

Graphs with no \bar{P}_7 -minor

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Abstract

Let \bar{P}_7 denote the complement of a path on seven vertices. We determine all 4-connected graphs that do not contain \bar{P}_7 as a minor.

1 Introduction

In this paper, a graph G is called H -free, where H is a graph, if no minor of G is isomorphic to H . Many important problems in graph theory can be formulated in terms of H -free graphs. For instance, the four color theorem can be equivalently stated as: all K_5 -free graphs are 4-colorable. To solve problems involving H -free graphs, it is often desirable to explicitly determine all H -free graphs. In this area, the two most famous open problems are to determine K_6 -free and Petersen-free graphs. Notice that both graphs have fifteen edges.

For each 3-connected graph H with at most eleven edges, all H -free graphs have been completely determined. A survey of these results can be found in [3]. For 3-connected graphs with twelve edges, the characterization problem is solved for the cube [5], the octahedron [2, 6], and the Wagner graph V_8 [7]. In addition, 4-connected Oct^+ -free graphs are also determined [4], where Oct^+ is the unique 13-edge graph obtained from the octahedron by adding an edge. In this paper we consider \bar{P}_7 -free graphs, where \bar{P}_7 , a 15-edge graph, is the complement of a path on seven vertices. Our result makes \bar{P}_7 the large graph H for which 4-connected H -free graphs are completely determined. In contrast, 6-connected K_6 -free graphs are not determined (although there is a conjecture on these graphs) and nothing is known about 6-connected Petersen-free graphs.

To state our main result we need to define a few classes of graphs. For each integer $n \geq 3$, let DW_n denote a *double-wheel*, which is a graph on $n + 2$ vertices obtained from a cycle C_n by adding two adjacent vertices and connecting them to all vertices on the cycle. Let $DW = \{DW_n : n \geq 3\}$. For each integer $n \geq 5$, let C_n^2 be a graph obtained from a cycle C_n by joining all pairs of vertices of distance two on the cycle. Notice that $C_5^2 = DW_3 = K_5$, and C_n^2 is nonplanar when n is odd. Let $\mathcal{C}_0 = \{C_{2n}^2 : n \geq 3\}$, $\mathcal{C}_1 = \{C_{2n+1}^2 : n \geq 2\}$, and $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$. Let \mathcal{K} consist of graphs that are 4-connected nonplanar minors of some $K_{4,n}$. In other words, these are 4-connected nonplanar graphs obtained from some $K_{4,n}$ ($n \geq 1$) by adding edges to the color class of size four. It is routine to check that \mathcal{K} contains exactly one graph (K_5) on five vertices, two ($K_6 \setminus e, DW_4$) on six vertices, six ($K_{4,3}^1, K_{4,3}^2, K_{4,3}^3, K_{4,3}^4, K_{4,3}^5, K_{4,3}^6$ in Figure 4.2) on seven vertices, and eleven on n ($n \geq 8$)

32 vertices. Given a graph G , the *line graph* of G , denoted by $L(G)$, is the graph with vertex set $E(G)$ and
 33 edge set $\{xy : x, y \in E(G) \text{ are adjacent in } G\}$. Our main result is the following.

34 **Theorem 1.1.** *A 4-connected graph G is \bar{P}_7 -free if and only if either G is planar or G belongs to $\mathcal{DW} \cup$
 35 $\mathcal{C}_1 \cup \mathcal{K} \cup \{K_6, L(K_{3,3}), \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$, where $\Gamma_1, \dots, \Gamma_5$ are the five graphs shown below.*

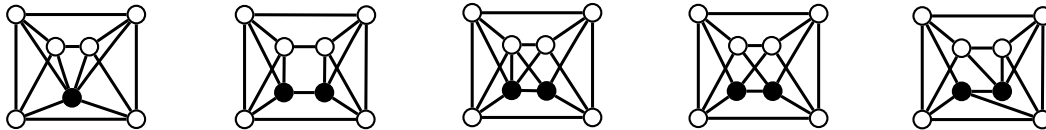


Figure 1.1: Graphs $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$

36 This theorem implies the following.

37 **Corollary 1.2.** *A 4-connected graph G is \mathcal{C}_7^2 -free if and only if either G is planar or G belongs to $\mathcal{DW} \cup$
 38 $\mathcal{K} \cup \{K_6, L(K_{3,3}), \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$.*

39 We remark that Theorem 1.1 is not a complete characterization of \bar{P}_7 -free graphs, since we do not know
 40 those \bar{P}_7 -free graphs that have a low connectivity. As observed in [2], \bar{P}_7 -free graphs are precisely graphs
 41 constructed by 0-, 1-, 2-, and 3-sums starting from K_1, K_2, K_3, K_4 , and internally 4-connected \bar{P}_7 -free
 42 graphs. It follows that we need to determine all internally 4-connected \bar{P}_7 -free graphs in order to obtain a
 43 complete characterization. Theorem 1.1 determines all 4-connected \bar{P}_7 -free graphs, but it seems that there
 44 are still many internally 4-connected ones that are not 4-connected. For instance, consider graphs obtained
 45 from two disjoint copies of $K_{2,n}$ by adding a perfect matching in between. All such graphs are internally
 46 4-connected and \bar{P}_7 -free.

47 We close this section by providing an outline of the rest of the paper. In the next section we explain how
 48 our approach works. In particular, we introduce a chain theorem for 4-connected graphs, which says that all
 49 4-connected graphs are “extensions” of certain basic graphs. Our proof of Theorem 1.1 will be divided into
 50 two parts. First, in Section 3, we determine \bar{P}_7 -free extensions of every basic graph that is not K_5 . Then,
 51 in Section 4, we determine \bar{P}_7 -free extensions of K_5 . Finally, we prove Theorem 1.1 and Corollary 1.2 in the
 52 end of Section 4.

53 2 Basic lemmas

54 Our main tool is a chain theorem for 4-connected graphs. To explain this result, we need a few definitions.
 55 A 3-connected cubic graph with at least six vertices is called *internally 4-connected* if it has four disjoint
 56 paths between any two cycles of the graph. Let \mathcal{L} denote the class of line graphs of internally 4-connected
 57 cubic graphs.

58 All graphs considered in this paper are simple. In particular, we use G/e to denote the graph obtained
 59 from G by first contracting e and then deleting all but one parallel edges from each parallel family. The
 60 reverse operation of this modified contraction is called *splitting* a vertex, which is formally defined as follows.
 61 Let v be a vertex of a graph G . Let $N_G(v)$ denote the set of vertices of G that are adjacent to v , which are

62 also known as *neighbors* of v . Let $X, Y \subseteq N_G(v)$ such that $X \cup Y = N_G(v)$ and $|X|, |Y| \geq 3$. Let G' be
63 obtained from $G \setminus v$ by adding two adjacent vertices x, y and then joining x to all vertices in X and y to all
64 vertices in Y . We call G' a *split* of G . Now we can state the chain theorem [8] that we will use.

65 **Theorem 2.1.** *Every 4-connected graph can be obtained from a graph in $\mathcal{C} \cup \mathcal{L}$ by repeatedly splitting vertices.*

66 We also make the following observation.

67 **Lemma 2.2.** *If G' is obtained from a 4-connected graph G by splitting a vertex v , then G' is also 4-connected.*

68 **Proof.** Suppose, to the contrary, that G' has a vertex cut S of size at most three. Let x, y and X, Y be as
69 in the definition of vertex split. Since $G = G'/xy$ is 4-connected, exactly one of x, y , say x , is in S . Then,
70 for the same reason, y is an isolated vertex in $G' - S$, which contradicts the assumption $|Y| \geq 3$. \square

71 The above two results suggest an algorithm for generating all 4-connected graphs. We begin with graphs
72 in $\mathcal{C} \cup \mathcal{L}$, which are known to be 4-connected. In the general step, we split each vertex of each constructed
73 graph in all possible ways. Theorem 2.1 implies that graphs generated by this procedure include all 4-
74 connected graphs, and Lemma 2.2 ensures that the generated graphs are precisely all 4-connected graphs.
75 We will follow this algorithm to generate all 4-connected \bar{P}_7 -free graphs.

76 When analyzing cubic graphs we will need the following version of Menger theorem, which can be found
77 in Section 3.3 of [1].

78 **Lemma 2.3.** *Let G be a graph and let \mathcal{P} be a set of k disjoint paths of G between disjoint $A, B \subseteq V(G)$. If
79 G has a set \mathcal{Q} of $k + 1$ disjoint paths between A and B , then \mathcal{Q} can be chosen so that each end of a path in
80 \mathcal{P} is also an end of a path in \mathcal{Q} .*

81 A graph G is a *subdivision* of a graph H if G is obtained from H by repeatedly deleting an edge xy , and
82 adding a new vertex z and two new edges zx, zy . The next is an easy lemma which was also observed in [6].

83 **Lemma 2.4.** *If a subdivision of H is a subgraph of G then $L(H)$ is a minor of $L(G)$.*

84 We also need the following result from [4].

85 **Theorem 2.5.** *If a nonplanar graph G is obtained from a 4-connected planar graph by splitting a vertex,
86 then G contains \bar{P}_7 as a minor.*

87 3 Extensions of large graphs

88 Let $Ext(G)$ be the class of \bar{P}_7 -free graphs that are either G or obtained from G by repeatedly splitting
89 vertices. By Theorem 2.1 we need to determine $Ext(G)$ for every $G \in \mathcal{C} \cup \mathcal{L}$. In this section we consider
90 extension of graphs in $(\mathcal{C} - \{K_5\}) \cup \mathcal{L}$, and we will consider $Ext(K_5)$ in the next section.

91 We first consider planar graphs in $\mathcal{C} \cup \mathcal{L}$. The result follows from Theorem 2.5 and Lemma 2.2.

92 **Lemma 3.1.** *If $G \in \mathcal{C} \cup \mathcal{L}$ is planar then all graphs in $Ext(G)$ are planar.*

93 Next we consider nonplanar graphs in \mathcal{L} .

94 **Lemma 3.2.** $L(K_{3,3})$ is C_7^2 -free.

95 **Proof.** Since $L(K_{3,3})$ is connected, if C_7^2 is a minor of $L(K_{3,3})$, the minor can be obtained by contracting
 96 two edges e, f and then deleting some edges. Let $e = xy$ and let xyz be the unique triangle containing e (see
 97 Figure 3.1). Notice that z is cubic in $L(K_{3,3})/e$, so f has to be incident with z . If f is not in the triangle xyz
 98 then $L(K_{3,3})/e/f$ has a cubic vertex, which cannot contain C_7^2 . If f is in the triangle xyz then $L(K_{3,3})/e/f$
 99 is isomorphic to Γ_1 . To obtain C_7^2 , we have to delete one edge from Γ_1 . However, any edge deletion results
 in a cubic vertex, which implies that $L(K_{3,3})$ is C_7^2 -free. \square

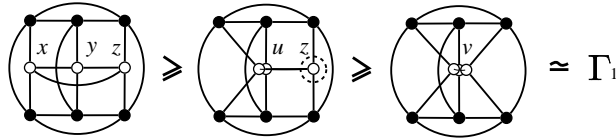


Figure 3.1: Contracting edges of $L(K_{3,3})$

100

101 **Lemma 3.3.** If G is an internally 4-connected nonplanar cubic graph, then either $G = K_{3,3}$ or G contains
 102 a subdivision of V_8 .

103 **Proof.** This result follows from the characterization of V_8 -free graphs [7]. However, instead of explaining
 104 this characterization, we can provide a short proof of this lemma.

105 Since G is cubic and nonplanar, G contains a subgraph H that is a subdivision of $K_{3,3}$. Let $x_1, x_2, x_3, y_1,$
 106 y_2, y_3 be the cubic vertices of H and let P_{ij} , where $i, j \in \{1, 2, 3\}$, be the $x_i y_j$ -path of H corresponding to
 107 edge $x_i y_j$ of $K_{3,3}$. If $|V(H)| = 6$ then $G = K_{3,3}$ since G is connected. So we may assume that P_{11} has
 108 interior vertices. Since G is 3-connected, $G - \{x_1, y_1\}$ has a path Q between $P_{11} - \{x_1, y_1\}$ and $H - V(P_{11})$.
 109 If an end of Q is on P_{ij} for some $i, j \in \{2, 3\}$ then $H \cup Q$ is a subdivision of V_8 . So we may assume that Q
 110 has an end on P_{12} . Let A be the cycle contained in $P_{11} \cup P_{12} \cup Q$ and let B be the union of P_{ij} for $i = 2, 3$
 111 and $j = 1, 2, 3$. By Lemma 2.3, since G is internally 4-connected, G has four disjoint paths Q_1, Q_2, Q_3, Q_4
 112 between A, B and such that y_i ($i = 1, 2, 3$) is an end of Q_i . Now it is easy to check that the union of A, B
 113 and Q_1, Q_2, Q_3, Q_4 contains a subdivision of V_8 . \square

114 **Lemma 3.4.** If $G \in \mathcal{L}$ is nonplanar then $Ext(G) = \emptyset$, unless $G = L(K_{3,3})$, and in this case $Ext(G) = \{G\}$.

115 **Proof.** We first observe that the line graph of any planar cubic graph is planar. So if $G \in \mathcal{L}$ is nonplanar
 116 and $G = L(H)$, then H is nonplanar. If H is not $K_{3,3}$, by Lemma 3.3, H contains a subdivision of V_8 . Notice
 117 that $L(V_8)$ contains a \bar{P}_7 -minor (see Figure 3.2), so we deduce from Lemma 2.4 that G contains a \bar{P}_7 -minor,
 which proves $Ext(G) = \emptyset$.

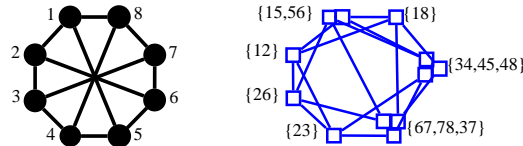


Figure 3.2: V_8 and a \bar{P}_7 -minor of $L(V_8)$

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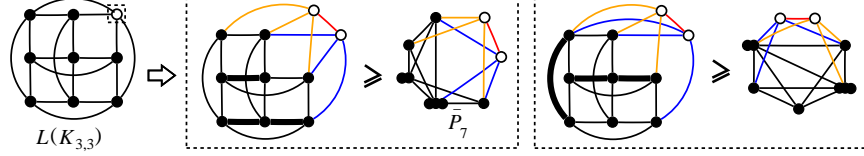


Figure 3.3: Up to symmetry, $L(K_{3,3})$ has two minimal splits, both of which contain a \bar{P}_7 -minor by contracting the thick edges.

119 Next we consider the case $H = K_{3,3}$. Since \bar{P}_7 can be obtained from C_7^2 by joining two nonadjacent
 120 vertices, by Lemma 3.2, $L(K_{3,3})$ is P_7^2 -free. To complete the proof, we show that any split of $L(K_{3,3})$
 121 contains a \bar{P}_7 -minor. Clearly, we only need to consider the cases that both the two new vertices have degree
 122 four, because other splits contain these special splits. Up to symmetry, there are two such splits, and both
 123 of them contains a \bar{P}_7 -minor, as illustrated in Figure 3.3. \square

124 Finally we consider nonplanar graphs in $\mathcal{C} - \{K_5\}$, which are exactly graphs in $\mathcal{C}_1 - \{K_5\}$.

125 **Lemma 3.5.** *For every integer $n \geq 3$, C_{2n+1}^2 is \bar{P}_7 -free.*

126 **Proof.** If a simple connected graph $G = (V, E)$ has an embedding in the projective plane, then by Euler
 127 formula, the embedding has $k = |E| - |V| + 1$ faces. If the size of the faces are f_1, f_2, \dots, f_k , then $2|E| =$
 128 $f_1 + f_2 + \dots + f_k \geq 3k = 3|E| - 3|V| + 3$, which implies $|E| \leq 3|V| - 3$.

129 For any graph G , let $G + u$ denote the graph obtained from G by adding a new vertex u and joining u
 130 to all vertices of G . Then $\bar{P}_7 + u$ is not projective since it has 8 vertices and $22 > 3|V| - 3$ edges. However,
 131 it is easy to see that C_{2n+1}^2 can be embedded in the Möbius strip with all vertices on the boundary, hence
 132 $C_{2n+1}^2 + u$ admits is projective embedding. As a result, $G + u$ is projective for every minor G of C_{2n+1}^2 and
 133 thus \bar{P}_7 is not a minor of C_{2n+1}^2 . \square

134 **Lemma 3.6.** *For any $n \geq 3$, $Ext(C_{2n+1}^2) = \{C_{2n+1}^2\}$.*

135 **Proof.** By Lemma 3.5, we only need to show that very split of C_{2n+1}^2 ($n \geq 3$) contains a \bar{P}_7 -minor. We
 136 prove this by induction on n . First, every split of C_7^2 contains a \bar{P}_7 -minor because it contains a split, that
 both new vertices have degree four, as shown in Figure 3.4. Next, suppose $n > 3$. We claim that every split

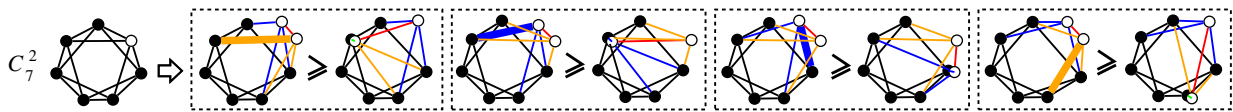


Figure 3.4: Up to symmetry, C_7^2 has four minimal splits, all of which contain a \bar{P}_7 -minor by contracting the thick edges.

137 of C_{2n+1}^2 contains a split of C_{2n-1}^2 as a minor. Let $\{v_1, v_2, \dots, v_{2n+1}\}$ be the vertex set of C_{2n+1}^2 such that
 138 for all $1 \leq i \leq 2n+1$, $N(v_i) = \{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$, where the indices are taken modulo $2n+1$. Since
 139 all vertices of C_{2n+1}^2 are symmetric, we choose to split vertex v_4 and let G' be the resulted graph. Then we
 140 contract edges $v_{2n-1}v_{2n+1}$ and $v_{2n}v_1$. Let G'' be the resulted graph, and let v'_{2n-1}, v'_1 be the new vertices
 141 obtained from contracting $v_{2n-1}v_{2n+1}$ and $v_{2n}v_1$, respectively. Then $N_{G''}(v'_{2n-1}) = \{v_{2n-3}, v_{2n-2}, v'_1, v_2\}$,
 142 and $N_{G''}(v'_1) = \{v_{2n-2}, v'_{2n-1}, v_2, v_3\}$. Since $n > 3$, $v_{2n-1}, v_{2n+1}, v_{2n}, v_1$ are not adjacent to v_4 . So G'' is a

144 split of C_{2n-1}^2 , which proves the claim. Now the induction hypothesis implies that G' contains a \bar{P}_7 -minor,
 145 which completes our induction and thus the lemma is proved. \square

146 4 Extensions of K_5

147 In this section we determine all graphs in $Ext(K_5)$.

148 **Lemma 4.1.** $Ext(K_5) = \mathcal{DW} \cup \mathcal{K} \cup \{K_6, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$

149 We divide the proof of Lemma 4.1 into a sequence of lemmas.

150 **Lemma 4.2.** *Every graph in \mathcal{DW} is C_7^2 -free.*

151 **Proof.** Each double-wheel has at most two vertices whose deletion results in a graph of maximum degree
 152 at most two. This is a property preserved by all its minors. It is easy to check that C_7^2 does not have this
 153 property, so it is not a minor of any double-wheel. \square

154 **Lemma 4.3.** *Every graph in \mathcal{K} is C_7^2 -free.*

155 **Proof.** Every $K_{4,n}$ has at most four vertices that cover all edges of the graph. This is a property preserved
 156 by all its minors. It is easy to check that C_7^2 does not have this property, so it is not a minor of any $G \in \mathcal{K}$
 157 since G is a minor of some $K_{4,n}$. \square

158 **Lemma 4.4.** *All graphs in $\{K_6, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$ are C_7^2 -free.*

159 **Proof.** It is clear that K_6 is C_7^2 -free since K_6 has fewer vertices than C_7^2 . Since $|V(\Gamma_1)| = |V(C_7^2)|$,
 160 $|E(\Gamma_1)| = |E(C_7^2)| + 1$, and $\Gamma_1 \setminus e$ has a cubic vertex for every edge e , it follows that Γ_1 is C_7^2 -free. For
 161 $i = 2, 3, 4, 5$, notice that Γ_i has eight vertices, see Figure 4.1. If Γ_i contains a \bar{P}_7 -minor, we may assume
 162 that one of its edges is contracted. Some edges of Γ_i cannot be contracted since its contraction destroys
 163 the 4-connectivity of the graph. In Γ_2 , we can contract only edge 14, 25, 36, or 78, and the resulted graph
 164 is isomorphic to Γ_1 . In Γ_3 , we can contract only edge 14, 25, 28, 36, 57, or 78, and the resulted graph is
 165 isomorphic to Γ_1 or $K_{4,3}^3$, where $K_{4,3}^3$ is a graph in \mathcal{K} as shown in Figure 4.2. In Γ_4 , we can contract only
 166 edge 14, 25, 28, 36, 57, or 78, and the resulted graph is isomorphic to Γ_1 or $K_{4,3}^4$, where $K_{4,3}^4$ is a graph in
 167 \mathcal{K} as shown in Figure 4.2. In Γ_5 , we can contract only edge 14 or 78, and the resulted graph is isomorphic
 to Γ_1 . By Lemma 4.3, $\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ are C_7^2 -free. \square

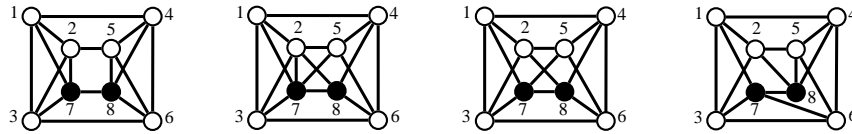


Figure 4.1: $\Gamma_2, \Gamma_3, \Gamma_4$, and Γ_5

168

169 In the next few lemmas we determine all extensions of K_5 . The process is illustrated in Figure 4.2. We
 170 first determine all three splits of K_5 . Then we determine all \bar{P}_7 -free splits of each of these three. We will
 171 repeat this procedure and further determine all \bar{P}_7 -free splits of these 7-vertex graphs. Finally, we show that

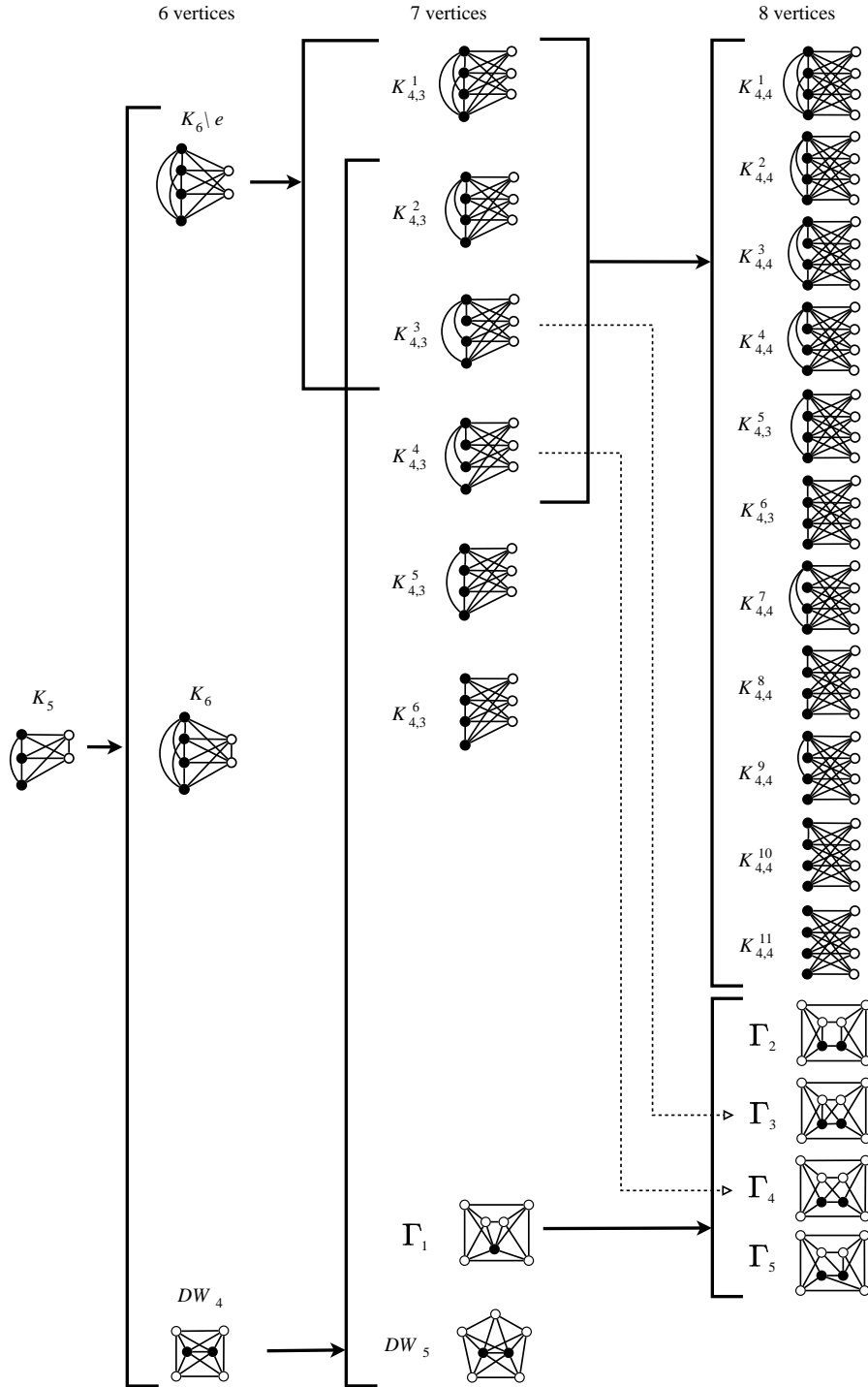


Figure 4.2: \bar{P}_7 -free splits of K_5 on 6, 7, and 8 vertices

172 any further split will create a \bar{P}_7 -minor except for graphs in $DW \cup \mathcal{K}$. In the following, we will denote graphs
 173 in \mathcal{K} with seven or eight vertices by $K_{4,i}^j$ as shown in Figure 4.2.

174 **Lemma 4.5.** *The only splits of K_5 are K_6 , $K_6 \setminus e$, and DW_4 .*

175 **Proof.** When a vertex of K_5 is split, the degree sum of the two new vertices could be 8, 9, or 10, and these
 176 correspond to DW_4 , $K_6 \setminus e$, and K_6 , which proves the lemma. \square

177 **Lemma 4.6.** *Every split of K_6 contains a \bar{P}_7 -minor.*

178 **Proof.** To prove this lemma, we may assume that both the two new vertices have degree four. Up to
 179 symmetry, K_6 has only one such split, which contains \bar{P}_7 as a spanning subgraph (this is more clear if we
 180 consider the complements of the two graphs). \square

181 The proof of this lemma is easy since the conclusion is simple. In proving the remainder lemmas we will
 182 see more cases. Typically, when we split a vertex we first consider the case when both the two new vertices
 183 have degree four. Then we view other splits as obtained from these minimal splits by adding edges. The
 184 following is a useful lemma for this approach. Let $G + e$ denote a graph obtained from a graph G by adding
 185 an edge e between two nonadjacent vertices.

186 **Lemma 4.7.** (i) $DW_5 + e$ contains a \bar{P}_7 -minor;

187 (ii) $\Gamma_i + e$ contains a \bar{P}_7 -minor, unless $i = 4$ and $\Gamma_4 + e$ is isomorphic to Γ_3 ;

188 (iii) $K_{4,3}^i + e$, where e is added to the color class of size 3, contains a \bar{P}_7 -minor, unless $i = 4$;

189 (iv) $K_{4,4}^i + e$, where e is added to the color class of size 4, contains a \bar{P}_7 -minor, unless $i = 11$.

190 **Proof.** Part (i). Notice that the complement of $DW_5 + e$ is P_5 together with two isolated vertices, which is
 191 a subgraph of P_7 , so $DW_5 + e$ contains \bar{P}_7 as a subgraph.

192 Part (ii). Notice that the complement of $\Gamma_1 + e$ is P_6 together with one isolated vertex, which is a
 193 subgraph of P_7 , so $\Gamma_1 + e$ contains \bar{P}_7 as a subgraph. For $i = 2, 3, 4, 5$, Γ_i can be obtained from Γ_1 by
 194 splitting the degree-5 vertex, as shown in Figure 4.1, where the two new vertices are 7 and 8. If e is incident
 195 to neither 7 nor 8, then $\Gamma_i + e$ contains a $\Gamma_1 + e$ -minor by contracting 78. So in each Γ_i , we only need to
 196 consider the addition of those missing edges e that are incident with every edge f of Γ_i with Γ_i/f isomorphic
 197 to Γ_1 . We use labels in Figure 4.1. In Γ_2 , since Γ_2/f is isomorphic to Γ_1 for $f \in \{14, 25, 36, 78\}$, $\Gamma_2 + e$
 198 contains a \bar{P}_7 -minor. In Γ_3 , since Γ_3/f is isomorphic to Γ_1 for $f \in \{25, 28, 57, 78\}$, we only consider adding
 199 $e = 58$. The resulted graph contains a \bar{P}_7 -minor because it contains $\Gamma_2 + e$ as a spanning subgraph. In Γ_4 ,
 200 since Γ_4/f is isomorphic to Γ_1 for $f \in \{25, 28, 57, 78\}$, we only consider adding $e \in \{27, 58\}$ and the resulted
 201 graph is isomorphic to Γ_3 . In Γ_5 , since Γ_5/f is isomorphic to Γ_1 for $f \in \{14, 78\}$, we only consider adding
 202 $e \in \{18, 47\}$. Notice that these two additions are isomorphic. Suppose $e = 18$, then by contracting 23, the
 203 resulted graph is isomorphic to \bar{P}_7 .

204 Part (iii). Since $K_{4,3}^4 + e$ is isomorphic to $K_{4,3}^3$, by Lemma 4.3, $K_{4,3}^4 + e$ is \bar{P}_7 -free. For $i = 1, 2, 3, 5$, $K_{4,3}^i$
 205 contains $K_{4,3}^6$ as a subgraph. We only need to consider $K_{4,3}^6 + e$. Notice that the complement of $K_{4,3}^6 + e$
 206 consists of P_4 with one edge and one isolated vertex, which is a subgraph of P_7 , so $K_{4,3}^6 + e$ contains \bar{P}_7 as
 207 a subgraph.

208 Part (iv). Since $K_{4,4}^{11} + e$ is isomorphic to $K_{4,4}^{10}$, by Lemma 4.3, $K_{4,4}^{11} + e$ is \bar{P}_7 -free. For $i = 1, \dots, 9$, $K_{4,4}^i$
 209 contains $K_{4,4}^{10}$ as a subgraph. We only need to consider $K_{4,4}^{10} + e$. Notice that there is an edge f in $K_{4,4}^{10} + e$

210 incident to two degree-4 vertices in $K_{4,4}^{10} + e$. Then $K_{4,4}^{10} + e$ contains a $K_{4,3}^6 + e$ -minor by contracting f . So
 211 $K_{4,4}^{10} + e$ contains a \bar{P}_7 -minor. \square

212 **Lemma 4.8.** *The only \bar{P}_7 -free splits of $K_6 \setminus e$ are $K_{4,3}^1$, $K_{4,3}^2$, and $K_{4,3}^3$.*

213 **Proof.** We first claim that splitting a degree-4 vertex of $K_6 \setminus e$ must result in a \bar{P}_7 -minor. To prove this
 214 we may assume that both the two new vertices have degree four. Up to symmetry, $K_6 \setminus e$ has only one such
 215 split. The complement of the split is P_5 together with two isolated vertices, which is a subgraph of P_7 , so
 216 the split contains \bar{P}_7 as a subgraph.

217 Next we consider splitting a degree-5 vertex of $K_6 \setminus e$. Note that this vertex has two degree-4 and three
 218 degree-5 neighbors. Suppose both the two new vertices, x^1, x^2 , have degree four. Up to symmetry, there are
 219 only three such splits, which we denote by G_1, G_2 , and G_3 . The first two splits, G_1 and G_2 , contain \bar{P}_7 as a
 220 spanning subgraph, as shown in Figure 4.3(a). The third split, G_3 is isomorphic to $K_{4,3}^3$, as shown in Figure
 221 4.3(b), which is \bar{P}_7 -free by Lemma 4.3. Notice that one of the two new vertices, say x^1 , belongs to the set
 222 of four vertices that cover all edges of $K_{4,3}^3$.

223 Now suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by
 224 G , is obtained from G_1, G_2 , or G_3 by adding edges. If G contains G_1 or G_2 then G contains a \bar{P}_7 -minor. So
 225 we assume that G is obtained from G_3 by adding edges. If we only add edges incident with x^1 , then G is
 226 isomorphic to $K_{4,3}^1$ or $K_{4,3}^2$, see Figure 4.3(c), both of which are \bar{P}_7 -free by Lemma 4.3. If we also add edges
 227 incident with x^2 then, by Lemma 4.7(iii), G contains a \bar{P}_7 -minor because G contains $K_{4,3}^i + e$ ($i = 1, 2, 3$) as
 a spanning subgraph. Hence, the only \bar{P}_7 -free splits of $K_6 \setminus e$ are $K_{4,3}^1, K_{4,3}^2$, and $K_{4,3}^3$. \square

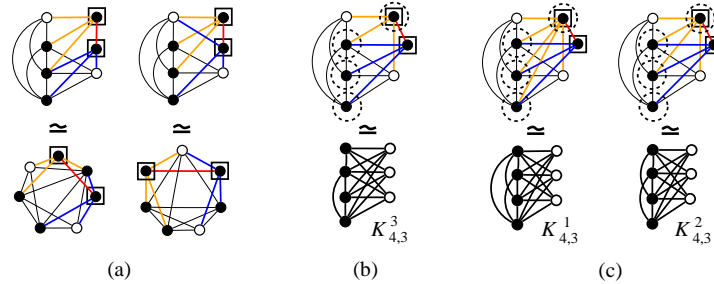


Figure 4.3: (a) Splits of a degree-5 vertex in $K_6 \setminus e$, in which both the two new vertices have degree four and they have a minor \bar{P}_7 ; (b) a split of a degree-5 vertex in $K_6 \setminus e$, which is isomorphic to $K_{4,3}^3$; (c) splits of a degree-5 vertex in $K_6 \setminus e$, which are isomorphic to $K_{4,3}^1$ and $K_{4,3}^2$.

228

229 Proofs for Lemmas 4.9, 4.11, and 4.12 are of the same flavor. We generate all possible splits and we
 230 identify the ones that are \bar{P}_7 -free. Since these lemmas only talk about graphs with fewer than ten vertices,
 231 the conclusions can be verified by a computer. We did reproduce the process (splitting vertices and testing for
 232 minors) using Mathematica and we found that our conclusions do agree with results produced by computer.
 233 So readers who are comfortable with computer-assisted proofs can fast forward to Lemma 4.13 for a summary
 234 and then move on to the next lemma where we will deal with graphs of unbounded size.

235 **Lemma 4.9.** *The only \bar{P}_7 -free splits of DW_4 are $\Gamma_1, DW_5, K_{4,3}^2, K_{4,3}^3, K_{4,3}^4, K_{4,3}^5$, and $K_{4,3}^6$.*

236 **Proof.** We first consider splitting a degree-4 vertex of DW_4 . Note that this vertex has two degree-4 and two

237 degree-5 neighbors in DW_4 . Suppose both the two new vertices have degree four. Up to symmetry, there are
 238 only three such splits, which we denote by G_1, G_2 , and G_3 . The first two splits, G_1 and G_2 , contain \bar{P}_7 as a
 239 spanning subgraph, as shown in Figure 4.4(a). The third split, G_3 is isomorphic to DW_5 , as shown in Figure
 240 4.4(b), which is \bar{P}_7 -free by Lemma 4.2. Notice that both the two new vertices in this split are in the cycle
 241 C_5 of DW_5 . Now suppose at least one of the two new vertices has degree exceeding four. Then this split,
 242 denoted by G , is obtained from G_1, G_2 , or G_3 by adding edges. If G contains G_1 or G_2 then G contains a
 243 \bar{P}_7 -minor. So we assume that G is obtained from G_3 by adding edges. Then G contains a \bar{P}_7 minor because
 244 G contains $DW_5 + e$ in Lemma 4.7(i) as a spanning subgraph.

245 Next we consider splitting a degree-5 vertex. Note that this vertex has four degree-4 and one degree-5
 246 neighbors in DW_4 . Suppose both the two new vertices, x^1, x^2 , have degree four. Up to symmetry, there are
 247 only four such splits, which we denote by G_1, G_2, G_3 , and G_4 . The first split, G_1 , contains \bar{P}_7 as a spanning
 248 subgraph, as shown in Figure 4.4(c). The last three splits, G_2, G_3 , and G_4 , are isomorphic to $\Gamma_1, K_{4,3}^4$, and
 249 $K_{4,3}^6$, respectively, as shown in Figure 4.4(d), which are \bar{P}_7 -free by Lemmas 4.3 and 4.4.

250 Now suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by
 251 G , is obtained from G_1, G_2, G_3 , or G_4 by adding edges. If G contains G_1 then G contains a \bar{P}_7 -minor. If G is
 252 obtained from G_2 by adding edges, then G contains $\Gamma_1 + e$ in Lemma 4.7(ii) because x^1 and x^2 are degree-4
 253 vertices in Γ_1 . So G contains a \bar{P}_7 -minor. We assume that G is obtained from G_3 or G_4 by adding edges.
 254 Notice that one of the two new vertices, say x^1 , belongs to the set of four vertices that cover all edges of $K_{4,3}^i$
 255 ($i = 4, 6$). If we only add edges incident with x^1 , then G is isomorphic to $K_{4,3}^2, K_{4,3}^3$, or $K_{4,3}^5$, as shown in
 256 Figure 4.4(e), which are \bar{P}_7 -free by Lemma 4.3. If we also add edges incident to x^2 , then G contains $K_{4,3}^i + e$
 257 ($i = 2, \dots, 6$) in Lemma 4.7(iii) as a spanning subgraph. So G contains a \bar{P}_7 -minor. Hence, the only \bar{P}_7 -free
 splits of DW_4 are Γ_1 , and $K_{4,3}^i$ for $i = 2, \dots, 6$. \square

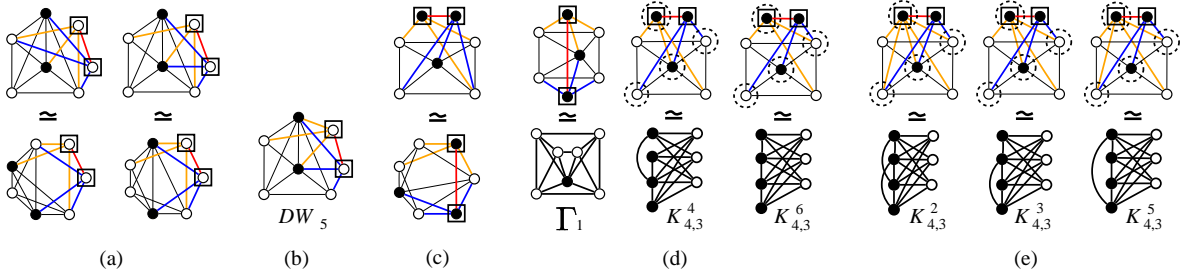


Figure 4.4: (a) Splits of a degree-4 vertex in DW_4 , in which both the two new vertices have degree four and they have a minor \bar{P}_7 ; (b) a split of a degree-4 vertex in DW_4 , which is isomorphic to DW_5 ; (c) a split of a degree-5 vertex in DW_4 , which has a minor \bar{P}_7 ; (d) splits of a degree-5 vertex in DW_4 , which are isomorphic to $\Gamma_1, K_{4,3}^4$, and $K_{4,3}^6$; (e) splits of a degree-5 vertex in DW_4 , which are isomorphic to $K_{4,3}^2, K_{4,3}^3$, and $K_{4,3}^5$.

258

259 **Lemma 4.10.** *Let x be a degree-4 vertex of a graph G , which does not have a 4-cycle with vertex set $N_G(x)$.
 260 Then every split G' of G at x contains a $G+e$ -minor, for some e between two nonadjacent vertices of $N_G(x)$.*

261 **Proof.** We only need to consider the case that both the two new vertices x^1 and x^2 have degree four, because
 262 other splits contain these special splits. Then there are $y_1, y_2 \in N_G(x)$ such that $y_1 \in N_{G'}(x^1) - N_{G'}(x^2)$,
 263 and $y_2 \in N_{G'}(x^2) - N_{G'}(x^1)$. Since x has degree four in G , we let $N_{G'}(x^1) \cap N_{G'}(x^2) = \{u_1, u_2\}$. since

264 $y_1u_1y_2u_2$ is not a 4-cycle of G , we may assume by symmetry that y_1u_1 is not an edge in G . So G' contains
 265 a $G + y_1u_1$ -minor by contracting x^1y_1 . \square

266 **Lemma 4.11.** *The only \bar{P}_7 -free splits of Γ_1 are $\Gamma_2, \Gamma_3, \Gamma_4$, and Γ_5 , and no split of Γ_i ($i = 2, \dots, 5$) is
 267 \bar{P}_7 -free.*

268 **Proof.** We first claim that splitting a degree-4 vertex of Γ_1 must result in a \bar{P}_7 -minor. Let x be a degree-4
 269 vertex of Γ_1 . Since Γ_1 does not have a 4-cycle with vertex set $N_{\Gamma_1}(x)$, by Lemma 4.10, every split G' of Γ_1 at
 270 x contains a $\Gamma_1 + e$ -minor. By Lemma 4.7(ii), G' contains a \bar{P}_7 -minor. Next we consider splitting a degree-5
 271 vertex of Γ_1 . Suppose both the two new vertices have degree four. Up to symmetry, there are only three
 272 such splits, which we denote by G_1, G_2 , and G_3 . These splits are isomorphic to Γ_2, Γ_4 , and Γ_5 , respectively,
 273 as shown in Figure 4.1, where the two new vertices are 7 and 8. By Lemma 4.4, these splits are \bar{P}_7 -free.
 274 Now suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by G ,
 275 is obtained from G_1, G_2 , or G_3 by adding edges. If G is obtained from G_1 or G_3 by adding edges, then G
 276 contains $\Gamma_i + e$ for some $i = 2, 5$ in Lemma 4.7(ii) as a spanning subgraph. So G contains a \bar{P}_7 -minor. We
 277 assume that G is obtained from G_2 by adding edges. Since G_2 is isomorphic to Γ_4 , G contains $\Gamma_4 + e$ as a
 278 spanning subgraph. By Lemma 4.7(ii), either G is isomorphic to Γ_3 or G contains a \bar{P}_7 -minor. So the only
 279 \bar{P}_7 -free splits of Γ_1 are $\Gamma_2, \Gamma_3, \Gamma_4$, and Γ_5 .

280 Next, we consider splits of $\Gamma_i, i = 2, \dots, 5$. For $i = 2, 5$, every vertex x in Γ_i has degree four and Γ_i does
 281 not have a 4-cycle with vertex set $N_{\Gamma_i}(x)$. By Lemma 4.10, every split G' of Γ_i at x contains a $\Gamma_i + e$ -minor.
 282 Then by Lemma 4.7(ii), G' contains a \bar{P}_7 -minor.

283 In Γ_4 , every vertex x in Γ_4 has degree four and Γ_4 does not have a 4-cycle with vertex set $N_{\Gamma_4}(x)$.
 284 By Lemma 4.10, every split G' of Γ_4 at x contains a $\Gamma_4 + e$ -minor. We first claim that G' have a minor
 285 $\Gamma_4 + e$, which is not isomorphic to Γ_3 . To prove this we may assume that both the two new vertices, x^1
 286 and x^2 , have degree four. We use labels in Figure 4.1. By symmetry, G' is obtained from Γ_4 by splitting
 287 either 4 or 5. For splitting 4, $N_{\Gamma_4}(4) = \{1, 5, 6, 8\}$ and we may assume that $1 \in N_{G'}(x^1)$. Notice that
 288 $N_{G'}(x^1) - \{1, x^2\} \subseteq \{5, 6, 8\}$, and in Γ_4 , 1 is not adjacent to any vertices in this set. Since x^1 and x^2
 289 have degree four, there is $y \in N_{G'}(x^1) - N_{G'}(x^2)$. Then G'/x^1y contains $\Gamma_4 + 1u$ for some $u \in \{5, 6, 8\}$
 290 as a spanning subgraph, and $\Gamma_4 + 1u$ is not isomorphic to Γ_3 . For splitting 5, $N_{\Gamma_4}(5) = \{2, 4, 6, 7\}$ and
 291 for each $i = 1, 2$, either $\{2, 7\} \subseteq N_{G'}(x^i)$ or $\{4, 6\} \subseteq N_{G'}(x^i)$. Since x^1 and x^2 have degree four, there is
 292 $y \in N_{G'}(x^1) - N_{G'}(x^2)$. Notice that in Γ_4 , $\{2, 7\}$ is not adjacent to $\{4, 6\}$. Then G'/x^1y contains $\Gamma_4 + uv$
 293 for some $u \in \{2, 7\}$ and $v \in \{4, 6\}$ as a spanning subgraph, and $\Gamma_4 + uv$ is not isomorphic to Γ_3 . By Lemma
 294 4.7(ii), G' contains a \bar{P}_7 -minor.

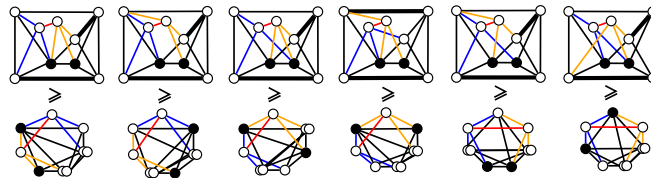


Figure 4.5: Splits of a degree-5 vertex in Γ_3 , in which both the two new vertices have degree four, contain a \bar{P}_7 -minor by contracting the thick edges.

295 In Γ_3 , for every degree-4 vertex x , Γ_3 does not have a 4-cycle with vertex set $N_{\Gamma_3}(x)$. By Lemmas 4.7(ii)
 296 and 4.10, every split of Γ_3 at x contains a \bar{P}_7 -minor. For splitting a degree-5 vertex in Γ_3 , we claim that

297 such splits contain a \bar{P}_7 -minor. To prove this we may assume that both the two new vertices have degree
 298 four. Up to symmetry, Γ_3 has only six such splits, which contain a \bar{P}_7 -minor, as shown in Figure 4.5. \square

299 In Lemma 4.12 and Lemma 4.15, we will consider splits of graphs in \mathcal{K} . We will use the following
 300 terminology in both cases. For any graph K in \mathcal{K} , let X be a set of four vertices that cover all edges of K ,
 301 and let $Y = V(K) - X$. Let G be a split of K , where a vertex x in X is split into x^1, x^2 . Then there are
 302 two possibilities. If x^1 or x^2 is adjacent to no vertex in Y then we call G a *clean* split of K . It is clear that
 303 if G is clean then G belongs to \mathcal{K} . A non-clean split is called a *mixed* split. In other words, G is a mixed
 304 split if both x^1 and x^2 have a neighbor in Y .

305 **Lemma 4.12.** *Let G be a split of $K_{4,3}^i$, $i = 1, \dots, 6$.*

306 (i) *If G is obtained by splitting a vertex in Y then G contains a \bar{P}_7 -minor.*

307 (ii) *If G is mixed then either G contains a \bar{P}_7 -minor or G is isomorphic to Γ_j , for some $j = 3, 4$.*

308 **Proof.** Part (i). Suppose G is obtained by splitting some $y \in Y$ into y^1 and y^2 . We may assume that y^1
 309 and y^2 have degree four. Then G has a minor \bar{P}_7 because G contains a split as shown in Figure 4.6(a) as a
 310 subgraph.

311 Part (ii). Suppose G is mixed. For the case $i = 1, 2, 5, 6$, we show that G contain a \bar{P}_7 -minor. To prove
 312 this we may assume that x^1 and x^2 have degree four. Since a subgraph of $K_{4,3}^i$ induced by X is connected,
 313 x^1 or x^2 must be adjacent to a vertex in $X \setminus x$, say x^1 . Notice that there is a path of $K_{4,3}^i$ with vertex set X .
 314 Then there is an edge in $X \setminus x$. By considering common neighbors of x^1 and x^2 in Y , there are six special
 315 splits of $K_{4,3}^i$ that G contains at least one of them as a subgraph, see Figure 4.6, where in Figure 4.6(a), x^1
 316 and x^2 are x and y^1 , respectively. As shown in Figure 4.6, these splits contains a \bar{P}_7 -minor and so does G .

317 For the case $i = 3$, x has either degree four, five, or six in $K_{4,3}^3$. We first claim that if x has degree four
 318 or five then G contains a \bar{P}_7 -minor. To prove this we may assume that x^1 and x^2 have degree four. If x has
 319 degree four, then $N_G(x) = Y \cup \{x_1\}$ for some $x_1 \in X - \{x\}$. Up to symmetry, there are only two such splits
 320 that G can be isomorphic to, and each split contains a graph in Figure 4.6(b) or (c) as a subgraph. So G
 321 contains a \bar{P}_7 -minor. If x has degree five, then $N_G(x) = Y \cup \{x_1, x_2\}$ for some $x_1, x_2 \in X - \{x\}$, and x_1 is
 322 adjacent to x_2 . Up to symmetry, there are only three such splits that G can be isomorphic to, and each split
 323 contains a graph in Figure 4.6(a), (d), or (e) as a subgraph. So G contains a \bar{P}_7 -minor. We now consider
 324 the case that x has degree six in $K_{4,3}^3$. Suppose x^1 and x^2 have degree four. Then $N_G(x^1) \cap N_G(x^2) = \emptyset$.
 325 Since G is mixed, up to symmetry, there are only two such splits that G can be isomorphic to, which we
 326 denote by G_1 and G_2 . Then G_1 contains a graph in Figure 4.6(f) as a subgraph, and G_2 is isomorphic to
 327 Γ_3 . Note that x^1 and x^2 are corresponding to vertices 3 and 6 of Γ_3 in Figure 4.1, respectively. Suppose
 328 x^1 or x^2 has degree exceeding four. Then G is obtained from G_1 or G_2 by adding edges. If G contains G_1
 329 then G contains a \bar{P}_7 -minor. So we assume that G is obtained from G_2 by adding edges. Then G contains
 330 a \bar{P}_7 -minor because G contains $\Gamma_3 + e$ in Lemma 4.7(ii) as a subgraph.

331 For the case $i = 4$, x has either degree four or six in $K_{4,3}^4$. We first claim that if x has degree four then
 332 G contains a \bar{P}_7 -minor. To prove this we may assume that x^1 and x^2 have degree four. Up to Symmetry,
 333 there are only two such splits that G can be isomorphic to, and each split contains a graph in Figure 4.6(b)
 334 or (c) as a subgraph. So G contains a \bar{P}_7 -minor. We now consider the case that x has degree six in $K_{4,3}^4$.
 335 Suppose x^1 and x^2 have degree four. Then $N_G(x^1) \cap N_G(x^2) = \emptyset$. Since G is mixed, up to symmetry, there
 336 is only one such split and it is isomorphic to Γ_4 . Note that x^1 and x^2 are corresponding to vertices 3 and 6
 337 of Γ_4 in Figure 4.1, respectively. Suppose x^1 or x^2 has degree exceeding four. Then G is obtained from Γ_4

by adding edges. By Lemma 4.7(ii), either G is isomorphic to Γ_3 or G contains a \bar{P}_7 -minor. □

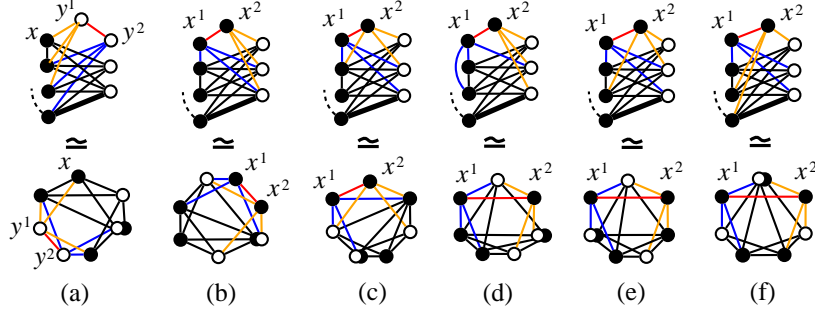


Figure 4.6: A black vertex denotes a vertex in X . (a) A split of a vertex $y \in Y$ of $K_{4,3}^i$, which has a minor \bar{P}_7 by contracting the thick edge; (b-f) splits of a vertex $x \in X$ of $K_{4,3}^i$, which have a minor \bar{P}_7 by contracting the thick edges.

338

339 **Lemma 4.13.** *Every $G \in \text{Ext}(K_5)$ satisfies at least one of the following.*

- 340 (i) $G \in \{K_6, DW_4, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$;
- 341 (ii) $G \in \mathcal{K}$ with $|V(G)| \leq 7$;
- 342 (iii) $G \in \text{Ext}(DW_5)$;
- 343 (iv) $G \in \text{Ext}(K)$ for some $K \in \mathcal{K}$ with $|V(K)| = 8$.

344 **Proof.** The result follows from Lemmas 4.5, 4.6, 4.8, 4.9, 4.11, and 4.12. □

345 In the last two lemmas of this section we consider the two infinite families contained in $\text{Ext}(K_5)$.

346 **Lemma 4.14.** $\text{Ext}(DW_5) = \{DW_n : n \geq 5\}$.

347 **Proof.** Since C_7^2 is a subgraph of \bar{P}_7 , by Lemma 4.2, every double-wheel is \bar{P}_7 -free. Thus it is enough for us
 348 to show that, for each $n \geq 5$, the only \bar{P}_7 -free split of DW_n is DW_{n+1} . Suppose DW_n is constructed from
 349 cycle $v_1 v_2 \dots v_n$ and two adjacent vertices u_1, u_2 . Let G be a split of DW_n .

350 Suppose G is obtained by splitting u_i . Let u_i^1 and u_i^2 be the two new vertices. If $n \geq 6$ then at least one
 351 of u_i^1, u_i^2 , say u_i^1 , has degree exceeding four. Let v_j be a neighbor of u_i^1 . If possible we choose j such that v_j is
 352 not a neighbor of u_i^2 . Then $G' = G/v_j v_{j+1}$ is a split of DW_{n-1} , because $G'/u_i^1 u_i^2 = DW_{n-1}$ and both u_i^1, u_i^2
 353 have degree at least four in G' . By repeating this process we see that G contains a minor that is obtained
 354 from DW_5 by splitter a degree-6 vertex. Note that DW_5 has two possible splits at u_i and both contain a
 \bar{P}_7 -minor (see Figure 4.7). Therefore, G contains a \bar{P}_7 -minor. Now suppose G is obtained by splitting v_i .

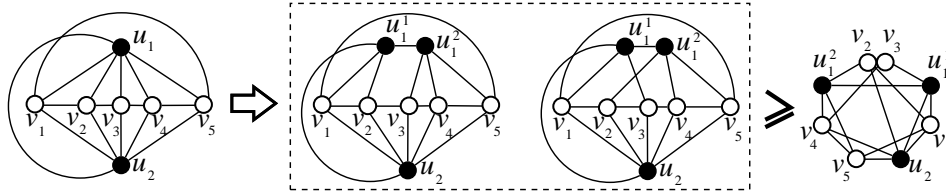


Figure 4.7: Splitting a degree-5 vertex u_1 of DW_5 such that both new vertices have degree five.

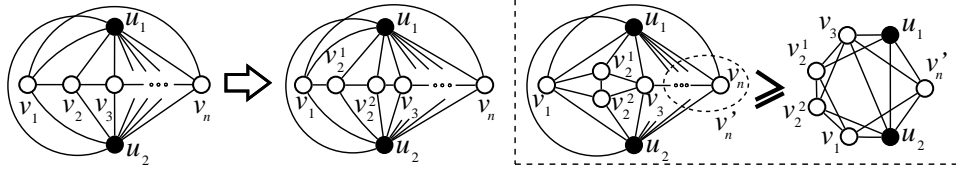


Figure 4.8: Splitting a degree-4 vertex v_2 of DW_n gives either DW_{n+1} or a graph containing a \bar{P}_7 -minor.

355

356 If $G \setminus u_1 u_2$ is nonplanar, by applying Theorem 2.5 to $G \setminus u_1 u_2$ we deduce that G contains a \bar{P}_7 -minor. So we
 357 assume that $G \setminus u_1 u_2$ is planar. Then there are two cases as shown in Figure 4.8 with $i = 2$. So either G is
 358 isomorphic to DW_{n+1} or G contains a \bar{P}_7 -minor. \square

359 **Lemma 4.15.** *If $K \in \mathcal{K}$ has at least eight vertices and G is a \bar{P}_7 -free split of K , then G belongs to \mathcal{K} .*

360 **Proof.** Suppose the lemma is false. Then there exist $K \in \mathcal{K}$ with $|V(K)| \geq 8$ and a \bar{P}_7 -free split G of K with
 361 $G \notin \mathcal{K}$. We choose such a K with $|V(K)|$ as small as possible. Recall that we denote by $X = \{x_1, x_2, x_3, x_4\}$
 362 a set of four vertices that cover all edges of K and we let $Y = V(K) - X = \{y_1, y_2, \dots, y_n\}$. If G is obtained
 363 by splitting some y_j , say, $j = 1$, we consider $G' = G / \{x_1 y_4, x_1 y_5, \dots, x_1 y_n\}$. Since $n \geq 4$, G' satisfies the
 364 assumption in Lemma 4.12(i), which implies that G' contains a \bar{P}_7 -minor. This is a contradiction since G is
 365 \bar{P}_7 -free. This contradiction shows that G is not obtained by splitting any vertex in Y , and thus G is obtained
 366 by splitting a vertex $x_i \in X$. Since G is not in \mathcal{K} , G must be a mixed split. Without loss of generality, let
 367 $i = 1$, let x_1^1 and x_1^2 be the two new vertices, and let y_j ($j = 1, 2$) be a neighbor of x_1^j .

368 We first claim that in each $G/x_i y_j$ ($i = 2, 3, 4$ and $j = 3, 4, \dots, n$), at least one of x_1^1, x_1^2 has degree
 369 smaller than four. Suppose on the contrary that both x_1^1 and x_1^2 have degree at least four in $G/x_i y_j$. Since
 370 $(G/x_i y_j)/x_1^1 x_1^2 = (G/x_1^1 x_1^2)/x_i y_j = K/x_i y_j \in \mathcal{K}$, it follows that $G/x_i y_j$ is a mixed split of $K/x_i y_j$. To
 371 proceed we consider two cases. First, assume $n = 4$. Then $K/x_i y_j = K_{4,3}^k$ for some $k = 1, 2, \dots, 6$. By
 372 Lemma 4.12(ii) and our assumption that G is \bar{P}_7 -free, it follows that $G/x_i y_j$ is isomorphic to either Γ_3 or
 373 Γ_4 . However, by Lemma 4.11, G contains a \bar{P}_7 -minor, a contradiction. Next, assume $n \geq 5$. Observe that
 374 $G/x_i y_j \notin \mathcal{K}$, because if Z is a set of four vertices covering all edges of $G/x_i y_j$, then some vertex in $Y - \{y_j\}$
 375 is not in Z , which implies $\{x_2, x_3, x_4\} \subseteq Z$, and thus a contradiction since we have to cover $x_1^1 y_1$ and $x_1^2 y_2$
 376 with only one vertex. This observation shows that $K/x_i y_j$ is a smaller counterexample, which contradicts
 377 the choice of G and K , and this contradiction completes the proof of our claim.

378 The above claim implies that for each edge $x_i y_j$ ($i = 2, 3, 4$ and $j = 3, 4, \dots, n$), there exists $k \in \{1, 2\}$
 379 such that $x_1^k x_i y_j$ is a triangle and x_1^k is a degree-4 vertex in G . In particular, each x_i ($i = 2, 3, 4$) is adjacent
 380 to at least one of x_1^1, x_1^2 . Consequently, $d_G(x_1^1) + d_G(x_1^2) \geq |Y| + (|X| - 1) + 2 \geq 9$, which implies that
 381 only one of x_1^1, x_1^2 , say, x_1^1 , has degree four. Now we conclude that $x_1^1 x_i y_j$ is a triangle for all $i = 2, 3, 4$ and
 382 $j = 3, 4$, which contradicts $d_G(x_1^1) = 4$ and this contradiction proves the lemma. \square

383 **Proof of Lemma 4.1.** The result follows from Lemmas 4.13, 4.14, and 4.15. \square

384 **Proof of Theorem 1.1.** By Theorem 2.1, Lemmas 3.1, 3.4, and 3.6, we only need to determine $Ext(K_5)$.
 385 Then the result follows from Lemma 4.1. \square

386 **Proof of Corollary 1.2.** Since C_{2n+1} ($n \geq 4$) contains a C_{2n-1}^2 -minor, by Lemmas 3.2, 4.2, 4.3, and
387 4.4, the result follows from Theorem 1.1. \square

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