Characterizing binary matroids with no P_9-minor

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Abstract

In this paper, we give a complete characterization of binary matroids with no P_9-minor. A 3-connected binary matroid M has no P_9-minor if and only if M is one of the internally 4-connected non-regular minors of a special 16-element matroid Y_{16}, a 3-connected regular matroid, a binary spike with rank at least four, or a matroid obtained by 3-summing copies of the Fano matroid to a 3-connected cographic matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 2)$. Here the simple graphs $K'_{3,n}, K''_{3,n}$, and $K'''_{3,n}$ are obtained from $K_{3,n}$ by adding one, two, or three edges in the color class of size three, respectively.

1 Introduction

It is well known that the class of binary matroids consists of all matroids without any $U_{2,4}$-minor, and the class of regular matroids consists of matroids without any $U_{2,4}, F_7$ or F_7^*-minor. Kuratowski's Theorem states that a graph is planar if and only if it has no minor that is isomorphic to $K_{3,3}$ or K_5. These examples show that characterizing a class of graphs and matroids without certain minors is often of fundamental importance. We say that a matroid is *N-free* if it does not contain a minor that is isomorphic to N. A 3-connected matroid M is said to be internally 4-connected if for any 3-separation of M, one side of the separation is either a triangle or a triad.

There is much interest in characterizing binary matroids without small 3-connected minors. Since non-3-connected matroids can be constructed by 3-connected matroids using 1-, 2-sum operations, one needs only determine the 3-connected members of a minor closed class. There is exactly one 3-connected binary matroid with 6-elements, namely, W_3 where W_n denotes both the wheel graph with n-spokes and the cycle matroid of W_n. There are exactly two 7-element binary 3-connected matroids, F_7 and F_7^*. There are

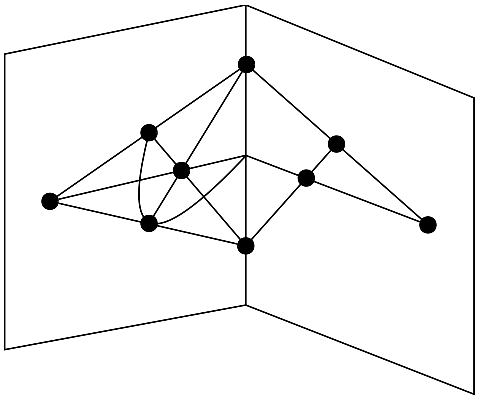

Figure 1: A geometric representation of P_9

three 8-element binary 3-connected matroids, W_4, S_8 and $AG(3,2)$, and there are eight 9-element 3-connected binary matroids: $M(K_{3,3})$, $M^*(K_{3,3})$, Prism, $M(K_5\backslash e)$, P_9, P_9^*, binary spike Z_4 and its dual Z_4^*.

$\lvert E(M)\rvert$	Binary 3-connected matroids
6	W_3
7	F_7, F_7^*
8	$W_4, S_8, AG(3,2)$
9	$M(K_{3,3}), M^*(K_{3,3}), M(K_5\backslash e), Prism, P_9, P_9^*, Z_4, Z_4^*$

For each matroid N in the above list with less than nine elements, with the exception of $AG(3,2)$, the problem of characterizing 3-connected binary matroids with no N-minor has been solved. Since every 3-connected binary matroid having at least four elements has a W_3-minor, the class of 3-connected binary matroids excluding W_3 contains only the trivial 3-connected matroids with at most three elements. Seymour in [11] determined all 3-connected binary matroids with no F_7-minor (F_7^*-minor). Any such matroid is either regular or is isomorphic to F_7^* (F_7). In [8], Oxley characterized all 3-connected binary W_4-free matroids. These are exactly $M(K_4)$, F_7, F_7^*, binary spikes Z_r, Z_r^*, $Z_r\backslash t$, or $Z_r\backslash y_r$ ($r \geq 4$) plus the trivial 3-connected matroids with at most three elements. It is well known that F_7, F_7^*, and $AG(3,2)$ are the only 3-connected binary non-regular matroids without any S_8-minor.

In the book [3], Mayhew, Royle and Whittle characterized all internally 4-connected binary $M(K_{3,3})$-free matroids. Mayhew and Royle [5], and independently Kingan and Lemos [7], determined all internally 4-connected binary Prism-free (therefore $M(K_5\backslash e)$-free) matroids. For each matroid N in the above list with exactly nine elements, the problem of characterizing 3-connected binary matroids with no N-minor is still unsolved yet. The problem of characterizing internally 4-connected binary $AG(3,2)$-free matroids is also open. Since Z_4 has an $AG(3,2)$-minor, characterizing internally 4-connected binary Z_4-free matroids is an even harder problem. Oxley [8] determined all 3-connected binary matroids with no P_9- or P_9^*-minor:

Theorem 1.1. *Let M be a binary matroid. Then M is 3-connected having no minor isomorphic to P_9 or P_9^* if and only if*

(i) M is regular and 3-connected;

(ii) M is a binary spike $Z_r, Z_r^*, Z_r\backslash y_r$ or $Z_r\backslash t$ for some $r \geq 4$; or

(iii) $M \cong F_7$ or F_7^*.

P_9 is a very important matroid and it appears frequently in the structural matroid theory (see, for example, [4, 8, 13]). In this paper, we give a complete characterization of the 3-connected binary matroids with no P_9-minor. Before we state our main result, we describe a class of non-regular matroids. First let $\mathcal{K}$ be the class 3-connected cographic matroids $N = M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 2)$. Here the simple graphs $K'_{3,n}, K''_{3,n}$, and $K'''_{3,n}$ are obtained from $K_{3,n}$ by adding one, two, or three edges in the color class of size three, respectively. Note that when $n = 2$, $N \cong W_4$, or the cycle matroid of the prism graph. From now on, we will use Prism to denote the prism graph as well as its cycle matroid. Take any t disjoint triangles $T_1, T_2, \ldots, T_t$ $(1 \leq t \leq n)$ of N and t copies of F_7. Perform 3-sum operations consecutively starting from N and F_7 along the triangles T_i $(1 \leq i \leq t)$. Any resulting matroid in this infinite class of matroids is called a (multi-legged) starfish. Note that each starfish is not regular since at least one Fano was used (and therefore the resulting matroid has an F_7-minor) in the construction. The class of starfishes and the class of spikes have empty intersection as spikes are W_4-free, while each starfish has a W_4-minor.

Our next result, the main result of this paper, generalizes Oxley's Theorem 1.1 and completely determines the 3-connected P_9-free binary matroids. The matroid Y_{16}, a single-element extension of $PG(3,2)^*$, in standard representation without the identity matrix is given in Figure 2.

Theorem 1.2. *Let M be a binary matroid. Then M is 3-connected having no minor isomorphic to P_9 if and only if one of the following is true:*

(i) M is one of the 16 *internally 4-connected non-regular minors of* Y_{16}*; or*

(ii) M is regular and 3-connected; or

(iii) *M is a binary spike* $Z_r, Z_r^*, Z_r\backslash y_r$ *or* $Z_r\backslash t$ *for some* $r \geq 4$*; or*

(iv) *M is a starfish.*

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure 2: A binary standard representation for Y_{16}

The next result, which follows easily from the last theorem, characterizes all binary P_9-free matroids.

Theorem 1.3. *Let M be a binary matroid. Then M has no minor isomorphic to* P_9 *if and only if M can be constructed from internally 4-connected non-regular minors of* Y_{16}*, 3-connected regular matroids, binary spikes, and starfishes using the operations of direct sum and 2-sum.*

Proof. Since every matroid can be constructed from 3-connected proper minors of itself by the operations of direct sum and 2-sum, by Theorem 1.2, the forward direction is true. Conversely, suppose that $M = M_1 \oplus M_2$, or $M = M_1 \oplus_2 M_2$, where M_1 and M_2 are both P_9-free. As P_9 is 3-connected, by [9, Proposition 8.3.5], M is also P_9-free. Thus if M is constructed from internally 4-connected non-regular minors of Y_{16}, 3-connected regular matroids, binary spikes, and starfishes using the operations of direct sum and 2-sum, then M is also P_9-free. □

Our proof does not use Theorem 1.1 except we use the fact that all spikes are P_9-free which can be proved by an easy induction argument. In Section 2, we determine all internally 4-connected binary P_9-free matroids. These are exactly the 16 internally 4-connected non-regular minors of Y_{16}. These matroids are determined using the Sage matroid package and the computation is confirmed by the matroid software Macek. Most of the work is in Section 3, which is to determine how the internally 4-connected pieces can be put together to avoid a P_9-minor.

For terminology we follow [9]. Let M be a matroid. The *connectivity function* λ_M of M is defined as follows. For $X \subseteq E$ let

$$\lambda_M(X) = r_M(X) + r_M(E - X) - r(M). \quad (1)$$

Let $k \in \mathbb{Z}^+$. Then both X and $E - X$ are said to be *k-separating* if $\lambda_M(X) = \lambda_M(E-X) < k$. If X and $E-X$ are k-separating and $\min\{|X|, |E-X|\} \geq k$, then $(X, E - X)$ is said to be a k-separation of M. Let $\tau(M) = \min\{ j : M \text{ has a } j\text{-separation}\}$ if M has a k-separation for some k; otherwise let $\tau(M) = \infty$. M is *k-connected* if $\tau(M) \geq k$. Let $(X, E-X)$ be a k-separation of M. This separation is said to be a *minimal k-separation* if $\min\{|X|, |E-X|\} = k$. The matroid M is called *internally 4-connected* if and only if M is 3-connected and the only 3-separations of M are minimal (in other words, either X or Y is a triangle or a triad).

2 Characterizing internally 4-connected binary P_9-free matroids

In this section, we determine all internally 4-connected binary P_9-free matroids.

Theorem 2.1. *A binary matroid M is internally 4-connected and P_9-free if and only if*

(i) M is internally 4-connected graphic or cographic; or

(ii) M is one of the 16 internally 4-connected non-regular minors of Y_{16}; or

(iii) M is isomorphic to R_{10}.

Sandra Kingan recently informed us that she also obtained the internally 4-connected binary P_9-free matroids as a consequence of a decomposition result for 3-connected binary P_9-free matroids.

The following two well-known theorems of Seymour [11] will be used in our proof.

Theorem 2.2. *(Seymour's Splitter Theorem) Let N be a 3-connected proper minor of a 3-connected matroid M such that $|E(N)| \geq 4$ and if N is a wheel, it is the largest wheel minor of M; while if N is a whirl, it is the largest whirl minor of M. Then M has a 3-connected minor M' which is isomorphic to a single-element extension or coextension of N.*

Theorem 2.3. *If M is an internally 4-connected regular matroid, then M is graphic, cographic, or is isomorphic to R_{10}.*

The following result is due to Zhou [13, Corollary 1.2].

Theorem 2.4. *A non-regular internally 4-connected binary matroid other than F_7 and F_7^* contains one of the following matroids as a minor: N_{10}, $\widetilde{K_5}$, $\widetilde{K_5}^*$, $T_{12}\backslash e$, and T_{12}/e.*

The matrix representations of these matroids can be found in [13]. We use X_{10} to denote the matroid $\widetilde{K_5}^*$. It is straightforward to verify that among the five matroids in Theorem 2.4, only X_{10} has no P_9-minor. We use $\mathcal{L}$ to denote the set of matroids consisting of the following matroids in reduced standard representation, in addition to F_7, F_7^* and Y_{16}. From the matrix representations of these matroids, it is straightforward to check that each matroid in $\mathcal{L}$ is a minor of Y_{16}, and each has an X_{10}-minor. Indeed, It is clear that (i) each X_i is a single-element co-extension of X_{i-1} for $11 \leq i \leq 15$; (ii) each Y_i is a single-element extension of X_{i-1} for $11 \leq i \leq 16$; (iii) each Y_i is a single-element co-extension of Y_{i-1} for $11 \leq i \leq 16$, and it is easy to check that (iv) in the list $X_{10}, X'_{11}, X'_{12}, X_{13}$, each matroid is a single-element coextension of its immediate predecessor. Therefore, X_{10} is a minor of all matroids in $\mathcal{L}$, and each is a minor of Y_{16}. From these matrices, it is also routine to check that the only matroid of $\mathcal{L}$ having a triangle is F_7 (this can also be easily verified by using the Sage matroid package).

$$X_{10}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 1&0&0&1 \end{pmatrix} X_{11}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&0\\ 1&0&0&1 \end{pmatrix} X'_{11}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&1\\ 1&0&0&1 \end{pmatrix} Y_{11}: \begin{pmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&0&1&1&0\\ 0&0&1&1&1\\ 0&1&0&1&1\\ 1&0&0&1&1 \end{pmatrix}$$

$$X_{12}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&0\\ 0&1&1&1\\ 1&0&0&1 \end{pmatrix} X'_{12}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&1\\ 1&0&0&1\\ 1&1&1&1 \end{pmatrix} Y_{12}: \begin{pmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&0&1&1&0\\ 0&0&1&1&1\\ 0&1&0&1&1\\ 0&1&1&0&1\\ 1&0&0&1&1 \end{pmatrix}$$

$$X_{13}:\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&0&1&1\\0&1&0&1\\0&1&1&0\\0&1&1&1\\1&0&0&1\\1&0&1&0\end{pmatrix}\quad Y_{13}:\begin{pmatrix}1&1&1&0&0\\1&1&0&1&0\\1&0&1&1&0\\0&0&1&1&1\\0&1&0&1&1\\0&1&1&0&1\\0&1&1&1&0\\1&0&0&1&1\end{pmatrix}\quad X_{14}:\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&0&1&1\\0&1&0&1\\0&1&1&0\\0&1&1&1\\1&0&0&1\\1&0&1&0\\1&1&0&0\end{pmatrix}\quad Y_{14}:\begin{pmatrix}1&1&1&0&0\\1&1&0&1&0\\1&0&1&1&0\\0&0&1&1&1\\0&1&0&1&1\\0&1&1&0&1\\0&1&1&1&0\\1&0&0&1&1\\1&0&1&0&1\end{pmatrix}$$

$$X_{15}\cong PG(3,2)^*:\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&0&1&1\\0&1&0&1\\0&1&1&0\\0&1&1&1\\1&0&0&1\\1&0&1&0\\1&1&0&0\\1&1&1&1\end{pmatrix}\qquad Y_{15}:\begin{pmatrix}1&1&1&0&0\\1&1&0&1&0\\1&0&1&1&0\\0&0&1&1&1\\0&1&0&1&1\\0&1&1&0&1\\0&1&1&1&0\\1&0&0&1&1\\1&0&1&0&1\\1&1&0&0&1\end{pmatrix}\tag{2}$$

Proof of Theorem 2.1: If M is one of the matroids listed in (i) to (iii), then M is internally 4-connected. All matroids in (i) or (iii) are regular, thus are P_9-free. Using the Sage matroid package, it is easy to verify that Y_{16} is P_9-free, hence all matroids in (ii) are also P_9-free. Let M be an internally 4-connected binary matroid with no P_9-minor. If M is regular, then by Theorem 2.3, M is either graphic, cographic, or isomorphic to R_{10}, which is regular. Therefore, we need only show that an internally 4-connected matroid M is non-regular and P_9-free if and only if M is a non-regular minor of Y_{16}. Suppose that M is an internally 4-connected non-regular and P_9-free matroid. If M has exactly seven elements, then $M \cong F_7$ or $M \cong F_7^*$. Suppose that M has at least eight elements. By Theorem 2.4, M has an N_{10}, X_{10}, X_{10}^*, $T_{12}\backslash e$, or T_{12}/e-minor. Since all but X_{10} has a P_9-minor among these matroids, M must have an X_{10}-minor. We use the Sage matroid package (by writing simple Python scripts) and the matroid software Macek independently to do our computation and have obtained the same result. Excluding P_9, we extend and coextend X_{10} seven times and found only thirteen 3-connected binary matroids. These matroids are X_{11}, $X'_{11}, Y_{11}, X_{12}, X'_{12}, Y_{12}, X_{13}, Y_{13}, X_{14}, Y_{14}, X_{15} \cong PG(3,2)^*, Y_{15},$

and Y_{16}; each having at most 16 elements; each being a minor of Y_{16}; and each being internally 4-connected. As X_{10} is neither a wheel nor a whirl, by the Splitter Theorem (Theorem 2.2), M is one of the matroids in $\mathcal{L}$, each of which is a non-regular internally 4-connected minor of Y_{16}. Note that all non-regular internally 4-connected minors of Y_{16} are P_9-free, hence $\mathcal{L}$ consists of all internally 4-connected non-regular minors of Y_{16}. □

3 Characterizing 3-connected binary P_9-free matroids

In this section, we will prove our main result. We begin with several lemmas. Let G be a graph with a specified triangle $T = \{e_1, e_2, e_3\}$. By a *rooted K_4''-minor* using T we mean a loopless minor H of G such that $si(H) \cong K_4$; $\{e_1, e_2, e_3\}$ remains a triangle of H; and $H\backslash\{e_i, e_j\}$ is isomorphic to K_4, for some distinct $i, j \in \{1, 2, 3\}$. By a *rooted K_4'-minor* using T we mean a loopless minor H of G such that $si(H) \cong K_4$; $\{e_1, e_2, e_3\}$ remains a triangle of H; and $H\backslash e_i$ is isomorphic to K_4, for some $i \in \{1, 2, 3\}$. Let T be a specified triangle of a matroid M. We can define a rooted $M(K_4')$-minor using T and a rooted $M(K_4'')$-minor using T similarly. Moreover, in the following proof, any K_4' is obtained from K_4 by adding a parallel edge to an element in the common triangle T used in the 3-sum specified in the context.

Lemma 3.1. *([12]) Let T be a triangle of 3-connected binary matroid M with at least four elements. Then T is contained in a $M(K_4)$-minor of M.*

Lemma 3.2. *([1]) Let T be a triangle of a binary non-graphic matroid M. Then the following are true:*

(i) If M is non-regular, then T is contained in a F_7-minor;

(ii) If M is regular but not graphic, then T is contained in a $M^*(K_{3,3})$-minor.

Let M_1 and M_2 be matroids with ground sets E_1 and E_2 such that $E_1 \cap E_2 = T$ and $M_1|T = M_2|T = N$. The following result of Brylawski [2] about the generalized parallel connection can be found in [9, Propsition 11.4.14].

Lemma 3.3. *The generalized parallel connection $P_N(M_1, M_2)$ has the following properties:*

(i) $P_N(M_1, M_2)|E_1 = M_1$ and $P_N(M_1, M_2)|E_2 = M_2$.

(ii) If $e \in E_1 - T$, then $P_N(M_1, M_2)\backslash e = P_N(M_1\backslash e, M_2)$.

(iii) If $e \in E_1 - cl_1(T)$, then $P_N(M_1, M_2)/e = P_N(M_1/e, M_2)$.

(iv) If $e \in E_2 - T$, then $P_N(M_1, M_2)\backslash e = P_N(M_1, M_2\backslash e)$.

(v) *If* $e \in E_2 - cl_2(T)$, *then* $P_N(M_1, M_2)/e = P_N(M_1, M_2/e)$.

(vi) *If* $e \in T$, *then* $P_N(M_1, M_2)/e = P_{N/e}(M_1/e, M_2/e)$.

(vii) $P_N(M_1, M_2)/T = (M_1/T) \oplus (M_2/T)$.

In the rest of this paper, we consider the case when the generalized parallel connection is defined across a triangle T, where T is the common triangle of the binary matroids M_1 and M_2. Then $P_N(M_1, M_2) = P_N(M_2, M_1)$ (see [9, Propsition 11.4.14]). Moreover, $N = M_1|T = M_2|T \cong U_{2,3}$. We will use T to denote both the triangle and the submatroid $M_1|T$. Thus we use $P_T(M_1, M_2)$ instead of $P_N(M_1, M_2)$ for the rest of the paper.

Lemma 3.4. *Let* $M = P_T(M_1, P_S(M_2, M_3))$ *where* M_i *is a binary matroid* $(1 \le i \le 3)$*;* S *is the common triangle of* M_2 *and* M_3*;* T *is the common triangle of* M_1 *and* M_2*. Then the following are true:*

(i) *if* $E(M_1) \cap (E(M_3) \backslash E(M_2)) = \emptyset$, *then* $M = P_S(P_T(M_1, M_2), M_3)$*;*

(ii) *if* $E(M_1) \cap E(M_3) = \emptyset$, *then* $M_1 \oplus_3 (M_2 \oplus_3 M_3) = (M_1 \oplus_3 M_2) \oplus_3 M_3$.

Proof. (i) As $E(M_1) \cap (E(M_3) \backslash E(M_2)) = \emptyset$, $T = E(M_1) \cap E(P_S(M_2, M_3))$, and T is the common triangle of M_1 and $P_S(M_2, M_3)$. Moreover, $S = E(M_3) \cap E(P_T(M_1, M_2))$, and S is the common triangle of M_3 and $P_T(M_1, M_2)$. By [9, Proposition 11.4.13], a set F of M is a flat if and only if $F \cap E(M_1)$ is a flat of M_1 and $F \cap E(P_S(M_2, M_3))$ is a flat of $P_S(M_2, M_3)$. The latter is true if and only if $[F \cap (E(M_2) \cup E(M_3))] \cap E(M_i) = F \cap E(M_i)$ is a flat of M_i for $i = 2, 3$. Therefore, F is a flat of M if and only if $F \cap E(M_i)$ is a flat of M_i for $1 \le i \le 3$. The same holds for $P_S(P_T(M_1, M_2), M_3)$. Thus $M = P_S(P_T(M_1, M_2), M_3)$.

(ii) As $E(M_1) \cap E(M_3) = \emptyset$, we deduce that $S \cap T = \emptyset$, and the conclusion of (i) holds. Therefore,

$$P_T(M_1, P_S(M_2, M_3)) \backslash (S \cup T) = P_S(P_T(M_1, M_2), M_3) \backslash (S \cup T).$$

By Lemma 3.3, we conclude that

$$P_T(M_1, P_S(M_2, M_3) \backslash S) \backslash T = P_S(P_T(M_1, M_2) \backslash T, M_3) \backslash S.$$

That is, $M_1 \oplus_3 (M_2 \oplus_3 M_3) = (M_1 \oplus_3 M_2) \oplus_3 M_3$. □

Lemma 3.5. *Let* $M = P_T(M_1, M_2)$ *where* M_i *is a binary matroid* $(1 \le i \le 2)$ *and* T *is the common triangle of* M_1 *and* M_2*. Then* C^* *is a cocircuit of* M *if and only if one of the following is true:*

(i) C^* *is a cocircuit of* M_1 *or* M_2 *avoiding* T*;*

(ii) $C^* = C_1^* \cup C_2^*$ *where* C_i^* *is a cocircuit of* M_i *such that* $C_1^* \cap T = C_2^* \cap T$, *which has exactly two elements.*

Proof. By [9, Proposition 11.4.13], a set F of M is a flat if and only if $F \cap E(M_i)$ is a flat of M_i for $1 \leq i \leq 2$. Moreover, for any flat F of M, $r(F) = r(F \cap E(M_1)) + r(F \cap E(M_2)) - r(F \cap T)$ (see, for example, [9, (11.23)]). Let C^* be a cocircuit of M and $H = E(M) - C^*$. As M is binary, $|C^* \cap T| = 0, 2$, and thus $|H \cap T| = 3, 1$. First assume that $|C^* \cap T| = 0$. As $r(H) = r(H \cap E(M_1)) + r(H \cap E(M_2)) - r(H \cap T)$, then $r(M) - 1 = r(M_1) + r(M_2) - 3 = r(H) = r(H \cap E(M_1)) + r(H \cap E(M_2)) - 2$. Thus,

$$r(M_1) + r(M_2) - 1 = r(H \cap E(M_1)) + r(H \cap E(M_2)).$$

Therefore, either $r(H \cap E(M_1)) = r(M_1) - 1$ and $r(H \cap E(M_2)) = r(M_2)$, or $r(H \cap E(M_2)) = r(M_2) - 1$ and $r(H \cap E(M_1)) = r(M_1)$. In the former case, as $H \cap E(M_1)$ and $H \cap E(M_2)$ are flats of M_1 and M_2 respectively, we deduce that $H \cap E(M_2) = E(M_2)$; $H \cap E(M_1)$ is a hyperplane of M_1 and thus $C^* \subseteq E(M_1)$ is a cocircuit of M_1 avoiding T. The latter case is similar.

If $|C^* \cap T| = 2$, then $|H \cap T| = 1$. As $r(H) = r(H \cap E(M_1)) + r(H \cap E(M_2)) - r(H \cap T)$, we deduce that $r(M) - 1 = r(M_1) + r(M_2) - 3 = r(H) = r(H \cap E(M_1)) + r(H \cap E(M_2)) - 1$. We conclude that

$$r(M_1) + r(M_2) - 2 = r(H \cap E(M_1)) + r(H \cap E(M_2)).$$

Now, for $1 \leq i \leq 2$, $H \cap E(M_i)$ is a proper flat of M_i, so that $r(H \cap E(M_i)) \leq r(M_i) - 1$. Therefore, $r(H \cap E(M_1)) = r(M_1) - 1$ and $r(H \cap E(M_2)) = r(M_2) - 1$. We conclude that $C_i^* = E(M_i) - H$ is a cocircuit of M_i and $C^* = C_1^* \cup C_2^*$ such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements. Note that the converse of the above arguments is also true, thus the proof of the lemma is complete. □

The following corollary might be of independent interest.

Corollary 3.6. *Let M_1 and M_2 be a binary matroids and $M = M_1 \oplus_3 M_2$ such that M_1 and M_2 have the common triangle T. Then the following are true:*

(i) any cocircuit C^* of M is either a cocircuit of M_1 or M_2 avoiding T, or $C^* = C_1^* \Delta C_2^*$ where C_i^* is a cocircuit of M_i $(i = 1, 2)$ such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements.

(ii) if C^* is either a cocircuit of M_1 or M_2 avoiding T, then C^* is also a cocircuit of M. Moreover, suppose that C_i^* is a cocircuit of M_i such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements. Then either $C_1^* \Delta C_2^*$ is a cocircuit of M, or $C_1^* \Delta C_2^*$ is a disjoint union of two cocircuits R^* and Q^* of M, where R^* and Q^* meet both M_1 and M_2.

Proof. As $M = M_1 \oplus_3 M_2 = P_T(M_1, M_2)\backslash T$, the cocircuits of M are the minimal non-empty members of the set $\mathcal{F} = \{D - T\colon\ D$ is a cocircuit of $P_T(M_1, M_2)\}$. If C^* is a cocircuit of M, then $C^* = D - T$ for some cocircuit D of $P_T(M_1, M_2)$. By the last lemma, either (a) D is a cocircuit of M_1 or M_2 avoiding T, or (b) $D = C_1^* \cup C_2^*$ where C_i^* is a cocircuit of M_i $(i = 1, 2)$ such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements. In (a), $C^* = D$, and in (b), $C^* = C_1^* \Delta C_2^*$. Hence either (i) or (ii) holds in the lemma.

Conversely, if C^* is either a cocircuit of M_1 or M_2 avoiding T, then clearly C^* is also a cocircuit of M, as $C^* = C^* - T$ is clearly a non-empty minimal member of the set $\mathcal{F}$. Now suppose that C_i^* $(i = 1, 2)$ is a cocircuit of M_i such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements. If $C_1^* \Delta C_2^*$ is not a cocircuit of M, then it contains a cocircuit R^* of M which is a proper subset of $C_1^* \Delta C_2^*$. Clearly, R^* must meet both C_1^* and C_2^*. By (i), $R^* = R_1^* \Delta R_2^*$, where R_i^* is a cocircuit of M_i $(i = 1, 2)$ such that $R_1^* \cap T = R_2^* \cap T$, which has exactly two elements. Suppose that $C_1^* \cap T = C_2^* \cap T = \{x, y\}$, then $R_1^* \cap T = R_2^* \cap T = \{x, z\}$ or $\{y, z\}$, say the former. Moreover, $R_i^* \backslash T$ is a proper subset of $C_i^* \backslash T$ for $i = 1, 2$ as T does not contain any cocircuit of either M_1 or M_2. As both M_1 and M_2 are binary, $Q_i^* = C_i^* \Delta R_i^*$ $(i = 1, 2)$ contains, and indeed, is a cocircuit of M_i such that $Q_1^* \cap T = Q_2^* \cap T = \{y, z\}$. Now it is straightforward to see that $Q_1^* \Delta Q_2^*$ is a minimal non-empty member of $\mathcal{F}$ and thus is a cocircuit of M. As $C^* = R^* \cup Q^*$, (ii) holds. □

The 3-sum of two cographic matroids may not be cographic. However, the following is true.

Lemma 3.7. *Suppose that $M_1 = M^*(G_1)$ and $M_2 = M^*(G_2)$ are both cographic matroids with u and v being vertices of degree three in G_1 and G_2, respectively. Label both uu_i and vv_i as e_i $(1 \le i \le 3)$ so that $T = E(M_1) \cap E(M_2) = \{e_1, e_2, e_3\}$ is the common triangle of M_1 and M_2. Then $P_T(M_1, M_2) = M^*(G)$, where G is obtained by adding a matching $\{u_1v_1, u_2v_2, u_3v_3\}$ between $G_1 - u$ and $G_2 - v$. In particular, $M^*(G_1) \oplus_3 M^*(G_2) = M^*(G/e, f, g)$ is also cographic.*

Proof. We need only show that $P_T(M_1, M_2)$ and $M^*(G)$ have the same set of cocircuits. By Lemma 3.5, C^* is a cocircuit of $M = P_T(M_1, M_2)$ if and only if one of the following is true:

(i) C^* is a cocircuit of M_1 or M_2 avoiding T. In other words, C^* is either a circuit of G_1 or a circuit of G_2 which does not meet T (i.e., C^* is a circuit of either $G_1 - u$ or a circuit of $G_2 - v$);

(ii) $C^* = C_1^* \cup C_2^*$ where C_i^* is a cocircuit of M_i such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements. In other words, $C^* = C_1^* \cup C_2^*$ where C_i^* $(i = 1, 2)$ is a circuit of G_i containing u and v respectively, such that

$C_1^* \cap T = C_2^* \cap T$, which contains exactly two edges. Now it is easily seen that the set of cocircuits of M is exactly equal to the set of circuits of $M(G)$ (or the set of cocircuits of $M^*(G)$). In particular, $M^*(G_1) \oplus_3 M^*(G_2) = P_T(M^*(G_1), M^*(G_2)) \backslash T = M^*(G) \backslash T = M^*(G/e, f, g)$ is cographic. This completes the proof of the lemma. □

The following consequence of the last lemma will be used frequently in the paper.

Corollary 3.8. *Suppose that $M^*(K_{3,m}), M^*(K'_{3,m}), M^*(K_{3,n}) \in \mathcal{K}$ $(m, n \geq 2)$. Then the following are true:*

(i) $M^*(K_{3,m}) \oplus_3 M^*(K_{3,n})) \cong M^*(K_{3,m+n-2})$;

(ii) $M^*(K'_{3,m}) \oplus_3 M^*(K_{3,n}) \cong M^*(K'_{3,m+n-2})$;

(iii) $P(M^*(K_{3,m}), M(K_4))$ is cographic and is isomorphic to $M^*(G)$ where G is obtained by putting a 3-edge matching between the 3-partite set of $K_{3,m-1}$ and the three vertices of K_3.

(iv) $M^*(K_{3,m}) \oplus_3 M(K'_4)) \cong M^*(K'_{3,m})$ where K'_4 is obtained from K_4 by adding a parallel edge to an element in the common triangle T used in the 3-sum.

(v) if $M_1 \cong M^*(K'_{3,m})$, and $M_2 \cong M(K'_4)$, then depending on which element in T is in a parallel pair in $M(K'_4)$ and which extra edge was added to $K'_{3,m}$ from $K_{3,m}$, the matroid $M_1 \oplus_3 M_2$ is either isomorphic to $M^*(K''_{3,m})$ or $M^*(G)$, where G is obtained from $K'_{3,m}$ by adding an edge parallel to the extra edge.

(vi) if $M_1 \in \mathcal{K}$ and $M_2 \cong M(K'_4)$, then either $M_1 \oplus_3 M_2 \in \mathcal{K}$ or $M_1 \oplus_3 M_2 \cong M^*(G)$, where G has a parallel pair which does not meet any triad of G.

(vii) if $M_1 \in \mathcal{K}$ and $M_2 \in \mathcal{K}$, then either $M_1 \oplus_3 M_2 \in \mathcal{K}$ or $M_1 \oplus_3 M_2 \cong M^*(G)$, where G has at least one parallel pair which does not meet any triad of G.

Proof. (i)-(v) are direct consequences of Lemma 3.7. Suppose that $M_1 \in \mathcal{K}$ and is isomorphic to $M^*(K_{3,m})$, $M^*(K'_{3,m})$, $M^*(K''_{3,m})$, or $M^*(K'''_{3,m})$. Then either $M_1 \oplus_3 M_2 \cong M^*(K'_{3,m}), M^*(K''_{3,m})$ or $M^*(K'''_{3,m})$ and thus is in $\mathcal{K}$ (in this case, M_1 is not isomorphic to $M^*(K'''_{3,m})$), or isomorphic to $M^*(G)$, where G is obtained from $K'_{3,m}, K''_{3,m}$, or $K'''_{3,m}$ by adding an edge in parallel to an existing edge added between two vertices of the 3-partite set of $K_{3,m}$. Clearly, this parallel pair does not meet any triad of G. We omit the straightforward and similar proof of (vii). □

Corollary 3.9. *Let M be a binary matroid and $M = M_1 \oplus_3 M_2$ where M_1 is a starfish. Suppose that M_2 is a starfish, or $M_2 \cong M(K_4')$, or $M_2 \cong M^*(G) \in \mathcal{K}$: $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ $(n \geq 2)$. Then either M is also a starfish, or M has a 2-element cocircuit which does not meet any triangle of M.*

Proof. Suppose that the starfish M_1 uses s Fano matroids and M_2 uses t Fano matroids where $s \geq 1$ and $t \geq 0$. Clearly, in the starfish M_1, any triangle is a triad in the corresponding 3-connected graph $G_1 \cong K_{3,m}$, $K'_{3,m}$, $K''_{3,m}$, or $K'''_{3,m}$ $(m \geq 2)$ used to construct M_1. We assume that first $s = 1$ and $t = 0$. Then by the definition of the starfish, $M_1 \cong F_7 \oplus_3 N_1$, where $N_1 \cong M^*(G_1)$, and either $M_2 \cong M(K_4')$, or $M_2 \cong M^*(G)$; G is 3-connected where $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ $(n \geq 2)$. By Lemma 3.4, we have that $M = (F_7 \oplus_3 N_1) \oplus_3 M_2 \cong F_7 \oplus_3 (N_1 \oplus_3 M_2)$ (the condition of the lemma is clearly satisfied). By Corollary 3.8, we deduce that either $N_1 \oplus_3 M_2 \in \mathcal{K}$, or it has a 2-element cocircuit avoiding any triangle of $N_1 \oplus_3 M_2$. In the former case, we conclude that M is a starfish. In the latter case, by Corollary 3.6, M has a 2-element cocircuit avoiding any triangle of M. The general case follows from an easy induction argument using Lemmas 3.4 and Corollaries 3.6 and 3.8. □

Lemma 3.10. *Suppose that $M \cong M^*(G)$ for a 3-connected graph $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ $(n \geq 2)$, or M is a starfish. Then for any triangle T of M, there are at least two elements e_1, e_2 of T, such that for each e_i $(i = 1, 2)$, there is a rooted K_4'-minor using both T and e_i such that e_i is in a parallel pair.*

Proof. Suppose that $M \cong M^*(G)$ for a 3-connected graph $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ $(n \geq 2)$. When $n \geq 3$, the proof is straightforward. When $n = 2$, then $G \cong W_4$ or $K_5 \backslash e$, and the result is also true.

Now suppose that M is a starfish constructed by starting from $N \cong M^*(G)$ for a 3-connected graph $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ $(n \geq 2)$ with t $(1 \leq t \leq n)$ copies of F_7 by performing 3-sum operations. Choose an element f_i of $E(M)$ in each copy of F_7 $(1 \leq i \leq t)$. By the definition of a starfish, and by using Lemma 3.3(iii),(v), $M/f_1, f_2, \ldots f_t$ is isomorphic to N containing T. Now the result follows from the above paragraph. □

We will need the following result [11, 11.1].

Lemma 3.11. *Let e be an edge of a simple 3-connected graph G on more than four vertices. Then either $G \backslash e$ is obtained from a simple 3-connected graph by subdividing edges or G/e is obtained from a simple 3-connected graph by adding parallel edges.*

Let $G = (V, E)$ be a graph and let x, y be distinct elements of $V \cup E$. By *adding an edge between* x, y we obtain a graph G' defined as follows. If x and y are both in V, we assume $xy \notin E$ and we define $G' = (V, E \cup \{xy\})$; if x is in V and $y = y_1y_2$ is in E, we assume $x \notin \{y_1, y_2\}$ and we define $G' = (V \cup \{z\}, (E \backslash \{y\}) \cup \{xz, y_1z, y_2z\})$; if $x = x_1x_2$ and $y = y_1y_2$ are both in E, we define $G' = (V \cup \{u, v\}, (E \backslash \{x, y\}) \cup \{ux_1, ux_2, uv, vy_1, vy_2\})$

Lemma 3.12. *Let G be a simple 3-connected graph with a specified triangle T. Then G has a rooted K_4''-minor unless G is K_4, W_4, or Prism.*

Proof. Suppose the Lemma is false. We choose a counterexample $G = (V, E)$ with $|E|$ as small as possible. Let x, y, z be the vertices of T. We first prove that $G - \{x, y, z\}$ has at least one edge.

Suppose $G - \{x, y, z\}$ is edgeless. Since G is 3-connected, every vertex in $V - \{x, y, z\}$ must be adjacent to all three of x, y, z, which means that $G = K_{3,n}'''$ for a positive integer n. Since G is a counterexample, G cannot be K_4 and thus G contains $K_{3,2}'''$, which contains a rooted K_4''-minor. This contradicts the choice of G and thus $G - \{x, y, z\}$ has at least one edge.

Let $e = uv$ be an edge of $G - \{x, y, z\}$. By Lemma 3.12, there exists a simple 3-connected graph H such that at least one of the following holds:

Case 1. $G \backslash e$ is obtained from H by subdividing edges;
Case 2. G/e is obtained from H by adding parallel edges.

Since H is a proper minor of G and H still contains T, by the minimality of G, H has to be K_4, W_4, or Prism, because otherwise H (and G as well) would have a rooted K_4''-minor. Now we need to deduce a contradiction in Case 1 and Case 2 for each $H \in \{K_4, W_4, \mathit{Prism}\}$.

Let P^+ be obtained from Prism by adding an edge between two nonadjacent vertices. Before we start checking the cases we make a simple observation: with respect to any of its triangles, P^+ has a rooted K_4''-minor. As a result, G cannot contain a *rooted P^+-minor*: a P^+-minor in which T remains a triangle.

We first consider Case 1. Note that G is obtained from H by adding an edge between some $\alpha, \beta \in V \cup E$. By the choice of e, we must have $\alpha, \beta \notin V(T) \cup E(T)$. If $H = K_4$ then $G = Prism$, which contradicts the choice of G. If $G = W_4$ or $Prism$, then it is straightforward to verify that G contains a rooted P^+-minor (by contracting at most two edges), which is a contradiction by the above observation.

Next, we consider Case 2. Let w be the new vertex created by contracting e. Then G/e is obtained from H by adding parallel edges incident with w. Observe that w has degree three in H, for each choice of H. Consequently, as G is simple, G has four, three, or two more edges than H. Suppose G has four or three more edges than H. Then H is $G-u$ or $G-v$. Without loss of generality, let $H=G-u$. Choose three paths P_x, P_y, P_z in H from v to x, y, z, respectively, such that they are disjoint except for v. Now it is not difficult to see that a rooted K_4''-minor of G can be produced from the union of the triangle T, the three paths P_x, P_y, P_z, and the star formed by edges incident with u. This contradiction implies that G has exactly two more edges than H. Equivalently, G is obtained from H by adding an edge between a neighbor s of w and an edge wt with $t \neq s$.

If $H = K_4$ then $G = W_4$, which contradicts the choice of G. If $H = W_4$ then $G = W_5$ or P^+. In both cases, G contain a rooted K_4''-minor, no matter where the special triangle is. Finally, if $H = Prism$ then G contains a rooted P^+-minor, which is impossible by our early observation. In conclusion, Case 2 does not occur, which completes our proof. □

Lemma 3.13. *Let $M = M^*(G)$ be a 3-connected cographic matroid with a specified triangle T. Then M has a rooted K_4''-minor using T unless $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 1$. In particular, if $M^*(G)$ is not graphic, then $n \geq 3$.*

Proof. Suppose that M does not contain rooted K_4''-minor using T. Note that $M^*(G)$ does not have a rooted K_4''-minor using T if and only if G does not have a minor obtained from K_4 (where T is cocircuit) by subdividing two edges of T. Now we show that T is a vertex triad (which corresponds to a star of degree three). Otherwise, let $G-E(T) = X \cup Y$, where T is a 3-element edge-cut but not a vertex triad. If $G \cong Prism$, then clearly $M^*(G)$ has a rooted K_4''-minor; a contradiction. If G is not isomorphic to a Prism, we can choose a cycle in one side and a vertex in another side which is not incident with any edge of T. Then we can get a rooted K_4''-minor; a contradiction again. Hence the edges of T are all incident to a common vertex v of degree three with neighbors v_1, v_2, and v_3. A rooted K_4''-minor using T exists if and only if G has a cycle missing v and at least two of v_1, v_2, and v_3. Hence every cycle of $G-v$ contains at least two of v_1, v_2, and v_3, and thus $G-v-v_i-v_j$ is a tree for $1 \leq i \neq j \leq 3$. Moreover, $G-v-v_1-v_2-v_3$ has to be an independent set. Otherwise, it is a forest. Take two pedants in a tree, each of which has at least two neighbors in v_1, v_2, or v_3. Thus $G-v$ contains a cycle missing at least two vertices of v_1, v_2, and v_3. This contradiction shows that $G-v-v_1-v_2-v_3$ is an independent set and thus G is $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 1$. In particular, if $M^*(G)$ is not graphic, then $n \geq 3$. □

Lemma 3.14. *Let M be a 3-connected binary P_9-free matroid and $M = M_1 \oplus_3 M_2$ where M_1 is non-regular, and M_1 and M_2 have the common triangle T. Then* *(i) if M_2 is graphic, then either $M_2 \cong M(G)$ where G is W_4 or the Prism, or $M_2 \cong M(K_4^{'})$ where $M(K_4^{'})$ is obtained from $M(K_4)$ (which contains T) by adding an element parallel to an element of T ;*

(ii) if M_2 is cographic but not graphic, then $M_2 \cong M^*(G)$, where $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 3$.

Proof. Suppose that $M = P(M_1, M_2)\backslash T$, where T is the common triangle of M_1 and M_2. As M is 3-connected, by [11, 4.3], both $si(M_1)$ and $si(M_2)$ are 3-connected, and only elements of T can have parallel elements in M_1 or M_2. Then by Lemma 3.2, T is contained in a F_7-minor in $si(M_1)$. Now M_2 does not contain a rooted $K_4^{''}$-minor using T, where $K_4^{''}$ is obtained from this K_4 by adding a parallel element to any two of the three elements of T (otherwise, the 3-sum of M_1 and M_2 contains a P_9-minor).

If M_2 is graphic, then by Lemma 3.12, $si(M_2) \cong M(G)$ where G is either W_3, W_4 or the Prism. When G is either W_4 or the Prism, then it is easily seen that M_2 has to be simple, and thus $M_2 \cong W_4$ or Prism. If $G \cong W_3$, then as M is P_9-free and M_2 has at least seven elements (from the definition of 3-sum), it is easily seen that $M_2 \cong M(K_4^{'})$.

If M_2 is cographic but not graphic, then by Lemma 3.13, $si(M_2) \cong M^*(G)$, where G is $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 3$. If M_2 is not simple, then it is straightforward to find a rooted $M(K_4^{''})$-minor using T in M_2, thus a P_9-minor in M; a contradiction. This completes the proof of the lemma. □

Lemma 3.15. *Let M be a 3-connected regular matroid with at least six elements and T be a triangle of M. Then M has no rooted $M(K''_4)$-minor using T if and only if M is isomorphic to a 3-connected matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, $M^*(K'''_{3,n})$ for some $n \geq 1$.*

Proof. If M is isomorphic to a 3-connected matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, $M^*(K'''_{3,n})$ ($n \geq 1$), then it is straightforward to check for any triangle T, M has no rooted $M(K''_4)$-minor using T.

Conversely, suppose that M is a 3-connected regular matroid with at least six elements and T is a triangle of M, such that M has no rooted $M(K''_4)$-minor using T. If M is internally 4-connected, then by Theorem 2.3, M is either graphic, cographic, or is isomorphic to R_{10}. The result follows from Lemmas 3.12 and 3.13, and the fact that R_{10} is triangle-free. So we may assume that M is not internally 4-connected and has a 3-separation (X, Y) where $|X|, |Y| \geq 4$. We may assume that $|X \cap T| \geq 2$.

Suppose that $Y \cap T$ has exactly one element e. Then as T is a triangle, $(X \cup e, Y\backslash e)$ is also a 3-separation. If $|Y| = 4$, then $Y - e$ is a triangle or a triad. Moreover, $r(Y) + r^*(Y) - |Y| = 2$. As M is 3-connected and binary,

$r(Y), r^*(Y) \geq 3$, and thus $r(Y) = r^*(Y) = 3$. If $Y - e$ is a triangle, then it is not a triad, and thus Y contains a cocircuit which contains e. This is a contradiction as this cocircuit meets T with exactly one element. Hence $Y - e$ is a triad, and from $r(Y) = 3$, there is an element $f \in T, f \neq e$ such that $Y - f$ is a triangle. In other words, Y forms a 4-element fan. We conclude that $M \cong M_1 \oplus_3 M(K_4')$ by [11, 2.9] where S is the common triangle of M_1 and $M(K_4')$, and $M(K_4')$ is obtained from $M(K_4)$ (containing T) by adding an element e_1 in parallel to an element e of S. By switching the label of e_1 to e in M_1, we obtain a matroid M_1' ($\cong M_1$) which is isomorphic to a minor of M having triangle T. By [11, 4.3], $si(M_1)$ is 3-connected. Hence by induction, $si(M_1)$ is isomorphic to a 3-connected matroid $M^*(K_{3,m})$, $M^*(K'_{3,m})$, $M^*(K''_{3,m})$, $M^*(K'''_{3,m})$ for some $m \geq 1$. As M has no rooted $M(K_4'')$-minor using T, we have that $r_{M_1}(S \cup T) > 2$. Moreover, the element e_1 is in two triangles of $si(M_1)$, so $m \leq 3$. Now using Lemma 3.7, it is straightforward to verify that $M \cong W_4 \cong M^*(K''_{3,2})$ and thus the Lemma holds. Hence we may assume that $|Y| \geq 5$ and thus $|Y \backslash e| \geq 4$.

Therefore we may assume that M has a separation (X, Y) such that $T \subseteq X$, and both X and Y have at least four elements. Hence by [11, (2.9)], $M = M_1 \oplus_3 M_2$ where M_1 and M_2 are isomorphic to minors of M having the common triangle S, and T is a triangle of M_1. Moreover, $|E(M_i)| < |E(M)|$ for $i = 1, 2$, and both $si(M_1)$ and $si(M_2)$ are 3-connected [11, (4.3)]. First assume that each element of S is parallel to an element of T in M_1. Then by Lemma 3.1, $si(M_1)$ contains a rooted $M(K_4)$-minor using T. As each element of T in M_1 is in a parallel pair, we conclude that M has a rooted $M(K_4'')$-minor using T; a contradiction.

So we may assume that at least one element of T is not parallel to an element of S (as M is binary, there are at least two such elements). As $si(M_1)$ is a 3-connected minor of M, it has no rooted $M(K_4'')$-minor using T. By induction, $si(M_1) \cong M^*(K_{3,s})$, $M^*(K'_{3,s})$, $M^*(K''_{3,s})$, $M^*(K'''_{3,s})$ for some $s \geq 2$, or $si(M_1) \cong M(K_4)$. Remove all elements of M_1 not in the set $S \cup T$ in $P_S(M_1, M_2)$. Then every element of $T \backslash S$ is parallel to an element of $S \backslash T$. Contracting all elements of $S \backslash T$, we obtained a minor of M isomorphic to M_2 and T is a triangle of this minor. By induction again, $si(M_2) \cong M^*(K_{3,t})$, $M^*(K'_{3,t})$, $M^*(K''_{3,t})$, $M^*(K'''_{3,t})$ for some $t \geq 2$, or $si(M_2) \cong M(K_4)$. Suppose that $si(M_i) \cong M(K_4)$ for some $i = 1, 2$. Then as M_i have at least seven elements and M has no rooted $M(K_4'')$-minor using T, we deduce that $M_i \cong M(K_4')$. As M has no $M(K_4'')$-minor containing T, and M is 3-connected, using Corollary 3.8, it is routine to verify that $M \cong M^*(K_{3,n}), M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ for some $n \geq 2$. □

Corollary 3.16. *Let M be a 3-connected binary non-regular P_9-free matroid. Suppose that $M = M_1 \oplus_3 M_2$ such that M_1 and M_2 have the common triangle*

T. If M_2 is regular, then M_2 is isomorphic to a 3-connected matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ ($n \geq 2$), or $M_2 \cong M(K'_4)$ where $M(K'_4)$ is obtained from $M(K_4)$ (containing T) by adding an element in parallel to an element of T.

Proof. As M is 3-connected, by [11, 4.3], both $si(M_1)$ and $si(M_2)$ are 3-connected, and only elements of T can have parallel elements in M_1 or M_2. As M is non-regular and M_2 is regular, $si(M_1)$ is non-regular and thus (by Lemma 3.2) has a F_7-minor containing the common triangle T of M_1 and M_2. As M is P_9-free, M_2 has no rooted $M(K''_4)$-minor using T. By Lemma 3.15, $si(M_2)$ is isomorphic to a 3-connected matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, $M^*(K'''_{3,n})$ ($n \geq 2$), or $M(K_4)$. Now using Lemma 3.10, it is straightforward to check that either $M_2 \cong M(K'_4)$, or M_2 is simple, and $M_2 \cong M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ ($n \geq 2$). □

Now we are ready to prove our main theorem.

Proof of Theorem 1.2. Suppose that a starfish M is constructed from a 3-connected cographic matroid N by consecutively applying the 3-sum operations with t copies of F_7, where $N \cong M^*(G)$; $G \cong K_{3,n}, K'_{3,n}, K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 2$. First we show that M is 3-connected. We use induction on t. When $t = 0$, N is 3-connected. Suppose that M is 3-connected for $t < k \leq n$. Now suppose that $t = k$. Then $M = M_1 \oplus_3 F$, where $F \cong F_7$ and M_1 and F share the common triangle T. Take an element f of $E(F) \cap E(M)$. Then by Lemma 3.3, $M/f = P(M_1, F/e)\backslash T \cong M_1$, which is a starfish with $t = k - 1$, and thus is 3-connected by induction. If M is not 3-connected, then f is either in a loop of M, or is in a cocircut of size one or two. Clearly, M does not have any loop, thus f is in a cocircuit C^* of M with size one or two. As $P(M_1, F)$ is 3-connected, it does not contain any cocircuit of size less than three. Hence $C^* \cup T$ contains a cocircuit D^* of $P(M_1, F)$. As $P(M_1, F)$ is binary, $D^* \cap T$ has exactly two elements, and thus D^* has at most four elements. As T contains no cocircuit of either M_1 or F, by Lemma 3.5, $F \cong F_7$ has a cocircuit of size at most three meeting two elements of T. This contradiction shows that M is 3-connected.

Next we show that if M is one of the matroid listed in (i)-(iv), then M is P_9-free. By Theorem 2.1 and the fact that all spikes and regular matroids are P_9-free, we need only show that any starfish is P_9-free. We use induction on the number of elements of the starfish M. By the definition, the unique smallest starfish has nine elements, and is isomorphic to P^*_9. Clearly, P^*_9 is P_9-free. Suppose that any starfish with less than n (≥ 10) elements is P_9-free. Now suppose that we have a starfish M with n elements. Suppose, on the contrary, that M has a P_9-minor. Then by the Splitter Theorem (Theorem 2.2), there is an element e in M such that either $M\backslash e$ or M/e is 3-connected having a P_9-minor. Note that the elements of a starfish consists of two types: those are subsets of $E(N)$ (denote this set by K), or those are in part of copies of F_7 (denote this set by F). Then $E(M) = K \cup F$. First we assume that $e \in F$. Then $M = M_1 \oplus_3 M_2$, where M_1 is either one of $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$, or a starfish with fewer elements; $M_2 \cong F_7$, and $e \in E(M_2)$. By the construction of the starfish and Lemma 3.3, $M/e \cong M_1$ and is either cographic or a smaller starfish and therefore does not contain a P_9-minor; a contradiction. Therefore $M\backslash e$ is 3-connected and contains a P_9-minor. But then by Lemma 3.4, $M\backslash e \cong P(M_1, M(K_4))\backslash T$. By Corollary 3.8, as $M\backslash e$ is 3-connected, we conclude that $M\backslash e$ is a smaller starfish and therefore is P_9-free. This contradiction shows that $e \in K$.

If e is in a triangle of M, then M/e is not 3-connected, and thus $M\backslash e$ is 3-connected and contains a P_9-minor. Each triangle of M is corresponding to a triad in G. By Lemmas 3.3 and 3.4 again, we can do the deletion $N\backslash e$ first, then perform the 3-sum operations with copies of F_7. Note that $N\backslash e \cong M^*(G/e)$ where $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ $(n \geq 2)$. As $M\backslash e$ is 3-connected and thus simple, we deduce that $n \geq 3$, $N \cong M^*(K_{3,n})$ or $M^*(K'_{3,n})$, and $N\backslash e \cong M^*(K''_{3,n-1})$, or $M^*(K'''_{3,n-1})$. Therefore, $M\backslash e$ is another starfish and does not contain any P_9-minor by induction; a contradiction. Finally assume that $e \in K$ is not in any triangle of M. Then e is not in any triad of G. Hence if $n = 2$, then $G \cong K'''_{3,2}$. As G/e has parallel elements, the matroid $N\backslash e$ has serial-pairs, and thus $M\backslash e$ is not 3-connected, we conclude that M/e is 3-connected having a P_9-minor. Note that $N \cong M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 2)$, and thus $N/e \cong M^*(K_{3,n})$, $M^*(K'_{3,n})$, or $M^*(K''_{3,n})$, which is still 3-connected. We conclude again, by Lemma 3.3, that M/e is a smaller starfish than M, thus cannot contain any P_9-minor. This contradiction completes the proof of the first part.

Now suppose that M is a 3-connected binary matroid with no P_9-minor. We may assume that M is not regular. If M is internally 4-connected, then the theorem follows from Theorem 2.1. Now suppose that M is neither regular nor internally 4-connected. We show that M is either a spike or a starfish. Suppose that $|E(M)| \leq 9$. As M is not internally 4-connected, M is not F_7 or F_7^*. Hence $|E(M)| \geq 8$. Then M is $AG(3,2), S_8, Z_4, Z_4^*$ (all spikes), or P_9^*, which is the 3-sum of F_7 and $W_4 = M^*(K''_{3,2})$, thus is a starfish. We conclude that the result holds for $|E(M)| \leq 9$. Now suppose that $|E(M)| \geq 10$. As M is not internally 4-connected, $M = M_1 \oplus_3 M_2 = P(M_1, M_2)\backslash T$, where M_1 and M_2 are isomorphic to minors of M ([11, 4.1]) and $T = \{x, y, z\}$ is the common triangle of M_1 and M_2. Moreover, $|E(M_i)| < |E(M)|$ for $i = 1, 2$, and both $si(M_1)$ and $si(M_2)$ are 3-connected [11, (4.3)]. The only possible parallel element(s) of either M_1 or M_2 are those in the common triangle. As M has no P_9-minor, and M_1 and M_2 are isomorphic to minors of M, we deduce that neither $si(M_1)$ nor $si(M_2)$ has a P_9-minor. By induction, the theorem holds for both $si(M_1)$ and $si(M_2)$. As M is not regular, at least one of $si(M_1)$ and $si(M_2)$, say $si(M_1)$, is not regular.

Claim: M_1 (and M_2) is simple unless both $si(M_1)$ and $si(M_2)$ are spikes.

Suppose not and we may assume that x in T has a parallel element x_1 in M_1. By Lemma 3.2, T is in a F_7-minor of M_1 plus a parallel element x_1. By induction, $si(M_2)$ is either regular and 3-connected, or one of the 16 internally 4-connected non-regular minors of Y_{16} (thus is F_7 since it has a triangle); or is a spike or a starfish. Moreover, $si(M_1)$ is either one of the 16 internally 4-connected non-regular minors of Y_{16} (thus is F_7); or is a spike or a starfish. Suppose that $si(M_2)$ is not a spike. Then either $si(M_2)$ is regular or is a starfish. By Lemmas 3.10 and 3.16, either $M_2 \cong M(K_4^{'})$ where $M(K_4^{'})$ is obtained from $M(K_4)$ (which contains T) by adding an element parallel to an element of T, or T is in a rooted $M(K_4')$-minor of M_2 using T (obtained from $M(K_4)$ containing T by adding an element parallel to either y or z). In either case, as M is simple, we conclude that M contains a P_9-minor, a contradiction. Hence $si(M_2)$ is a spike thus contains an F_7-minor containing T. Now if $si(M_1)$ is not a spike, then $si(M_1)$ is a starfish. Again using Lemma 3.10, it is easily checked that M has a P_9-minor; a contradiction. Therefore M_1 is simple unless both $si(M_1)$ and $si(M_2)$ are spikes. A similar argument shows that M_2 is also simple unless both $si(M_1)$ and $si(M_2)$ are spikes.

Case 1: $si(M_2)$ is regular. By Lemma 3.16, M_2 is either graphic or cographic. Moreover,

(i) if M_2 is graphic, then either $M_2 \cong M(G)$ where G is W_4 or the Prism, or $M_2 \cong M(K_4')$ where $M(K_4^{'})$ is obtained from $M(K_4)$ (which contains T) by adding an element parallel to an element of T; and

(ii) if M_2 is cographic but not graphic, then $M \cong M^*(G)$, where $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 3$.

By the above claim, both M_1 and M_2 are simple. Moreover, M_1 is 3-connected, non-regular, and P_9-free. By induction, M_1 is either one of the 16 internally 4-connected non-regular minors of Y_{16} (therefore is F_7 as M_1 has a triangle); or M_1 is a spike or a starfish. That is, either M_1 is a spike or a starfish. If M_1 is a starfish, by Lemma 3.9, $M = M_1 \oplus_3 M_2$ is also a starfish. Thus we may assume that M_1 is a spike which contains a triangle. Then M_1 is either F_7, S_8, Z_s $(s \geq 4)$ or $Z_s \backslash y_s$ for some $s \geq 5$. Suppose that M_1 is F_7. Then $M = F_7 \oplus_3 M_2$ is either S_8 (not possible as M has at least 10 elements) or a starfish by the definition of a starfish. Suppose that M_1 is Z_s $(s \geq 4)$ or $Z_s \backslash y_s$ for some $s \geq 5$ and suppose that M_2 is not isomorphic to $M(K_4')$. Then M_1 has a Z_4-restriction containing T. Clearly, such restriction contains a F_7'-minor which is obtained from F_7 (which contains T) by adding an element parallel to the tip of the spike, say x in T. By Lemma 3.10, T is in a $M(K_4')$- minor of M_2 which is obtained from K_4 containing T by adding an element parallel to an element $z \neq x$ of T. Thus we can find a P_9-minor in M, a contradiction. Suppose that M_1 is Z_s ($s \geq 4$) or $Z_s \backslash y_s$ for some $s \geq 5$ and suppose that $M_2 \cong M(K_4')$. If the extra element e of $M(K_4')$ added to $M(K_4)$ is not parallel to x in M_2, then using the previously mentioned F_7'-minor of M_1 containing T and the $M(K_4')$-minor containing e, we obtain a P_9-minor of M; a contradiction. Now it is straightforward to see that $M \cong Z_{s+2} \backslash y_{s+2}$ ($s \geq 4$) which is a spike, or $Z_{s+2} \backslash y_s, y_{s+2}$ ($s \geq 5$). The latter case does not happen as $\{y_s, y_{s+2}\}$ would be a 2-element cocircuit, but M is 3-connected. Finally we assume that $M_1 \cong S_8 = F_7 \oplus_3 M(K_4')$ with tip x. Then $M = (F_7 \oplus_3 M(K_4')) \oplus_3 M_2$. By Lemma 3.4, $M = F_7 \oplus_3 (M(K_4') \oplus_3 M_2)$. By Corollary 3.16, M_2 is isomorphic to a 3-connected cographic matroid $M^*(K_{3,n})$, $M^*(K_{3,n}')$, $M^*(K_{3,n}'')$, or $M^*(K_{3,n}''')$ ($n \geq 2$), or $M_2 \cong M(K_4')$. If $M_2 \cong M(K_4')$, then $|E(M)| = 9$; a contradiction. Thus M_2 is not isomorphic to $M(K_4')$. By Corollary 3.8, $M(K_4') \oplus_3 M_2 \cong M^*(G)$, where $G \cong K_{3,n}'$, $K_{3,n}''$, or $K_{3,n}'''$ for some $n \geq 2$, or $M(K_4') \oplus_3 M_2$ contains a 2-element cocircuit which does not meet any triangle of $M(K_4') \oplus_3 M_2$. In this case, by Corollary 3.6, this 2-element cocircut would also be a cocircuit of M. As M is 3-connected, we conclude that the latter does not happen, and that M is still a starfish.

Case 2: Neither M_1 nor M_2 is regular. By induction and the fact that both M_1 and M_2 have a triangle, that $si(M_1)$ is either a spike containing a triangle or a starfish, and so is $si(M_2)$.

Case 2.1: Both $si(M_1)$ and $si(M_2)$ are starfishes. By the above claim, both M_1 and M_2 must be simple matroids. Now by Lemma 3.9, M is also a starfish.

Case 2.2: One of $si(M_1)$ and $si(M_2)$, say the former, is a spike. Suppose that $si(M_2)$ is a starfish. By the claim, both M_1 and M_2 are simple. As M_1 contains the triangle T, it is either Z_s ($s \geq 3$) or $Z_s \backslash y_s$ for some $s \geq 4$. If $M_1 \cong Z_3 \cong F_7$, by the definition of a starfish, M is also a starfish. If $M_1 \cong Z_s$ ($s \geq 4$) or $Z_s \backslash y_s$ for some $s \geq 5$, then M_1 contains a Z_4 as a restriction which contains T. But Z_4 contains a F_7'-minor containing T where F_7' is obtained from F_7 by adding an element in parallel to the tip x of M_1. By Lemma 3.10, T is in a $M(K_4')$-minor of M_2 which is obtained from $M(K_4)$ containing T by adding an element parallel to y or z. We conclude that M contains a P_9-minor, a contradiction. Now suppose that $M_1 \cong Z_4 \backslash y_4 \cong S_8 = F_7 \oplus_3 M(K_4')$ with tip x. Then $M = (F_7 \oplus_3 M(K_4')) \oplus_3 M_2$. By Lemma 3.4, $M = F_7 \oplus_3 (M(K_4') \oplus_3 M_2)$. By Corollary 3.9, $M(K_4') \oplus_3 M_2$ is either a starfish, or $M(K_4') \oplus_3 M_2$ and thus M contains a 2-element cocircuit. As M is 3-connected, we conclude that the latter does not happen, and that M is still a starfish by the definition of a starfish.

Hence we may assume that $si(M_2)$ is also a spike. As $si(M_2)$ contains a triangle also, it is either Z_t $(t \geq 3)$ or $Z_t \backslash y'_t$ for some $t \geq 4$. Suppose that $si(M_1)$ and $si(M_2)$ do not share a common tip, say $si(M_1)$ has tip x and $si(M_2)$ has tip z. Then neither matroid is isomorphic to F_7 as any element of T can be considered as a tip then. We first assume either $si(M_1)$ or $si(M_2)$, say $si(M_1)$, has at least nine elements. Then M_1 has a Z_4-restriction containing T, thus has a F'_7-minor (with a parallel pair containing x) containing T. The matroid $si(M_2)$ has a S_8-restriction, thus has a $M(K'_4)$-minor (with a parallel pair containing z) containing T. By Lemma 3.3, we conclude that M has a P_9-minor; a contradiction. Hence both $si(M_1)$ and $si(M_2)$ have exactly eight elements and both are isomorphic to S_8. Now if either M_1 or M_2 is not simple, then similar to the argument above, one can get a P_9-minor; a contradiction. Hence both matroid are simple. Now it is straightforward to see that $M \cong F_7 \oplus_3 W_4 \oplus_3 F_7$, which is a starfish.

Therefore we may assume that $si(M_1)$ and $si(M_2)$ share a common tip, say x. First assume that a non-tip element in T, say y, is in a parallel pair of either M_1 or M_2, say M_1. As M is both simple and P_9-free, it is easily seen that M_2 has to be simple. Since any element of T can be considered as a tip in F_7, we deduce that both $si(M_1)$ and M_2 have at least 8 elements. If one of these two matroids has at least 9 elements, then it contains a Z_4-restriction containing T. Such a restriction contains a F'_7-minor containing T with x being in a parallel pair. At the same time, $si(M_i)$ contains a $M(K_4)$-minor containing T for $i = 1, 2$. Noting that y is in a parallel pair of M_1, we deduce that M contains a P_9-minor; a contradiction. Hence we may assume that both $si(M_1)$ and M_2 contain exactly 8 elements. Now it is easily seen that M_1 contains a F'_7-minor containing T with y being in a parallel pair. At the same time, $si(M_2)$ contains a $M(K'_4)$-minor containing T with x being in a parallel pair. This is a contradiction as M now contains a P_9-minor.

So from now on we may assume that if M_1 or M_2 is not simple, then only x could be in a parallel pair. Indeed, as M is simple, at most one of M_1 and M_2 is not simple. Suppose that one of M_1 and M_2, say M_1, is not simple, then either $M \cong Z_{s+t}$, $M \cong Z_{s+t} \backslash y_s$, $M \cong Z_{s+t} \backslash y'_t$, or $M \cong Z_{s+t} \backslash y_s, y'_t$, all of which are spikes except the last matroid. The last matroid, $M \cong Z_{s+t} \backslash y_s, y'_t$, however, contains a cocircuit $\{y_s, y'_t\}$, contradicting to the fact that M is 3-connected. Finally assume that both M_1 and M_2 are simple. Then $M \cong Z_{s+t} \backslash x$, $M \cong Z_{s+t} \backslash x, y_s$, $M \cong Z_{s+t} \backslash x, y'_t$, or $M \cong Z_{s+t} \backslash x, y_s, y'_t$, all of which are spikes except the last matroid. The last matroid, $M \cong Z_{s+t} \backslash x, y_s, y'_t$, again, contains a cocircuit $\{y_s, y'_t\}$; a contradiction. This completes the proof of Case 2.2, thus the proof of the theorem. □

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