Packing of bipartite graphs

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March 1993

ABSTRACT

Let $H_0 = (X_0, Y_0, E_0)$ be a connected bipartite graph. The complexity of the following decision problem is investigated. For any given bipartite graph $H = (X, Y, E)$, can the vertex set $X \cup Y$ of H be partitioned into $X_1 \cup Y_1$, ..., $X_n \cup Y_n$, such that the subgraph of H induced by each $X_i \cup Y_i$ is isomorphic to H_0, and each of these isomorphism σ_i can be chosen so that $\sigma_i(X_i) = X_0$? It is proved in this paper that this problem is NP-complete if and only if H_0 is not a star $K_{1,d}$. A few variations of this problem are also considered.

1 Introduction

Consider the following decision problem.

Problem A. Does a given graph have a spanning subgraph for which all the connected components are isomorphic to K_2?

This is the well-known 1-factor problem. For any given graph, Tutte gave a necessary and sufficient condition for the existence of a 1-factor, and Edmonds give a polynomial time algorithm to find a 1-factor, if it exists at all. Thus we consider Problem A as a well solved problem.

It is natural to replace K_2 in Problem A by K_3, K_4, or even a general fixed connected graph G. For that purpose, let us call a subgraph J of a graph H a *G-subgraph* if, for every connected component J' of J, J' is an induced subgraph of H and J' is isomorphic to G. Then we have the following decision problem.

Problem B. Does a given graph have a spanning G-subgraph?

Schaefer [2] proved that, if G is K_3, then Problem B is NP-complete. By extending Schaefer's techniques, Kirkpatrick and Hell [3] proved that, if G has at least three vertices, then Problem B is NP-complete. Notice that Problem B is trivial if G has only one vertex, and it is precisely Problem A if G has two vertices. Thus we may say that the complexity of Problem B is determined completely.

In this paper, we first consider the following decision problem, where G is a fixed bipartite graph.

Problem C. Does a given bipartite graph have a spanning G-subgraph?

Clearly, Problem C is the restriction of Problem B into the class of bipartite graphs. By applying the key lemma in [3], we prove the following theorem and thus we improve the main result in [3] for bipartite graphs.

Theorem 1. *Problem C is NP-complete if G has at least three vertices.*

Again, notice that Problem C is trivial if G has only one vertex, and it is a special case of Problem A if G has two vertices. Thus we may also say that the complexity of Problem C is determined completely.

Next, we shall go one step further. Let us assume that, together with every given bipartite graph H, there is already a 2-coloring of the vertices of H, and these two color classes are denoted by $X(H)$ and $Y(H)$, respectively. Let G and H be bipartite graphs and let G be connected. A G-subgraph J of H is *consistent* if, for each connected component J' of J, there is an isomorphism σ from G to J' such that $\sigma(X(G)) \subseteq X(H)$. Now we have the following decision problem.

Problem D. Does a given bipartite graph have a spanning consistent G-subgraph?

By extending the techniques used in [3], we prove the following theorem.

Theorem 2. *Problem D is NP-complete if G is not a star $K_{1,d}$.*

If G is a star, we make the following discussions. Let us call a consistent $K_{1,d}$-subgraph a *d-1 matching.* If J is a d-1 matching, then the vertices in $V(J)$ are said to be *covered* by J.

Problem D1. Dose a given bipartite graph H have a d-1 matching that covers $X(H)$?
Problem D2. Dose a given bipartite graph H have a d-1 matching that covers $Y(H)$?
Problem D3. Given a positive integer n and a bipartite graph H, does H have a d-1 matching that covers n vertices in $X(H)$?

For these three problems with $d \geq 3$, we have the following conclusions.

Theorem 3. *Problem D1 and Problem D are solvable in polynomial time. Problem D2 and Problem D3 are NP-complete.*

For the case when $d \leq 2$, it has been proved in [1] that Problem D3 is solvable in polynomial time and thus Problem D, Problem D1 and Problem D2 are all solvable in polynomial time. Therefore, we may say that the complexity of Problem D is determined completely.

2 On Theorem 1.

Let m and p be positive integers and let $Z_p = \{1, ..., p\}$. Let $(Z_p)^m$ be the set of all m-tuples of elements of Z_p. Then, for any subset U of $(Z_p)^m$, a *perfect matching* of U is a subset W of U of cardinality p, such that no two elements of W agree in any coordinate. Let m-DM (m-Dimensional Matching) denotes the following decision problem: does a given subset U of $(Z_p)^m$ have a perfect matching? It is well known [2] that m-DM is NP-complete if m exceeds two. In this section, we shall prove Theorem 1 by transforming m-DM to Problem C. For that purpose, we need to construct a bipartite graph H, according to U, such that U has a perfect matching if and only if H has the required subgraph. First, we have some preparations on the construction of H.

A *module* M is a pair (F, A), where F is a graph and A is a set of independent vertices of F. Members of A and $V(F)-A$ are called *connector vertices* and *interior vertices* of M, respectively. Let H be a graph with $V(H) \cap V(F) = \emptyset$ and let B_1 be a set of independent vertices of H with $|B_1| \leq |A|$. Let B_2 be a set of new vertices and let $B = B_1 \cup B_2$. Suppose that $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_k\}$. Then, *gluing* M on to B means that, taking the disjoint union of H, F and B_2 first, and then, identifying the pairs $\{a_1, b_1\}$, ..., $\{a_k, b_k\}$ respectively. The resulting graph shall be called a *modular extension* of M. Let G be a connected graph and let J be a spanning G-subgraph of a modular extension of M. Suppose that $J_1, ..., J_n$ are the connected components of J. Then a vertex v of F is said to be *bound* to M by J if $V(J_i) \subseteq V(F)$, where i is the index for which $v \in V(J_i)$.

Let v be a vertex of G and let G_v be the graph obtained from G by deleting v, and then, adding two non-adjacent new vertices v' and v'' each of which is adjacent to all neighbors of v in G. Clearly, if G is a bipartite graph, then G_v is also a bipartite graph, and v' and v'' are both contained in either $X(G_v)$ or $Y(G_v)$. We shall denote by $D(G, v)$ the module $(G_v, \{v', v''\})$. Next, from $D(G, v)$ we define another module. Let $V(G) = \{v_1, ..., v_n\}$ and let $\{u_1, ..., u_n\}$, which is denoted by V_1, be a set of vertices disjoint from $V(G)$. Then, glue n copies of $D(G, v)$ on to $\{v_1, u_1\}$, ..., $\{v_n, u_n\}$, respectively. Let the resulting graph be denoted by G_1 and let $S(G, v)$ be the module (G_1, V_1). Clearly, if G is a bipartite graph, then G_1 is also a bipartite graph.

A vertex v of G is said to have property (*) if v is not a cut-vertex and the block that contains v contains at most one cut-vertex. It is not difficult to see that every connected graph has a vertex with property (*). The following is the key lemma in [3] with which we shall prove Theorem 1.

Lemma 2.1. *Let G be a connected graph and let v be a vertex of G with property (*). Then,*

(i) every spanning G-subgraph of every modular extension of $S(G, v)$ binds to $S(G, v)$ either all or none of its connector vertices, and

(ii) both G_1 and $G_1 - V_1$, where $(G_1, V_1) = S(G, v)$, have a spanning G-subgraph.

The spanning G-subgraphs of G_1 and $G_1 - V_1$ shall be called subgraphs of $S(G, v)$ of *type I* and *type II*, respectively.

Proof of Theorem 1. Obviously, Problem C belongs to NP. Thus we need only show the completeness. Let $|V(G)| = m$. We shall transform m-DM to Problem C.

Let v be a vertex of G with property (*) and let $S(G, v) = (G_1, V_1)$. Recall that G_1 is a bipartite graph. Thus we may define $V_x = V_1 \cap X(G_1)$ and $V_y = V_1 \cap Y(G_1)$. Let $|V_x| = m_x$ and $|V_y| = m_y$. Then $m_x + m_y = m$. For any given subset U of $(Z_p)^m$, let

$$I_x = \{(i,j) : 1 \leq i \leq m_x,\ 1 \leq j \leq p\}, \quad I_y = \{(i,j) : m_x + 1 \leq i \leq m,\ 1 \leq j \leq p\}$$

and $I = I_x \cup I_y$. Then, for each $(u_1, ..., u_m)$ in U, glue a copy of $S(G,v)$ on to $\{(1,u_1), ..., (m,u_m)\}$ such that vertices in V_x are identified with vertices in I_x. Let us denote the resulting graph by H. Since G_1 is a bipartite graph, it follows from its construction that H is also a bipartite graph. Next, we prove that U has a perfect matching if and only if H has a spanning G-subgraph.

Suppose that $\{u^1, ..., u^p\}$ is a perfect matching of U. Then, for each u^i, there is a copy S^i of $S(G,v)$ that corresponds to u^i. Let S^{p+1}, ..., S^q be copies of $S(G,v)$ that corresponds to other members of U. Then we define J as the disjoint union of type I subgraphs of S^i for $1 \leq i \leq p$ and type II subgraphs of S^i for $p < i \leq q$. Clearly, J is a G-subgraph of H. Moreover, from the construction of H it is not difficult to see that J is a spanning G-subgraph of H.

Suppose now that H has a spanning G-subgraph J. Since H is a modular extension of each copy S of $S(G,v)$, it follows, by Lemma 2.1, that J binds to S either all or none of its connector vertices. We say S is "chosen" by J if J binds to S all of its connector vertices. Since J is spanning, we deduce that every vertex in I is bound to exactly one "chosen" module S. Let W be the set of members of U that corresponds to those "chosen" modules. It follows from the construction of H that W is a perfect matching of U.

Finally, since $|V(H)| \leq m(m+1)|U|$, it follows that our reduction can be done in polynomial time. Thus the proof of Theorem 1 is complete. ■

Remark. This proof is actually a refinement of the proof of the main result in [3].

3 On Theorem 2.

We break the proof of Theorem 2 into the proofs of a few lemmas. Throughout this section, we assume that G is a connected bipartite graph with at least three vertices. We shall denote Problem D by $P(G)$.

Lemma 3.1. *If there are vertices x^* in $X(G)$ and y^* in $Y(G)$ with property (*), then $P(G)$ is NP-complete.*

To prove this lemma, we need the following result from [3].

Lemma 3.2. *If v is a vertex of G with property (*), then, every spanning G-subgraph of every modular extension of $D(G,v)$ binds to $D(G,v)$ all its interior vertices and exactly one of its connector vertices.*

Proof of Lemma 3.1. Let $X(G) = \{x_1, ..., x_{m_x}\}$ and $Y(G) = \{y_1, ..., y_{m_y}\}$. Let V_x and V_y be two sets of new vertices with $V_x = \{x'_1, ..., x'_{m_x}\}$ and $V_y = \{y'_1, ..., y'_{m_y}\}$. Then let G_1 be the graph obtained by gluing m_x copies of $D(G,x^*)$ on to each $\{x_i, x'_i\}$, and m_y copies of $D(G,y^*)$ on to each $\{y_i, y'_i\}$. From its construction it is easy to see that G_1 is a bipartite graph. Let us define $X(G_1)$ and $Y(G_1)$ in such a way that $V_x \subseteq X(G_1)$ and $V_y \subseteq Y(G_1)$. Also, let $V_1 = V_x \cup V_y$. Then, it is easy to verify that both G_1 and $G_1 - V_1$ have spanning consistent G-subgraphs. They should be called "special" subgraphs of $S(G,x^*,y^*)$, where $S(G,x^*,y^*)$ is the module (G_1,V_1). We first prove the following proposition which is analogous with Lemma 2.1(i).

Proposition. *Every spanning G-subgraph of every modular extension of $S(G,x^*,y^*)$ binds to $S(G,x^*,y^*)$ either all or none of the vertices in V_1.*

Proof. Let H be a modular extension of $S(G,x^*,y^*)$ and let J be a spanning G-subgraph of H. For $1 \leq i \leq m_x$, let D^i be the copy of $D(G,x^*)$ that is glued on to $\{x_i, x'_i\}$, and, for $1 \leq i \leq m_y$, let D^{m_x+i} be the copy of $D(G,y^*)$ that is glued on to $\{y_i, y'_i\}$. Since H is also a modular extension of each D^i, it follows from Lemma 3.2 that J binds to $S(G,x^*,y^*)$ all the interior vertices of each D^i and at least one connector vertex of each D^i. Let Z be the set of vertices in $V_0 = \{x_1, ..., x_{m_x}, y_1, ..., y_{m_y}\}$ that are not bound by J to any D^i. If $Z = \emptyset$, then J binds to $S(G,x^*,y^*)$ none of the vertices in V_1 and thus we are done. If $Z \neq \emptyset$, let $z \in Z$ and let J_z be the connected component of J that contains z. From Lemma 3.2 and the choice of z it is clear that $V(J_z)$ does not contain any interior vertex of any D^i or any vertex in $V_0 - Z$. Since J_z is connected, it follows that $V(J_z) \subseteq Z$. Consequently, $|Z| \geq m_x + m_y$ and thus $Z = V_0$. Therefore, J binds to $S(G,x^*,y^*)$ all the vertices in V_1 as required. ■

Now we continue the proof of Lemma 3.1. Clearly, $P(G)$ belongs to NP. Thus we only need to prove the completeness. Let $m_x + m_y = m$. We shall transform m-DM to $P(G)$. Construct H the same way as we did in the proof of Theorem 1, except that $S(G,v)$ is replaced by $S(G,x^*,y^*)$. Then the same argument shows that the reduction can be done in polynomial time, and H has a spanning G-subgraph if and only if U has a perfect matching. Moreover, from Lemma 3.2 and the above proposition we conclude that every spanning G-subgraph of H is the disjoint union of "special" subgraphs of each copy of $S(G,x^*,y^*)$. It follows from the construction of H that every spanning G-subgraph of H is consistent and thus the proof of Lemma 3.1 is complete. ■

Corollary 1. *If $\delta(G) \geq 2$, where $\delta(G)$ is the minimum degree of G, then $P(G)$ is NP-complete.*

Proof. Let v be a vertex of G with property (*). Without lose of generality, let us assume that v is in $X(G)$. By Lemma 3.1, we only need to show that there is a vertex in $Y(G)$ with property (*). Let G_1 be a block of G such that $V(G_1)$ contains at most one cut-vertex of G. Since $\delta(G) \geq 2$, we must have $X(G_1) \neq \emptyset$ and $|Y(G_1)| \geq 2$. It follows that $Y(G_1)$ contains a vertex u such that u is not a cut-vertex. Therefore, u is a vertex with property (*). ■

Let v be a vertex of G of degree one and let u be the unique vertex adjacent to v. Let G_0 be the graph obtained from G by deleting v, and then, adding non-adjacent vertices $v_1, ..., v_l$ each of them is only adjacent to u. Let $V_0 = \{v_1, ..., v_l\}$. Clearly, G_0 is a bipartite graph and V_0 is included in either $X(G_0)$ or $Y(G_0)$. We shall denote by $D_l(G, v)$ the module (G_0, V_0). Let n be a positive integer with $n \leq l$ and let V_1 be a set of new vertices with $|V_1| = l$. Then, let G_1 be the graph obtained by gluing n copies of $D_l(G, v)$ on to V_1. We denote the module (G_1, V_1) by $S_{l,n}(G, v)$. From its construction it is clear that G_1 is a bipartite graph and, for every subset Z of V_1 with $|Z| = l - n$, the graph $G_1 - Z$ has a spanning consistent G-subgraph.

Lemma 3.3. *Every spanning G-subgraph of every modular extension of $S_{l,n}(G, v)$ binds to $S_{l,n}(G, v)$ all its interior vertices and exactly n connector vertex.*

Proof. Since every modular extension of $S_{l,n}(G, v)$ is also a modular extension of each of the n copies of $D_l(G, v)$, thus, from the construction of $S_{l,n}(G, v)$ it follows that, to prove Lemma 3.3, we need only show that every spanning G-subgraph J of every modular extension H of $D_l(G, v)$ binds to $D_l(G, v)$ all its interior vertices and exactly one connector vertex.

Let u be the unique vertex of G that is adjacent to v and let J_1 be a connected component of J that contains an interior vertex of $D_l(G, v)$. Since the connected components of $H - u$ that contain interior vertices of $D_l(G, v)$ have at most $|V(G)| - 2$ vertices, it follows that u belongs to $V(J_1)$ and thus J_1 contains all the interior vertices of $D_l(G, v)$. Since $D_l(G, v)$ has $|V(G)| - 1$ interior vertices and J_1 is connected, the other vertex of J_1 must be a connector vertex of $D_l(G, v)$. Therefore, J binds to $D_l(G, v)$ all its interior vertices and exactly one connector vertex as required. ■

Lemma 3.4. *Let Y_0 be the set of vertices in $Y(G)$ with degree one and let $G_0 = G - Y_0$. Then $P(G_0) \leq P(G)$, ie. $P(G_0)$ can be transformed to $P(G)$ in polynomial time.*

Proof. Clearly, we only need to consider the case when $0 < |Y_0| < |Y(G)|$. Let H_0 be a bipartite graph which is an instance of $P(G_0)$. Without lose of generality, we assume that there is a positive integer k such that $|X(H_0)| = k|X(G)|$ and $|Y(H_0)| = k|Y(G)|$. We shall construct another bipartite graph H such that H has a spanning consistent G-subgraph if and only if H_0

has a spanning consistent G_0-subgraph.

Let $|X(G)| = m_x$, $|Y(G)| = m_y$ and $|Y_0| = m_0$. Let $X(H_0) = \{x_1, ..., x_{km_x}\}$. For each x_i, let $Y_i = \{y_{i,1}, ..., y_{i,m_0}\}$ and $E_i = \{x_i y_{i,j} : 1 \leq j \leq m_0\}$. Then we define

$$H_1 = (X(H),\ Y(H) \cup Y,\ E(H) \cup E), \quad \text{where} \quad Y = \bigcup_1^{km_x} Y_i \quad \text{and} \quad E = \bigcup_1^{km_x} E_i.$$

Finally, we defined H to be the graph obtained by gluing $S_{l,n}(G, v)$ on to Y, where $l = km_0 m_x$, $n = km_0(m_x - 1)$ and $v \in Y_0$.

Clearly, the transformation can be done in polynomial time. In the following, we prove that H has a spanning consistent G-subgraph if and only if H_0 has a spanning consistent G_0-subgraph.

Suppose that H_0 has a spanning consistent G_0-subgraph J. Let $J_1, ..., J_k$ be the connected components of J. Then, it is clear from the construction of H that, for each J_i, there is a subset Z_i of $\bigcup\{Y_j : x_j \in X(J_i)\}$ such that the subgraph J_i' of H induced by $V(J_i) \cup Z_i$ is isomorphic to G. Let Z be the union of these Z_i. Then $|Z| = km_0$ and $|Y - Z| = km_0 m_x - km_0 = n$. Let $S_{l,n}(G, v) = (G_1, Y)$. Then $G_1 - Z$ has a spanning consistent G-subgraph J_0'. Therefore, the disjoint union of J_0' and $J_1', ..., J_k'$ is a spanning consistent G-subgraph of H.

Suppose now that H has a spanning consistent G-subgraph J. Since H is a modular extension of $S_{l,n}(G, v)$, we conclude from Lemma 3.3 that there are n connected components $J_1, ..., J_n$ of J such that $V(J_1) \cup ... \cup V(J_n)$ consists of the interior vertices and n connector vertices of $S_{l,n}(G, v)$. Let $V = V(J_1) \cup ... \cup V(J_n)$ and let $H' = H - V$. Then $|Y(H') \cap Y| = l - n = km_0$ and, by the construction of H, all the vertices in $Y(H') \cap Y$ have degree one in H'. Let $J_0 = J - V$. Then J_0 is a spanning consistent G-subgraph of H'. Since $Y(J_0)$ has exactly km_0 vertices of degree one, it follows that these vertices are exactly the vertices in $Y(H') \cap Y$ and thus $J_0 - Y$ is a spanning consistent G_0-subgraph of H_0. ■

Corollary 2. *$P(G)$ is NP-complete if G is not a tree.*

Proof. If $\delta(G) \geq 2$, then the result follows from Corollary 1. If $\delta(G) = 1$, then we may apply Lemma 3.4 to G repeatedly. Since G is not a tree, this reduction eventually reduces G into a graph G' with $\delta(G') \geq 2$. Then, again, the result follows from Corollary 1. ■

In the rest of this section, we consider the case when G is a tree. We call a tree T *elementary* if there is a vertex y_0 in $Y(T)$ of degree at least two such that $Y(T) - \{y_0\}$ is the set of leaves of T. An elementary tree T is depict in Figure 1.

Lemma 3.5. *If T is elementary, then $P(T)$ is NP-complete.*

Proof. Obviously, $P(T)$ belongs to NP. Thus we only need to show the completeness. Let $X(T) = \{x_1, ..., x_m\}$. We shall transform (m+1)-DM to $P(T)$. For that purpose, we first have some preparations.

For each x_i in $X(T)$, let Y_i be the set of leaves of T that are adjacent to x_i and let D^i be a copy of $D(T - Y_i, x_i)$. Let y_1 be a vertex in Y_1 and let D^0 be a copy of $D(T, y_1)$. Now let $V_1 = \{x'_1, ..., x'_m, y'_0\}$ be a set of new vertices and let T_1 be the tree obtained by gluing each D^i on to $\{x_i, x'_i\}$ and D^0 on to $\{y_0, y'_0\}$. We shall denote by M the module (T_1, V_1). From the construction of T_1 it is clear that $T_1 - V_1$ has a (unique) spanning consistent T-subgraph. This subgraph shall be called the *type I* subgraph of M. Next, for each Y_i, let Y'_i be a copy of Y_i and let $E_i = \{x'_i y' : y \in Y'_i\}$. Then we define $T_1^+ = (X(T_1),\ Y(T_1) \cup Y',\ E(T_1) \cup E)$, where $Y' = \bigcup_1^m Y'_i$ and $E = \bigcup_1^m E_i$. It is not difficult to see that T_1^+ has a (unique) spanning consistent T-subgraph. This subgraph shall be called the *type II* subgraph of M (and, of T_1^+).

Let a bipartite graph H be a modular extension of M and let J be a spanning consistent T-subgraph of H. Clearly, H is also a modular extension of $D(T, y_1)$. Since y_1 has property (*), it follows from Lemma 3.2 that J binds to $D(T, y_1)$ exactly one of y_0 and y'_0.

(1) *If y_0 is bound to $D(T, y_1)$ by J, then there is a subgraph J' of J such that J' is a type I subgraph of M.*

(2) *If y'_0 is bound to $D(T, y_1)$ by J, then there is a subgraph J' of J such that J' is a type II subgraph of M.*

To see these, for $i = 1, ..., m$, let $D^i = (T^i, \{x_i, x'_i\})$ and let y_0^i be the unique vertex in $Y(T^i)$ that is not a leaf of T^i. Also, let J'_i and J''_i be the connected components of J that contain y_0^i and x_i, respectively. Then, it follows from the definition of J that J'_i contains all the interior vertices of D^i and J''_i contains all the vertices in Y_i. Now, if y_0 is bound to $D(T, y_1)$ by J, then $J'_i = J''_i$ for all i in Z_m, and hence (1) holds. On the other hand, if y'_0 is bound to $D(T, y_1)$ by J, then $J'_i \neq J''_i$ for all i in Z_m, and hence (2) holds.

Next, we transform (m+1)-DM to $P(T)$. Let $U \subseteq (Z_p)^{m+1}$ and let $I = Z_{m+1} \times Z_p$. For each $(u_1, ..., u_{m+1})$ in U, glue a copy of M on to $\{(1, u_1), ..., (m+1, u_{m+1})\}$ such that each x'_i is glued on to (i, u_i) and y'_0 is glued on to $(m+1, u_{m+1})$. Let H_1 be the resulting graph. Then it is clear that H_1 is a bipartite graph. Let the vertices of H_1 be two colored in such a way that $(m+1, u_{m+1})$ belongs to $Y(H_1)$. For $i = 1, ..., m$, let $|Y_i| = n_i$. Now we define $H = (X(H_1),\ Y(H_1) \cup Y,\ E(H_1) \cup E)$ where $Y = \{y_{i,j,k} : 1 \leq i \leq m,\ 1 \leq j \leq p,\ 1 \leq k \leq n_i\}$ and $E = \{((i,j), y_{i,j,k}) : 1 \leq i \leq m,\ 1 \leq j \leq p,\ 1 \leq k \leq n_i\}$. It is easy to see from its

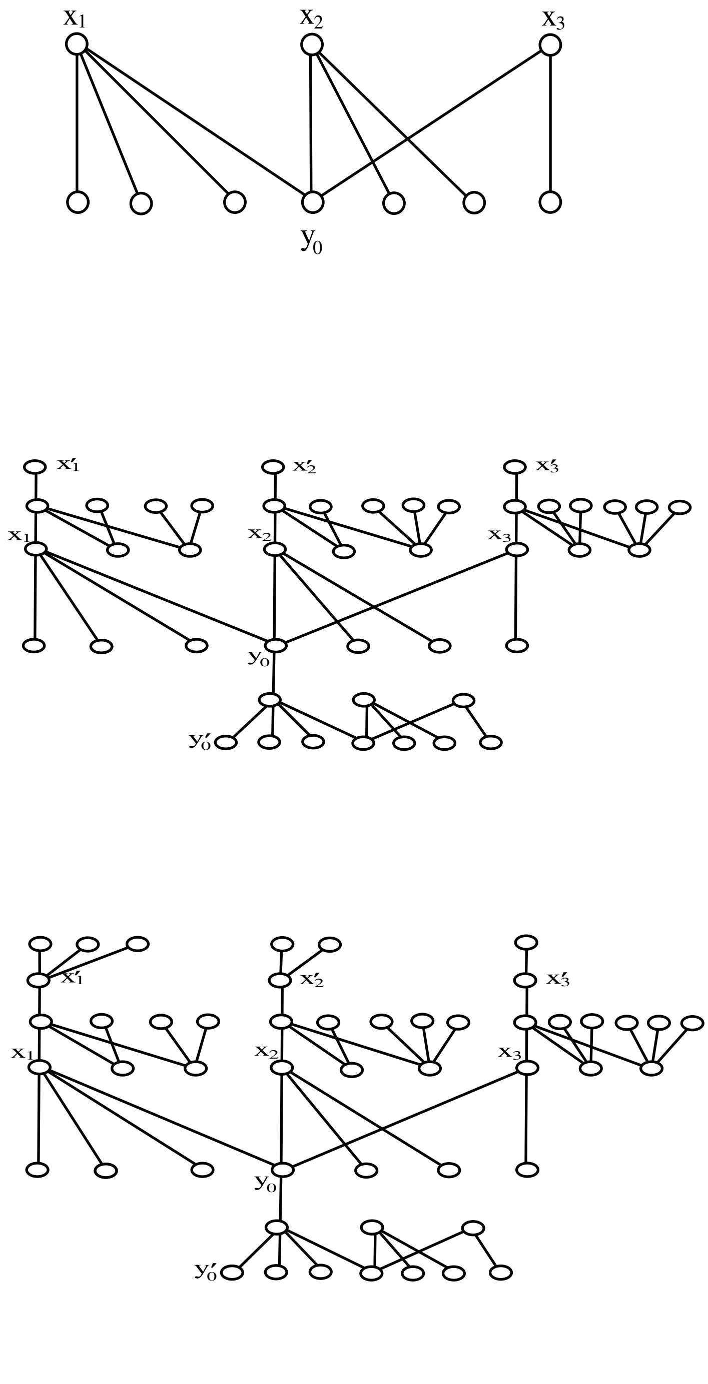

Figure 1. T, T_1 and T_1^+ (where m=3)

construction that H is a bipartite graph. Moreover, this construction can be done in polynomial time. Thus we only need to show that U has a perfect matching if and only if H has a spanning consistent T-subgraph.

Suppose that $\{u^1, ..., u^p\}$ is a perfect matching of U. Then, let $u^{p+1}, ..., u^q$ be the rest of the members of U. For each $l = 1, ..., q$, let $M^l = (T_1^l, V_1^l)$ be the copy of M that corresponds to u^l. Let $u^l = (u_{l_1}, ..., u_{l_{m+1}})$ and let $Y^l = \{y_{i,j,k} : 1 \le i \le m,\ j = u_{l_i},\ 1 \le k \le n_i\}$. Let H^l be the subgraph of H induced by $V(T_1^l) \cup Y^l$. Let J be the disjoint union of the type II subgraph of each H^l (for $1 \le l \le p$) and the type I subgraph of each M^l (for $p+1 \le l \le q$). Then it is clear that J is a spanning consistent T-subgraph of H.

Suppose now that H has a spanning consistent T-subgraph J. Let M^1, ..., M^r be those copies of M for which J binds y_0' to it. Let M^{r+1}, ..., M^q be the rest of the copies of M. For each vertex $(m+1, j)$, since it is a connector vertex of each $D(T, y_1)$, it follows that $(m+1, j)$ is bound to some M^i by J. As a consequence, we have $r = p$. Therefore, we conclude from (1) and (2) that $\{u^1, ..., u^p\}$, where each u^i is the member of U that corresponds to M^i, is a perfect matching of U. ■

Proof of Theorem 2. By Corollary 2, we may assume that G is a tree. If G is elementary, then the result follows from Lemma 3.5 immediately. If G is not elementary, since G is not a star, we may apply Lemma 3.4 to G and reduce G to an elementary tree. Then, by Lemma 3.5 again, the result follows. ■

4 On Theorem 3.

To prove Theorem 3, it is clear that we only need to prove the following two lemmas.

Lemma 4.1. *Problem D1 is solvable in polynomial time.*

Lemma 4.2. *Problem D2 is NP-complete.*

Proof of Lemma 4.1. We prove that Problem D1 can be transformed into a network-flow problem. Let $H = (X, Y, E)$ be the given bipartite graph. Then we defined a digraph $D = (V, A)$ as follows. Let s and t be two new vertices and let $V = X \cup Y \cup \{s, t\}$. Let $A_1 = \{(s, x) : x \in X\}$, $A_2 = \{(y, t) : y \in Y\}$, $A_3 = \{(x, y) : x \in X, y \in Y$, and $xy \in E\}$ and $A = A_1 \cup A_2 \cup A_3$. We also define the capacity function c as follows. For each arc a in A_1, let $c(x) = d$, and for each other arc a in A, let $c(a) = 1$. Obviously, H has a d-1 matching that covers X if and only if the network (D, c) has a flow from s to t of value $d|X|$. Since the network-flow problem can be solved in polynomial time, it follows that Problem D1 can also be solved in polynomial time. ■

Proof of Lemma 4.2. It is clear that Problem D2 belongs to NP, thus we only need to show the completeness. Let $S = K_{1,d}$ and let L be the set of leaves of S. Let M be the module (S, L). Next we transform d-DM to Problem D2.

Let U be a subset of $(Z_p)^d$ and let $Y = Z_d \times Z_p$. Then, for each member $u = (u_1, ..., u_d)$ of U, glue a copy (S_u, L) of M on to $\{(1, u_1), ..., (d, u_d)\}$. Let H be the resulting graph. Clearly, H is a bipartite graph with $Y(H) = Y$. Moreover, it is easy to see from the construction of H that $\{u^1, ..., u^p\}$ is a perfect matching of U if and only if the disjoint union of $S_{u^1}, ..., S_{u^p}$ is a d-1 matching of H that covers Y. Finally, observe that the reduction can be done in polynomial time. Thus Lemma 4.2 is proved. ■

A final Remark.

Let us modify the definition of a G-subgraph as follows. Let us call a subgraph J of a graph H a *G-subgraph* if every connected component of J is isomorphic to G. That is, let us drop the requirement that each connected component of J is an induced subgraph of H. Then we still have Problem C and Problem D. It is not difficult to see that the proofs given in this paper still work under the new definition of a G-subgraph. Therefore, the three theorems still hold under the new definition of a G-subgraph.

Acknowledgment This work was done while the author was a graduate student at the Institute of System Science, Academia Sinica. He thanks his advisors K.C. Hsu, Zhongfan Ma, and Maocheng Cai for their inspiration and academic guidance. The author also thanks Professor Zhenhong Liu for reading the first version of this paper. This research was partially supported by a grant from the Louisiana Education Quality Support Fund through the Board of Regents.

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