta(v_6)\} \subseteq \Delta$, say the former, letting $\Delta_1 = \{S_5\} \cup (\{\delta(v_6)\} \cap \Delta)$ and $\Delta_2 = \{S_6\} \cup (\{\delta(v_5\} \cap \Delta)$, then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Thus, by (2) and symmetry, we may assume that $\{\delta(v_1), \delta(v_6)\} \subseteq \Delta$ and $\{\delta(v_2), \delta(v_5)\} \cap \Delta = \emptyset$. Let C be the even cycle induced by $V_4\backslash\{v_2, v_5\}$ in G_4, let (R_1, R_2) be the bipartition of C, and let $\Delta_i = \delta(R_i)$ for $i = 1, 2$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ; this contradiction to (5a) justifies (4).

(5) $\mathcal{O} \neq \emptyset$.

Assume the contrary: $\mathcal{O} = \emptyset$. Then $\delta(X) \not\subseteq \Delta$, for otherwise, let $\Delta_1 = \{S_1, S_4\} \cup (\delta(A_2 \cup B_1) \cap \Delta)$ if both P_1 and P_2 are odd and $\Delta_1 = \{S_1, S_3\} \cup (\delta(A_2 \cup B_1) \cap \Delta)$ otherwise, and let $\Delta_2 = (\Delta - \delta(X)) \cup (\cup_{i=1}^4 \{S_i\}) - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

By symmetry, we may assume that $\delta(v_6) \notin \Delta$. If $\delta(v_5) \notin \Delta$, then $\{\delta(v_1), \delta(v_2)\} \subseteq \Delta$ by (2) and $\{\delta(v_4), \delta(v_7)\} \cap \Delta = \emptyset$ by Lemma 5.2(iii). Hence $\Delta \subseteq \{\delta(v_1), \delta(v_2), \delta(v_3), \delta(v_8)\}$ by (1). Let $\Delta_1 = \{S_1\} \cup (\{\delta(v_3)\} \cap \Delta)$ and $\Delta_2 = \{S_2\} \cup (\{\delta(v_8)\} \cap \Delta)$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, this contradiction to (5a) implies that $\delta(v_5) \in \Delta$. It follows from Lemma 5.2 (iii) and (1) that $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$, contradicting (4).

(6) $|\mathcal{O}| \geq 2$.

Assume the contrary. Then $|\mathcal{O}| = 1$ by (5). Let $\mathcal{O} = \{S_i\}$. Symmetry allows us to distinguish among the following subcases:

• $i = 3$. In this subcase, by (3) and symmetry, we may assume that $\delta(v_6) \notin \Delta$. Observe that $\{\delta(v_1), \delta(v_2)\} \not\subseteq \Delta$, for otherwise, let $\Delta_1 = \{S_1\} \cup ((\{\delta(v_5)\} \cup \delta(A_2 \cup B_1)) \cap \Delta)$ if both P_1 and P_2 are odd and $\Delta_1 = \{S_1, S_3\} \cup (\delta(A_2 \cup B_1) \cap \Delta)$ otherwise, and let $\Delta_2 = (\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\} - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

If $\{\delta(v_1), \delta(v_2)\} \cap \Delta = \emptyset$, then $(\{S_3\}, \Delta - \{S_3\})$ would be an equitable subpartition by Lemma 5.2(iii) and (1). Thus symmetry allows us to assume that $\delta(v_1) \in \Delta$ and $\delta(v_2) \notin \Delta$. It follows from Lemma 5.2(iii) that $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$. Moreover, at least one of $\delta(v_5)$ and $\delta(v_6)$ is in Δ. By (3), we assume that $\delta(v_6) \in \Delta$ and $\delta(v_5) \notin \Delta$. Consequently, $(\{S_3, S_6\}, \{S_8\})$ is an equitable subpartition of Δ, contradicting (5a).

• $i = 5$. In this subcase, let $\Delta_1 = \{S_5, \delta(v_6)\} \cap \Delta$ and $\Delta_2 = \Delta - \Delta_2$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ; this contradiction to (5a) proves (6).

Using Lemma 5.4(i), it is routine to obtain the following statement.

(7) If $\{S_i, S_j\} \subseteq \mathcal{O}$ with $1 \leq i < j \leq 8$, then $\{i, j\}$ is one of the following pairs:

$$\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{s,t\}$$

with $s \in \{1,2,3,4\}$ and $t \in \{5,6,7,8\}$.

(8) $|\mathcal{O}| \geq 3$.

Assume the contrary. Then $|\mathcal{O}| = 2$ by (6). Let $\mathcal{O} = \{S_i, S_j\}$. In view of (7), we distinguish among the following subcases:

• $\{i, j\} \in \{\{1,3\}, \{2,4\}\}$. By symmetry, we may assume that $\{i, j\} = \{1,3\}$. By (3), we have $\delta(v_i) \notin \Delta$ nor $\delta(v_j) \notin \Delta$ for $i = 1$ or 2 and $j = 5$ or 6. Symmetry allows us to further assume that $\{\delta(v_2), \delta(v_6)\} \cap \Delta = \emptyset$. Let $\Delta_1 = \{S_3\} \cup ((\{\delta(v_1)\} \cup \delta(A_1 \cup B_2)) \cap \Delta)$ if both P_1 and P_2 are odd and $\Delta_1 = \{S_1, S_3\} \cup (\delta(A_2 \cup B_1) \cap \Delta)$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then $(\Delta_1, \Delta - \Delta_1)$ is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} \in \{\{1,4\}, \{2,3\}\}$. By symmetry, we may assume that $\{i, j\} = \{1,4\}$. By (3), we have $\delta(v_i) \notin \Delta$ nor $\delta(v_j) \notin \Delta$ for $i = 1$ or 2 and $j = 5$ or 6. Symmetry allows us to further assume that $\{\delta(v_2), \delta(v_6)\} \cap \Delta = \emptyset$. Let $\Delta_1 = \{S_1, S_4\} \cup (\delta(A_2 \cup B_1) \cap \Delta)$ if both P_1 and P_2 are of odd path and $\Delta_1 = \{S_1\} \cup (\delta(A_2 \cup B_1)) \cap \Delta)$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} \in \{\{1,5\}, \{1,6\}, \{2,5\}, \{2,6\}, \{3,7\}, \{3,8\}, \{4,7\}, \{4,8\}\}$. By symmetry, we may assume that $\{i, j\} = \{1,5\}$. By (3), we may further assume that $\delta(v_2) \notin \Delta$. Let $\Delta_1 = \{S_1, S_5, \delta(v_6)\} \cap \Delta$ if $\{\delta(v_1), \delta(v_8)\} \subseteq \Delta$ and $\Delta_1 = \{S_5, \delta(v_6)\} \cap \Delta$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i,j\} \in \{\{1,7\},\{1,8\},\{2,7\},\{2,8\},\{3,5\},\{3,6\},\{4,5\},\{4,6\}\}$. By symmetry, we may assume that $\{i,j\}=\{1,7\}$. By (3), we have $\delta(v_t)\notin\Delta$ for $t=1$ or 2. Let $\Delta_1=\mathcal{O}$ if $\delta(v_2)\in\Delta$ and $\Delta_1=\{S_7,\delta(v_1)\}\cap\Delta$ otherwise, and let $\Delta_2=\Delta-\Delta_1$. Then (Δ_1,Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining above observations, we see that (8) holds.

(9) $\mathcal{O}$ is $\{S_i,S_j,S_k\}$ for some $i\in\{1,2\}$, $j\in\{3,4\}$, and $k\in\{5,6,7,8\}$.

To justify this, let H be the graph with vertex set $\{S_1,S_2,\ldots,S_8\}$ and with all edges $\{S_i,S_j\}$ as described in (7). Since H contains no K_4, we have $|\mathcal{O}|<4$ and hence $|\mathcal{O}|=3$ by (8). The triangles in H are all displayed in (9), so the statement follows.

By (9) and symmetry, we may assume that $\mathcal{O}=\{S_1,S_3,S_5\}$. By (3), we may further assume that $\delta(v_2)\notin\Delta$. Let $\Delta_1=\mathcal{O}$ if $\{\delta(v_1),\delta(v_5),\delta(v_8)\}\subseteq\Delta$, let $\Delta_1=\{S_1,S_5,\delta(v_6)\}\cap\Delta$ if $\{\delta(v_1),\delta(v_8)\}\subseteq\Delta$ and $\delta(v_5)\notin\Delta$, let $\Delta_1=\{S_3,S_5\}$ if $\{\delta(v_1),\delta(v_8)\}\not\subseteq\Delta$ and $\delta(v_5)\in\Delta$, let $\Delta_1=\{S_5,\delta(v_6)\}\cap\Delta$ otherwise, and let $\Delta_2=\Delta-\Delta_1$. Then (Δ_1,Δ_2) is an equitable subpartition of Δ, contradicting (5a). Therefore G_4 is ESP if Case 1 occurs.

Case 2. P_1 and P_2 have different parities.

By symmetry, we may assume that P_1 is of odd length and P_2 is of even length. Let $U_9=V_4\backslash\{v_1,v_5\}$, $U_{10}=V_4\backslash\{v_2,v_6\}$, $U_{11}=V_4\backslash\{v_1,v_6\}$, $U_{12}=V_4\backslash\{v_2,v_5\}$, and $U_{13}=V_4$. Then $S_i=E[U_i]$ is an odd set in G_4 for $9\le i\le 13$.

(10) If $S_i\in\Delta$ for $i=1$ or 3, then $\delta(v)\notin\Delta$ for some $v\in V(P_1\cup P_2)$. Otherwise, $\delta(P_1)\cup\delta(P_2)\subseteq L$. By symmetry, we may assume that $S_1\in\Delta$. Let $\Delta'=(\Delta-(\{S_1\}\cup\delta(A_2\cup B_1)))\cup\{S_{13}\}$. Then Δ' dominates Δ and $f(\Delta')>f(\Delta)$, contradicting Lemma 5.2(i)

(11) If $\{\delta(v_1),\delta(v_2)\}\subseteq\Delta$, then $\delta(v)\notin\Delta$ for some $v\in V(P_1\cup P_2)$. Otherwise, let $\Delta^*=(\Delta-\{\delta(v_1),\delta(v_2)\})\cup\{S_1,S_2\}$. Then Δ^* dominates Δ. By using the same proof employed in the preceding paragraph (with Δ^* in place of Δ), we reach a contradiction to Lemma 5.2(i).

(12) $\mathcal{O}\neq\emptyset$.

Assume the contrary: $\mathcal{O}=\emptyset$. Observe that $\delta(X)\not\subseteq\Delta$, for otherwise, $\delta(v)\notin\Delta$ for some $v\in V(P_1\cup P_2)$ by (11). So $\delta(v)\notin\Delta$ for all $v\in V(P_1)\backslash Y$ or for all $v\in V(P_2)\backslash Y$ by (1). Let $\Delta_1=\{S_2,S_4\}\cup((\{\delta(v_7),\delta(v_8)\}\cup\delta(B_2))\cap\Delta)$ if $\delta(v)\notin\Delta$ for all $v\in V(P_1)\backslash Y$ and $\Delta_1=\{S_2,S_3\}\cup((\{\delta(v_4)\}\cup\delta(A_1))\cap\Delta)$ otherwise, and let $\Delta_2=((\Delta-\delta(X))\cup(\cup_{i=1}^4\{S_i\}))-\Delta_1$. Then (Δ_1,Δ_2) is an equitable subpartition of Δ, contradicting (5a).

By symmetry, we may assume that $\delta(v_6)\notin\Delta$. Then $\delta(v_5)\notin\Delta$, for otherwise, $\delta(P_1)\cup\delta(P_2)\subseteq\Delta$ by Lemma 5.2(iii), contradicting (11). Let $\Delta_1=\{S_1\}\cup(\{\delta(v_3)\}\cap\Delta)$ and $\Delta_2=\{S_2\}\cup(\{\delta(v_8)\}\cap\Delta)$. Then (Δ_1,Δ_2) is an equitable subpartition of Δ, contradicting (5a).

(13) $|\mathcal{O}|\ge 2$.

Assume the contrary. Then $|\mathcal{O}|=1$ by (12). Let $\mathcal{O}=\{S_i\}$. Symmetry allows us to distinguish among following subcases:

• $i=3$. In this subcase, $\delta(v_5)$ or $\delta(v_6)\notin\Delta$ by (3), say the latter. Moreover, $\delta(P_t)\subseteq\Delta$ for $t=1,2$ if $\delta(v)\in\Delta$ for some $v\in V(P_t)\backslash Y$ by (1). Observe that $\{\delta(v_1),\delta(v_2)\}\cap\Delta\neq\emptyset$, for otherwise, $(S_3,\Delta-\{S_3\})$ would be an equitable subpartition of Δ by Lemma 5.2(iii), a contradiction. By (10) and Lemma 5.2(iii), we further obtain $\{\delta(v_1),\delta(v_2)\}\subseteq\Delta$. Let $\Delta_1=\{S_2\}\cup((\{\delta(v_5),\delta(v_7),\delta(v_8)\}\cup\delta(B_2))\cap\Delta)$ if $\delta(v)\notin\Delta$ for all $v\in V(P_1)\backslash Y$ and $\Delta_1=\{S_2,S_3\}\cup((\{\delta(v_4)\}\cup\delta(A_1))\cap\Delta)$ if $\delta(v)\notin\Delta$ for all $v\in V(P_2)\backslash Y$ (see (11)), and let $\Delta_2=((\Delta-\{\delta(v_1),\delta(v_2)\})\cup\{S_1,S_2\})-\Delta_1$. Then (Δ_1,Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $i = 4$. In this subcase, $\{\delta(v_1), \delta(v_2)\} \cap \Delta \neq \emptyset$, for otherwise, $(\{S_4\}, \Delta - \{S_4\})$ would be an equitable subpartition of Δ by Lemma 5.2(iii), a contradiction. Observe that $\{\delta(v_1), \delta(v_2)\} \subseteq \Delta$, for otherwise, we may assume that $\delta(v_1) \in \Delta$ and $\delta(v_2) \notin \Delta$ by symmetry. Thus $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$ by Lemma 5.2(iii). Let $\Delta_1 = \{\delta(v_1), S_4\} \cup \delta(A_2 \cup B_2)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, a contradiction. Let $\Delta_1 = \{S_2, S_4\} \cup ((\{\delta(v_7), \delta(v_8)\} \cup \delta(B_2)) \cap \Delta)$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1) \backslash Y$ and $\Delta_1 = \{S_1, S_4\} \cup ((\{\delta(v_3)\} \cup \delta(A_2)) \cap \Delta)$ if $\delta(v) \notin \Delta$ for all $v \in V(P_2) \backslash Y$ (see (11)), and let $\Delta_2 = (\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\} - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $i = 9$. In this subcase, let $\Delta_1 = \{S_9\} \cup (\{\delta(v_1), \delta(v_6)\} \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $i = 13$. In this subcase, let $\Delta_1 = \{S_{13}\}$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining above observations, we see that (13) holds.

(14) If $\{S_i, S_j\} \subseteq \mathcal{O}$ with $i \neq j$, then $\{i, j\}$ is one of the following pairs:

$$\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{s,13\}$$

with $s \in \{1,2,3,4\}$.

To justify this, note that

• $\{i,j\} \notin \{\{1,2\}, \{3,4\}, \{9,11\}, \{9,12\}, \{10,11\}, \{10,12\}\} \cup \{\{s,t\} : 1 \leq s \leq 4,\ 9 \leq t \leq 12\}$ by Lemma 5.2(iii).

• $\{i,j\} \notin \{\{s,13\} : 9 \leq s \leq 12\}$. Otherwise, by symmetry we may assume that $\{i,j\} = \{9,13\}$. Let $\Delta' = (\Delta - \{S_9, S_{13}\}) \cup \delta(U_9)$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

• $\{i,j\} \notin \{\{9,10\}, \{11,12\}\}$. Otherwise, by symmetry we may assume that $\{i,j\} = \{9,10\}$. Let $\Delta' = (\Delta - \{S_9, S_{10}\}) \cup \delta(P_1 \cup P_2)$. Then Δ' dominates Δ and $\rho(\Delta') < \rho(\Delta)$, contradicting (5a).

Combining above observations, we see that (14) holds.

(15) $|\mathcal{O}| \geq 3$.

Assume the contrary. Then $|\mathcal{O}| = 2$ by (13). Let $\mathcal{O} = \{S_i, S_j\}$. In view of (14), we distinguish between the following subcases.

• $\{i,j\} \in \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}$. In this subcase, by symmetry we may assume that $\{i,j\} = \{1,3\}$ and that $\{\delta(v_2), \delta(v_6)\} \cap \Delta = \emptyset$ (see (3)). Observe that $\delta(P_1) \subseteq \Delta$, for otherwise, let $\Delta_1 = \{S_1, S_3\} \cup (\delta(B_1) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, a contradiction. Hence, by (1) and (10), we obtain $\delta(v) \notin \Delta$ for all $v \in V(P_2) \backslash Y$. Let $\Delta_1 = \{S_1\} \cup \delta(A_2) \cup (\{\delta(v_3), \delta(v_5)\} \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i,j\} \in \{\{1,13\}, \{2,13\}, \{3,13\}, \{4,13\}\}$. In this subcase, by symmetry we may assume that $\{i,j\} = \{1,13\}$ and that $\delta(v_2) \notin \Delta$ (see (3)). Let $\Delta_1 = \mathcal{O}$ if $\{\delta(v_1), \delta(v_8)\} \subset \Delta$ and $\Delta_1 = S_{13}$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining above observations, we see that (15) holds.

(16) $\mathcal{O}$ is $\{S_1, S_3, S_{13}\}$, $\{S_1, S_4, S_{13}\}$, $\{S_2, S_3, S_{13}\}$, or $\{S_2, S_4, S_{13}\}$.

To justify this, let H be the graph with vertex set $\{S_1, S_2, S_3, S_4, S_9, \ldots S_{13}\}$ and with all edges $\{S_i, S_j\}$ as described in (14). Since H contains no K_4, we have $|\mathcal{O}| < 4$ and hence $|\mathcal{O}| = 3$ by (15). The triangles in H are all displayed in (16), so the statement holds.

By (16) and symmetry, we may assume that $\mathcal{O} = \{S_1, S_3, S_{13}\}$. Symmetry and (3) allow us to further assume that $\{\delta(v_2), \delta(v_5)\} \cap \Delta = \emptyset$. Let $\Delta_1 = \mathcal{O}$ if $\{\delta(v_1), \delta(v_6), \delta(v_7), \delta(v_8)\} \subseteq \Delta$, let $\Delta_1 = \{S_1, S_{13}\}$ if $\{\delta(v_1), \delta(v_8)\} \subseteq \Delta$ and $\{\delta(v_6), \delta(v_7)\} \not\subseteq \Delta$, let $\Delta_1 = \{S_3, S_{13}\}$ if $\{\delta(v_1), \delta(v_8)\} \not\subseteq \Delta$ and $\{\delta(v_6), \delta(v_7)\} \subseteq \Delta$, let $\Delta_1 = \{S_{13}\}$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Therefore G_4 is also ESP if Case 2 occurs. This completes the proof of the present lemma. ∎

Lemma 5.11. *The graph $G_5 = (V_5, E_5)$ (see Figure 14) is ESP.*

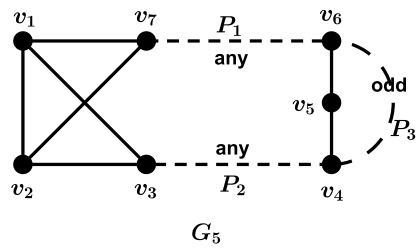


Figure 14: The primitive graph G_5

Proof. Suppose on the contrary that G_5 is not ESP. Let Δ be a collection of stars and odd sets in G_5 as specified by (5a-d) (with G_5 in place of G). By Lemma 5.2(i), we have $m_\Delta(K) = 1$ for all $K \in \Delta$. Let $U_1 = \{v_1, v_2, v_7\}$, $U_2 = \{v_1, v_2, v_3\}$, and $U_3 = \{v_5\} \cup V(P_3)$. Then $S_i = E[U_i]$ is an odd set in G_5 for $i = 1, 2, 3$. Throughout this proof, we reserve

- $\mathcal{O}$ for the family consisting of all odd sets in Δ;
- X for $\{v_3, v_4, v_6, v_7\}$;
- (A_1, A_2) (resp. (A_3, A_4)) for the bipartition of P_1 (resp. P_2) with $v_7 \in A_1$ (resp. $v_3 \in A_3$);
- (B_1, B_2) for the bipartition of P_3 with $v_4 \in B_1$.

Repeated application of Lemma 5.2(iii) yields

(1) for $i = 1, 2, 3$, if no odd set in Δ contains P_i and $\delta(v) \in \Delta$ for some $v \in V(P_i)\backslash X$, then $\delta(P_i) \subseteq \Delta$.

(2) If $\delta(P_3) \subseteq \Delta$, then $\delta(v_5) \notin \Delta$. Otherwise, let Δ' be obtained from Δ by replacing $\{\delta(v_5)\} \cup \delta(B_1\backslash v_4)$ with S_3. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

(3) $\{S_i, \delta(v_1), \delta(v_2)\} \not\subseteq \Delta$ for $i = 1, 2$. Otherwise, by symmetry we may assume that $\{\delta(v_1), \delta(v_2), S_1\} \subseteq \Delta$. Let $\Delta' = (\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\}$. Then Δ' dominates Δ and $m_{\Delta'}(S_1) \geq 2$, contradicting Lemma 5.2(i).

Depending on the parities of P_1 and P_2, we consider two cases.

Case 1. P_1 and P_2 have the same parity.

Let $U_4 = V_5$, $U_5 = \{v_1, v_2, v_5\} \cup V(P_1 \cup P_2)$, $U_6 = V_5\backslash\{v_1, v_5\}$, $U_7 = V_5\backslash\{v_2, v_5\}$. Then $S_i = E[U_i]$ is an an odd set in G_5 for $4 \leq i \leq 7$. Note that $S_4 = S_5$ if $|V(P_3)| = 2$. So we implicitly assume that $|V(P_3)| \geq 3$ if S_5 occurs in our proof.

(4) If $S_3 \in \Delta$ and $\{\delta(v_1), \delta(v_2)\} \cap \Delta \neq \emptyset$, then $\delta(v) \notin \Delta$ for some $v \in V(P_1 \cup P_2)$.

Otherwise, $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$. By symmetry, we may assume that $\delta(v_1) \in \Delta$. Let $\Delta' = (\Delta - (\{S_3\} \cup \delta(A_1 \cup A_3))) \cup \{S_4\}$ if both P_1 and P_2 are odd and $\Delta' = (\Delta - (\{\delta(v_1), S_3\} \cup \delta(A_2 \cup A_4))) \cup \{S_4\}$ otherwise. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

(5) If $\{S_i\} \cup \delta(P_1) \cup \delta(P_2) \subseteq \Delta$ for $i = 1$ or 2, then $\delta(v) \notin \Delta$ for all $v \in \{v_5\} \cup V(P_3)\backslash X$.

Assume the contrary: $\delta(v) \in \Delta$ for some $v \in \{v_5\} \cup V(P_3)\backslash X$. By symmetry, we may assume that $S_1 \in \Delta$. Observe that $v \neq v_5$, for otherwise, if both P_1 and P_2 are odd, letting $\Delta' = (\Delta - \{\delta(v_5), S_1\} \cup \delta(A_3) \cup \delta(A_2\backslash v_6)) \cup \{S_5\}$, then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii). Similarly, we can reach a contradiction if both P_1 and P_2 are even. It follows from (1) that $\delta(P_3) \subseteq \Delta$. If both P_1 and P_2 are odd, letting Δ' by obtained from Δ by replacing $\{S_1\} \cup \delta(A_2 \cup A_3 \cup B_2)$ with S_4, then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii). Similarly, we can reach a contradiction if both P_1 and P_2 are even.

(6) $\mathcal{O} \neq \emptyset$.

Assume the contrary: $\mathcal{O} = \emptyset$. By (1), (2) and Lemma 5.2(iii), we have $\{\delta(v_1), \delta(v_2)\} \cap \Delta \neq \emptyset$. Furthermore, $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$ if $\delta(v_5) \in \Delta$. Observe that $\{\delta(v_1), \delta(v_2)\} \not\subseteq \Delta$, for otherwise, let $\Delta' = (\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\}$. Then Δ' dominates Δ. If $\delta(v_5) \in \Delta$, then $\{S_1\} \cup \delta(P_1) \cup \delta(P_2) \subseteq \Delta'$, and thus we can reach a contradiction to Lemma 5.2(ii) by using the same argument as employed in the proof of (5). If $\delta(v_5) \notin \Delta$, then $\delta(v) \notin \Delta$ for all $v \in V(P_1 \cup P_2 \cup P_3)\backslash\{v_3, v_7\}$ by (1), (5) and Lemma 5.2(iii). Let $\Delta_1 = \{S_1\} \cup (\{\delta(v_3)\} \cap \Delta)$ and $\Delta_2 = \Delta' - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

By symmetry, we may assume that $\delta(v_1) \in \Delta$ and $\delta(v_2) \notin \Delta$. Then $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$ by (1) and Lemma 5.2(iii). Consider the subcase when $\delta(v_5) \in \Delta$. Now $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$ by (1) and (2). Let $\Delta_1 = \{\delta(v_1)\} \cup \delta(A_2 \cup A_4)$ if both P_1 and P_2 are odd and $\Delta_1 = \{\delta(v_1), \delta(v_5)\} \cup \delta(A_2 \cup A_4)$ otherwise. Then $(\Delta_1, \Delta - \Delta_1)$ is an equitable subpartition of Δ, a contradiction. It remains to consider the subcase when $\delta(v_5) \notin \Delta$. Now $\delta(v) \in \Delta$ for all $v \in V_5\backslash\{v_2, v_5\}$ by (1) and Lemma 5.2(iii). Thus $(\{S_4\}, \{S_7\})$ is an equitable subpartition of Δ, a contradiction. Therefore (6) is established.

(7) $|\mathcal{O}| \geq 2$.

Assume the contrary. Then $|\mathcal{O}| = 1$ by (6). Let $\mathcal{O} = \{S_i\}$. Symmetry allows us to distinguish among the following subcases.

• $i = 1$. In this subcase, we may assume that $\delta(v_2) \notin \Delta$ by (3) and symmetry. If $\delta(v_3) \in \Delta$, then $\delta(P_2) \subseteq \Delta$; furthermore, $\delta(P_3) \subseteq \Delta$ or $\delta(v_5) \in \Delta$ by Lemma 5.2(iii). It follows that $\delta(P_1) \subseteq \Delta$, contradicting (5). So $\delta(v_3) \notin \Delta$, which implies that $\delta(v) \notin \Delta$ for all $v \in V_5\backslash\{v_1, v_2, v_7\}$ by (1) and Lemma 5.2(iii). Thus $(\{S_1\}, \Delta - \{S_1\})$ is an equitable subpartition of Δ, contradicting (5a).

• $i = 3$. In this subcase, if $\{\delta(v_1), \delta(v_2)\} \cap \Delta = \emptyset$, then $(\{S_3\}, \Delta - \{S_3\})$ is an equitable subpartition of Δ by (1) and Lemma 5.2(iii). So $\{\delta(v_1), \delta(v_2)\} \cap \Delta \neq \emptyset$. By (1), (4) and symmetry, we may assume that $\delta(v) \notin \Delta$ for all $V(P_1)\backslash X$, which implies $\{\delta(v_1), \delta(v_2)\} \subseteq \Delta$ by Lemma 5.2(iii). Let $\Delta' = (\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\}$, and let $\Delta_1 = \{S_1, S_3\} \cup (\delta(A_3) \cap \Delta')$ if both P_1 and P_2 are odd and $\Delta_1 = \{S_2, S_3\} \cup ((\{\delta(v_7)\} \cup \delta(A_4)) \cap \Delta')$ otherwise. Then $(\Delta_1, \Delta' - \Delta_1)$ is an equitable subpartition of Δ, contradicting (5a).

• $i = 4$ or 5. In this subcase, observe that if $i = 5$, then $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$ by (1) and Lemma 5.4(ii). Thus $(\{S_i\}, \Delta - \{S_i\})$ is an equitable subpartition of Δ for $i = 4, 5$, contradicting (5a).

• $i = 6$. In this subcase, let $\Delta_1 = \{\delta(v_1), \delta(v_5), S_6\} \cap \Delta$. Then $(\Delta_1, \Delta - \Delta_1)$ is an equitable subpartition of Δ, contradicting (5a).

Combining above observations, we see that (7) holds.

(8) If $\{S_i, S_j\} \subseteq \mathcal{O}$ with $1 \le i < j \le 7$, then $\{i, j\}$ is one of the following pairs:

$$\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{1,5\}, \{2,5\}, \{3,4\}, \{4,5\}.$$

To justify this, note that

• $\{i,j\} \notin \{\{1,2\},\{1,6\},\{1,7\},\{2,6\},\{2,7\},\{3,6\},\{3,7\},\{6,7\}\}$ by Lemma 5.4(i).

• $\{i,j\} \neq \{3.5\}$. Otherwise, let $\Delta' = (\Delta - \{S_3, S_5\}) \cup \{S_4, \delta(v_5)\}$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

• $\{i,j\} \notin \{\{4,6\},\{4,7\}\}$. Otherwise, by symmetry we may assume that $\{i,j\} = \{4,6\}$. Let Δ' be obtained from Δ by replacing $\{S_4, S_6\}$ with $\delta(U_6)$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

• $\{i,j\} \notin \{\{5,6\},\{5,7\}\}$. Otherwise, by symmetry we may assume that $\{i,j\} = \{5,6\}$. Let Δ' be obtained from Δ by replacing $\{S_5, S_6\}$ with $\delta(U_5 \cap U_6) \cup \delta(B_1 \backslash v_4)$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

Combining above observations, we see that (8) holds.

(9) If $\{S_i, S_3\} \subseteq \Delta$ for $i = 1$ or 2, then $\delta(v) \notin \Delta$ for some $v \in V(P_1 \cup P_2)$.

Assume the contrary: $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$. By symmetry, we may assume that $S_1 \in \Delta$. Let $\Delta' = (\Delta - (\{S_1, S_3\} \cup \delta(A_3 \cup A_1 \backslash v_7)) \cup \{S_4\}$ if both P_1 and P_2 are odd. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii). Similarly, we can reach a contradiction if both P_1 and P_2 are even.

(10) $|\mathcal{O}| \ge 3$.

Assume the contrary. Then $|\mathcal{O}| = 2$ by (7). Let $\mathcal{O} = \{S_i, S_j\}$. In view of (8), we distinguish among the following subcases.

• $\{i,j\} \in \{\{1,3\},\{2,3\}\}$. By symmetry, we may assume that $\{i,j\} = \{1,3\}$. By (9) and (1), we have $\delta(v) \notin \Delta$ for all $v \in V(P_1)\backslash X$ or for all $v \in V(P_2)\backslash X$. Let $\Delta_1 = \{S_1, S_3\} \cup \delta(A_3) \cap \Delta$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1)\backslash X$ and $\Delta_1 = (\{\delta(v_3), \delta(v_5), S_1\} \cup \delta(A_2) \cup \delta(P_3)) \cap \Delta$ if $\delta(v) \notin \Delta$ for all $v \in V(P_2)\backslash X$. Then $(\Delta_1, \Delta - \Delta_1)$ is an equitable subpartition of Δ if both P_1 and P_2 are odd. Similarly, we can reach a contradiction to (5a) if both P_1 and P_2 are even.

• $\{i,j\} \in \{\{1,4\},\{2,4\},\{1,5\},\{2,5\}\}$. By symmetry, we may assume that $i = 1$ and $\delta(v_2) \notin \Delta$ (see (3)). Observe that if $S_5 \in \Delta$, then $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$ by Lemma 5.4(ii) and (1). Let $\Delta_1 = \mathcal{O}$ and $\Delta_2 = \Delta - \Delta_1$ if $\{\delta(v_1), \delta(v_7)\} \subseteq \Delta$, and let $\Delta_1 = \{S_j\}$ and $\Delta_2 = \Delta - \Delta_1$ for $j = 4, 5$ otherwise. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i,j\} = \{3,4\}$. Observe that if $\delta(P_3) \subseteq \Delta$, letting $\Delta_1 = \mathcal{O}$ and $\Delta_2 = \Delta - \Delta_1$, then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Thus $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$ by Lemma 5.3. It follows that $(\{S_4\}, \Delta - \{S_4\})$ is an equitable subpartition of Δ, contradicting (5a).

• $\{i,j\} = \{4,5\}$. Observe that $\delta(P_i) \subseteq \Delta$ for $i = 1, 2$ if $\delta(v) \in \Delta$ for some $v \in V(P_i)\backslash X$ by Lemma 5.3. If $\{\delta(v_1), \delta(v_2)\} \not\subseteq \Delta$, say $\delta(v_2) \notin \Delta$, letting $\Delta_1 = (\{\delta(v_5), S_4\} \cup \delta(A_1 \cup A_3)) \cap \Delta$ if both P_1 and P_2 are odd and $\Delta_1 = (\{\delta(v_1), S_4\} \cup \delta(A_2 \cup A_4)) \cap \Delta$, then $(\Delta_1, \Delta - \Delta_1)$ is an equitable subpartition of Δ, contradicting (5a). So $\{\delta(v_1), \delta(v_2)\} \subseteq \Delta$. If $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$, then $(\mathcal{O}, \Delta - \mathcal{O})$ is an equitable subpartition of Δ, a contradiction. Hence $\delta(v) \notin \Delta$ for all $v \in V(P_1)\backslash X$ or all $\delta(v) \notin \Delta$ for all $v \in V(P_2)\backslash X$ by Lemma 5.3. Consider the subsubcase when both P_1 and P_2 are odd. Let $\Delta_1 = \{S_2, S_4\} \cup ((\{\delta(v_5)\} \cup \delta(A_1)) \cap \Delta)$ if $\delta(v) \notin \Delta$ for all $v \in V(P_2)\backslash X$ and $\Delta_1 = \{S_1, S_4\} \cup ((\{\delta(v_5)\} \cup \delta(A_3)) \cap \Delta)$ otherwise, and let $\Delta_2 = ((\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\}) - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Similarly, we can reach a contradiction if both P_1 and P_2 are of even length.

Combining above observations, we see that (10) holds.

(11) $\mathcal{O}$ is $\{S_1, S_3, S_4\}$, $\{S_1, S_4, S_5\}$, $\{S_2, S_3, S_4\}$, or $\{S_2, S_4, S_5\}$.

To justify this, let H be the graph with vertex set $\{S_1, S_2, \ldots, S_7\}$ and with all edges $\{S_i, S_j\}$ as described in (8). Since H contains no K_4, we have $|\mathcal{O}| < 4$ and hence $|\mathcal{O}| = 3$ by (10). The triangles in H are all displayed in (11), so the statement holds.

By (11) and symmetry, we only need to consider the following subcases.

• $\mathcal{O} = \{S_1, S_3, S_4\}$. In this subcase, observe that if $\delta(P_3) \not\subseteq \Delta$, then $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$ by Lemma 5.3. Let $\Delta_1 = \mathcal{O}$ if $\{\delta(v_1), \delta(v_7)\} \cup \delta(P_3) \subseteq \Delta$, let $\Delta_1 = \{S_1, S_4\}$ if $\{\delta(v_1), \delta(v_7)\} \subseteq \Delta$ and $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$, let $\Delta_1 = \{S_3, S_4\}$ if $\{\delta(v_1), \delta(v_7)\} \not\subseteq \Delta$ and $\delta(P_3) \subseteq \Delta$, let $\Delta_1 = \{S_4\}$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\mathcal{O} = \{S_1, S_4, S_5\}$. In this subcase, by (3) and symmetry we may assume that $\delta(v_2) \notin \Delta$. When both P_1 and P_2 are odd, let $\Delta_1 = \{S_4, S_5\}$ if $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$, let $\Delta_1 = \{S_1, S_4\} \cup (\delta(A_3) \cap \Delta)$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1)\backslash X$, let $\Delta_1 = (\{\delta(v_1), S_4\} \cup \delta(A_1)) \cap \Delta$ if $\delta(v) \notin \Delta$ for all $v \in V(P_2)\backslash X$, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Similarly, we can reach a contradiction if both P_1 and P_2 are even.

Combining above subcases, we conclude that G_5 is ESP if Case 1 occurs.

Case 2. P_1 and P_2 have different parities.

By symmetry, we may assume that P_1 is odd and P_2 is even. Let $U_8 = V_5\backslash v_5$, $U_9 = V_5\backslash v_1$, $U_{10} = V_5\backslash v_2$, $U_{11} = \{v_2, v_5\} \cup V(P_1 \cup P_2)$, $U_{12} = \{v_1, v_5\} \cup V(P_1 \cup P_2)$. Then $S_i = E[U_i]$ is an odd set in G_5 for $8 \le i \le 12$. Note that $S_9 = S_{11}$ and $S_{10} = S_{12}$ if $|V(P_3)| = 2$. So we implicitly assume that $|V(P_3)| \ge 3$ if S_{11} or S_{12} occurs in our proof.

(12) $\mathcal{O} \neq \emptyset$.

Assume the contrary: $\mathcal{O} = \emptyset$. Let us first consider the subcase when $\delta(v_5) \in \Delta$. By (1) and (2), we have $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$. From Lemma 5.2(iii), we further deduce that $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$ and that $\{\delta(v_1), \delta(v_2)\} \cap \Delta \neq \emptyset$. When $\delta(v_1) \in \Delta$, let $\Delta' = (\Delta - (\{\delta(v_1), \delta(v_5)\} \cup \delta(A_1 \cup A_4))) \cup \{S_1, S_{12}\}$. Then Δ' dominates Δ. Set $\Delta_1 = \{\delta(v_2), S_{12}\} \cap \Delta'$ and $\Delta_2 = \Delta' - \Delta_1$. When $\delta(v_1) \notin \Delta$, let $\Delta' = (\Delta - (\{\delta(v_2), \delta(v_5)\} \cup \delta(A_1 \cup A_4))) \cup \{S_1, S_{11}\}$. Then Δ' dominates Δ. Set $\Delta_1 = \{S_{11}\}$ and $\Delta_2 = \Delta' - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

It remains to consider the subcase when $\delta(v_5) \notin \Delta$. If $\delta(v) \notin \Delta$ for some $v \in V(P_1 \cup P_2 \cup P_3)$, then $\{\delta(v_1), \delta(v_2)\} \subseteq \Delta \subseteq \{\delta(v_1), \delta(v_2), \delta(v_3), \delta(v_7)\}$ by Lemma 5.2(iii). Let $\Delta_1 = \{S_1\} \cup (\{\delta(v_3)\} \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). So $\delta(P_i) \subseteq \Delta$ for $i = 1, 2, 3$, which implies $\{\delta(v_1), \delta(v_2)\} \cap \Delta \neq \emptyset$. By symmetry, we may assume that $\delta(v_1) \in \Delta$. Let $\Delta_1 = \{S_8\}$ and $\Delta_2 = \{S_{10}, \delta(v_2)\} \cap \Delta$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

(13) $|\mathcal{O}| \ge 2$.

Assume the contrary. Then $|\mathcal{O}| = 1$ by (12). Let $\mathcal{O} = \{S_i\}$. Symmetry allows us to distinguish among the following subcases.

• $i = 1$. In this subcase, observe that if $\delta(v_5) \in \Delta$, then $\delta(P_1 \cup P_2) \subseteq \Delta$ by (1) and $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$ by (2). Let $\Delta_1 = \{S_1\} \cup \delta(A_2 \cup A_3)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2)

is an equitable subpartition of Δ, contradicting (5a). Hence $\delta(v_5) \notin \Delta$. If $\delta(v) \notin \Delta$ for some $v \in V(P_1 \cup P_2 \cup P_3)$, then $(\{S_1\}, \Delta - \{S_1\})$ is an equitable subpartition of Δ; this contradiction implies that $\delta(P_i) \subseteq \Delta$ for $i = 1, 2, 3$. Thus $\delta(v_i) \in \Delta$ for $i = 1$ or 2. Let $\Delta_1 = \{S_8\}$ and $\Delta_2 = \{S_1, S_{10}\}$ if $i = 1$ and let $\Delta_1 = \{S_8\}$ and $\Delta_2 = \{S_1, S_9\}$ if $i = 2$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $i = 3$. In this subcase, observe that $\{\delta(v_1), \delta(v_2)\} \not\subseteq \Delta$, for otherwise, let $\Delta' = (\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\}$, and let $\Delta_1 = \{S_2, S_3\} \cup (\delta(A_1 \cup A_4) \cap \Delta)$ and $\Delta_2 = \Delta' - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, a contradiction. If $\{\delta(v_1), \delta(v_2)\} \cap \Delta = \emptyset$, then $(\{S_3\}, \Delta - \{S_3\})$ is an equitable subpartition of Δ by Lemma 5.2(iii). Thus precisely one of $\delta(v_1)$ and $\delta(v_2)\}$ belongs to Δ, which implies $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$. Let $\Delta_1 = \{S_{10}\}$ and $\Delta_2 = (\Delta - (\{\delta(v_1), S_3\} \cup \delta(A_1 \cup A_4))) \cup \{S_1\}$ if $\delta(v_1) \in \Delta$, and let $\Delta_1 = \{S_9\}$ and $\Delta_2 = (\Delta - (\{\delta(v_2), S_3\} \cup \delta(A_1 \cup A_4))) \cup \{S_1\}$ if $\delta(v_2) \in \Delta$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $i = 8$. In this subcase, let $\Delta_1 = \{S_8, \delta(v_5)\} \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $i = 9$. In this subcase, let $\Delta_1 = \{S_9, \delta(v_1)\} \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $i = 11$. In this subcase, $\delta(v) \notin \Delta$ for all $v \in V(P_3)\backslash X$ by Lemma 5.4(ii) and (1). Let $\Delta_1 = \{S_{11}, \delta(v_1)\} \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining above subcases, we see that (13) holds.

(14) If $\{S_i, S_j\} \subseteq \Delta$, then $\{i, j\}$ is one of the following pairs:

$$\{1,3\}, \{1,8\}, \{1,9\}, \{1,10\}, \{2,3\}, \{2,8\}, \{2,9\}, \{2,10\}, \{3,9\}, \{3,10\}, \{9,11\}, \{10,12\}.$$

To justify this, note that

• $\{i, j\} \notin \{\{1,2\}, \{1,11\}, \{1,12\}, \{2,11\}, \{2,12\}, \{9,10\}, \{11,12\}\}$ by Lemma 5.4(i).

• $\{i, j\} \neq \{3, 8\}$. Otherwise, let Δ' be obtained from Δ by replacing $\{S_3, S_8\}$ with $\{S_1\} \cup \delta(A_2 \cup A_3) \cup \delta(P_3)$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

• $\{i, j\} \notin \{\{3,11\}, \{3,12\}\}$. Otherwise, by symmetry we may assume that $\{i, j\} = \{3, 11\}$. Let $\Delta' = (\Delta - \{S_3, S_{11}\}) \cup \{\delta(v_5), S_9\}$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

• $\{i, j\} \notin \{\{8,11\}, \{8,12\}\}$. Otherwise, by symmetry we may assume that $\{i, j\} = \{8, 11\}$. Let Δ' be obtained from Δ by replacing $\{S_8, S_{11}\}$ with $\delta(U_{11}\backslash v_5) \cup \delta(B_1\backslash v_4)$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

• $\{i, j\} \notin \{\{9,12\}, \{10,11\}\}$. Otherwise, by symmetry we may assume that $\{i, j\} = \{9, 12\}$. Let Δ' be obtained from Δ by replacing $\{S_9, S_{12}\}$ with $\delta(U_{12}\backslash v_1) \cup \delta(B_1\backslash v_4)$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

• $\{i, j\} \notin \{\{8,9\}, \{8,10\}\}$. Otherwise, by symmetry we may assume that $\{i, j\} = \{8, 9\}$. Let Δ' be obtained from Δ by replacing $\{S_8, S_9\}$ with $\delta(U_8 \cap U_9)$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

Combining above observations, we see that (14) holds.

(15) $|\mathcal{O}| \geq 3$.

Assume the contrary. Then $|\mathcal{O}| = 2$ by (13). Let $\mathcal{O} = \{S_i, S_j\}$. In view of (14), we distinguish among the following subcases.

• $\{i, j\} = \{1, 3\}$. Let $\Delta_1 = \{S_1\} \cup (\delta(A_2 \cup A_3) \cup \delta(P_3)) \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} = \{2, 3\}$. Let $\Delta_1 = \{S_2, S_3\} \cup (\delta(A_1 \cup A_4) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} \in \{\{1, 8\}, \{2, 8\}\}$. By symmetry, we may assume that $\{i, j\} = \{1, 8\}$ and that $\delta(v_2) \notin \Delta$ (see (3)). Let $\Delta_1 = \{S_1, S_8, \delta(v_5)\} \cap \Delta$ if $\{\delta(v_1), \delta(v_7)\} \subseteq \Delta$ and $\Delta_1 = \{S_8, \delta(v_5)\} \cap \Delta$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} \in \{\{1, 9\}, \{1, 10\}, \{2, 9\}, \{2, 10\}\}$. By symmetry, we may assume that $\{i, j\} = \{1, 9\}$. Let $\Delta_1 = \mathcal{O}$ if $\delta(v_2) \in \Delta$ and $\Delta_1 = \{S_9, \delta(v_1)\} \cap \Delta$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} \in \{\{3, 9\}, \{3, 10\}\}$. By symmetry, we may assume that $\{i, j\} = \{3, 9\}$. From Lemma 5.3, we see that $\delta(v) \notin \Delta$ for all $v \in V(P_3) \backslash X$ if $\delta(P_3) \nsubseteq \Delta$. Let $\Delta_1 = \{\delta(v_1), S_1, S_9\} \cap \Delta$ if $\delta(P_3) \subseteq \Delta$ and $\Delta_1 = \{\delta(v_1), \delta(v_5), S_9\}$, and let $\Delta_2 = \Delta - \Delta_2$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\{i, j\} \in \{\{9, 11\}, \{10, 12\}\}$. By symmetry, we may assume that $\{i, j\} = \{9, 11\}$. Observe that $\delta(v_1) \notin \Delta$, for otherwise, let $\Delta' = (\Delta - \{\delta(v_1), S_9\}) \cup \{\delta(v_5), S_8\}$. Then Δ' dominates Δ and satisfies (5a-d). Since $\{S_8, S_9\} \subseteq \Delta'$, we reach a contradiction to (14). Let $\Delta_1 = \mathcal{O}$ if $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$, let $\Delta_1 = (\{S_9, \delta(v_2), \delta(v_5)\} \cup \delta(A_4)) \cap \Delta$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1) \backslash X$, let $\Delta_1 = (\{S_9, \delta(v_3), \delta(v_5)\} \cup \delta(A_1)) \cap \Delta$ if $\delta(v) \notin \Delta$ for all $v \in V(P_2) \backslash X$, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining above subcases, we see that (15) holds.

(16) $\mathcal{O}$ is $\{S_1, S_3, S_9\}$, $\{S_1, S_3, S_{10}\}$, $\{S_2, S_3, S_9\}$, or $\{S_2, S_3, S_{10}\}$.

To justify this, let H be the graph with vertex set $\{S_1, S_2, S_3, S_8, \ldots, S_{12}\}$ and with all edges $\{S_i, S_j\}$ as described in (14). Since H contains no K_4, we have $|\mathcal{O}| < 4$ and hence $|\mathcal{O}| = 3$ by (15). The triangles in H are all displayed in (16), so the statement holds.

By (16) and symmetry, we only need to consider the subcase when $\mathcal{O} = \{S_1, S_3, S_9\}$. Let $\Delta_1 = \mathcal{O}$ if $\{\delta(v_2)\} \cup \delta(P_3) \subseteq \Delta$, let $\Delta_1 = \{S_1, S_9, \delta(v_5)\} \cap \Delta$ if $\delta(v_2) \in \Delta$ and $\delta(v) \notin \Delta$ for all $v \in V(P_3) \backslash X$, and let $\Delta_1 = \{S_3, S_9, \delta(v_1)\} \cap \Delta$ if $\delta(v_2) \notin \Delta$ and $\delta(P_3) \subseteq \Delta$, let $\Delta_1 = \{S_9, \delta(v_1), \delta(v_5)\}$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Therefore G_5 is also ESP if Case 2 occurs. This complete the proof of present lemma. ∎

Lemma 5.12. *The graph $G_6 = (V_6, E_6)$ (see Figure 15) is ESP.*

Proof. Suppose on the contrary that G_6 is not ESP. Let Δ be a collection of stars and odd sets in G_6 as specified by (5a-d) (with G_6 in place of G). By Lemma 5.2(i), we have $m_\Delta(K) = 1$ for all $K \in \Delta$. We use H to denote the fully subdivided graph in G_6. Throughout this proof, we reserve

• $\mathcal{O}$ for the family consisting of all odd sets in Δ;

• $\mathcal{P}$ for the family consisting of all paths connecting v_1 and v_2 in H; and

• (X, Y) for the bipartition of H with $\{v_1, v_2\} \subseteq X$.

Let $U_P = V(P)$ for each $P \in \mathcal{P}$. Then $S_P = E[U_P]$ is an odd set in G_6. We break the proof into a few observations.

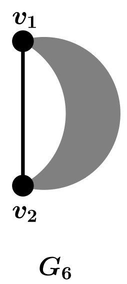


Figure 15: The primitive graph G_6

(1) Each $P \in \mathcal{P}$ contains a vertex $v \in Y$ with $\delta(v) \notin \Delta$. Otherwise, let Δ' be obtained from Δ by replacing $\delta(V(P) \cap Y)$ with S_P. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

(2) $\mathcal{O} \neq \emptyset$.

Assume the contrary: $\mathcal{O} = \emptyset$. By (1) and Lemma 5.5, Δ contains an equitable partition (Δ_1, Δ_2) of Δ with $|\Delta_i \cap \{\delta(v_1), \delta(v_2)\}| \leq 1$ for $i = 1, 2$ (with Δ in place of Ω), contradicting (5a).

(3) $|\mathcal{O}| = 1$.

Assume the contrary. Then $|\mathcal{O}| \geq 2$ by (2). Let $\{S_P, S_Q\} \subseteq \Delta$ with P, Q distinct in $\mathcal{P}$, let $W_1 = V(P \cup Q) \cap X$ and $W_2 = V(P) \cap V(Q) \cap Y$, and let Δ' be obtained from Δ by replacing $\{S_P, S_Q\}$ with $\delta(W_1 \cup W_2)$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

By (3), we have $\mathcal{O} = \{S_P\}$ for some $P \in \mathcal{P}$. Let $\Delta_1 = \{S_P\} \cup (\delta(Y \backslash V(P)) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Therefore G_6 is ESP. ∎

Lemma 5.13. *The graph $G_7 = (V_7, E_7)$ (see Figure 16) is ESP.*

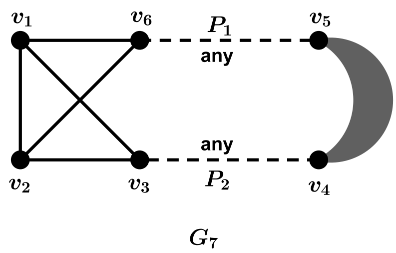


Figure 16: The primitive graph G_7

Proof. Suppose on the contrary that G_7 is not ESP. Let Δ be a collection of stars and odd sets in G_7 as specified by (5a-d) (with G_7 in place of G). By Lemma 5.2(i), we have $m_\Delta(K) = 1$ for all $K \in \Delta$. Let $U_1 = \{v_1, v_2, v_6\}$ and $U_2 = \{v_1, v_2, v_3\}$. Then $S_1 = E[U_1]$ and $S_2 = E[U_2]$ are two odd sets in G_7. We use H to denote the fully subdivided subgraph in G_7. Throughout this proof, we reserve

- $\mathcal{O}$ for the family consisting of all odd sets in Δ;
- $\mathcal{P}$ for the family consisting of all paths connecting v_4 and v_5 in H;
- (X, Y) for the bipartition of H with $\{v_4, v_5\} \subseteq X$;
- Z for $\{v_3, v_4, v_5, v_6\}$;
- Ω for $\delta(X \cup Y) \cap \Delta$; and
- (A_1, A_2) (resp. (A_3, A_4)) for the bipartition of P_1 (resp. P_2) with $v_6 \in A_1$ (resp. $v_3 \in A_3$).

Repeated application of Lemma 5.2(iii) yields

(1) for $i = 1, 2$, if no odd set in Δ contains P_i and $\delta(v) \in \Delta$ for some $v \in V(P_i) \backslash Z$, then $\delta(P_i) \subseteq \Delta$.

(2) $\{S_i, \delta(v_1), \delta(v_2)\} \nsubseteq \Delta$ for $i = 1, 2$. Otherwise, by symmetry we may assume that $\{\delta(v_1), \delta(v_2), S_1\} \subseteq \Delta$. Let $\Delta' = (\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\}$. Then Δ' dominates Δ and $m_{\Delta'}(S_1) \geq 2$, contradicting Lemma 5.2(i).

Depending on the parities of P_1 and P_2, we distinguish between two cases.

Case 1. P_1 and P_2 have the same parity.

Let $U_P = V(P_1 \cup P_2 \cup P) \cup \{v_1, v_2\}$ for each $P \in \mathcal{P}$. Then $S_P = E[U_P]$ is an odd set in G_7.

(3) If $\{\delta(v_1), \delta(v_2)\} \cap \Delta \neq \emptyset$ and $\delta(Y \cap V(P)) \subseteq \Delta$ for some $P \in \mathcal{P}$, then $\delta(v) \notin \Delta$ for some $v \in V(P_1 \cup P_2)$.

Assume the contrary: $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$. By symmetry, we may assume that $\delta(v_1) \in \Delta$. Let $\Delta' = (\Delta - \delta(A_1 \cup A_3 \cup (V(P) \cap Y))) \cup \{S_P\}$ if both P_1 and P_2 are odd and $\Delta' = (\Delta - (\{\delta(v_1)\} \cup \delta(A_2 \cup A_4 \cup (V(P) \cap Y)))) \cup \{S_P\}$ otherwise. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

(4) $\mathcal{O} \neq \emptyset$.

Assume the contrary: $\mathcal{O} = \emptyset$. Let us proceed by considering three subcases.

- $\{\delta(v_1), \delta(v_2)\} \subseteq \Delta$. In this subcase, observe that $\delta(v) \notin \Delta$ for some $v \in V(P_1 \cup P_2)$, for otherwise, (3) and Lemma 5.5 would guarantee the existence of an equitable partition (Ω_1, Ω_2) of Ω such that $\delta(v_4) \in \Omega_1$ and $\delta(v_5) \in \Omega_2$. Let $\Delta_1 = \{S_1\} \cup \delta((A_2 \backslash v_5) \cup A_3) \cup \Omega_2$ and $\Delta_2 = \{S_2\} \cup \delta(A_1 \cup (A_4 \backslash v_4)) \cup \Omega_1$ if both P_1 and P_2 are odd, and let $\Delta_1 = \{S_1\} \cup \delta(A_2 \cup (A_3 \backslash v_4)) \cup \Omega_1$ and $\Delta_2 = \{S_2\} \cup \delta((A_1 \backslash v_5) \cup A_4) \cup \Omega_2$ otherwise. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

When both P_1 and P_2 are odd, set $\Delta_1 = \{S_1\} \cup ((\delta(Y \cup A_3)) \cap \Delta)$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1) \backslash Z$ and $\Delta_1 = \{S_2\} \cup ((\delta(Y \cup A_1)) \cap \Delta)$ if $\delta(v) \notin \Delta$ for all $v \in V(P_2) \backslash Z$ (see (1)). When both P_1 and P_2 are even, set $\Delta_1 = \{S_2\} \cup ((\{\delta(v_6)\} \cup \delta(Y \cup A_4)) \cap \Delta)$ if $\delta(v) \notin \Delta$ for all $v \in V(P_1) \backslash Z$ and $\Delta_1 = \{S_1\} \cup ((\{\delta(v_3)\} \cup \delta(Y \cup A_2)) \cap \Delta)$ if $\delta(v) \notin \Delta$ for all $v \in V(P_2) \backslash Z$. Set $\Delta_2 = ((\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\}) - \Delta_1$. It is routine to check that (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

- $\{\delta(v_1), \delta(v_2)\} \cap \Delta = \emptyset$. In this subcase, $\Delta \subseteq \Omega$ by Lemma 5.2(iii) and (1). Let $\Delta_1 = \delta(X) \cap \Delta$ and $\Delta_2 = \delta(Y) \cap L$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

- $|\{\delta(v_1), \delta(v_2)\} \cap \Delta| = 1$. In this subcase, by symmetry we may assume that $\delta(v_1) \in \Delta$ and $\delta(v_2) \notin \Delta$. Let $\Delta_1 = \delta(Y \cup A_1 \cup A_3) \cap \Delta$ if both P_1 and P_2 are odd and $\Delta_1 = \{\delta(v_1)\} \cup \delta(Y \cup A_2 \cup A_4) \cap \Delta$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining the above subcases, we see that (4) holds.

(5) $|\mathcal{O}| \geq 2$.

Assume the contrary. Then $|\mathcal{O}| = 1$ by (4). Let $\mathcal{O} = \{S_i\}$. Symmetry allows us to distinguish between the following two subcases.

• $i = 1$. In this subcase, observe that $\delta(v_3) \in \Delta$, for otherwise, $\delta(v) \notin \Delta$ for all $v \in V(P_2)\backslash Z$ by (1) and Lemma 5.2(iii). Let $\Delta_1 = \delta(Y \cup A_1) \cap \Delta$ if both P_1 and P_2 are odd and $\Delta_1 = (\{S_1\} \cup \delta(Y \cup A_2)) \cap \Delta$, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, a contradiction. Thus, by symmetry and Lemma 5.2(iii), we may assume that $\delta(v_1) \in \Delta$. It follows that $\delta(v_2) \notin \Delta$ (see (2)) and that $\delta(P_2) \subseteq \Delta$ (see (1) and Lemma 5.2(iii)). If each path in $\mathcal{P}$ contains a vertex v with $\delta(v) \notin \Delta$, then Ω admits an equitable partition (Ω_1, Ω_2), with $\delta(v_4) \in \Omega_1$ if $\delta(v_4) \in \Delta$ and with $\delta(v_5) \in \Omega_2$, by Lemma 5.5. Let $\Delta_1 = (\{S_1\} \cup \delta((A_2\backslash v_5) \cup A_3) \cup \Omega_2) \cap \Delta$ and $\Delta_2 = (\{\delta(v_1)\} \cup \delta(A_1 \cup (A_4\backslash v_4)) \cup \Omega_1) \cap \Delta$ if both P_1 and P_2 are odd, and let $\Delta_1 = (\{S_1\} \cup \delta(A_2 \cup (A_3\backslash v_4)) \cup \Omega_1) \cap \Delta$ and $\Delta_2 = (\{\delta(v_1)\} \cup \delta((A_1\backslash v_5) \cup A_4) \cup \Omega_2) \cap \Delta$ otherwise. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Hence there exists $P \in \mathcal{P}$ such that $\delta(P) \subseteq \Delta$. Therefore, by (3) and the fact $\delta(P_2) \subseteq \Delta$, we obtain $\delta(v) \notin \Delta$ for all $v \in V(P_1)\backslash Z$. Let $\Delta_1 = (\{S_1\} \cup \delta(Y \cup A_3)) \cap \Delta$ if both P_1 and P_2 are odd and $\Delta_1 = (\{\delta(v_1), \delta(v_6)\} \cup \delta(Y \cup A_4)) \cap \Delta$ otherwise, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $i = P$ for some $P \in \mathcal{P}$. In this subcase, let $\Delta_1 = \{S_P\} \cup (\delta(Y\backslash V(P)) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining the above subcases, we see that (5) holds.

(6) If $S_P \in \Delta$ for some $P \in \mathcal{P}$, then $S_Q \notin \Delta$ for all $Q \in \mathcal{P}\backslash P$.

Assume the contrary: $S_Q \in \Delta$ for some $Q \in \mathcal{P}\backslash P$. Let $W_1 = V(P \cup Q) \cap X$, let $W_2 = V(P) \cap V(Q) \cap Y$, and let Δ' be obtained from Δ by replacing $\{S_P, S_Q\}$ with $\{\delta(v_1), \delta(v_2)\} \cup \delta(P_1\backslash v_5) \cup \delta(P_2\backslash v_4) \cup \delta(W_1 \cup W_2)$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

(7) If $\{S_i, S_j\} \subseteq \Delta$, then $i \in \{1, 2\}$ and $j \in \mathcal{P}$.

To justify this, note that

• $\{i, j\} \neq \{1, 2\}$ by Lemma 5.4(i).

• $\{i, j\} \neq \{P, Q\}$ for any distinct P and Q in $\mathcal{P}$ by (6).

Combining these two observations, we see that (7) holds.

(8) $\mathcal{O}$ is $\{S_1, S_P\}$ or $\{S_2, S_P\}$ for some $P \in \mathcal{P}$.

Let K be the graph with vertex set $\{S_1, S_2\} \cup \{S_P : P \in \mathcal{P}\}$ and with edges $\{S_i, S_j\}$ as described in (7). Since K contains no triangle, we have $|\mathcal{O}| < 3$ and hence $|\mathcal{O}| = 2$ by (5). Thus the statement follows instantly.

By (8) and symmetry, we only need to consider the subcase when $\mathcal{O} = \{S_1, S_P\}$ for some $P \in \mathcal{P}$. Symmetry and (2) allows us to assume that $\delta(v_2) \notin \Delta$. Let $\Delta_1 = \{S_1, S_P\} \cup (\delta(Y\backslash V(P)) \cap \Delta)$ if $\{\delta(v_1), \delta(v_6)\} \subseteq \Delta$ and $\Delta_1 = \{S_P\} \cup (\delta(Y\backslash V(P)) \cap \Delta)$ and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Therefore G_7 is ESP if Case 1 occurs.

Case 2. P_1 and P_2 have different parities.

By symmetry, we may assume that P_1 is an odd path and P_2 is an even path. For each $P \in \mathcal{P}$, let $U_P = \{v_2\} \cup V(P_1 \cup P_2 \cup P)$ and $U'_P = \{v_1\} \cup V(P_1 \cup P_2 \cup P)$, and let $T_P = E[U'_P]$ and $T'_P = E[U'_P]$. Then T_P and T'_P are odd sets in G_7.

(9) $\mathcal{O} \neq \emptyset$.

Assume the contrary: $\mathcal{O} = \emptyset$. Let us proceed by considering three subcases.

• $\{\delta(v_1), \delta(v_2)\} \subseteq \Delta$. In this subcase, let $\Delta_1 = \{S_2\} \cup (\delta(Y \cup A_1 \cup A_4) \cap \Delta)$ and $\Delta_2 = ((\Delta - \{\delta(v_1), \delta(v_2)\}) \cup \{S_1, S_2\}) - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a)

• $\{\delta(v_1), \delta(v_2)\} \cap \Delta = \emptyset$. In this subcase, $\Delta \subseteq \delta(X) \cup \delta(Y)$ by Lemma 5.2(iii) and (1). Let $\Delta_1 = \delta(X) \cap \Delta$ and $\Delta_2 = \delta(Y) \cap \Delta$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $|\{\delta(v_1), \delta(v_2)\} \cap \Delta| = 1$. In this subcase, $\delta(P_1) \cup \delta(P_2) \subseteq \Delta$ by Lemma 5.2(iii). By symmetry, we may assume that $\delta(v_1) \in \Delta$. Observe that $\delta(P) \not\subseteq \Delta$ for any $P \in \mathcal{P}$, for otherwise, let $\Delta_1 = \{T'_P\} \cup (\delta(Y \backslash V(P)) \cap \Delta)$ and $\Delta_2 = ((\Delta - \{\delta(v_1)\} \cup \delta(A_1 \cup A_4 \cup (Y \cap V(P))) \cup \{S_1, T'_P\}) - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Thus Lemma 5.5 guarantees the existence of an equitable partition (Ω_1, Ω_2) of Ω with $\delta(v_4) \in \Omega_1$ and $\delta(v_5) \in \Omega_2$. Let $\Delta_1 = \{\delta(v_1)\} \cup \delta((A_2 \backslash v_5) \cup A_4) \cup \Omega_2$ and $\Delta_2 = \delta(A_1 \cup (A_3 \backslash v_4)) \cup \Omega_1$. Then then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining the above observations, we see that (9) holds.

(10) $|\mathcal{O}| = 1$.

To justify this, observe that

• Δ contains none of the following pairs

$$\{S_1, S_2\}, \{S_1, T_P\}, \{S_1, T'_P\}, \{S_2, T_P\}, \{S_2, T'_P\}, \{T_P, T'_P\}$$

for any $P \in \mathcal{P}$ by Lemma 5.4(i).

• Δ contains neither $\{T_P, T_Q\}$ nor $\{T'_P, T'_Q\}$ for any distinct P, Q in $\mathcal{P}$. Otherwise, let $W_1 = V(P \cup Q) \cap X$ and $W_2 = V(P) \cap V(Q) \cap Y$, and let Δ' be obtained from Δ by replacing $\{T_P, T_Q\}$ with $\{\delta(v_2)\} \cup \delta(P_1 \backslash v_5) \cup \delta(P_2 \backslash v_4) \cup \delta(W_1 \cup W_2)$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

• Δ contains no $\{T_P, T'_Q\}$ for any distinct P, Q in $\mathcal{P}$. Otherwise, let $W_1 = V(P \cup Q) \cap X$ and $W_2 = V(P) \cap V(Q) \cap Y$, and let Δ' be obtained from Δ by replacing $\{T_P, T'_Q\}$ with $\delta(P_1 \backslash v_5) \cup \delta(P_2 \backslash v_4) \cup \delta(W_1 \cup W_2)$. Then Δ' dominates Δ and $\rho(\Delta') < \rho(\Delta)$, contradicting Lemma 5.2(ii).

Let K be the graph with vertex set $\{S_1, S_2\} \cup (\cup_{P \in \mathcal{P}} \{T_P, T'_P\})$ and with all edges which are not excluded above. Then the degree of each vertex in K is zero, which implies that $|\mathcal{O}| < 2$, so (10) is established.

By symmetry and (10), we only need to consider the following subcases

• $\mathcal{O} = \{S_1\}$. In this subcase, let $\Delta_1 = \{\delta(v_i)\} \cup (\delta(Y \cup A_1 \cup A_4) \cap \Delta)$ if $\delta(v_i) \in \Delta$ for $i = 1$ or 2 (see (2)), and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\mathcal{O} = \{S_2\}$. In this subcase, let $\Delta_1 = \{S_2\} \cup (\delta(Y \cup A_1 \cup A_4) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• $\mathcal{O} = \{T_P\}$ for some $P \in \mathcal{P}$. In this subcase, let $\Delta_1 = \{\delta(v_1), T_P\} \cup (\delta(Y \backslash V(P)) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining the above subcases, we conclude that G_7 is also ESP if Case 2 occurs. This completes the proof of the present lemma. ∎

Lemma 5.14. *The graph G_8 (see Figure 17) is ESP.*

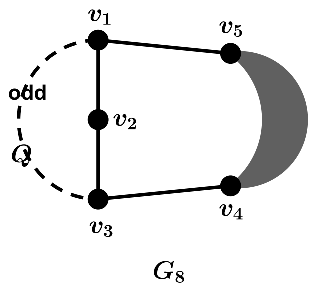


Figure 17: The primitive graph G_8

Proof. Suppose on the contrary that G_8 is not ESP. Let Δ be a collection of stars and odd sets in G_8 as specified by (5a-d) (with G_8 in place of G). By Lemma 5.2(i), we have $m_\Delta(K) = 1$ for all $K \in \Delta$. Let H denote the fully subdivided subgraph in G_8. Throughout this proof, we reserve

- $\mathcal{O}$ for the family consisting of all odd sets in Δ;
- (X, Y) for the bipartition of H with $\{v_4, v_5\} \subseteq X$;
- $\mathcal{P}$ for the family consisting of all path in H connecting v_4 and v_5;
- Ω for $\delta(X \cup Y) \cap \Delta$; and
- (A_1, A_2) for the bipartition of Q with $v_1 \in A_1$.

Let $U_1 = \{v_2\} \cup V(Q)$ and $U_P = V(P \cup Q)$ for each $P \in \mathcal{P}$. Then $S_1 = E[U_1]$ and $S_P = E[U_P]$ are odd sets in G_8. We break the proof into a series of observations.

(1) If $\delta(Q) \subseteq \Delta$, then $\delta(v_2) \notin \Delta$. Otherwise, let Δ' be obtained from Δ by replacing $\{\delta(v_2)\} \cup \delta(A_1 \backslash v_1)$ with S_1. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

(2) If no odd set contains Q and $\delta(v) \in \Delta$ for some $v \in V(Q)$, then $\delta(Q) \subseteq \Delta$ by Lemma 5.2(iii).

(3) $\mathcal{O} \neq \emptyset$.

Assume the contrary: $\mathcal{O} = \emptyset$. Observe that if $\delta(v_2) \in \Delta$, then $\delta(v) \notin \Delta$ for all $v \in V(Q) \backslash \{v_1, v_3\}$ by (1) and (2). Let $\Delta_1 = (\delta(Y) \cup \{\delta(v_1), \delta(v_3)\}) \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ; this contradiction implies that $\delta(v_2) \notin \Delta$. If $\{\delta(v_1), \delta(v_3)\} \cap \Delta = \emptyset$, letting $\Delta_1 = \delta(X) \cap \Delta$ and $\Delta_2 = \delta(Y) \cap \Delta$, then (Δ_1, Δ_2) would be an equitable subpartition of Δ, a contradiction again. So Δ contains $\delta(v_1)$ or $\delta(v_3)$. From Lemma 5.2(iii) and (2), it follows that $\{\delta(v_4), \delta(v_5)\} \cup \delta(Q) \subseteq \Delta$. We claim that $\delta(V(P) \cap Y) \not\subseteq \Delta$ for any $P \in \mathcal{P}$, for otherwise, let $\Delta' = (\Delta - (\delta(Y \cap V(P)) \cup \delta(Q))) \cup \{S_1, S_P\}$, let $\Delta_1 = \{S_P\} \cup (\delta(Y \backslash V(P)) \cap \Delta)$, and let $\Delta_2 = \Delta' - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a). Our claim and Lemma 5.5 guarantee the existence of an equitable partition (Ω_1, Ω_2) of Ω with $\delta(v_4) \in \Omega_1$ and $\delta(v_5) \in \Omega_2$. Let $\Delta_1 = \delta(A_1) \cup \Omega_1$ and $\Delta_2 = \delta(A_2) \cup \Omega_2$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

(4) $|\mathcal{O}| = 1$.

To justify this, observe that

- $\{S_1, S_P\} \not\subseteq \Delta$ for any $P \in \mathcal{P}$ by Lemma 5.4(i).

• $\{S_{P_1}, S_{P_2}\} \not\subseteq \Delta$ for any distinct P_1, P_2 in $\mathcal{P}$. Otherwise, let $W_1 = (V(P_1 \cup P_2)) \cap X$ and $W_2 = V(P_1) \cap V(P_2) \cap Y$, and let Δ' be obtained from Δ by replacing $\{S_{P_1}, S_{P_2}\}$ with $\delta(Q) \cup \delta(W_1 \cup W_2)$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(iii).

Let K be the graph with vertex set $\{S_1\} \cup \{S_P : P \in \mathcal{P}\}$ and with all edges that are not excluded above. Then the degree of each vertex in K is zero, so $|\mathcal{O}| < 2$ and hence $|\mathcal{O}| = 1$ by (3).

By (1) and symmetry, we only need to consider the following two subcases.

• $\mathcal{O} = \{S_1\}$. In this subcase, let $\Delta_1 = \{S_1\} \cup (\delta(X) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition, contradicting (5a).

• $\mathcal{O} = \{S_P\}$ for some $P \in \mathcal{P}$. In this subcase, let $\Delta_1 = \{S_P\} \cup ((\{\delta(v_2)\} \cup \delta(Y \backslash V(P)) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining the above observations, we conclude that G_8 is ESP. ∎

In the proof of the next lemma, we need the following Lovász' Open Ear Decomposition Theorem.

Theorem 5.15. (Lovász [13]) *Let H be a 2-connected factor-critical graph. Then H can be decomposed as $P_0 + P_1 + \cdots + P_r$, where P_0 is an odd cycle and P_{i+1} is an odd path having only its two ends in common with $P_0 + P_1 + \cdots + P_i$ for any $0 \le i \le r-1$.*

Lemma 5.16. *The graph $G_9 = (V_9, E_9)$ (see Figure 18) is ESP.*

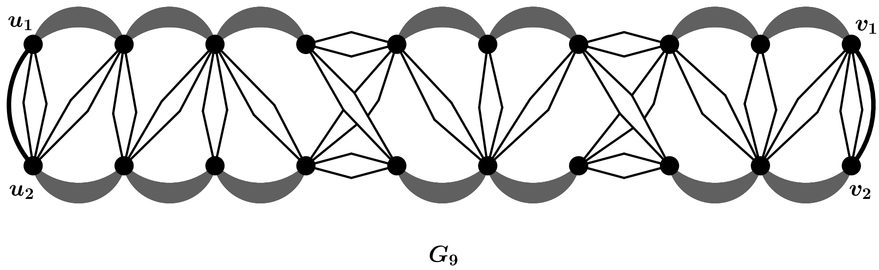


Figure 18: The primitive graph G_9

Proof. Suppose on the contrary that G_9 is not ESP. Let Δ be a collection of stars and odd sets in G_9 as specified by (5a-d) (with G_9 in place of G). By Lemma 5.2(i), $m_\Delta(K) = 1$ for all $K \in \Delta$. Recall the definitions of ladder and plump ladder in Subsection 4.1,

(1) G_9 is obtained from a ladder H with top u_1u_2, bottom v_1v_2, and outer cycle C by

- replacing each chord e of C in H with a complete bipartite graph $L_e = K_{2,n}$ for some $n \ge 1$, in which one color class consists of the two ends of e only; and
- replacing each edge f in $C \backslash \{u_1v_1, u_2v_2\}$ with a fully subdivided graph L_f, in which both ends of f belong to the color 1 class, where $L_f = K_{2,t}$ for some $t \ge 1$ if f is contained in a 4-cycle induced by two crossing chords.

For convenience, we assume that u_1, v_1, v_2, u_2 occur on C in clockwise cyclic order, and view $V(C)$ as a vertex subset of G_9; that is, $V(C) \subseteq V_9$. As introduced in Section 4, for each vertex

u on C, we use u^- (resp. u^+) to denote the vertex preceding (resp. succeeding) u on C in the clockwise direction. Let Z_e be the color class of L_e disjoint from $V(C)$ for each chord e of C in H, let Z be the set of all these Z_e, and let $\phi(C) = |\delta(Z) \cap \Delta|$.

Suppose a_1b_1 and a_2b_2 are two crossing chords of C in H, with both a_1 and a_2 on $C[u_1, v_1]$. Then, by the definition of ladder, a_1a_2 and b_1b_2 are two edges of C. Let C' be obtained from C by replacing $\{a_1a_2, b_1b_2\}$ with $\{a_1b_1, a_2b_2\}$. Observe that H is also a ladder with top u_1u_2, bottom v_1v_2, and outer cycle C'. We call the operation of replacing C by C' a *switching* with respect to a_1b_1 and a_2b_2, and assume that

(2) C is an outer cycle of H with the minimum $\phi(C)$ under switching operations with respect to crossing chords.

Throughout the proof, for each edge f in $C \backslash \{u_1u_2, v_1v_2\}$, we reserve

- (X_f, Y_f) for the bipartition of L_f, with two ends of f contained in X_f;
- Ω_f for $\delta(X_f \cup Y_f) \cap \Delta$;
- C_f (resp. C'_f) for the longest cycle in H containing the edge u_1u_2 (resp. v_1v_2), precisely one end of f, and precisely one chord of C; and
- Θ_f (resp. Θ'_f) for the set of all chords of C with two ends on C_f (resp. C'_f).

Moreover, we reserve

- X_1 (resp. Y_1) for $\cup_{f \in C[u_1,v_1]} X_f$ (resp. $\cup_{f \in C[u_1,v_1]} Y_f$);
- X_2 (resp. Y_2) for $\cup_{f \in C[v_2,u_2]} X_f$ (resp. $\cup_{f \in C[v_2,u_2]} Y_f$);
- X for $X_1 \cup X_2$ and Y for $Y_1 \cup Y_2$ (so $Z = V_9 \backslash (X \cup Y)$); and
- $\mathcal{O}$ for the family consisting of all odd sets in Δ.

Since $G_9 \backslash \{u_1u_2, v_1v_2\}$ is a bipartite graph, the following statement follows instantly from Theorem 5.15.

(3) Every odd set $S = E[U]$ in G_9 contains at least one of u_1u_2 and v_1v_2. Furthermore, if S contains precisely one of these two edges, then $G[U] = P_0$ is an odd cycle. If S contains both of them, then $G[U] = P_0 + P_1$, where P_0 is an odd cycle containing u_1u_2, and P_1 is an odd path containing v_1v_2 and having only its two ends in common with P_0.

Let $S = E[U]$ be an odd set in G_9. We say that S is of *Type* 1 if it contains precisely one of u_1u_2 and v_1v_2 and is of *Type* 2 otherwise. We also say that S passes through an edge e in $H \backslash \{u_1u_2, v_1v_2\}$ if $|U \cap (V(L_e) \backslash X)| \geq 1$. By (3), each odd set in G_9 is either of Type 1 or of Type 2.

For each odd set $S = E[U]$ in G_9 of Type 1 with $u_1u_2 \in S$ (resp. $v_1v_2 \in S$), there exist vertices a on $C[u_1, v_1]$ and b on $C[v_2, u_2]$ such that no vertex in $C(a, b)$ (resp. $C(b, a)$) is contained in U. From (3) and the definition of ladder, we see that ab is a chord of C in H and S passes through ab. We call ab the *representing chord* of C for S. Moreover, the following statement holds.

(4) Let $S = E[U]$ be an odd set in G_9 of Type 1, with representing chord ab and with a on $C[u_1, v_1]$. If $u_1u_2 \in S$ (resp. $v_1v_2 \in S$), then all vertices on $C[b, a]$ (resp. $C[a, b]$) are contained in U. Moreover, if S passes through one of two crossing chords of C other than ab, then it also passes through the other.

By (3) and (4), we get the following structural property.

(5) Let $S = E[U]$ be an odd set in G_9 of Type 2, let P_0 and P_1 be as defined in (3), and let ab be the representing chord of P_0. Then the ends of P_1 are $\{a, b\}$ or $\{a^-, a\}$ or $\{b, b^+\}$, and $V(C) \subseteq U$.

(6) If Δ contains two distinct odd sets $E[U_1]$ and $E[U_2]$ with $|U_1 \cap U_2| \geq 2$ and $U_1 \backslash U_2 \neq \emptyset \neq U_2 \backslash U_1$, then both $E[U_1]$ and $E[U_2]$ are of Type 1. Furthermore, $u_1u_2 \in E[U_i]$ and $v_1v_2 \in E[U_{3-i}]$ for $i = 1$ or 2.

Suppose the contrary. Let Δ' be obtained from Δ by replacing $\{E[U_1], E[U_2]\}$ with $\delta((U_1 \cup U_2) \cap X) \cup \delta(U_1 \cap U_2 \cap (Y \cup Z))$. Using (1) and (3)-(5), it is a routine matter to check that Δ' dominates Δ. Since $g(\Delta') < g(\Delta)$, we reach a contradiction to Lemma 5.2(ii) and hence establish (6).

(7) If Δ contains two distinct odd sets $E[U_1]$ and $E[U_2]$, then $|U_1 \cap U_2| \leq 1$ or $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

Assume the contrary: $|U_1 \cap U_2| \geq 2$ and $U_1 \backslash U_2 \neq \emptyset \neq U_2 \backslash U_1$. By (6), both $E[U_1]$ and $E[U_2]$ are of Type 1. Furthermore, $u_1u_2 \in E[U_i]$ and $v_1v_2 \in E[U_{3-i}]$ for $i = 1$ or 2, say the former. By (3), U_j induces an odd cycle C_j in G_9 for $j = 1, 2$. Let $e_1 = a_1b_1$ be the representing chord of C for $E[U_1]$ with a_1 on $C[u_1, v_1]$. Let c and d be two vertices in $V(C_1) \cap V(C_2)$ such that $C_2[c, d]$ contains v_1v_2 and $C_2(c, d)$ has no vertex in common with C_1. From the definition of ladder H, we see that $\{c, d\}$ is $\{a_1, b_1\}$ or $\{a_1^-, a_1\}$ or $\{b_1, b_1^+\}$. Set $A = U_1 \cup V(C_2(c, d))$ and $B = V(C_2[d, c])$. Let Δ' be obtained from Δ by replacing $\{E[U_1], E[U_2]\}$ with $\{E[A]\} \cup \delta(B \cap (Y \cup Z))$. Then Δ' dominates Δ and $g(\Delta') < g(\Delta)$, contradicting Lemma 5.2(ii).

For each edge f in $H \backslash \{u_1u_2, v_1v_2\}$, let $\mathcal{P}_f$ be the set of all paths in L_f connecting the ends of f in H hereafter. We call f *saturated* if there exists $P \in \mathcal{P}_f$ with $\delta(V(P) \backslash X) \subseteq \Delta$, and *unsaturated* otherwise. Furthermore, we call an edge f in $C \backslash \{u_1u_2, v_1v_2\}$ *strongly unsaturated* with respect to u_1u_1 (resp. v_1v_2) if f and chords of C in Θ_f (resp. Θ'_f) are all unsaturated.

(8) Let $e = ab$ be a chord of C in H with $a \in C[u_1, v_1]$. If all edges in $C[b, a] \backslash u_1u_2$ or all edges in $C[a, b] \backslash v_1v_2$ are saturated, then e is unsaturated. (In particular, e is unsaturated if it is parallel to u_1u_2 or v_1v_2.)

Assume the contrary: $\delta(t) \in \Delta$ for some t in $V(L_e) \backslash X$. By symmetry, we may assume that all edges f in $C[b, a] \backslash u_1u_2$ are saturated. Let P_f be a path in $\mathcal{P}_f$ with $\delta(V(P_f) \backslash X) \subseteq \Delta$ for each such edge f, let U be the union of $V(P_f)$ for all these f, and let Δ' be obtained from Δ by replacing $\delta(U \cap Y) \cup \{\delta(t)\}$ with $E[U \cup \{t\}]$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$, contradicting Lemma 5.2(ii).

A saturated chord $e = ab$ of C in H, with $a \in C[u_1, v_1]$, is called *u_1u_2-minimal* (resp. *v_1v_2-minimal*) if there is no saturated chord $e' = a'b'$ of C in H, with $a' \in C[u_1, v_1]$, such that $C[b', a']$ (resp. $C[a', b']$) is a proper subpath of $C[b, a]$ (resp. of $C[a, b]$).

(9) Let $e = ab$ be a saturated chord of C in H that is u_1u_2-minimal (resp. v_1v_2-minimal), with $a \in C[u_1, v_1]$. Then $C[b, a]$ (resp. $C[a, b]$) contains a strongly unsaturated edge with respect to u_1u_2 (resp. v_1v_2).

Assume the contrary: $C[b, a]$, say, contains no strongly unsaturated edge with respect to u_1u_2. By (8), there exists an unsaturated edge on $C[b, a] \backslash u_1u_2$. Let f be an arbitrary unsaturated edge on $C[u_1, a]$, if any. Since f is not strongly unsaturated, there exists a saturated chord g in Θ_f. From the minimality assumption on e, we deduce that e and g are crossing chords of C. By the definition of ladder H, we thus obtain $f = a^-a$ and $g = a^-b^-$. Similarly, if there exists an unsaturated edge f' in $C[b, u_2]$, then $f' = bb^+$ and $g' = a^+b^+$ is a saturated chord of C. From the definition of ladder H, we see that g and g' cannot exist simultaneously (because they are crossing and do not form a 4-cycle). Hence $C[b, a] \backslash u_1u_2$ contains precisely one unsaturated edge by (8). If g exists, then b^-b is an unsaturated edge, using (8) with respect to $C[b^-, a^-]$. Let C'

be obtained from C by switching with respect to crossing chords e and g. Then $\phi(C') > \phi(C)$, contradicting (2). Similarly, we can reach a contradiction if g' exists.

(10) Let $e = ab$ be a chord of C in H, with $a \in C[u_1, v_1]$, such that $C[b, a]$ (resp. $C[a, b]$) contains an unsaturated edge. Then $C[b, a]$ (resp. $C[a, b]$) contains a strongly unsaturated edge with respect to u_1u_2 (resp. v_1v_2).

Assume the contrary: no unsaturated edge in $C[b, a]\backslash u_1u_2$, say, is strongly unsaturated with respect to u_1u_2. Symmetry allows us to assume that $C[u_1, a]$ contains unsaturated edges; let f be such an arbitrary edge. Since f is not strongly unsaturated, there exists a saturated chord $g = cd$ in Θ_f that is u_1u_2-minimal, with c on $C[u_1, a)$. By (9), $C[d, c]$ contains a strongly unsaturated edge h. By assumption, h is outside $C[b, a]$. It follows that e and g are crossing chords of C in H, and hence $f = a^-a$, $g = a^-b^-$ and $h = bb^-$ by the definition of ladder H. Let C' be obtained from C by switching with respect to crossing chords e and g. Then $\phi(C') > \phi(C)$, contradicting (2).

(11) Let $e = ab$ be a saturated chord of C in H, with $a \in C[u_1, v_1]$. Then $C[b, a]$ contains a strongly unsaturated edge f with respect to u_1u_2, and $C[a, b]$ contains a strongly unsaturated edge g with respect to v_1v_2, such that $g \notin C_f$ and $f \notin C'_g$.

To justify this, note that $C[b, a]$ (resp. $C[a, b]$) contains a strongly unsaturated edge f (resp. g) with respect to u_1u_2 (resp. v_1v_2) by (8) and (10). Suppose on the contrary that $g \in C_f$ or $f \in C'_g$, say the former. By symmetry, we may assume that f is on $C[u_1, a]$ and g is on $C[v_2, b]$. Let h be the unique chord of C contained in C_f. Then e and h are crossing chords of C in H. By the definition of ladder H, we thus obtain $f = a^-a$, $g = bb^-$ and $h = a^-b^-$. Let C' be obtained from C by switching with respect to crossing chords e and h. Then $\phi(C') > \phi(C)$, contradicting (2).

For each odd set $S = E[U]$ in G_9 of Type 1, define $S^* = \{f \in C\backslash\{u_1u_2, v_1v_2\} : |V(L_f) \cap U| \leq 1\}$. Then $S^* \neq \emptyset$, because $G[U]$ is an odd cycle containing precisely one of the edges u_1u_2 and v_1v_2 by (1) and (3). Note that S^* is actually the edge set of $C[a, b]$ (resp. $C[b, a]$) if $u_1u_2 \in S$ (resp. $v_1v_2 \in S$), where ab is the representing chord of C for S with $a \in C[u_1, v_1]$.

(12) S^* contains an unsaturated edge for each odd set $S = E[U]$ of Type 1 in Δ.

Otherwise, for each $f \in S^*$, there exists $P_f \in \mathcal{P}_f$ such that $\delta(V(P_f)\backslash X) \subseteq \Delta$. Let $K = \cup_{f \in S^*} V(P_f)$ and let Δ' be obtained from Δ by replacing $\{E[U]\} \cup \delta(K \cap Y)$ with $E[U \cup K]$. Then Δ' dominates Δ and $f(\Delta') > f(\Delta)$; this contradiction to Lemma 5.2(ii) justifies (12).

In view of (10) and (12), we get

(13) S^* contains a strongly unsaturated edge with respect to v_1v_2 (resp. u_1u_2) for each odd set $S = E[U]$ of Type 1 in Δ if $u_1u_2 \in S$ (resp. $v_1v_2 \in S$).

(14) If $\mathcal{O} = \emptyset$, then $\delta(Z) \cap \Delta \neq \emptyset$.

Otherwise, let (A, B) be the bipartition of $G[X \cup Y]$. Then $(\delta(A) \cap \Delta, \delta(B) \cap \Delta)$ is an equitable subpartition of Δ, contradicting (5a).

(15) $\mathcal{O} \neq \emptyset$.

Assume the contrary: $\mathcal{O} = \emptyset$. By (14), we have $\delta(Z) \cap \Delta \neq \emptyset$; let $e = ab$ be a saturated chord of C in H, with $a \in C[u_1, v_1]$. By (11), $C[b, a]$ contains a strongly unsaturated edge $f = rr^+$ with respect to u_1u_2, and $C[a, b]$ contains a strongly unsaturated edge $g = ss^+$ with respect to v_1v_2, such that $g \notin C_f$ and $f \notin C'_g$. By symmetry, we may assume that f is on $C[u_1, a]$. By Lemma 5.5, Ω_f (resp. Ω_g) admits an equitable partition (Ω_f^1, Ω_f^2) (resp. (Ω_g^1, Ω_g^2)), with $\delta(r) \in \Omega_f^1$, $\delta(r^+) \in \Omega_f^2$, $\delta(s) \in \Omega_g^1$ and $\delta(s^+) \in \Omega_g^2$, if the corresponding star exists in Δ.

Observe that g is on $C[v_2,b]$, for otherwise, let Π_1 be the union of $\delta(X_h)$ for all edges $h \in C[u_1,r] \cup C[s^+,v_1]$, let Π_2 be the union of $\delta(Y_h)$ for all $h \in C[r^+,s] \cup C[v_2,u_2]$, let $\Delta_1 = (\Pi_1 \cup \Pi_2 \cup \Omega_f^1 \cup \Omega_g^2 \cup \delta(Z)) \cap \Delta$, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Let Π_3 be the union of $\delta(X_h)$ for all edges $h \in C[u_1,r] \cup C[v_2,s]$, let Π_4 be the union of $\delta(Y_h)$ for all edges $h \in C[r^+,v_1] \cup C[s^+,u_2]$, let $\Delta_1 = (\Pi_3 \cup \Pi_4 \cup \Omega_f^1 \cup \Omega_g^1 \cup \delta(Z)) \cap \Delta$, and let $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a) again.

(16) $|\mathcal{O}| \geq 2$.

Assume the contrary. Then $|\mathcal{O}| = 1$ by (15). Let $S = E[U]$ be the unique odd set in $\mathcal{O}$. Depending on the type of S, we consider two cases.

• S is of Type 1. In this case, symmetry allows us to assume that $u_1u_2 \in E[U]$. Let ab be the representing chord of C for S with a on $C[u_1,v_1]$. By (13), $C[a,b]$ contains a strongly unsaturated edge $g = ss^+$ with respect to v_1v_2. By symmetry, we may assume that g is on $C[a,v_1]$. By Lemma 5.5, Ω admits an equitable partition (Ω_g^1, Ω_g^2) of Ω_g with $\delta(s) \in \Omega_g^1$ and $\delta(s^+) \in \Omega_g^2$, if the corresponding star exists in Δ. Let Π_1 be the union of $\delta(X_h)$ for all edges $h \in C[s^+,v_1]$, let Π_2 be the union of $\delta(Y_h)$ for all edges $h \in C[a,s] \cup C[v_2,b]$, and let Π_3 be the union of $\delta(Y_h \backslash U)$ for all edges $h \in C[u_1,a] \cup C[b,u_2]$. Set $\Delta_1 = (\{S\} \cup \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Omega_g^2 \cup \delta(Z \backslash U)) \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$. Clearly, (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

• S is of Type 2. In this case, let $\Delta_1 = (\{S\} \cup \delta((Y \cup Z) \backslash U)) \cap \Delta$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining the above cases, we see that (16) holds.

The following statement follows instantly from (3)-(5) and (7).

(17) Let $S_1 = E[U_1]$ and $S_2 = E[U_2]$ be two odd sets in $\mathcal{O}$. Then one of the following two cases occurs:

• S_1 and S_2 are both of Type 1 and $|U_1 \cap U_2| \leq 1$;

• S_i is of Type 1, S_{3-i} is of Type 2, and $U_i \subseteq U_{3-i}$ for $i = 1$ or 2.

(18) $|\mathcal{O}| = 2$.

Assume the contrary: $|\mathcal{O}| \geq 3$. Let $S_i = E[U_i]$ for $i = 1, 2, 3$ be three odd sets in $\mathcal{O}$. By (17), we may assume that S_1 and S_2 are of Type 1, with $u_1u_2 \in S_1$ and $v_1v_2 \in S_2$, while S_3 is of Type 2. In view of (3) (with S_2 in place of S), $S_3 = P_0 + P_1$, where P_0 is an odd cycle containing u_1u_2, and P_1 is an odd path containing v_1v_2 and having only its two ends c and d in common with P_0. Let Q stand for the cd-subpath of $P_0 \backslash u_1u_2$. From (3) and (17), we obtain $G[U_1] = P_0$ and $G[U_2] = Q \cup P_1$. Thus $V(Q) \subseteq U_1 \cap U_2$ and hence $|U_1 \cap U_2| \geq 2$, contradicting (17).

Let $\mathcal{O} = \{E[U_1], E[U_2]\}$. By (17) and symmetry, we may assume that $E[U_1]$ is of Type 1 and contains u_1u_2. If $E[U_2]$ is of Type 1, then v_1v_2 is contained in $E[U_2]$ by (17). Let $\Delta_1 = \{E[U_1], E[U_2]\} \cup (\delta((Y \cup Z) \backslash (U_1 \cup U_2)) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) is an equitable subpartition of Δ; this contradiction to (5a) implies that $E[U_2]$ is of Type 2. Hence $U_1 \subseteq U_2$ by (17).

(19) $\delta(y) \notin \Delta$ for some $y \in U_1 \cap Y$. Otherwise, $\delta(N(y)) \subseteq \Delta$ for all $y \in U_1 \cap Y$ by Lemma 5.3 as $U_1 \subseteq U_2$. Let $\Delta_1 = \{E[U_1], E[U_2]\} \cup (\delta((Y \cup Z) \backslash U_2) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Then (Δ_1, Δ_2) an equitable subpartition of Δ, contradicting (5a).

(20) Let f be an edge on $C \backslash \{u_1u_2, v_1v_2\}$ such that $\delta(y) \notin \Delta$ for some $y \in U_1 \cap Y_f$. Then f is unsaturated.

Assume the contrary: $\delta(V(P)\backslash X) \subseteq \Delta$ for some $P \in \mathcal{P}_f$. Let Q be the path in $\mathcal{P}_f$ with $V(Q) \subseteq U_1$. Then $P \neq Q$. Let $U_1' = (U_1 \backslash V(Q)) \cup V(P)$, and let Δ' be obtained from Δ by replacing $\{E[U_1]\} \cup \delta(V(P) \cap Y)$ with $\{E[U_1']\} \cup \delta(V(Q) \cap Y)$. Then Δ' dominates Δ and satisfies (5a-d). Since $E[U_1']$ is of Type 1 and $U_1' \not\subseteq U_2$, we reach a contradiction to (17) (with Δ' in place with Δ).

Let ab be the representing chord of C for $E[U_1]$ with a on $C[u_1, v_1]$. By (19) and (20), $C[b, a]$ contains an unsaturated edge, and hence contains a strongly unsaturated edge $g = ss^+$ with respect to $u_1 u_2$ by (10).

By Lemma 5.5, Ω_g admits an equitable partition (Ω_1, Ω_2) with $\delta(s) \in \Omega_1$ and $\delta(s^+) \in \Omega_2$, if the corresponding star exists in Δ. By symmetry, we may assume that g is on $C[u_1, a]$. Let Π_1 be the union of $\delta(X_h)$ for all edges $h \in C[u_1, s]$, let Π_2 be the union of $\delta(Y_h)$ for all edges $h \in C[s^+, a] \cup C[b, u_2]$, and let Π_3 be the union of $\delta(Y_h \backslash U_2)$ for all edges $h \in C[a, v_1] \cup C[v_2, b]$. Set $\Delta_1 = \{E[U_2]\} \cup ((\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Omega_1 \cup \delta(Z)) \cap \Delta)$ and $\Delta_2 = \Delta - \Delta_1$. Clearly, (Δ_1, Δ_2) is an equitable subpartition of Δ, contradicting (5a).

Combining the above subcases, we conclude that G_9 is ESP. ∎

6 Proof of Theorem 1.4

In Section 3 we have established the "if" part of Theorem 1.4 (see Theorem 3.2). In Section 5, we have derived the "only if" part of Theorem 1.4 when G is i-2-c. To complete the proof, we may lift the connectivity of G in the opposite case using the following two summing operations.

Let H_1 and H_2 be two graphs. As usual, the 0-sum of H_1 and H_2 is their disjoint union. The 1-*sum* of H_1 and H_2 is obtained by first choosing an edge $a_i b_i$ of H_i for $i = 1, 2$ such that b_i has degree one in H_i, then deleting b_i from H_i, and finally identifying a_1 and a_2 (let a be the resulting vertex); see Figure 19 for an illustration.

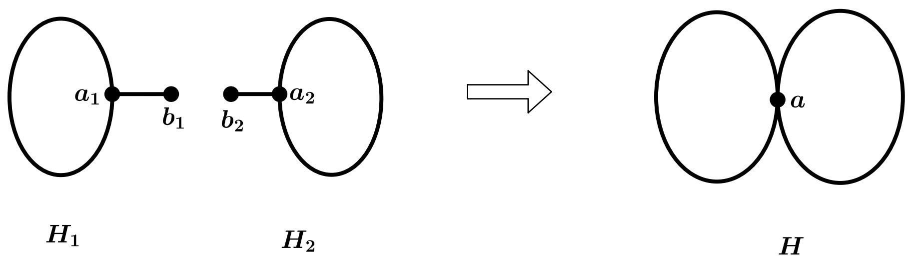


Figure 19: The 1-sum of two graphs

Lemma 6.1. *Let H be the 0-sum of H_1 and H_2. If both $\sigma(H_1)$ and $\sigma(H_2)$ are box-TDI, then so is $\sigma(H)$.*

Proof. Write the linear system $\sigma(H_i)$ as $A_i \boldsymbol{x} \leq \boldsymbol{b}_i$, $\boldsymbol{x} \geq \boldsymbol{0}$ for $i = 1, 2$, and write $\sigma(H)$ as $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$. Since H is the 0-sum of H_1 and H_2, by definition $U \subseteq \mathcal{T}(H)$ if and only if

$U \subseteq \mathcal{T}(H_i)$ for $i = 1$ or 2. Thus

$$A = \begin{bmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_2 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}.$$

Therefore the statement holds trivially. ∎

Lemma 6.2. *Let H be the 1-sum of H_1 and H_2. If both $\sigma(H_1)$ and $\sigma(H_2)$ are box-TDI, then so is $\sigma(H)$.*

Proof. Recall the definition: $H = (V, E)$ is obtained from $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ by first choosing an edge a_ib_i of H_i for $i = 1, 2$ such that b_i has degree one in H_i, then deleting b_i from H_i, and finally identifying a_1 and a_2 (let a be the resulting vertex). Write the linear system $\sigma(H)$ as $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \mathbf{0}$. Assume on the contrary that $\sigma(H)$ is not box-TDI. Then there exist $\boldsymbol{l} \in \mathbb{Q}_+^E$ and $\boldsymbol{u} \in (\mathbb{Q}_+ \cup \{+\infty\})^E$ with $\boldsymbol{l} \leq \boldsymbol{u}$, such that $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}$, $\boldsymbol{x} \geq \mathbf{0}$ is not a TDI-system; subject to this, we assume that

(1) $L(a) = \sum_{e \in \delta(a)} l(e)$ is maximized.

With a slight abuse of notation, we write $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ for both the linear program $\max\{\boldsymbol{w}^T\boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}, \boldsymbol{x} \geq \mathbf{0}\}$ and its optimal value, and write $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ for both the linear program $\min\{\boldsymbol{\alpha}^T\boldsymbol{b} - \boldsymbol{\beta}^T\boldsymbol{l} + \boldsymbol{\gamma}^T\boldsymbol{u} : \boldsymbol{\alpha}^T A - \boldsymbol{\beta}^T + \boldsymbol{\gamma}^T \geq \boldsymbol{w}^T, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \geq \mathbf{0}\}$ and its optimal value. For a detailed description of this primal-dual pair, refer to the paragraph below Lemma 3.1. By the definition of TDI systems, there exists $\boldsymbol{w} \in \mathbb{Z}^E$ such that $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ has finite optimum, but has no integral optimal solution. Observe that

(2) for any optimal solution $\boldsymbol{x}$ to $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, we have $x(e) = l(e)$ for all $e \in \delta(a)$.

Suppose the contrary: there exists an optimal solution $\boldsymbol{x}$ to $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ such that $x(f) > l(f)$ for some $f \in \delta(a)$. Let $\theta = (x(f) - l(f))/2$. Then $\theta > 0$. Let $\bar{\boldsymbol{l}}$ be obtained from $\boldsymbol{l}$ by replacing $l(f)$ with $l(f) + \theta$. Then $\boldsymbol{x}$ remains to be an optimal solution to $\mathrm{Max}(A, \boldsymbol{b}, \bar{\boldsymbol{l}}, \boldsymbol{u}, \boldsymbol{w})$, because the feasible region of $\mathrm{Max}(A, \boldsymbol{b}, \bar{\boldsymbol{l}}, \boldsymbol{u}, \boldsymbol{w})$ is a subset of that of $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$. So $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}) = \mathrm{Max}(A, \boldsymbol{b}, \bar{\boldsymbol{l}}, \boldsymbol{u}, \boldsymbol{w})$. Since $\bar{L}(a) = \sum_{e \in \delta(a)} \bar{l}(e) > L(a)$, there exists an integral optimal solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ to $\mathrm{Min}(A, \boldsymbol{b}, \bar{\boldsymbol{l}}, \boldsymbol{u}, \boldsymbol{w})$ by (1) and assumption. Note that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is also feasible to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$. Furthermore, $\beta(f) = 0$ by complementary slackness as $x(f) > \bar{l}(f)$. Thus $\boldsymbol{\alpha}^T\boldsymbol{b} - \boldsymbol{\beta}^T\boldsymbol{l} + \boldsymbol{\gamma}^T\boldsymbol{u} = \boldsymbol{\alpha}^T\boldsymbol{b} - \boldsymbol{\beta}^T\bar{\boldsymbol{l}} + \boldsymbol{\gamma}^T\boldsymbol{u}$, which implies that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is an integral optimal solution to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$; this contradiction justifies (2).

Set $\varepsilon_i = \sum_{e \in \delta(a) \cap E_i} l(e)$ for $i = 1, 2$. Then

(3) $\varepsilon_1 + \varepsilon_2 \leq 1$.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$. From the restricted Edmonds system, we see that $\sum_{e \in \delta(a)} x(e) \leq 1$. By (2), we obtain $\sum_{e \in \delta(a)} x(e) = \varepsilon_1 + \varepsilon_2$. Thus (3) follows.

Let $H' = (V', E')$ be the 0-sum of H_1 and H_2. By Lemma 6.1, $\sigma(H')$ is box-TDI. Write $\sigma(H')$ as $A\boldsymbol{x}' \leq \boldsymbol{b}', \boldsymbol{x}' \geq \mathbf{0}$ and define

(4) $\boldsymbol{l}' \in \mathbb{Q}_+^{E'}$, $\boldsymbol{u}' \in (\mathbb{Q}_+ \cup \{+\infty\})^{E'}$, and $\boldsymbol{w}' \in \mathbb{Z}^{E'}$ such that

- $l'(e) = l(e)$, $u'(e) = u(e)$, $w(e) = w(e)$ for all $e \in E' \backslash \{a_1b_1, a_2b_2\}$,
- $l'(a_1b_1) = \varepsilon_2$, $l'(a_2b_2) = \varepsilon_1$, $u'(a_1b_1) = u'(a_2b_2) = +\infty$, and $w'(a_1b_1) = w'(a_2b_2) = 0$.

Since no constraint $x'(e) < +\infty$ appears in $\mathrm{Max}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$, neither $\gamma'(a_1b_1)$ nor $\gamma'(a_2b_2)$ is introduced in $\mathrm{Min}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$ by (4).

(5) $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}) = \mathrm{Max}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, and let $\boldsymbol{x}' \in \mathbb{R}^{E'}$ be defined by $x'(e) = x(e)$ for all $e \in E' \backslash \{a_1b_1, a_2b_2\}$, $x'(a_1b_1) = \varepsilon_2$, and $x'(a_2b_2) = \varepsilon_1$. In view of (2) and (4), $\boldsymbol{x}'$ is a feasible solution to $\mathrm{Max}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$ with $(\boldsymbol{w}')^T\boldsymbol{x}' = \boldsymbol{w}^T\boldsymbol{x}$. So $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}) \le \mathrm{Max}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$.

Assume on the contrary that $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}) < \mathrm{Max}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be optimal solutions to $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ and $\mathrm{Max}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$, respectively. By (4), we have $\sum_{e \in E_i \backslash \{a_ib_i\}} w'(e)x'(e) > \sum_{e \in E_i \backslash \{a_ib_i\}} w(e)x(e)$ for $i = 1$ or 2, say the former. Let $\bar{\boldsymbol{x}} \in \mathbb{R}^E$ be defined by $\bar{x}(e) = x'(e)$ for all $e \in E_1 \backslash \{a_1b_1\}$ and $\bar{x}(e) = x(e)$ for all $e \in E_2 \backslash \{a_2b_2\}$. Note that $\sum_{e \in \delta(a)} \bar{x}(e) = \sum_{e \in E_1 \backslash \{a_1b_1\}} x'(e) + \sum_{e \in E_2 \backslash \{a_2b_2\}} x(e) \le 1 - x'(a_1b_1) + \sum_{e \in E_2 \backslash \{a_2b_2\}} x(e) \le 1 - \varepsilon_2 + \varepsilon_2 = 1$, where the last inequality follows from (2) and (4). So $\bar{\boldsymbol{x}}$ is a feasible solution to $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, with $\boldsymbol{w}^T\bar{\boldsymbol{x}} > \boldsymbol{w}^T\boldsymbol{x}$; this contradiction establishes (5).

Since $\sigma(H')$ is box-TDI, $\mathrm{Min}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$ has an integral optimal solution $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}')$. For this solution the constraints corresponding to edges in $\delta(a_1) \cup \delta(a_2)$ read, respectively,

(6) $\sum_{e \in \delta(v)} \alpha'(v) + \sum_{e \in E[U]} \alpha'(U) - \beta'(e) + \gamma'(e) \ge w'(e)$ for all $e \in \delta(a_1) \cup \delta(a_2) \backslash \{a_1b_1, a_2b_2\}$;

(7) $\alpha'(a_i) - \beta'(a_ib_i) \ge 0$ for $i = 1, 2$.

We may assume that both equalities in (7) hold with equalities; that is,

(8) $\alpha'(a_i) - \beta'(a_ib_i) = 0$ for $i = 1, 2$.

Otherwise, let $\theta_i = \alpha'(a_i) - \beta'(a_ib_i)$. Then at least one of θ_1 and θ_2 is positive. Let $\boldsymbol{\beta}''$ be obtained from $\boldsymbol{\beta}'$ by replacing $\beta'(a_1b_1)$ with $\beta'(a_1b_1) + \theta_1$ and replacing $\beta'(a_2b_2)$ with $\beta'(a_2b_2) + \theta_2$. It is easy to see that $(\boldsymbol{\alpha}', \boldsymbol{\beta}'', \boldsymbol{\gamma}')$ is a feasible solution to $\mathrm{Min}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$, with $(\boldsymbol{\alpha}')^T\boldsymbol{b}' - (\boldsymbol{\beta}'')^T\boldsymbol{l}' + (\boldsymbol{\gamma}')^T\boldsymbol{u}' \le (\boldsymbol{\alpha}')^T\boldsymbol{b}' - (\boldsymbol{\beta}')^T\boldsymbol{l}' + (\boldsymbol{\gamma}')^T\boldsymbol{u}'$. So $(\boldsymbol{\alpha}', \boldsymbol{\beta}'', \boldsymbol{\gamma}')$ is also an optimal solution to $\mathrm{Min}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$. Hence we may assume (8), otherwise replace $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}')$ with $(\boldsymbol{\alpha}', \boldsymbol{\beta}'', \boldsymbol{\gamma}')$.

(9) $\varepsilon_1 + \varepsilon_2 = 1$.

Otherwise, $\varepsilon_1 + \varepsilon_2 < 1$ by (3). Let $\boldsymbol{x}$ be an optimal solution to $\mathrm{Max}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$. By (2), we have $\sum_{e \in \delta(a)} x(e) = \varepsilon_1 + \varepsilon_2 < 1$. Let $\boldsymbol{x}' \in \mathbb{R}^{E'}$ be defined by $x'(e) = x(e)$ for all $e \in E' \backslash \{a_1b_1, a_2b_2\}$, $x'(a_1b_1) = \varepsilon_2$, and $x'(a_2b_2) = \varepsilon_1$. In view of (2) and (3), $\boldsymbol{x}'$ is a feasible solution to $\mathrm{Max}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$. Note that $\boldsymbol{w}^T\boldsymbol{x} = (\boldsymbol{w}')^T\boldsymbol{x}'$ by (4), so $\boldsymbol{x}'$ is an optimal solution to $\mathrm{Max}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$ by (5). Since $\sum_{e \in \delta(a_i)} x'(e) = \sum_{e \in E_i \backslash \{a_ib_i\}} x(e) + \varepsilon_{3-i} = \varepsilon_1 + \varepsilon_2 < 1$, we deduce from complementary slackness that $\alpha'(a_1) = \alpha'(a_2) = 0$. Hence $\beta'(a_1b_1) = \beta'(a_2b_2) = 0$ by (8). Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ be defined by $\alpha(u) = \alpha'(u)$ for all $u \in I(H) \cup \mathcal{T}(H) \backslash \{a\}$, $\alpha(a) = 0$, $\beta(e) = \beta'(e)$, $\gamma(e) = \gamma'(e)$ for all $e \in E$. It is routine to check check that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is a feasible solution to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, with $(\boldsymbol{\alpha}')^T\boldsymbol{b}' - (\boldsymbol{\beta}')^T\boldsymbol{l}' + (\boldsymbol{\gamma}')^T\boldsymbol{u}' = \boldsymbol{\alpha}^T\boldsymbol{b} - \boldsymbol{\beta}^T\boldsymbol{l} + \boldsymbol{\gamma}^T\boldsymbol{u}$. By (5), $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is an integral optimal solution to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, contradicting our assumption.

We may further assume that

(10) $\alpha'(a_1) = \alpha'(a_2)$. So $\beta'(a_1b_1) = \beta'(a_2b_2)$ by (8).

Otherwise, symmetry allows us to assume that $\alpha'(a_1) > \alpha'(a_2)$. Set $\theta = \alpha'(a_1) - \alpha'(a_2)$. Let $\boldsymbol{\alpha}''$ be obtained from $\boldsymbol{\alpha}'$ by replacing $\alpha'(a_2)$ with $\alpha'(a_2) + \theta$, and let $\boldsymbol{\beta}''$ be obtained from $\boldsymbol{\beta}'$ by replacing $\beta'(e)$ with $\beta'(e) + \theta$ for all $e \in \delta(a_2)$. It is easy to see that $(\boldsymbol{\alpha}'', \boldsymbol{\beta}'', \boldsymbol{\gamma}')$ satisfies the constraints corresponding to (6) and (7), which implies that $(\boldsymbol{\alpha}'', \boldsymbol{\beta}'', \boldsymbol{\gamma}')$ is a feasible solution to $\mathrm{Min}(A', \boldsymbol{b}', \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$. By (4) and (9), we obtain $(\boldsymbol{\alpha}'')^T\boldsymbol{b}' - (\boldsymbol{\beta}'')^T\boldsymbol{l}' + (\boldsymbol{\gamma}')^T\boldsymbol{u}' = [(\boldsymbol{\alpha}')^T\boldsymbol{b}' + \theta] - [(\boldsymbol{\beta}')^T\boldsymbol{l}' + \theta\sum_{e \in \delta(a_2)} l'(e)] + (\boldsymbol{\gamma}')^T\boldsymbol{u}' = (\boldsymbol{\alpha}')^T\boldsymbol{b}' - (\boldsymbol{\beta}')^T\boldsymbol{l}' + (\boldsymbol{\gamma}')^T\boldsymbol{u}' + [\theta - \theta(\varepsilon_1 + \varepsilon_2)] = (\boldsymbol{\alpha}')^T\boldsymbol{b}' - (\boldsymbol{\beta}')^T\boldsymbol{l}' + (\boldsymbol{\gamma}')^T\boldsymbol{u}'$. So $(\boldsymbol{\alpha}'', \boldsymbol{\beta}'', \boldsymbol{\gamma}')$ is also an optimal solution to $\mathrm{Min}(A', \boldsymbol{b}, \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{w}')$.

Hence we may assume (10), otherwise replace $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}')$ with $(\boldsymbol{\alpha}'', \boldsymbol{\beta}'', \boldsymbol{\gamma}')$.

Let us now construct an integral optimal solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ by setting

- $\alpha(u) = \alpha'(u)$ for $u \in (I(H) \cup \mathcal{T}(H)) \backslash a$;
- $\alpha(a) = \alpha'(a_1)$;
- $\beta(e) = \beta'(e)$ and $\gamma(e) = \gamma'(e)$ for all $e \in E$,

From (10) it is easy to see that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is feasible to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$.

(11) $\boldsymbol{\alpha}^T\boldsymbol{b} - \boldsymbol{\beta}^T\boldsymbol{l} + \boldsymbol{\gamma}^T\boldsymbol{u} = (\boldsymbol{\alpha}')^T\boldsymbol{b}' - (\boldsymbol{\beta}')^T\boldsymbol{l}' + (\boldsymbol{\gamma}')^T\boldsymbol{u}'$.

Indeed, by direct computation we obtain

$$\begin{aligned}
& \boldsymbol{\alpha}^T\boldsymbol{b} - \boldsymbol{\beta}^T\boldsymbol{l} + \boldsymbol{\gamma}^T\boldsymbol{u} \\
= \;& \sum_{v \in I(H)} \alpha(v) + \sum_{U \in \mathcal{T}(H)} \lfloor \frac{1}{2}|U| \rfloor \alpha(U) - \sum_{e \in E} l(e)\beta(e) + \sum_{e \in E} u(e)\gamma(e) \\
= \;& \sum_{v \in I(H)} \alpha(v) + \sum_{U \in \mathcal{T}(H)} \lfloor \frac{1}{2}|U| \rfloor \alpha(U) - \sum_{e \in E} l(e)\beta(e) + \sum_{e \in E} u(e)\gamma(e) + (\alpha'(a_2) - \beta'(a_2b_2)) \\
= \;& \alpha'(a_1) + \alpha'(a_2) - \varepsilon_2\beta'(a_1b_1) - \varepsilon_1\beta'(a_2b_2) + \sum_{v \in I(H') \backslash \{a_1, a_2\}} \alpha'(v) + \sum_{U \in \mathcal{T}(H')} \lfloor \frac{1}{2}|U| \rfloor \alpha'(U) \\
& - \sum_{e \in E' \backslash \{a_1b_1, a_2b_2\}} l'(e)\beta'(e) + \sum_{e \in E' \backslash \{a_1b_1, a_2b_2\}} u'(e)\gamma'(e) \\
= \;& \alpha'(a_1) + \alpha'(a_2) + \sum_{v \in I(H') \backslash \{a_1, a_2\}} \alpha'(v) + \sum_{U \in \mathcal{T}(H')} \lfloor \frac{1}{2}|U| \rfloor \alpha'(U) - \sum_{e \in E'} l'(e)\beta'(e) \\
& + \sum_{e \in E'} u'(e)\gamma'(e) \\
= \;& (\boldsymbol{\alpha}')^T\boldsymbol{b}' - (\boldsymbol{\beta}')^T\boldsymbol{l}' + (\boldsymbol{\gamma}')^T\boldsymbol{u}',
\end{aligned}$$

where the second and third equalities follow from (8)-(10).

Combining (5) and (11), we conclude that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is an integral optimal solution to $\mathrm{Min}(A, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, contradicting our assumption. Therefore $\sigma(H)$ is box-TDI. ∎

We are ready to establish the main result of this paper.

Proof of Theorem 1.4. The "if" part follows from Theorem 3.2 . It remains to derive the "only if" part. We apply induction on $|V(G)|$. The case $|V(G)| = 1$ is trivial, so we proceed to the induction step. By Lemmas 3.1, 6.1 and 6.2, we may assume that G cannot be represented as the k-sum ($k = 0, 1$) of two smaller graphs (otherwise we are done). Thus G is i-2-c. From Theorem 4.1, we deduce that G is a bipartite graph or is a subgraph of one of the nine graphs $G_1, G_2, \ldots, G_9$ (see Figure 4). By Lemmas 5.7-5.16 and Lemma 5.1, $\sigma(K)$ is ESP and hence box-TDI, by Theorem 1.8, if K is a bipartite graph or one of $G_1, G_2, \ldots, G_9$. In view of Lemma 3.1, $\pi(K)$ is also box-TDI. From Lemma 3.8, we thus conclude that $\pi(G)$ is a box-TDI system.

This completes the proof of our theorem.

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