EXCLUDED-MINOR CHARACTERIZATION OF APEX-OUTERPLANAR
GRAPHS

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Abstract. The class of outerplanar graphs is minor-closed and can be characterized by two excluded minors: $K_4$ and $K_{2,3}$. The class of graphs that contain a vertex whose removal leaves an outerplanar graph is also minor-closed. We provide the complete list of 57 excluded minors for this class.

1. Introduction

A graph is outerplanar if it can be embedded in the plane (with no edges crossing) with all vertices incident to one common face. We say that a graph $G$ is apex-outerplanar if there exists $v \in V(G)$ such that $G - v$ is outerplanar. Such a vertex, if it exists, is called an apex vertex of $G$.

We let $O$ and $O^*$ denote the classes of outerplanar and apex-outerplanar graphs, respectively.

Given graphs $H$ and $G$, $H$ is a minor of $G$, denoted by $H \leq_m G$, or $G \geq_m H$, if $H$ can be obtained from a subgraph of $G$ by contracting edges. A class $C$ of graphs is minor-closed if for every $G \in C$ all the minors of $G$ are also in $C$. Examples of minor-closed classes are: planar graphs, outerplanar graphs, series-parallel graphs, graphs embeddable in a fixed surface, and graphs of tree-width bounded by a fixed constant.

Let $C$ be a minor-closed class of graphs, and let $C^*$ be the class of graphs that contain a vertex whose removal leaves a graph in $C$. Hence, clearly $C \subseteq C^*$, and it is easy to check that $C^*$ is also minor-closed, thus in particular $O^*$ is minor-closed.

It is a landmark result of Robertson and Seymour (see [7]) that every proper minor-closed class of graphs $C$ can be characterized by its finite set of excluded minors, or obstructions, that is, minor-minimal graphs not in $C$. We call this set obstruction set of $C$, and denote it by $\text{ob}(C)$. For example, it is a well-known fact that $\text{ob}(O) = \{K_4, K_{2,3}\}$. Equivalently, $G$ is outerplanar if and only if it does not contain a subdivision of $K_4$ nor a subdivision of $K_{2,3}$ as a subgraph. This equivalence follows from the known fact that if $H \leq_m G$ and $\Delta(H) \leq 3$, then $G$ contains an $H$-subdivision.

Let $S$ be the set of graphs in Figure 1, $\mathcal{T}$ be the set of graphs in Figure 4, $\mathcal{G}$ be the set of graphs in Figure 5, $\mathcal{J}$ be the set of graphs in Figure 6, $\mathcal{H}$ be the set of graphs in Figure 7, and $\mathcal{Q}$ be the set of graphs in Figure 8.

The following is our main result.

Theorem 1.1. A graph is apex-outerplanar if and only if it does not contain any of the 57 graphs in the set $S \cup \mathcal{T} \cup \mathcal{G} \cup \mathcal{J} \cup \mathcal{H} \cup \mathcal{Q}$ as a minor. Equivalently, $\text{ob}(O^*) = S \cup \mathcal{T} \cup \mathcal{G} \cup \mathcal{J} \cup \mathcal{H} \cup \mathcal{Q}$.

The reader should be confident that the 57 graphs in Theorem 1.1 are indeed pairwise non-isomorphic members of $\text{ob}(O^*)$. We have checked this several times. In this paper, we will show that there are no more graphs in $\text{ob}(O^*)$ other than the 57, that is, the list is complete.
Our result can be regarded as a test approach to the long-standing open problem of finding the complete list of excluded minors for the class of apex-planar graphs, which plays an important role in Graph Theory (for example, see [8]). Significant progress on this problem has already been made by A. Kezdy [6] and his team since our work was completed and announced in [4]. For instance, they have found all of the obstructions of connectivity 0, 1, and 2, and many of the ones of connectivity 3, 4, and 5, altogether 396 obstructions.

While working on the problem we did not use a computer, the 57 obstructions were found “by hand”. We believe that this was an advantage, since we were able to control and understand the way in which the obstructions were being generated, and in which the proof should be organized. After we found \( \text{ob}(O^*) \) and proved its completeness, G.E. Turner [9] kindly informed us that the 57 graphs had been known to him, since he had found them with the aid of a computer. However, he did not know whether his list was complete.

We now present an outline of the rest of the paper, which constitutes the proof of Theorem 1.1. In Section 2, we provide a starting set of seven obstructions \( S \subseteq \text{ob}(O^*) \), and prove a key lemma (Lemma 2.2), which together allow us to conclude that any obstruction in \( \text{ob}(O^*) - S \) is planar and of connectivity 2 or 3. The search for the remaining obstructions begins.

The connectivity-three case is presented in Section 6. Here, we rely on the existence of contractible edges in 3-connected graphs and the minor-minimality of the obstructions to prove that there are no 3-connected obstructions in \( \text{ob}(O^*) \) other than the ones already in our starting set \( S \).

Most of the work is in the connectivity-two case. Our key lemma (Lemma 2.2) splits the proof of this case into five major subcases, presented in Sections 3, 4, and 5. The cases are split based on the complexity of each side of a 2-separation in \( G \in \text{ob}(O^*) - S \), as indicated by Lemma 2.2. In the following outline of the case structure, all of the 2-separations refer to 2-separations \( (L, R) \) in \( G \) over vertices \( \{x, y\} \). Also, \( P_2 \) and \( C_4 \) are as drawn in Figure 3, with vertices \( \{x, y\} \) as labelled in the Figure.

**Case 1:** There exists a 2-separation such that both \( L \notin O \) and \( R \notin O \) (Section 3);

**Case 2:** For each 2-separation, \( L = P_2 \) or \( C_4 \) (Sections 4 and 5);

- **Subcase 2.1:** There exists a 2-separation such that \( L = C_4 \) (Proposition 4.1);
  - **Subsubcase 2.1.1:** There exists a 2-separation such that \( L = C_4 \) and \( G - \{x, y\} \notin O \);
  - **Subsubcase 2.1.2:** There exists a 2-separation such that \( L = C_4 \) and for every such 2-separation \( G - \{x, y\} \in O \);

- **Subcase 2.2:** For each 2-separation, \( L = P_2 \) (Proposition 5.1);
  - **Subsubcase 2.2.1:** There exists a 2-separation such that \( L = P_2 \) and \( G - \{x, y\} \notin O \);
  - **Subsubcase 2.2.2:** For each 2-separation, \( L = P_2 \) and \( G - \{x, y\} \in O \).

Note that organizing the case analysis in this way restricts the structure of \( G \) more and more as we proceed through the cases. An outline of each case will be given at the beginning of the corresponding section.
In this section, we provide a starting set of seven obstructions $S \subseteq \text{ob}(O^*)$, and prove the key Lemma 2.2, which narrows down the structure of the remaining obstructions.

For two graphs $G_1$ and $G_2$, we let $G_1 \mid G_2$ denote their disjoint union.

Let $S := \{K_5, K_{3,3}, \text{Oct}, Q, 2K_4, K_4 \mid K_{2,3}, 2K_{2,3}\}$ be the set of graphs in the figure below.

![Figure 1. Starting list of excluded minors for $O^*$](image)

It is easy to check that $S \subseteq \text{ob}(O^*)$.

**Definition 2.1.** Let $G$ be a graph and $x, y \in V(G)$. A 1-separation of $G$ over $x$ (or across $x$) (respectively, a 2-separation of $G$ over $\{x, y\}$ (or across $\{x, y\}$)) is a pair $S = (L, R)$ of induced subgraphs $L$ and $R$ of $G$, called the sides of $S$, such that the following holds

1. $E(L) \cup E(R) = E(G)$;
2. $V(L) \cup V(R) = V(G)$ and $V(L) \cap V(R) = \{x\}$ (respectively, $V(L) \cap V(R) = \{x, y\}$);
3. $V(L) - V(R) \neq \emptyset$ and $V(R) - V(L) \neq \emptyset$.

Note that in definition 2.1, we require that $L$ and $R$ to be induced subgraphs, and that $x$ is necessarily a cut-vertex of $G$ (respectively, $\{x, y\}$ is a 2-cut of $G$). Also, if $S = (L, R)$ is a 2-separation of $G$ over $\{x, y\}$, then we often denote $L$ and $R$ by $L(x, y)$ and $R(x, y)$, respectively, for emphasis.

We define a $K$-graph to be a graph that contains a $K_4$- or $K_{2,3}$-subdivision (both of which we call $K$-subdivisions) as a subgraph. Equivalently, $K$-graphs are precisely non-outerplanar graphs. It is a known fact that if $G$ is 2-connected and contains a $K$-subdivision, then $G = K_4$ or $G$ contains a $K_{2,3}$-subdivision.

**Lemma 2.2.** If $G \in \text{ob}(O^*) - S$, then $G$ is planar and of connectivity 2 or 3. Moreover, if the connectivity of $G$ is 2, then for every 2-separation $S = (L, R)$ of $G$ over vertices $\{x, y\}$ the following holds:

1. If no side of $S$ is in $O$, then one side of $S$ is $L_1$, $L_2$, $L_3$, $L_4$, or $L_5$ with prescribed vertices $x$ and $y$, as shown in Figure 2.
(2) If one side of $S$ is in $O$, then $xy \notin E(G)$ and that side is $P_2$ or $C_4$, where $P_2$ is a path on two edges with endpoints $x$ and $y$, and $C_4$ is a cycle on four edges with $x$ and $y$ non-adjacent, as shown in Figure 3.

**Figure 2.** $K_4$ and $K_{2,3}$’s with prescribed vertices $x$ and $y$

**Figure 3.** $P_2$ and $C_4$

**Proof.** Since $G \not\prec_m K_5$ and $G \not\prec_m K_{3,3}$, it follows that $G$ is planar.

First, suppose that $G$ is disconnected, and let $G$ be a union of two disjoint (not necessarily connected) graphs $G_1$ and $G_2$. If one of them, say $G_1$ is outerplanar, then by the minor-minimality of $G$, $G_2 = G - G_1 \in O^*$, hence $G_2$ has a vertex $v$ such that $G_2 - v \in O$. Then, $G_1 | (G_2 - v) \in O$, hence $v$ is an apex vertex in $G$, a contradiction. Therefore, both $G_1$ and $G_2$ are not outerplanar, and so each contains $K_4$ or $K_{2,3}$ as a minor. Hence $G$ contains one of $2K_4, K_4 | K_{2,3}, 2K_{2,3}$ as a minor, a contradiction. Thus $G$ is connected.

Now, suppose that $G$ has a cut-vertex $x$ and let $(L, R)$ be the 1-separation across $x$. By the same argument as above, both $L$ and $R$ are not outerplanar, hence they both contain $K_4$ or $K_{2,3}$ as a minor. This implies that both $R - x$ and $L - x$ are outerplanar (for otherwise, $G$ would contain one of $2K_4, K_4 | K_{2,3}, 2K_{2,3}$ as a minor). Hence $x \notin O$, and so $G \in O^*$, a contradiction. Therefore, $G$ is 2-connected.

Now, suppose that $G$ is 4-connected. Then $\delta(G) \geq 4$, and so by the theorem of Halin and Jung from [5], which says that $G$ contains a $K_5$- or $Oct$-minor whenever $\delta(G) \geq 4$, it follows that the assumption that $G$ is 4-connected is not true, because $K_5$ and $Oct$ are in $S$. Therefore the connectivity of $G$ is 2 or 3.

Proof of (1). Suppose now that the connectivity of $G$ is 2 and that no side of $S$, neither $L$ nor $R$, is in $O$. Note that $G - \{x, y\} \in O$, for otherwise $G$ would contain two disjoint $K$-graphs (for instance, $L$ and $R - \{x, y\}$) which cannot happen because $G$ does not contain $2K_4, K_4 | K_{2,3}, 2K_{2,3}$ as a minor. Since $G \notin O^*$, none of its vertices is apex. In particular, since $x$ is not apex in $G$ and $y$ is a cut-vertex in $G - x$, it follows that $L - x$ or $R - x$, say $R - x$, contains a $K$-subdivision, call it $K'$, which contains $y$ (since $R - \{x, y\}$ is outerplanar). Similarly, $R - y$ contains a $K$-subdivision.
$K''$ (not $L - y$, because such a $K$-subdivision would be disjoint from $K'$), which contains $x$. $K'$ and $K''$ must intersect, otherwise $G$ would contain two disjoint $K$-graphs. Also, $L - x \in O$ since it is disjoint from $K''$, and $L - y \in O$ since it is disjoint from $K'$. Hence, $G$ must have the following structure:

![Diagram](image)

Note that, as long as $L \notin O$, a graph with the above structure does not belong to $O^*$. This is because none of its vertices is apex: $x$ is not apex, because of $K'$; $y$ is not apex, because of $K''$; if $v \in L - \{x, y\}$, then $v$ is not apex, because of $K'$ (or $K''$); finally if $v \in R - \{x, y\}$, then $v$ is not apex, because of $L$. Therefore, if $L \notin \{K_4, K_{2,3}\}$, then since $L \notin O$, it follows that $L$ contains an edge $e \neq xy$ such that either $L \setminus e \notin O$, or $L/e \notin O$. Hence, either $G \setminus e \notin O^*$ or $G/e \notin O^*$, a contradiction since $G$ is minor-minimal not in $O^*$. Therefore $L \in \{L_1, L_2, L_3, L_4, L_5\}$ with $x, y$ as prescribed in Figure 2.

Proof of (2). Without loss of generality, suppose that $L \in O$. Since $G \notin O^*$, none of its vertices are apex. In particular, since $x$ is not apex, it follows that $R - x$ contains a $K$-subdivision. Similarly, $R - y$ contains a $K$-subdivision. Since $G$ is 2-connected, it follows that $L$ is connected. We have two cases based on the number of blocks of $L$.

Case 1. $L$ has exactly one block.

Note that $L \neq K_2$, for otherwise $(L, R)$ is not a 2-separation. Hence $L$ is 2-connected.

Since $L$ is 2-connected and outerplanar, it follows that $L$ is a cycle $C$ with chords, which has a unique planar embedding such that all the vertices and edges of $C$ are incident with the outer face, and all the chords lie in the interior of the disk bounded by $C$. We now show that $L$ has no chords. So, suppose that $L$ does have a chord $e$. Let $s$ be an apex vertex in $G \setminus e$. Then, since $R - x$ and $R - y$ contain $K$-subdivisions, it follows that $s \in V(R - \{x, y\})$. Assume that $(G \setminus e) - s \in O$ is embedded in the plane so that all of its vertices are incident with the outer face. Then this embedding, restricted to the subgraph $L \setminus e$, is such that all the vertices and edges of $C$ are incident with the outer face. Therefore, by putting the chord $e$ back in, we obtain an embedding of $G - s$ with all of its vertices still incident with the outer face, hence $G - s$ is outerplanar, a contradiction. Hence, we have shown that $L$ has no chords, therefore $L = C$.

Now, suppose that $x$ and $y$ are consecutive vertices of $C$, that is $xy \in E(C)$. Let $s$ be an apex vertex in $G \setminus xy$. Then, again we have that $s \in V(R - \{x, y\})$. Assume that $(G \setminus xy) - s \in O$ is embedded in the plane so that all of its vertices are incident with the outer face. Since all the vertices of $C - \{x, y\}$ have degree $= 2$ in $(G \setminus xy) - s$, it follows that all the edges of $C$ except for $xy$ are incident with the outer face. Therefore, by putting the edge $xy$ back in, we obtain an embedding of $G - s$ with all of its vertices still incident with the outer face, a contradiction.

Therefore, $x$ and $y$ are non-consecutive, which implies that the length of $C$ is at least four. In fact $C = C_4$, for suppose that $C = C_n$ with $n \geq 5$. Then one of the two paths from $x$ to $y$ in $C$ must have length at least three. Let $f$ be an edge on that path with endpoints different from $x$ and
y. Let \( s \) be an apex vertex in \( G/f \). Then, again \( s \in V(R - \{x, y\}) \). Assume that \((G/f) - s \in \mathcal{O}\) is embedded in the plane so that all of its vertices are incident with the outer face. Since all the vertices of \((C/f) - \{x, y\}\) have degree = 2 in \((G/f) - s\), it follows that all the edges of \(C/f\) are incident with the outer face. Therefore, by uncontracting edge \( f \in E(C) \), we obtain an embedding of \( G - s \) with all of its vertices still incident with the outer face, hence \( G - s \) is outerplanar, a contradiction. Hence, we have shown that \( L = C = C_4 \).

Therefore, we have shown that if \( L \) has only one block, then \( L \) is 2-connected, and in fact \( L = C_4 \) with \( x \) and \( y \) non-adjacent. Now, we consider the more general case.

**Case 2.** \( L \) has at least two blocks.

Let \( B_x \) and \( B_y \) be two distinct blocks containing \( x \) and \( y \), respectively. Then the block tree of \( L \) is, in fact, a path from \( B_x \) to \( B_y \), for otherwise \( G \) would contain a cut-vertex. Every block on this path is either \( K_2 \) or is 2-connected. If \( L \) contains a block \( B \) that is 2-connected, then let \( s, t \in V(B) \) be the two cut-vertices in \( L \) (or in the case of \( B_x \) and \( B_y \), the associated pair is given by the corresponding cut-vertex, and \( x \) or \( y \), respectively). Then since \( G \) has a 2-separation \((B, R')\) over \( \{s, t\} \), it follows by the previous argument that \( B = C_4 \). Therefore, every block of \( L \) (which is a path) is either \( K_2 \) or \( C_4 \).

Now suppose that \( L \) contains a block \( B = C_4 \), and let \( B' \) be any other block. Denote by \( G/B' \) the graph obtained by contracting all the edges of \( B' \). Again, let \( s \) be an apex vertex in \( G/B' \).

Then again \( s \in V(R - \{x, y\}) \). Assume that \((G/B') - s \in \mathcal{O}\) is embedded in the plane so that all of its vertices are incident with the outer face. Since two of the non-adjacent vertices of \( B \) have degree = 2 in \((G/B') - s\) and since all the blocks are either \( K_2 \) or \( C_4 \), it follows that all the edges of \( B \) and, in fact, all the edges of \( L/B' \) are incident with the outer face. Therefore, by uncontracting block \( B' \), we obtain an embedding of \( G - s \) with all of its vertices still incident with the outer face, a contradiction. Hence, we have shown that \( L \) does not contain a block \( B = C_4 \), and therefore all the blocks of \( L \) are \( K_2 \)’s, or equivalently \( L \) is an induced path of length at least two from \( x \) to \( y \).

Then, in fact, \( L = P_2 \), for suppose that \( L = P_n \) with \( n \geq 3 \). Let \( f \) be an edge in \( L = P_n \) with endpoints different from \( x \) and \( y \). Let \( s \) be an apex vertex in \( G/f \). Then, again \( s \in V(R - \{x, y\}) \).

Assume that \((G/f) - s \in \mathcal{O}\) is embedded in the plane so that all of its vertices are incident with the outer face. Since all the vertices of \((L/f) - \{x, y\}\) have degree = 2 in \((G/f) - s\), it follows that all the edges of \(L/f\) are incident with the outer face. Therefore, by uncontracting edge \( f \), we obtain an embedding of \( G - s \) with all of its vertices still incident with the outer face, a contradiction. Hence, we have shown that \( L = P_2 \). This proves (2). \( \square \)

3. **Connectivity 2: No Side in \( \mathcal{O} \)**

In this section, we focus on Case 1 of the outline given in the Introduction. Namely, we prove Proposition 3.1, which says that if an obstruction \( G \in \text{ob}(\mathcal{O}^*) - \mathcal{S} \) has a 2-separation both sides of which are not outerplanar, then \( G \in \mathcal{T} \).

**Proposition 3.1.** If \( G \in \text{ob}(\mathcal{O}^*) - \mathcal{S} \) is of connectivity 2 and has a 2-separation no side of which is in \( \mathcal{O} \), then \( G \) is a member the family \( \mathcal{T} \).
Proof. Let \( S = (L, R) \) be a 2-separation of \( G \) over \( \{x, y\} \) no side of which is in \( \mathcal{O} \). Since \((R, L)\) is also a 2-separation of \( G \) with the same property, we may assume without loss of generality that \( L \in \{L_1, L_2, L_3, L_4, L_5\} \) (see Lemma 2.2). Note that \( R - \{x, y\} \) is outerplanar, for otherwise \( G \) contains two disjoint \( K \)-graphs. Since \( G \not\in \mathcal{O}^* \), none of its vertices is apex. In particular, since \( x \) is not apex, \( R - x \) contains a \( K \)-subdivision, which contains \( y \) (since \( R - \{x, y\} \) is outerplanar). Similarly, \( R - y \) contains a \( K \)-subdivision, which contains \( x \). These two \( K \)-subdivisions must intersect, otherwise \( G \) would contain two disjoint \( K \)-graphs. Hence, \( G \) must have the following structure:

![Diagram](diagram.png)

Note that each of the \( L_i \) \((i = 1, \ldots, 5)\) contains \( C_4 \) as a minor (with the vertices \( x \) and \( y \) preserved). Let \( G' \) be the graph obtained from \( G \) by reducing \( L \) (under the minor operation) to \( C_4 \), so that \((C_4, R)\) is a 2-separation of \( G' \) over \( \{x, y\} \). Note that \( G' \) is a proper minor of \( G \), hence by the minor-minimality of \( G \), it follows that \( G' \in \mathcal{O}^* \). If there are at least two internally disjoint paths in \( R \) from \( x \) to \( y \), then \( G' \) has no apex vertex, a contradiction.

![Diagram](diagram.png)

Hence, \( R \) has a cut-vertex \( z \). Note that \( R - z \in \mathcal{O} \), otherwise \( R \) contains two disjoint \( K \)-graphs.

Let \( R_1 \) and \( R_2 \) be the two sides of the 1-separation of \( R \) across \( z \), such that \( x \in R_1 \) and \( y \in R_2 \). By applying Lemma 2.2 to the 2-separation in \( G \) over \( \{x, z\} \), and to the 2-separation in \( G \) over \( \{y, z\} \), we conclude that both \( R_1, R_2 \in \{L_1, L_2, L_3, L_4, L_5\} \). Therefore, \( G \) is one of the 30 graphs \( \{T_1, T_2, \ldots, T_{30}\} \) listed in Figure 4. It is straightforward to verify that each \( T_i \) is minor-minimal \( \not\in \mathcal{O}^* \) satisfying the hypothesis of Case 1. Hence \( T_i \in \text{ob}(\mathcal{O}^*) \) for \( i = 1, \ldots, 30 \). \( \square \)
4. Connectivity 2: At Least One Side $C_4$

In this section and the next (Section 5) we focus on Case 2 of the outline given in the Introduction. Namely, we assume that every 2-separation of $G \in \text{ob}(O^*) - S$ has one side that is outerplanar, which by Lemma 2.2 implies that that side is $P_2$ or $C_4$. In this section, we focus on the case that $G$ has a 2-separation one side of which is $C_4$ (Subcase 2.1 of the outline given in the Introduction).

Figure 4. $\mathcal{T}$ family
We prove Proposition 4.1, which says that in this case \(G \in \mathcal{G} \cup \mathcal{J}\). In the next section, we analyze the case that every 2-separation of \(G\) has one side that is \(P_2\) (Subcase 2.2).

Before we state and prove Proposition 4.1, we introduce some necessary terminology and notation. If \(P = u_1, u_2, \ldots, u_n, u_{n+1}\) is a path on \(n\) vertices, then we define its length to be \(n\), and denote \(P\) by \(P_n\). We call the set \(\{u_2, u_3, \ldots, u_n\}\) the interior of \(P\) and denote it by \(\text{int}(P)\). Two paths \(P\) and \(Q\) are said to be internally disjoint if their interiors are disjoint. If \(C = u_1, u_2, \ldots, u_n, u_1\) is a cycle, then its length is \(n\), and we denote \(C\) by \(C_n\). An edge \(e \in E(C)\) with both endpoints in \(V(C)\) is called a chord of \(C\). If \(C = u_1, u_2, \ldots, u_n, u_1\) is a cycle embedded in the plane with vertices listed in the clockwise order around \(C\), then we denote by \(C[u_i, u_j]\) the set \(\{u_i, u_{i+1}, \ldots, u_j\}\) if \(i \leq j\), or the set \(\{u_i, u_{i+1}, \ldots, u_n, u_1, \ldots, u_j\}\) if \(i > j\). Similarly, \(C[u_i, u_j] := C[u_i, u_j] \setminus \{u_j\}\), \(C(u_i, u_j) := C[u_i, u_j] \setminus \{u_i\}\), and \(C(u_i, u_j) := C[u_i, u_j] \setminus \{u_i, u_j\}\). Also, if \(P = u_1, u_2, \ldots, u_n\) is a path, then we define \(P[u_i, u_j], P[u_i, u_j], P(u_i, u_j), \) and \(P(u_i, u_j)\) analogously, and so \(\text{int}(P) = P(u_1, u_n)\).

**Proposition 4.1.** If \(G \in \text{ob}(\mathcal{O}^*) – \mathcal{S}\) is of connectivity 2 and one side of every 2-separation of \(G\) is in \(\mathcal{O}\) and, moreover, if \(G\) has a 2-separation \(S\) over \(\{x, y\}\) one side of which is \(C_4\), then following holds true:

1. If \(G - \{x, y\} \notin \mathcal{O}\) for some such \(S\), then \(G\) is a member of the family \(\mathcal{G}\);
2. If \(G - \{x, y\} \in \mathcal{O}\) for every such \(S\), then \(G\) is a member of the family \(\mathcal{J}\).

**4.1. Proof of (1).** Let \(S = (L, R)\) be a 2-separation \(G\) over \(\{x, y\}\) such that one side of it, say, \(L\), is \(C_4\), and let \(G - \{x, y\} \notin \mathcal{O}\). Then \(R - \{x, y\} \notin \mathcal{O}\) and hence \(R - \{x, y\}\) contains a \(K\)-subdivision, call it \(K'\). Note that if \(R\) does not have at least two internally disjoint paths from \(x\) to \(y\), then \(R\) has a cut-vertex \(z\) separating \(x\) and \(y\), and hence \(G\) has a 2-separation \((L', R')\) over \(\{x, z\}\) or over \(\{y, z\}\) with the property that \(R' \notin \mathcal{O}\), and either \(L' \notin \mathcal{O}\) (violating the theorem hypothesis that one side of every 2-separation of \(G\) is in \(\mathcal{O}\)) or \(L' \in \mathcal{O}\) but with \(L'\) different from \(P_2\) and \(C_4\) (violating Lemma 2.2), a contradiction. Hence, \(1.\) \(R\) has at least two internally disjoint paths from \(x\) to \(y\).

Also, note that \(R\) does not have a path \(P\) from \(x\) to \(y\) disjoint from \(K'\), for otherwise \(G\) would contain two disjoint \(K\)-graphs (namely \(K'\) and the \(K_{2,3}\)-subdivision formed from the union of \(L\) and \(P\)). Therefore \(G\) has the following structure:

![Diagram of a graph](image)

Note that, \(2.\) A graph with the above structure does not belong to \(\mathcal{O}^*\).

This is because none of its vertices is apex: if \(v \in V(G) - V(K')\), then \(v\) is not apex, because of \(K'\); and if \(v \in V(K')\), then \(R - v\) has a path from \(x\) to \(y\), which along with \(L\) forms a \(K_{2,3}\)-subdivision in \(G - v\), hence \(v\) is not apex.

Fix a planar embedding of \(G\). Let \(C\) be the outer cycle of \(K'\). Let \(S_x \subseteq V(C)\) and \(S_y \subseteq V(C)\) be the sets of vertices of \(C\) from which there is a path to \(x\), or respectively to \(y\), that doesn’t
contain other vertices of \( C \). It follows, by 1, that \(|S_x| \geq 2\) and \(|S_y| \geq 2\), hence \(|S_x \cup S_y| \geq 2\).

However, if \(|S_x \cup S_y| = 2\) (see the following figure), then let \( \{a, b\} := S_x = S_y \), and note that \( G \) has a 2-separation \((L'', R'')\) over \( \{a, b\} \), where \( L'' = K' \notin O \) and \( R'' \) contains a subdivision of \( K_{2,4} \), hence \( R'' \notin O \), a contradiction because one side of \((L'', R'')\) must be in \( O \).

Hence, \(|S_x \cup S_y| \geq 3\). Also note that, by 2, the paths from \( S_x \) to \( x \) and \( S_y \) to \( y \) are actually simple edges, for otherwise we could perform a contraction along such a path, and by 2, the resulting graph would still be outside of \( O^* \), contradicting the minor-minimality of \( G \).

Since \( K' \) is a subdivision of either \( K_4 \) or \( K_{2,3} \), it follows that actually \( K' = K_4 \) or \( K' \) is a subdivision of \( K_{2,3} \). If \( K' = K_4 \), then in view of all the observations above, \( G \) is the following graph:

![Graph](image)

It is easy to verify that the above graph is minor-minimal \( \notin O^* \) satisfying the hypothesis of (1) and the initial hypothesis of Proposition 4.1. We label it \( G_1 \), and so \( G_1 \in \text{ob}(O^*) \).

So now, \( K' \neq K_4 \), and so \( K' \) is a subdivision of \( K_{2,3} \). Therefore \( K' \) consists of the outer cycle \( C \) and a path \( Q \) of length at least 2 connecting two non-adjacent vertices of \( C \). Note that \( Q \) has length exactly 2, for otherwise we could perform a contraction along \( Q \), and by 2, the resulting graph would still be outside of \( O^* \), contradicting the minor-minimality of \( G \). Let \( Q = a, c, b \), so that \( a, b \in V(C) \). Then since \( K' \) is a subdivision of \( K_{2,3} \), we have:

3. There is at least one vertex in \( C(a, b) \) and at least one in \( C(b, a) \).

Thus, \( G \) has the following structure:

![Graph](image)

It is straightforward to verify that the following graphs are minor-minimal \( \notin O^* \) satisfying the hypothesis of (1) and the initial hypothesis of Proposition 4.1 (except the second one, which is minor-minimal after contracting \( e \); the resulting graph is \( J_1 \in J \) from Figure 6). We label them \( G_2, G_3, G_4, G_5 \). Hence \( J_1, G_i \in \text{ob}(O^*) \) for \( i = 1, \ldots, 5 \).
In the remainder of the proof, we assume furthermore that \( G \notin \{ J_1, G_1, G_2, G_3, G_4, G_5 \} \). Let \( x_1, x_2 \in S_x \) and \( y_1, y_2 \in S_y \) in the clockwise order \( x_1, x_2, y_1, y_2 \) around \( C \). First, assume that all four can be chosen so that they are all distinct. Then, if \( a, b \in C[x_1, x_2] \) or \( a, b \in C[y_1, y_2] \), then by 3, \( G \geq_m J_1 \), a contradiction. If \( a, b \in C[x_2, y_1] \) or \( a, b \in C[y_2, x_1] \), then by 3, \( G \geq_m G_2 \), a contradiction. Finally, if \( a \) and \( b \) are in distinct segments among \( C(x_1, x_2) \), \( C(y_1, y_2) \), \( C(x_2, y_1) \), \( C(y_2, x_1) \), or if \( \{a, b\} = \{x_1, y_1\} \) or if \( \{a, b\} = \{x_2, y_2\} \), then \( G \geq_m G_5 \), a contradiction.

Therefore \( x_1, x_2, y_1, y_2 \) cannot be chosen to be all distinct. Since \( |S_x| \geq 2 \) and \( |S_y| \geq 2 \), and \( |S_x \cup S_y| \geq 3 \), it follows that \( |S_x \cup S_y| = 3 \). Hence, we let \( x_1 = y_2 \) and \( x_2 \neq y_1 \), as in the figure below.

Now, if \( a \) is in one of \( C(x_1, x_2) \) or \( C(y_1, x_1) \), say \( C(y_1, x_1) \), then: if \( b \in C[y_1, x_1] \), then by 3, \( G \geq_m J_1 \), a contradiction; if \( b \in C(x_1, x_2) \), then \( G \geq_m G_4 \); finally, if \( b \in C(x_2, y_1) \), then \( G \geq_m G_3 \). Hence, we have shown that neither \( a \) nor \( b \) can be in \( C(x_1, x_2) \cup C(y_1, x_1) \). If \( a = x_1 \), then if \( b = x_2 \) or \( y_1 \), then by 3, \( G \geq_m J_1 \), a contradiction; and if \( b \in C(x_2, y_1) \), then \( G \geq_m G_3 \), a contradiction. So finally, both \( a \) and \( b \) must be in \( C[x_2, y_1] \). But, then it follows by 3 that \( G \geq_m G_2 \), a contradiction. This concludes the proof of (1) of Proposition 4.1.

Figure 5 shows slightly different embeddings of the \( G_i \)’s from the ones above.
4.2. **Proof of (2).** It is straightforward to verify that the graphs in Figure 6 are minor-minimal \( \notin O^* \) satisfying the hypothesis of (2) and the initial hypothesis of Proposition 4.1. We label them \( J_1, J_2, J_3, J_4, J_5 \). Hence \( J_i \in \text{ob}(O^*) \) for \( i = 1, \ldots, 5 \).

In the remainder of the proof, we assume that \( G \notin \{J_1, J_2, J_3, J_4, J_5, Q_2\} \), where \( Q_2 \in Q \) from Figure 8. Since \( R - \{x, y\} \in O \), it follows by the same arguments as in the proof of Proposition 3.1, that \( G \) must have the following structure:

\[
G = \begin{array}{c}
\text{K}''
\end{array}
\]

where \( K' \) is a \( K \)-subdivision contained in \( R - x \) containing \( y \) (so that \( K' - y \in O \)), and \( K'' \) is a \( K \)-subdivision contained in \( R - y \) containing \( x \) (so that \( K'' - x \in O \)). Note that,

1. \( R \) does not have a path \( P \) from \( x \) to \( y \) that is internally disjoint from \( K' \cup K'' \).

For otherwise, \( G \) would have a 2-separation \( (L', R') \) over \( \{x, y\} \), with \( L' = L \cup P \notin O \) and \( R' = R \notin O \), contradicting the hypothesis that one side must be in \( O \).

Also, note that if \( R \) does not have at least two internally disjoint paths from \( x \) to \( y \), then \( R \) has a cut-vertex \( z \). Note that \( z \) lies at the intersection of \( K' \) and \( K'' \) (for otherwise \( K' \) and \( K'' \) would be disjoint, or \( R - \{x, y\} \) would not be outerplanar). But, \( R - z \in O \) (for otherwise \( K' \) and \( K'' \) would
be disjoint), therefore $G - z \in \mathcal{O}$, a contradiction. Hence, $R$ has at least two internally disjoint paths from $x$ to $y$.

Note that,

2. A graph with the above structure (on the right) does not belong to $\mathcal{O}^*$.

This is because none of its vertices is apex: if $v \in V(G) - V(K')$, then $v$ is not apex, because of $K'$; if $v \in V(G) - V(K'')$, then $v$ is not apex, because of $K''$; and if $v \in V(K') \cap V(K'')$, then $R - v$ has a path from $x$ to $y$, which along with $L$ forms a $K_{2,3}$-subdivision in $G - v$, hence $v$ is not apex.

Fix a planar embedding of $G$ with $x$ and $y$ incident with the outer face. Since $R$ does not have a cut-vertex, it is 2-connected. Let $C$ be the outer cycle of $R$, so that the rest of $R$ is embedded in the closed disk bounded by $C$. Let $P_1$ and $P_2$ be the two internally disjoint paths from $x$ to $y$ whose union is $C$. Note that neither $P_1$ nor $P_2$ is a simple edge, since $xy \notin E(G)$. Note that,

3. There must be a path $P_3$ between $int(P_1)$ and $int(P_2)$ such that $V(P_3) \cap V(C) = \{a, b\}$, where $a \in int(P_1)$ and $b \in int(P_2)$ are the endpoints of $P_3$.

For otherwise, one of $int(P_1)$ or $int(P_2)$ would be vertex-disjoint from $K' \cup K''$, contradicting 1.

Let $\mathcal{P}$ be the set of paths with property 3. By 3, it follows that $\mathcal{P}$ is non-empty. Let $l(\mathcal{P})$ be the length of the longest path in $\mathcal{P}$.

We first suppose that $l(\mathcal{P}) = 1$. Then, all of the paths in $\mathcal{P}$ are simple edges. Let $a_1, a_2, \ldots, a_s \in int(P_1)$ be the left endpoints of the paths in $\mathcal{P}$ in the order of vertices in $P_1$ from $x$ to $y$, and similarly let $b_1, b_2, \ldots, b_t \in int(P_2)$ be the right endpoints of the paths in $\mathcal{P}$ in the order of vertices in $P_2$ from $x$ to $y$. Note that, for any $i = 1, \ldots, s - 1$ (and for any $j = 1, \ldots, t - 1$), if $a_i a_{i+1}$ (or $b_j b_{j+1}$) is not a simple edge, then $G$ has a 2-separation $(L', R')$ over $\{a_i, a_{i+1}\}$ (or over $\{b_j, b_{j+1}\}$).

By the initial hypothesis of Proposition 4.1 and (2) of Lemma 2.2, $L' = P_2$ or $C_4$. However, by the hypothesis of (2) of Proposition 4.1, $L' \neq C_4$, because $G - \{a_i, a_{i+1}\}$ (and $G - \{b_j, b_{j+1}\}$) contains a $K_{2,3}$-subdivision. Hence,

4. For $i = 1, \ldots, s - 1$ and for $j = 1, \ldots, t - 1$, $a_i a_{i+1}$ and $b_j b_{j+1}$ are either simple edges or edges subdivided once.
Similarly, if \( xa_1, xb_1, ya_s, \) or \( yb_i \) is not a simple edge, then \( G \) has a 2-separation \((L', R')\) over the corresponding 2-vertex set, and by the initial hypothesis of Proposition 4.1 and Lemma 2.2, \( L' = P_2 \) or \( C_4 \). If \( L'(x, a_1) = C_4 \) and \( L'(y, a_s) = C_4 \) (or \( L'(x, b_1) = C_4 \) and \( L'(y, b_i) = C_4 \)), then \( G \triangleright_m J_3 \), a contradiction (see figure below). Similarly, \( L'(x, a_1) = C_4 \) and \( L'(y, b_i) = C_4 \) (or \( L'(x, b_1) = C_4 \) and \( L'(y, a_s) = C_4 \)), then \( G \triangleright_m J_1 \), a contradiction (see figure below).

\[
\begin{align*}
\text{Figure: } & \quad J_3 \quad \text{ and } \quad J_1
\end{align*}
\]

Therefore, for one of the sides, say the \( x \)-side, we must have that \( xa_1 \) and \( xb_1 \) are either simple edges, or edges subdivided once. Therefore, it follows by 4 that the vertex \( y \) is apex in \( G \), a contradiction since \( G \notin \mathcal{O}^* \). Thus we have proved that \( l(\mathcal{P}) \geq 2 \).

Let \( P = p_0p_1 \ldots p_n \) be a path in \( \mathcal{P} \) of length \( n := l(\mathcal{P}) \geq 2 \), with \( p_0 \in \text{int}(P_1) \) and \( p_n \in \text{int}(P_2) \).

Since \( G \triangleright_m J_1 \), it follows that:

- **5a.** For \( i = 0, 1, \ldots, n - 2 \), there is no path of length \( \geq 2 \) from \( p_i \) to \( \text{int}(P_1) \) that is internally disjoint from \( P \cup C \).

Note that, by choice of \( P \), the same holds true for \( i = n - 1 \). Similarly:

- **6a.** For \( i = 2, 3, \ldots, n \), there is no path of length \( \geq 2 \) from \( p_i \) to \( \text{int}(P_1) \) that is internally disjoint from \( P \cup C \).

And, by choice of \( P \), the above also holds true for \( i = 1 \). Therefore, equivalently:

- **5b.** For \( i = 0, 1, \ldots, n - 1 \), all the paths from \( p_i \) to \( \text{int}(P_2) \) that are internally disjoint from \( P \cup C \) are simple edges.

- **6b.** for \( i = 1, 2, \ldots, n \), all the paths from \( p_i \) to \( \text{int}(P_1) \) that are internally disjoint from \( P \cup C \) are simple edges.

Let \( P_{11} \) and \( P_{12} \) be the subpaths of \( P_1 \) from \( x \) to \( p_0 \), and from \( p_0 \) to \( y \), respectively. Similarly, let \( P_{21} \) and \( P_{22} \) be the subpaths of \( P_2 \) from \( x \) to \( p_n \), and from \( p_n \) to \( y \), respectively. Let \( C_y \) be the cycle formed from the union of the paths \( P, P_{11} \) and \( P_{21} \), and let \( C_y \) be the cycle formed from the union of the paths \( P, P_{12} \) and \( P_{22} \).

Again, since \( G \triangleright_m J_1 \), it follows that:

- **7.** All the paths in \( \mathcal{P} \) that are internally disjoint from \( P \) are simple edges.

It follows by **5b** and **6b**, that \( G \) does not have a non-trivial bridge (where by a *trivial* bridge, we understand a simple edge) with one foot in \( \text{int}(P) \) and another in \( \text{int}(P_1) \cup \text{int}(P_2) \). Also, if \( G \) has a non-trivial bridge with two feet in \( P \), then if the feet are consecutive vertices of \( P \), then this violates the choice of \( P \); and if they are non-consecutive, then \( G \triangleright_m J_1 \), a contradiction. Therefore:

- **8a.** The only non-trivial bridges of \( G \) that attach to \( \text{int}(P) \) have exactly two feet: one in \( \text{int}(P) \) and the other at \( x \) or \( y \).

Let \( B \) be a non-trivial bridge that attaches to \( \text{int}(P) \). Then, it follows by **8a** that \( B \) has one foot, call it \( p \), in \( \text{int}(P) \) and the other at \( x \) or \( y \), say \( x \). Then \( G \) has a 2-separation \((L', R')\) over \( \{x, p\} \), and it follows by the hypothesis of Proposition 4.1 and Lemma 2.2 that \( L' = P_2 \) or \( C_4 \). Hence, \( B - \{x, p\} \) is a single vertex, or a pair of non-adjacent vertices. We call such a bridge a \( P_2 \)-bridge, or a \( C_4 \)-bridge over \( \{x, p\} \), respectively. Thus we have shown:
8b. If $B$ is a non-trivial bridge with one foot $p \in \text{int}(P)$ and the other at $x$ (or $y$), then $B$ is a $P_2$- or $C_4$-bridge over $\{x,p\}$ (over $\{y,p\}$ respectively).

Let $F_0$ be the set of edges with one endpoint in $\text{int}(P_1) - \{p_0\}$ and the other in $\text{int}(P_2) - \{p_n\}$, and let $F_1$ be the set of edges whose both endpoints are non-consecutive vertices of $P$. Let $F_2$ be the set of edges with one endpoint in $\{p_0,p_1,\ldots,p_n\}$ and the other in $\text{int}(P_2) - \{p_n\}$, and let $F_3$ be the set of edges with one endpoint in $\{p_2,p_3,\ldots,p_n\}$ and the other in $\text{int}(P_1) - \{p_0\}$. Note that $F_0$, $F_1$, $F_2$, and $F_3$ are pairwise disjoint. Let $F := F_0 \cup F_1 \cup F_2 \cup F_3$ if $n \geq 3$. For shorthand, we will say that an edge or a vertex is embedded in the top or in the bottom, if it is embedded in the closed disk bounded by $C_x$ or in the closed disk bounded by $C_y$, respectively. We now prove the following:

9. If $F \neq \emptyset$, then all edges of $F$ can be embedded on one side: top or bottom.

$Pf$. First, suppose that the claim in 9 is not true due to two edges $e$ and $f$ of $F_1$. If the endpoints of $e = p_{i_0}p_{i_1}$ and $f = p_{i_2}p_{i_3}$ overlap, in the sense that $i_0 < i_2 < i_1 < i_3$, then $G \succeq_m J_1$, a contradiction (see figure below).

If the endpoints of $e$ and $f$ do not overlap (in the sense that $i_0 < i_1 < i_2 < i_3$) and, without loss of generality, $e$ is in the top and $f$ is in the bottom, then since $G$ does not have a 2-separation over $\{p_{i_0}, p_{i_1}\}$ (by the initial hypothesis of Proposition 4.1 and (2) of Lemma 2.2), and since the vertices $p_{i_0}$, $p_{i_1}$ are non-consecutive in $P$, there is a path from a vertex in $P(p_{i_0}, p_{i_1})$ to $P_{i_2}$ (note that if the path is to a vertex in $\text{int}(P_2)$, then $G \succeq_m J_1$ as in the overlapping case above; and similarly if the path is to a vertex $p_i \in P$ for some $i < i_0$ or $i > i_1$). Similarly, since $G$ does not have a 2-separation over $\{p_{i_2}, p_{i_3}\}$, there is a path from a vertex in $P(p_{i_2}, p_{i_3})$ to $P_{i_1}$. Therefore $G \succeq_m Q_2$, a contradiction (see figure below).

Second, suppose that the claim in 9 is not true due to two edges $e$ and $f$ of $F_2$ (the proof for $F_3$ is similar). Hence, both $e$ and $f$ have one endpoint in $\{p_0, p_1, \ldots, p_{n-2}\}$, however $e$ has the other endpoint in $\text{int}(P_2)$ and $f$ in $\text{int}(P_2)$. Then, $G - \{x,y\}$ contains a $K_{2,3}$-subdivision, contradicting the hypothesis that $G - \{x,y\}$ is in $\mathcal{O}$.

Third, suppose that the claim in 9 is not true due to an edge $e \in F_3$, embedded, say, in the bottom, and an edge $f \in F_3$ embedded in the top. Then $G$ contains the following minor, which contains a $Q_2$-minor, a contradiction (see figure below).

Fourth, suppose that the claim in 9 is not true due to an edge $e \in F_1$, embedded, say, in the bottom, and an edge $f \in F_2$ (the proof for $f \in F_3$ is similar) embedded in the top. Let $p_{i_0}q := f$
with \( i_0 \in \{0, 1, \ldots, n - 2\} \) and \( q \in \text{int}(P_{21}) \), and let \( p_{i_1}p_{i_2} := e \) with \( i_1 < i_2 \). If \( i_1 \geq i_0 \), then 
\[
G - \{x, y\} \text{ contains a } K_4\text{-subdivision, a contradiction. Hence, } i_1 < i_0. \]
If \( i_2 = n \), then since \( i_0 \in \{0, 1, \ldots, n - 2\} \), it follows that \( G - \{x, y\} \) contains a \( K_{2,3}\)-subdivision, a contradiction. If \( i_2 \in \{i_0, n - 1\} \), then \( G \geq_m J_1 \) (as in the overlapping case), a contradiction. Therefore, \( i_2 \leq i_0 \) and since \( G \) does not have a 2-separation over \( \{p_{i_1}, p_{i_2}\} \) (by the initial hypothesis of Proposition 4.1 and (2) of Lemma 2.2), there is a path from a vertex in \( P(p_{i_1}, p_{i_2}) \) to \( P_{12} - \{p_0\} \), and thus \( G \) contains the following minor, which contains a \( Q_2\)-minor, a contradiction (see figure below).

Finally, suppose that the claim 9 is not true due to an edge \( e \in F_0 \), embedded, say, in the top, and an edge \( f \in F_0 \cup F_1 \cup F_2 \cup F_3 \) embedded in the bottom (the case \( f \in F_0 \) is illustrated below). Then, it can easily be checked that \( G - \{x, y\} \) contains a \( K_4\)- or \( K_{2,3}\)-subdivision, a contradiction. This proves 9.

As in the \( l(\mathcal{P}) = 1 \) case, let \( a_1, a_2, \ldots, a_s \in \text{int}(P_1) \) be the left endpoints of the paths in \( \mathcal{P} \) in the order of vertices on \( P_1 \) from \( x \) to \( y \), and similarly let \( b_1, b_2, \ldots, b_t \in \text{int}(P_2) \) be the right endpoints of the paths in \( \mathcal{P} \) in the order of vertices on \( P_2 \) from \( x \) to \( y \). Similarly to 4, we have that:

**10.** For \( i = 1, \ldots, s - 1 \) and for \( j = 1, \ldots t - 1 \), \( a_ia_{i+1} \) and \( b_jb_{j+1} \) are either simple edges or edges subdivided once.

Similarly, if \( xa_1, xb_1, ya_s, \) or \( yb_t \) is not a simple edge, then \( G \) has a 2-separation \((L', R')\) over the corresponding 2-vertex set, and by the initial hypothesis of Proposition 4.1 and Lemma 2.2, \( L' = P_2 \) or \( C_4 \). Thus:

**11.** If \( xa_1, xb_1, ya_s, \) or \( yb_t \) is not a simple edge, then \( L'(x, a_1), L'(x, b_1), L'(y, a_s), L'(y, b_t) \in \{P_2, C_4\} \), respectively (equivalently, \( G \) has a \( P_2\)- or \( C_4\)-bridge over \( \{x, a_1\}, \{x, b_1\}, \{y, a_s\}, \) or \( \{y, b_t\} \), respectively).

We now have two possibilities: either \( F \neq \emptyset \) or \( F = \emptyset \). We consider them below as Cases 1 and 2, respectively.

**Case 1.** \( F \neq \emptyset \).

It follows from 9 that all the edges of \( F \) can be embedded, say, in the bottom (hence there are no edges of \( F \) embedded in the top). We will show that since \( G \) does not contain \( J_{l}\)-minor for \( i = 1, \ldots, 5 \), the vertex \( x \) will be apex in \( G \), obtaining a contradiction. To do this, we first prove the following.

**12.** The only vertices embedded in the bottom are those lying on the cycle \( C_y \).

**Pf.** We prove this claim by showing that there are no non-trivial bridges embedded in the interior of the disk bounded by \( C_y \). So assume that there is such a bridge \( B \). First, if \( B \) has a foot in \( \text{int}(P) \), then by 8a and 8b, it follows that the other foot of \( B \) is \( y \). Since \( F \neq \emptyset \), it contains an
edge $e \in F_i$ for some $i = 0, 1, 2, 3$. Actually, $e \notin F_0$, for otherwise $e$ would cross $B$, a contradiction.

If $e \in F_1$, then $G$ contains the following minor, which contains a $Q_2$-minor, contradiction (see figure below).

$$G > Q_2$$

And if $e \in F_2$ (the proof for $F_3$ is similar), then $G$ contains the following minor, which again contains a $Q_2$-minor, contradiction (see figure below).

$$G > Q_2$$

Therefore $B$ has its feet in $P_{12} \cup P_{22}$, but it cannot have a foot in $P_{12}$ and another in $P_{22}$, because this would contradict either 5b, 6b, or 7. Hence, $B$ has all of its feet in $P_{12}$ or all in $P_{22}$; by symmetry, we may assume that in $P_{12}$. Let $p$ and $q$ be the first and last feet of $B$ in the order of vertices on $P_{12}$. Then $G$ has a 2-separation $(L', R')$ over $\{p, q\}$, and by the initial hypothesis of Proposition 4.1 and Lemma 2.2, $L' = P_2$ or $C_4$, so that $B$ is a $P_{2}$- or $C_4$-bridge over $\{p, q\}$. Since $F \neq \emptyset$, it follows that $B \neq C_4$ for otherwise $G$ would contain a $J_2$-, $J_4$-, or $J_5$-minor (see figure below).

$$G > J_2 \quad G > J_4 \quad G > J_5$$

Hence, $B = P_2$, and so $B$ is a subgraph of $P_{12}$. This proves 12.

It follows by 12 that $L'(y, a_s) \neq C_4$ and $L'(y, b_t) \neq C_4$. Hence, $ya_s$ and $yb_t$ are either simple edges, or edges subdivided once. However, $L'(x, a_1)$ and $L'(x, b_1)$ could be either $P_2$ or $C_4$, or $xa_1$ and $xb_1$ could be simple edges.

By the fact that there are no edges of $F$ in the top, and from 8a, 8b, 10, and 11, it follows that the only possible edges in the top are:

- edges from $p_1$ to $P_{11}$;
- edges from $p_{n-1}$ to $P_{21}$;
- edges from $\text{int}(P)$ to $x$;
- edges that are part of the $P_2$- or $C_4$-bridges from $\text{int}(P)$ to $x$;
- edges that are part of the $P_2$- or $C_4$-bridges from $a_1$ or $b_1$ to $x$;
- edges of the cycle $C_x$;
Hence, the only possible vertices lying in the interior of the disk bounded by $C_x$ are those from the $P_2$- or $C_4$-bridges from $int(P) \cup \{a_1, b_1\}$ to $x$. Hence, from this and 12 it follows that $G - x$ is outerplanar (i.e. $x$ is an apex vertex of $G$), a contradiction.

**Case 2. $F = \emptyset$.**

Again, by the fact $F$ is empty, and from 8a, 8b, 10, and 11, it follows that the only possible edges in $G$ are:

- edges from $p_1$ to $P_1$;
- edges from $p_{n-1}$ to $P_2$;
- edges from $int(P)$ to $x$ or to $y$;
- edges that are part of the $+P_2$- or $C_4$-bridges from $int(P)$ to $x$ or to $y$;
- edges that are part of the $P_2$- or $C_4$-bridges from $a_1$ or $b_1$ to $x$, and from $a_s$ or $b_t$ to $y$;
- edges of the cycles $C_x$ and $C_y$.

If there are no $P_2$- or $C_4$-bridges from $int(P)$ to $x$ nor to $y$, then, just as in the proof of the $l(P) = 1$ case, if $L'(x, a_1) = C_4$ and $L'(y, a_s) = C_4$ (or $L'(x, b_1) = C_4$ and $L'(y, b_t) = C_4$), then $G \geq_m J_3$. Similarly, if $L'(x, a_1) = C_4$ and $L'(y, b_t) = C_4$ (or $L'(x, b_1) = C_4$ and $L'(y, a_s) = C_4$), then $G \geq_m J_1$. Therefore, for one of the sides, say the $x$-side, we must have that $xa_1$ and $xb_1$ are either simple edges, or edges subdivided once. Hence, $G - y$ is outerplanar, a contradiction.

Hence, there is a $P_2$- or $C_4$-bridge from $int(P)$ to $x$ or to $y$, but there cannot be such bridges to both $x$ and $y$, for otherwise $G$ would contain a $Q_2$-minor. Hence, there is a $P_2$- or $C_4$-bridge from $int(P)$ to, say $x$, but not to $y$. Then, $L'(y, a_s) \neq C_4$ and $L'(y, b_t) \neq C_4$, for otherwise $G \geq_m J_5$ (see figure below).

![Figure](image_url)

Therefore, $ya_s$ and $yb_t$ are either simple edges, or edges subdivided once. Hence, $G - x$ is outerplanar, a contradiction (see figure below).

![Figure](image_url)

This concludes the proof of (2) of Proposition 4.1.

5. **Connectivity 2: One Side Always $P_2$**

In this section, we focus on the case that every 2-separation of $G \in \text{ob}(O^*) - \mathcal{S}$ has one side that is $P_2$ (Subcase 2.2 of the outline given in the Introduction). We prove the following proposition, which says that, in this case, $G \in \mathcal{H} \cup \mathcal{Q}$.

**Proposition 5.1.** If $G \in \text{ob}(O^*) - \mathcal{S}$ is of connectivity 2 and for every 2-separation $S$ of $G$ over $\{x, y\}$ one side is $P_2$, then the following holds true:

1. If $G - \{x, y\} \notin \mathcal{O}$ for some such $S$, then $G$ is a member of the family $\mathcal{H}$;
(2) If $G - \{x, y\} \in \mathcal{O}$ for every such $S$, then $G$ is a member of the family $\mathcal{Q}$.

We define a few terms first. A graph $H$ is \textit{internally} 3-connected if it is 2-connected, and for every 2-cut $\{s, t\}$, $H - \{s, t\}$ has two connected components, one of which is a single vertex. We say that a vertex in $H$ is \textit{pendant} if its degree in $H$ is 1. Similarly, we say that an edge in $H$ is \textit{pendant} if it is incident with a pendant vertex. Before presenting a proof of Proposition 5.1, we first establish some preliminary observations based on the hypotheses of Proposition 5.1, which will be used later in the proof.

It follows from the hypothesis of Proposition 5.1 that $G$ is internally 3-connected. Let $(L, R)$ be a 2-separation over vertices $\{x, y\}$ such that $L = P_2$. Let $v$ be the third (middle) vertex of $L$. Since $G$ is minor-minimal $\notin \mathcal{O}^*$, $G/vy$ has an apex vertex $a$ (i.e. a such that $(G/vy) - a \in \mathcal{O}$). Note that $a \neq y$ and $a \neq x$, for otherwise $y$ (or $x$, respectively) is an apex vertex in $G$, a contradiction.

Since $\text{deg}(v) = 2$, it follows that $G/vy$ is also internally 3-connected. Hence, the only possible 1-separations in $(G/vy) - a$ are those that separate a pendant vertex. Call such 1-separations \textit{trivial}. Therefore, $(G/vy) - a$ is 2-connected up to trivial 1-separations (pendant edges), and outerplanar.

Fix a planar embedding of $G$ so that all the vertices of $(G/vy) - a \in \mathcal{O}$ and $a$ are incident with the outer face (i.e. infinite face). Since $(G/vy) - a$ is 2-connected up to trivial 1-separations, it follows that all the vertices of $(G/vy) - a \in \mathcal{O}$ lie along a cycle $C$, except (possibly) for the vertices of degree 1 in $(G/vy) - a$ that are adjacent to some vertex of $C$. Note that such vertices have degree 2 in $G/vy$ (and in $G$), and that no two of them are adjacent to the same vertex $c$ of $C$, for otherwise $G$ has a 2-separation $(L', R')$ over $\{c, a\}$ such that $L' = C_4$ or $L' \notin \mathcal{O}$ and $R' \notin \mathcal{O}$, contradicting the hypothesis of Proposition 5.1. Since $v \in G - a \notin \mathcal{O}$, it follows that $v$ is embedded in the interior of the disk bounded by $C$. We have

1. The edges of $G$ are:
   - edges of $C$;
   - chords of $C$, that is, edges not in $E(C)$ with both endpoints in $C$ (note that such edges are embedded in the interior of the disk bounded by $C$);
   - edges $vx$ and $vy$, with $x, y \in V(C)$;
   - edges with one endpoint in $C$ and the other at $a$ (or such edges subdivided once).

   Also note that there are no two consecutive vertices in $C$ of degree 2, since such vertices and their neighbors would induce a $P_3$ or a $C_4$ in $G$ giving rise to a 2-separation violating the hypothesis of Proposition 5.1.

   In this context, by a \textit{neighbor of} $a$, we mean a vertex $u$ in $C$ such that $au$ is actually an edge of $G$ or an edge subdivided once. As usual, we denote by $N(a)$ the set of neighbors of $a$. Since $xy \notin E(G)$, it follows that $G$ has vertices in both $C(x, y)$ and $C(y, x)$. Furthermore,

2. $a$ must have a neighbor in both $C(x, y)$ and $C(y, x)$.

   For otherwise, $G$ has a 2-separation over $\{x, y\}$ contradicting the hypothesis of Proposition 5.1.

   Note that a chord must have both of its endpoints in $C[x, y]$ or $C[y, x]$. We say that two chords $c := c_1c_2$ and $d := d_1d_2$ are \textit{non-overlapping} if their endpoints satisfy $c_1 < c_2 \leq d_1 < d_2$ in the cyclic order of $C$, and are said to be \textit{nested} if $c_1 < d_1 < d_2 < c_2$ or $d_1 < c_1 < c_2 < d_2$. It follows from 1 that:
3. If $c := c_1c_2$ is a chord with $c_1 < c_2$ (in the clockwise order restricted to $C[x,y]$ or $C[y,x]$), then $a$ has a neighbor in $C(c_1, c_2)$.

For otherwise, $G$ has a 2-separation over $\{c_1, c_2\}$ contradicting the hypothesis of Proposition 5.1.

Also,

4. Within a single segment $C[x,y]$ or $C[y,x]$, there are no non-overlapping chords (or equivalently, all the chords are nested).

Suppose that the chords $c := c_1c_2$ and $d := d_1d_2$ are non-overlapping with $c_1 < c_2 \leq d_1 < d_2$ within, say $C[x,y]$. Then, by 3, $a$ has a neighbor in $C(c_1, c_2)$ and in $C(d_1, d_2)$, and by 2, it has a neighbor in $C(y,x)$. Then, $G$ contains the following graph as a minor, which we label $Q_1$, and which can easily be verified to belong to $\text{ob(O}^*)$. This is a contradiction, since $G$ is minor-minimal $\not\in \text{O}^*$.

\[ Q_1 = \]

5.1. **Proof of (1).** Let $G - \{x,y\} \not\in \text{O}$ for a 2-separation $S = (L, R)$ of $G$ over $\{x,y\}$. Hence $G - \{x,y\}$ contains a $K$-subdivision as a subgraph, call it $K'$. By 1, it follows that $a$ is a cut-vertex in $G - \{x,y\}$, hence, without loss of generality, $K'$ is a subgraph of $G - C[y,x]$. Let $C'$ be the outer cycle of $K'$. Then, $|V(C') \cap C(x,y)| \geq 2$, for otherwise if $u := V(C') \cap C(x,y)$, then it follows by 1 that $G$ has a 2-separation $(L', R')$ over $\{a,u\}$ such that $L' \not\in \text{O}$ and $R' \not\in \text{O}$, contradicting the hypothesis that one side of $(L', R')$ must be $P_2$ (and so in $\text{O}$).

Let $s, t \in V(C') \cap C(x,y)$ be the first and last vertices, respectively, of $V(C') \cap C(x,y)$ in the clockwise order of $C(x,y)$. Note that $s \neq x$ and $t \neq y$. Also, since $G$ does not contain two disjoint $K$-graphs, it follows that:

5. $G$ does not have a chord with one endpoint in $C[x,s]$ and the other in $C[t,y]$.

It is straightforward to verify that the graphs in Figure 7 are minor-minimal $\not\in \text{O}^*$ satisfying the hypothesis of (1) and the initial hypothesis of Proposition 5.1. We label them $H_1, H_2, H_3, H_4, H_5$. Hence $H_i \in \text{ob(O}^*)$ for $i = 1, \ldots, 5$.

Therefore, if $a$ has at least two neighbors in $C(y,x)$, or one neighbor $z \in C(y,x)$ and $C(y,z) \neq \emptyset$ and $C(z,x) \neq \emptyset$, then it is easy to verify that $G$ contains an $H_i$-minor for some $i = 1, \ldots, 5$ (see figure below). Hence, let $z$ be the only neighbor of $a$ in $C(y,x)$. We only need to consider two cases: either both $C(y,z)$ and $C(z,x)$ are empty, or one of them is empty, say $C(y,z)$, and the other is not.
First, suppose that $C(y, z) = \emptyset$ and $C(z, x) \neq \emptyset$. So $yz \in E(G)$. Then $G$ has the following structure as a subgraph:

6. In $G/yz$, the only apex vertex is $s$.

This is because an apex vertex in $G/yz$ must destroy both $K'$ and the $K_{2,3}$-subdivision with outer cycle $C$. Hence it must be a vertex in $V(C') \cap C(x, y)$. If $u \in C(s, t]$ is apex, then since $s, t,$ and $a$ all lie on $C'$, it follows that in $G/yz - u$ there is a path $P'$ in $C'$ from $a$ to $s$; this path, combined with the (possibly subdivided) edge $ay$ ($= az$) and the path along $C$ from $y$ to $s$ form an outer cycle of a $K_{2,3}$-subdivision with inner path $x, v, y$. Hence, $G/yz - u$ contains a $K_{2,3}$-subdivision, a contradiction. This proves 6.

7. $y$ ($= z$) is a cut-vertex in $G/yz - s$.

Note that there are no edges (or edges subdivided once) from $a$ to $C(z, x)$ in $G/yz$, since $z$ is the only neighbor of $a$ in $C(y, x)$ in $G$. Also, note that there are no edges (or edges subdivided once) from $a$ to $C[x, s]$ in $G/yz$, for otherwise $G/yz - s$ contains a $K_{2,3}$-subdivision, contradicting 6. Finally, there are no chords from $C[x, s]$ to $C(s, t]$ in $G/yz$, for otherwise $G/yz - s$ contains a $K_{2,3}$-subdivision. These facts combined with 5 imply 7.

Therefore, it follows by 7 that after uncontracting edge $yz$ in $G/yz - s$, the resulting graph $G - s$ is also outerplanar, a contradiction since $G \notin \mathcal{O}^*$.  

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Now consider the other case that both \( C(y, z) \) and \( C(z, x) \) are empty (so that \( yz, zx \in E(G) \)). Recall that \( z \) is the only neighbor of \( a \) in \( C(y, x) \). Then \( G \) has the following structure as a subgraph:

\[
\begin{array}{c}
\text{a} \\
\text{s} \quad \text{t} \quad \text{y} \\
\text{K'}
\end{array}
\]

Similarly to 6, we obtain the following fact.

8. In \( G \setminus az \), the only possible apex vertices are \( s \) and \( t \).

We use the above to prove the following key fact.

9. One or both of the following hold:
   (i) \( xs \) is an edge of \( G \) (or an edge subdivided once) and \( \deg(x) = 3 \);
   (ii) \( yt \) is an edge of \( G \) (or an edge subdivided once) and \( \deg(y) = 3 \).

Note that if a vertex in \( C(x, s) \) or \( C(t, y) \) has degree \( \geq 3 \), then it is a neighbor of \( a \) or an endpoint of a chord. Similarly, if \( \deg(x) \geq 4 \) or \( \deg(y) \geq 4 \), then \( x \), respectively \( y \), is a neighbor of \( a \) or an endpoint of a chord. To prove 9, we first note that \( a \) does not have neighbors in both \( C[x, s] \) and \( C[t, y] \), for otherwise \( G \setminus az \) has no apex vertex (since neither \( s \) nor \( t \) is apex in \( G \setminus az \)), a contradiction. Hence, by symmetry, we may assume that \( a \) has no neighbors in \( C[x, s] \). Then, by 3, there are no chords with both endpoints in \( C[x, s] \). If \( a \) has a neighbor in \( C(t, y) \), then there are no chords with one endpoint in \( C[x, s] \) and the other in \( C(s, t) \), for otherwise \( G \setminus az \) has no apex vertex (note that the other endpoint cannot lie in \( C(t, y) \) by 5), and thus (i) holds. And if \( a \) has no neighbors in \( C(t, y) \) then, again by 3, there are no chords with both endpoints in \( C[t, y] \). Therefore, the only chords in \( G \) are those with one endpoint in \( C[x, s] \) and the other in \( C(s, t) \) (in which case (ii) holds), or those with one endpoint in \( C[s, t] \) and the other in \( C(t, y) \) (in which case (i) holds), but not both, since two such chords would either cross or would be non-overlapping, violating 4. This proves 9.

By symmetry, we may assume that (i) holds in 9, so that \( xs \) is an edge of \( G \) (or an edge subdivided once, in which case denote the subdividing vertex by \( w \)). In the remainder of the proof, by \( G/xs \) we mean the graph obtained from \( G \) by contracting the path (of length 1 or 2) along \( C \) from \( s \) to \( x \).

10. In \( G/xs \), the only apex vertex is \( s (= x) \), unless (ii) in 9 also holds, then \( t \) may also be apex.

If (ii) does not hold, then either \( a \) has a neighbor in \( C(t, y) \) or \( G \) has a chord with one endpoint in \( C[s, t] \) and the other in \( C(t, y) \). And in either case \( t \) is not apex in \( G/xs \).

Note that \( (G/xs) - s = G - \{x, s\} \) (or possibly \( (G/xs) - s = G - \{x, w, s\} \) if \( xs \) is subdivided). Re-embed the graph \( (G/xs) - s \in \mathcal{O} \) (if necessary), so that all of its vertices are incident with the outer face. In \( (G/xs) - s \), \( \deg(z) = 2 \) and \( \deg(v) = 1 \), hence edges \( zy \) and \( vy \) are also incident with the outer face. Since \( yz \) is a simple edge, by putting \( x \) (and possibly \( w \)) back in, we obtain an embedding of \( G - s \) in which all the vertices are still incident with the outer face, hence \( G - s \) is outerplanar, a contradiction (see figure below).
Finally, if \( t \) is also apex in \( G/xs \), then by the above, (ii) in 9 also holds, so that \( yt \) is an edge of \( G \) (or an edge subdivided once, in which case denote the subdividing vertex by \( u \)) and \( \text{deg}(y) = 3 \). Since \( (G/xs) - t \in \mathcal{O} \), there is a face \( f \) in the current embedding incident with all the vertices of \( (G/xs) - t \). Since the path (of length 1 or 2) from \( s \) to \( x \) can be uncontracted along \( C \), it follows that \( f \) is also incident with all the vertices of \( G - t \), a contradiction since \( G \notin \mathcal{O}^* \) (see figure below).

This concludes the proof of (1) in the case that both \( C(y, z) \) and \( C(z, x) \) are empty, as well as the proof of (1) of Proposition 5.1.

5.2. **Proof of (2).** It is straightforward to verify that the graphs in Figure 8 are minor-minimal \( \notin \mathcal{O}^* \) satisfying the initial hypothesis and the hypothesis of (2) of Proposition 5.1. We label them \( Q_1, Q_2, Q_3, Q_4, Q_5 \). Hence \( Q_i \in \text{ob}(\mathcal{O}^*) \) for \( i = 1, \ldots, 5 \).

In the remainder of the proof, we assume that \( G \notin \{Q_1, Q_2, Q_3, Q_4, Q_5\} \). Observe that now the vertex \( a \) from 2 satisfies

11. \( \text{deg}(a) \geq 3 \).

For otherwise, if \( \text{deg}(a) = 2 \), then let the two neighbors of \( a \) be \( a_1 \) and \( a_2 \) (in \( C(x, y) \) and \( C(y, x) \), respectively, by 2). Note that there is a chord with one endpoint in \( C[x, a_1] \) and the other
in $C(a_1, y]$, for otherwise, it follows by 1 that $a_1$ is apex in $G$, a contradiction. Similarly, there is a chord with one endpoint in $C[y, a_2)$ and the other in $C(a_2, x]$, for otherwise, it follows by 1 that $a_2$ is apex in $G$, a contradiction. Since $\deg(a) = 2$, it follows that $G$ has a 2-separation over \{a_1, a_2\} such that $G - \{a_1, a_2\}$ contains a $K_{2,3}$-subdivision, contradicting the hypothesis of (2) that $G - \{a_1, a_2\} \in \mathcal{O}$. This proves 11.

**Case 1.** $G$ has no chords.

**Subcase 1.1.** $|N(a) \cap C(x, y)| = 1$ and $|N(a) \cap C(y, x)| = 1$.

Then, there is a subdivided edge $ay$, for otherwise $x$ is apex. Also, there is a subdivided edge $ax$, for otherwise $y$ is apex, and hence $G \succeq_m Q_2$, a contradiction.

**Subcase 1.2.** $|N(a) \cap C(x, y)| = 2$ and $|N(a) \cap C(y, x)| = 1$.

First suppose that $x, y \notin N(a)$. Let $a_1 \in N(a) \cap C(y, x)$ and $a_2, a_3 \in N(a) \cap C(x, y)$ in the clockwise order around $C$. Then, there is a vertex in $C(a_2, a_3)$, for otherwise $a_1$ is apex. Edge $aa_3$ is subdivided, for otherwise $x$ is apex. Edge $aa_2$ is subdivided, for otherwise $y$ is apex. There is a vertex in $C(y, a_1)$, for otherwise $a_2$ is apex. Finally, there is a vertex in $C(a_1, x)$, for otherwise $a_3$ is apex, and hence $G \succeq_m Q_1$.

Next, suppose that $x \in N(a)$, but $y \notin N(a)$. Then, edge $aa_3$ is subdivided, for otherwise $x$ is apex. Edge $ax$ is not subdivided, for otherwise $G \succeq_m Q_2$. Edge $aa_2$ is subdivided, for otherwise $y$ is apex. Finally, there is a vertex in $C(a_1, x)$, for otherwise $a_3$ is apex, and hence $G \succeq_m J_1$.

Finally, suppose that $x, y \in N(a)$. Then, at least one of $aa_3$, $ay$ is subdivided, for otherwise $x$ is apex. Also, at least one of $aa_2$, $ax$ is subdivided, for otherwise $y$ is apex. If $aa_2$ and $aa_3$ are, then $G \succeq_m J_1$. If $ax$ and $ay$ are, then $G \succeq_m Q_2$. Finally, if $ax$ and $aa_3$ are, or $a_2$ and $ay$ are, then again $G \succeq_m Q_2$, a contradiction.

**Subcase 1.3.** $|N(a) \cap C(x, y)| \geq 3$ and $|N(a) \cap C(y, x)| = 1$.

Let $a_1 \in N(a) \cap C(y, x)$ and $a_2, a_3 \in N(a) \cap C(x, y)$ be such that $a_2$ is the vertex in $N(a) \cap C(x, y)$ closest to $x$, and $a_3$ is the vertex in $N(a) \cap C(x, y)$ closest to $y$. Note that if $u \in N(a) \cap C(a_2, a_3)$, then edge $au$ is not subdivided, for otherwise $G - \{x, y\}$ contains a $K_{2,3}$-subdivision, contradicting the hypothesis that $G - \{x, y\} \in \mathcal{O}$.

Therefore, at least one of $aa_3$, $ay$ (if $ay \in E(G)$) is subdivided, for otherwise $x$ is apex. Also, at least one of $aa_2$, $ax$ (if $ax \in E(G)$) is subdivided, for otherwise $y$ is apex. Hence, $G \succeq_m Q_2$, a contradiction.

**Subcase 1.4.** $|N(a) \cap C(x, y)| \geq 2$ and $|N(a) \cap C(y, x)| \geq 2$.

Let $a_1, a_2 \in N(a) \cap C(y, x)$ and $a_3, a_4 \in N(a) \cap C(x, y)$ be such that $a_1$ and $a_4$ are the two neighbors of a closest to $y$, and $a_2$ and $a_3$ are the two neighbors of $a$ closest to $x$. Note that if $u \in N(a) \cap (C(a_1, a_2) \cup C(a_3, a_4))$, then edge $au$ is not subdivided, for otherwise $G - \{x, y\}$ contains a $K_{2,3}$-subdivision, contradicting the hypothesis that $G - \{x, y\} \in \mathcal{O}$.

Therefore, at least one of $aa_1, aa_4$, $ay$ (if $ay \in E(G)$) is subdivided, for otherwise $x$ is apex. Also, at least one of $aa_2$, $aa_3$, $ax$ (if $ax \in E(G)$) is subdivided, for otherwise $y$ is apex. Hence, it follows from these two facts that if $ay \in E(G)$ and it is subdivided, then $G \succeq_m Q_2$, a contradiction. Similarly, if $ax \in E(G)$ and it is subdivided, then $G \succeq_m Q_2$, a contradiction. Hence, if $ax \in E(G)$ or $ay \in E(G)$, then they are not subdivided. Finally, if $aa_1$ and $aa_2$ are, or if $aa_3$ and $aa_4$ are,
then $G \geq_m Q_5$, a contradiction. And if $aa_1$ and $aa_3$ are, or if $aa_2$ and $aa_4$ are, then $G \geq_m Q_2$, a contradiction. This concludes the proof of (2) of Proposition 5.1 in Case 1.

**Case 2.** $G$ has a chord.

We first strengthen 3 to the following:

12. If $c := c_1c_2$ is a chord with $c_1 < c_2$ (in the clockwise order restricted to $C[x, y]$ or $C[y, x]$), then $a$ has a neighbor in $C(c_1, c_2)$. Furthermore, for any such neighbor $w$, the edge $aw$ is not subdivided.

For otherwise, $G$ would have a 2-separation over $\{a, w\}$ such that $G - \{a, w\}$ has $K_{2,3}$-subdivision contradicting the hypothesis that $G - \{a, w\} \in \mathcal{O}$.

The following two claims greatly limit the structure of $G$.

**Claim 1.** Let $c = c_1c_2$, with $c_1, c_2 \in C(x, y)$ in the clockwise order around $C$, be an innermost chord of $G$ (in the sense that there are no other chords with both endpoints in $C[c_1, c_2]$). Then $a$ does not have two neighbors in $C(c_1, c_2)$.

**Pf.** Suppose that $a$ does have two neighbors $a_1, a_2 \in C(c_1, c_2)$. By 12, edges $aa_1$ and $aa_2$ are not subdivided. Also, $a$ does not have any other neighbors in $C(x, y)$, for otherwise $G - \{x, y\}$ would contain a $K_4$-subdivision, violating the hypothesis of (2) that $G - \{x, y\} \in \mathcal{O}$. Also, $C(a_1, a_2) = \emptyset$, for otherwise $G - \{x, y\}$ would contain a $K_{2,3}$-subdivision, violating the hypothesis of (2). Note that possibly, edges $a_1a_2$ and $a_2c_2$ are subdivided once, but since $c$ is an innermost chord, there are no other vertices in $C(c_1, c_2)$. If $a$ has at least two neighbors in $C(y, x)$, then $G \geq_m Q_5$, a contradiction. Hence, let $z$ be the only neighbor of $a$ in $C(y, x)$.

We let $u$ be an apex vertex in $G\setminus a_1a_2$, and we assume that the graph $(G\setminus a_1a_2) - u \in \mathcal{O}$ is embedded in the plane with all of its vertices incident with the outer face. Note that $u \in \{z, c_1, c_2\}$, for otherwise: if $u \in \{a_1, a_2\}$, then clearly $u$ is apex in $G$, a contradiction; if $u \in \{a\} \cup C(c_1, a_1) \cup C(a_2, c_2)$, then $(G\setminus a_1a_2) - u$ contains a $K_{2,3}$-subdivision; and if $u \in \{v\} \cup C(c_2, z) \cup C(z, c_1)$, then $(G\setminus a_1a_2) - u$ contains a $K_4$-subdivision.

If $u = z$, then the only neighbors of $a$ are $a_1$, $a_2$ and $z$ (because if $x$ or $y$ is a neighbor of $a$ then $(G\setminus a_1a_2) - z$ contains a $K_{2,3}$-subdivision). Then, in $(G\setminus a_1a_2) - z$, $\deg(a) = 2$, hence edges $aa_1$ and $aa_2$ are incident with the outer face, and by putting the edge $a_1a_2$ back in, we obtain an embedding of $G - z$ in which all the vertices are still incident with the outer face, hence $G - z$ is outerplanar, a contradiction.

Finally, suppose that $u = c_1$ (the case $u = c_2$ is symmetric). If $c_1a_1$ is subdivided once, then let $b$ be the subdividing vertex. Then, in $(G\setminus a_1a_2) - c_1$, $\deg(a_1) = 1$ (except if $c_1a_1$ is subdivided by $b$, then $\deg(a_1) = 2$, but $a_1$ is adjacent to $b$ with $\deg(b) = 1$, that is $a_1b$ is a pendant edge), and $\deg(a_2) = 2$. Hence edges $aa_2$ and $aa_1$ (and possibly $a_1b$) are incident with the outer face, and since $aa_2$ is a simple edge, we can put edge $a_1a_2$ back in to obtain an embedding of $G - c_1$ in which all the vertices are still incident with the outer face, a contradiction. This proves Claim 1.

**Claim 2.** $G$ does not have a chord with both endpoints distinct from $x$ and $y$.

**Pf.** Suppose that $G$ does have a chord with endpoints $s, t \in C(x, y)$ in the clockwise order around $C$. We may assume, without loss of generality, that $st$ is the innermost chord, in the sense that there are no other chords with both endpoints in $C[s, t]$. By 12, there is a vertex $w \in N(a) \cap C(s, t)$ and the edge $aw$ is not subdivided. Also, by Claim 1, $N(a) \cap C(s, t) = \{w\}$. Also, $a$ does not have neighbors in both $C(x, s)$ and $C[t, y]$, for otherwise $G - \{x, y\}$ would contain a $K_4$-subdivision,
violating the hypothesis that \( G - \{x, y\} \in \mathcal{O} \). Also, by 4, \( G \) does not have chords with both endpoints in \( C[x, s] \) or both in \( C[t, y] \). Let \( z \in N(a) \cap C(y, x) \). First, we show Claim 2a and then Claim 2b. They are needed for the proof of Claim 2.

**Claim 2a.** Neither \( s \) nor \( t \) can be a neighbor of \( a \).

**Pf.** By symmetry, we may assume that \( t \) is a neighbor of \( a \), so that \( s \) is not. Then, \( C(x, s) \cap N(a) = \emptyset \). Also, \( C(w, t) = \emptyset \), for otherwise \( G - \{x, y\} \) would contain a \( K_{2,3} \)-subdivision. Also, edges \( sw \) and \( ta \) are possibly subdivided once, but by choice of chord \( c \), there are no other vertices in \( C(s, t) \). Hence \( G \) contains the following subgraph:

![Subgraph](image_url)

First, suppose that edge \( ta \) is subdivided by vertex \( u \). Then \( C(t, y) \cap N(a) = \emptyset \), for otherwise \( G \geq_m Q_3 \). For the same reason, we have that \( (C(y, z) \cup C(z, x)) \cap N(a) = \emptyset \). Hence, the only neighbor of \( a \) other than \( z \), \( w \) and \( t \) is possibly \( x \). Furthermore, if \( ax \in E(G) \) then it is not subdivided for otherwise \( G \geq_m Q_2 \). Also, note that the remaining chords whose endpoints lie in \( C[x, y] \) must have one of their endpoints at \( t \), and the other in \( C[x, s] \), for otherwise 4 is violated, or the subdivided edge \( ta \) violates 12. It follows from all of the above that if \( C(z, x) = \emptyset \), then \( t \) is apex in \( G \), a contradiction. Hence \( C(z, x) \neq \emptyset \). Then, if \( ax \in E(G) \), then \( G \geq_m J_1 \) (by contracting \( z \) to \( y \), contracting \( s \) to \( x \), and deleting \( ws \)). Thus \( ax \notin E(G) \). Therefore, since \( C(w, t) = \emptyset \), if \( G \) has no chords with one endpoint in \( C[y, z] \) and the other in \( C(z, x) \), then \( z \) is apex in \( G \), a contradiction. Hence, \( G \) does have at least one such chord \( c \). If \( c \) has one endpoint in \( C(z, x) \) and the other in \( C[y, z] \), then \( G \geq_m Q_3 \) (by contracting \( z \) to \( a \), and \( s \) to \( x \) ). Hence, \( c \) has one endpoint at \( x \) and the other in \( C(y, z) \), but then again \( G \geq_m Q_2 \) (by deleting \( st \), contracting \( z \) to \( a \), contracting \( s \) to \( x \), and contracting \( t \) to \( y \) ), a contradiction. Thus we have shown that \( ta \) is not subdivided, that is \( ta \in E(G) \).

We let \( p \) be an apex vertex in \( G \backslash wt \), and we assume that the graph \( (G \backslash wt) - p \in \mathcal{O} \) is embedded in the plane with all of its vertices incident with the outer face. Note that \( p \notin \{w, t\} \), for otherwise \( p \) is apex in \( G \). In fact, it is easy to see that if \( p \notin \{z\} \cup C[x, s] \), then \( p \) is not apex in \( G \backslash wt \), a contradiction. \( G \) and \( G \backslash wt \) contain the following subgraphs, respectively:

![Subgraph](image_url)

Suppose that \( p = z \). Then, \( a \) has no neighbors other than \( w, t, \) and \( z \), for otherwise \( (G \backslash wt) - p \) contains a \( K_4 \)-subdivision. Therefore, in the graph \( (G \backslash wt) - p \), \( \deg(a) = 2 \), hence edges \( aw \) and \( at \) are incident with the outer face, and we can put edge \( wt \) back in, to obtain an embedding of
\(G - z\) in which all the vertices are still incident with the outer face, hence \(G - z\) is outerplanar, a contradiction.

Therefore \(p \in C[x, s]\). Recall from above that \(C(x, s) \cap N(a) = \emptyset\). Note that there are no chords with one endpoint in \(C[x, p]\) and the other in \(C[t, y]\), for otherwise \((G \setminus wt) - p\) contains a \(K_{2,3}\)-subdivision. Also, if a chord has one endpoint in \(C[p, s]\), then its other endpoint is \(t\), for otherwise \((G \setminus wt) - p\) contains a \(K_4\)-subdivision. For simplicity, assume that \(c = ct\) is the only such chord with \(c_1 \neq s\). If there is more than one such chord, the argument is similar. Also, note that edges \(pc_1, c_1s,\) and \(sw\) may be subdivided once, but the subdividing vertices can be ignored for the purposes of this argument, as will be apparent soon. So for simplicity, we assume that \(pc_1, c_1s,\) and \(sw\) are simple edges. By the observations above, it follows that in \((G \setminus wt) - p\), \(\deg(w) = 2,\) and \(\deg(c_1) = 2\), hence edges \(wa, ws, c_1s\) and \(ct\) are incident with the outer face, which implies that edge \(st\) is not. Therefore, since in \((G \setminus wt) - p\), \(\deg(s) = 3\), it follows that we can put edge \(wt\) back in, to obtain an embedding of \(G - p\) in which all of the vertices are still incident with the outer face, hence \(G - p\) is outerplanar, a contradiction (see figure below). Finally, note that if edges \(pc_1, c_1s,\) and \(sw\) are subdivided once, then its subdividing vertices are still incident with the outer face in the above embedding of \(G - p\), since in the above argument edges \(c_1s\) and \(sw\) are incident with the outer face. This proves Claim 2a.

Therefore, neither \(s\) nor \(t\) is a neighbor of \(a\). We now show furthermore:

**Claim 2b.** \(a\) does not have a neighbor in \(C(x, s) \cup C(t, y)\).

**Pf.** By symmetry, suppose that \(N(a) \cap C(t, y) \neq \emptyset\), so that \(N(a) \cap C(x, s) = \emptyset\), and let \(t' \in N(a) \cap C(t, y)\). Then, all the chords that have an endpoint in \(C(t, y)\) have the other endpoint at \(x\), for otherwise 4 is violated, or \(G - \{x, y\}\) contains a \(K_4\)-subdivision. Also, \(C(w, t) = \emptyset\), for otherwise \(G - \{x, y\}\) would contain a \(K_{2,3}\)-subdivision, violating the hypothesis of (2) that \(G - \{x, y\} \in \mathcal{O}\).

First, suppose that edge \(t'a\) is subdivided by vertex \(a\). Then, \(C(t, t') \cap N(a) = \emptyset\), for otherwise \(G \geq_m Q_2\). Also, \(C(t', y) \cap N(a) = \emptyset\), for otherwise \(G \geq_m Q_3\). For the same reason, we have that \((C(y, z) \cup C(z, x)) \cap N(a) = \emptyset\). Hence, the only neighbor of \(a\) other than \(z, w\) and \(t'\) is possibly \(x\). Furthermore, if \(ax \in E(G)\) then it is not subdivided for otherwise \(G \geq_m Q_2\). Now consider what the remaining chords within \(C[x, y]\) are. Note that a chord cannot have an endpoint in \(C(t', y)\), since it would violate either 4 or 12. And it cannot have an endpoint at \(t\), since the other endpoint would be in \(C[x, s]\), and \(G\) would contain a \(Q_2\)-minor; and similarly it cannot have an endpoint at \(C(t, t')\) (and hence the other at \(x\)). Hence, all the remaining chords whose endpoints lie in \(C[x, y]\) have an endpoint at \(t'\). It follows from all of the above that if \(C(z, x) = \emptyset\), then \(t'\) is apex in \(G\), a contradiction. Hence \(C(z, x) \neq \emptyset\), and so \(G \geq_m Q_5\) (by contracting \(s\) to \(x\) and deleting all the chords incident with \(t'\)), a contradiction. Thus we have shown that \(t'a\) is not subdivided, that is \(t'a \in E(G)\).
We will now proceed to show, in a sequence of steps, that the only possible chords with both endpoints in $C[x, y]$ other than $st$ are the ones with one endpoint at $x$ and the other in $C[t', y]$.

Recall from above that:

A. All chords that have an endpoint in $C(t, y)$ have the other endpoint at $x$.

We prove

B. There is no chord with one endpoint at $t$ and the other in $C[x, s]$.

For otherwise, let $u \in C[x, s]$ be the other endpoint of such a chord, and choose $u$ to be the closest to $s$, in the sense that there are no chords with both endpoints in $C[u, t]$ other than $st$ and $ut$. Note that, $us$ and $sw$ are either edges of $G$ or edges subdivided once, but again we may assume, without loss of generality, that $us$ and $sw$ are just simple edges. Let $p$ be an apex vertex in $G \setminus st$. It is easy to see that $p \in C[x, u]$. If $p = u$, then there are no more chords with an endpoint at $t$, for otherwise $(G \setminus st) - u$ contains a $K_{2,3}$-subdivision. Hence, in $(G \setminus st) - u$, $deg(t) = 2$ and $deg(s) = 1$, hence edge $wt$ and the pendant edge $ws$ are incident with the outer face. Therefore, since $C(w, t) = \emptyset$ (equivalently, $wt \in E(G)$), we can put edge $st$ back into this embedding to obtain an outerplanar embedding of $G - u$, a contradiction. Therefore, we must have $p \in C[x, u]$. Also, if a chord has one endpoint in $C(p, u)$, then its other endpoint is $t$, for otherwise if the other endpoint is $y$, then $(G \setminus st) - p$ contains a $K_4$-subdivision. For simplicity, assume that $c = c_1t$ is the only such chord with $c_1 \neq s$. If there is more than one such chord, the argument is similar. Also, note that edges $pc_1$, $c_1u$, $us$ and $sw$ may be subdivided once, but again we may assume, without loss of generality, that they are all just simple edges (since they will turn out to be incident with the outer face in $(G \setminus st) - p$). By the observations above, it follows that in $(G \setminus st) - p$, $deg(s) = 2$, and $deg(c_1) = 2$, hence edges $su$, $sw$, $c_1u$ and $c_1t$ are incident with the outer face, which implies that edge $ut$ is not. Therefore, since in $(G \setminus st) - p$, $deg(u) = 3$, it follows that by putting edge $st$ back in, we can embed $G - p$ so that all the vertices are still incident with the outer face, hence $G - p$ is outerplanar, a contradiction (see figure below). This proves B.

C. There is no chord with one endpoint in $C(t, t')$ and the other at $x$.

Suppose the contrary, and let $u \in C(t, t')$ be the endpoint of such a chord. By B, there is no chord with one endpoint at $t$ and the other in $C[x, s]$, hence $xs$ is an edge, or an edge subdivided once. Note that, $xs$ and $sw$ are either edges of $G$ or edges subdivided once, but again we may assume, without loss of generality, that $xs$ and $sw$ are just simple edges. Let $p$ be an apex vertex
in $G\backslash pt$. It is easy to see that $p = x$. Hence, in $(G\backslash pt) - p$, $deg(t) = 2$ and $deg(s) = 1$, hence edge $ut$ and the pendant edge $ws$ are incident with the outer face. Therefore, by putting edge $st$ back into this embedding, we obtain an outerplanar embedding of $G - x$, a contradiction. This proves C.

D. There is no chord with one endpoint at $y$ and the other in $C(x, s)$.

Suppose the contrary, and let $u \in C(x, s)$ be the endpoint of such a chord, and choose $u$ to be the closest to $s$, in the sense that there is no other chords with one endpoint at $y$ and the other in $C(u, s)$. Therefore, $us$ and $sw$ are either edges of $G$ or edges subdivided once, but again we may assume, without loss of generality, that $us$ and $sw$ are just simple edges. It is easy to see that $u$ is the only possible apex vertex in $G\backslash wt$. First, if $u \in C(x, s)$, then in $(G\backslash wt) - u$, $deg(s) = 2$, hence edges $sw$ and $st$ are incident with the outer face. Therefore, by putting edge $wt$ back into this embedding, we obtain an outerplanar embedding of $G - u$, a contradiction. Finally if $u = s$, then in $(G\backslash wt) - s$, $deg(t') = 3$ and $deg(t) = 1 = deg(w)$, hence edge $t'a$ and pendant edges $tt'$, $aw$ are incident with the outer face. Therefore, since $at'$ is a simple edge, by putting edge $wt$ back into this embedding, we obtain an outerplanar embedding of $G - s$, a contradiction. This proves D.

It follows by A - D that:

E. The only possible chords with both endpoints in $C[x, y]$ other than $st$ are the ones with one endpoint at $x$ and the other in $C[t', y]$.

Hence, $xs$ and $sw$ are either edges of $G$ or edges subdivided once, but again we may assume, without loss of generality, that $xs$ and $sw$ are just simple edges. In the remainder of the proof of Claim 2b, by $G/xs$ we mean the graph obtained from $G$ by contracting the path (of length 1 or 2) along $C$ from $s$ to $x$. Let $p$ be an apex vertex in $G/xs$. It is easy to see that $p = x$ or $p = t'$. If $p = x$, then in $(G/xs) - x$, $deg(w) = 2 = deg(t)$, hence edge $wt$ is incident with the outer face. Therefore, by putting edges $ws$ and $st$ back into this embedding, we obtain an outerplanar embedding of $G - x$, a contradiction. And if $p = t'$, then observe the following facts. First, there are no chords with one endpoint at $x$ and the other in $C(t', y)$, therefore, by E, the only possible chord with both endpoints in $C[x, y]$ other than $st$ is $xt'$. Second, $a$ has no other neighbors, except possibly $x$, for otherwise $(G/xs) - t'$ contains a $K_4$-subdivision. And if $x \in N(a)$, then $xa$ is not subdivided. Third, $C(z, x) = \emptyset$, and the only edges left in $G$ are chords from $x$ to $C(y, z)$. These facts account for all the edges of $G$. Hence $t'$ is apex in $G$, a contradiction. This concludes the proof of Claim 2b.

We now finish the proof of Claim 2. Note that edges $sw$ and $wt$ are possibly subdivided, but again we may assume, without loss of generality, that they are simple edges. It follows from Claims 2a and 2b that $a$ does not have neighbors in $C(x, s) \cup C[t, y]$. Also, by 4, there are no chords with both endpoints in $C[x, s]$ or both in $C[t, y]$. Again, we let $p$ be an apex vertex in $G/wa$. It follows from 11 that besides $w$ and $z$, $a$ has another neighbor (in $C[y, x]$). Therefore $p \neq z$, since $(G/wa) - z$ contains a $K_4$-subdivision. In fact, it is easy to check that $p \in C[x, s] \cup C[t, y]$, for otherwise $(G/wa) - p$ contains a $K$-subdivision.
By symmetry, we may assume that \( p \in C[x, s] \). First, if \( p = s \), then all the chords whose endpoints lie in \( C[x, y] \) have an endpoint at \( s \), for otherwise \( (G/wa) - s \) contains a \( K_{2,3} \)-subdivision. Thus, in \( (G/wa) - s \), \( \deg(t) = 2 \), hence edge \( ta \) is incident with the outer face. Therefore, in the current embedding of \( (G/wa) - s \), we can subdivide edge \( ta \) by \( w \) to obtain an embedding of \( G - s \) in which all the vertices are still incident with the outer face, hence \( G - s \) is outerplanar, a contradiction.

Therefore, \( p \in C[x, s] \). Then, by Claims 2a and 2b, \( a \) has no neighbors in \( C[p, s] \). If a chord has an endpoint in \( C[p, s] \), then its other endpoint is \( t \), otherwise \( (G/wa) - p \) contains a \( K_4 \)-subdivision. For simplicity, assume that \( c = c_1t \) is the only such chord with \( c_1 \neq s \). If there is more than one such chord, the argument is similar. Again, the edges \( pc_1 \) and \( c_1s \) may be subdivided once, but the subdividing vertices can be ignored for the purposes of this argument. So for simplicity, we assume that \( pc_1 \) and \( c_1s \) are simple edges. By the observations above, it follows that in \( (G/wa) - p \), \( \deg(c_1) = 2 \), hence edges \( c_1s \) and \( c_1t \) are incident with the outer face, which implies that edge \( st \) is not. Therefore, since in \( (G/wa) - p \), \( \deg(s) = 3 \), it follows that \( sa \) is also incident with the outer face (and hence edge \( at \) is not, for otherwise the edges of the cycle \( a, t, c_1, s, a \) are all incident with the outer face, which implies that those are all the vertices in \( (G/wa) - p \), since \( (G/wa) - p \) has no non-trivial 1-separations, a contradiction). Therefore, it follows that in the current embedding of \( (G/wa) - p \), we can delete edge \( sa \), subdivide edge \( at \) by vertex \( w \), and add edge \( ws \) and obtain an embedding of \( G - p \) in which all the vertices are still incident with the outer face, hence \( G - p \) is outerplanar, a contradiction (see figure below). This concludes the proof of Claim 2.

We now finish the proof Case 2 (“\( G \) has a chord”) and thus the entire proof of (2) of Proposition 5.1. By 4 and Claim 2, it follows that within each of the two segments \( C[x, y] \) and \( C[y, x] \) all the chords have an endpoint at \( x \) or all the chords have an endpoint at \( y \). We have three subcases (of Case 2: “\( G \) has a chord”):

**Subcase 2.1.** There are chords within \( C[x, y] \) and within \( C[y, x] \), and the ones within \( C[x, y] \) have an endpoint at \( y \), and the ones within \( C[y, x] \) have an endpoint at \( x \).

Let \( c_1y \) and \( d_1x \) be innermost chords within \( C[x, y] \) and \( C[y, x] \), respectively. By 12, \( a \) has a neighbor \( w \in C(c_1, y) \), and a neighbor \( z \in C(d_1, x) \), and edges \( aw \) and \( az \) are not subdivided.

First, suppose that \( a \) has a neighbor \( u \) such that edge \( au \) is subdivided. Then, by 12, \( u \notin C(c_1, y) \cup C(d_1, x) \). If \( u \in C(x, c_1) \) or \( u \in C(y, d_1) \), then \( G \geq_m Q_3 \) (by contracting \( za \) or \( wa \), respectively). Therefore, \( u \in \{x, y\} \), so by symmetry \( u = x \). Since \( G \not\geq_m Q_2 \), it follows that

![Diagram](image-url)
Therefore, \( x \) is apex in \( G \), a contradiction.

Therefore, for all neighbors \( u \) of \( a \), \( au \) is a simple edge. Note that if \( a \) has no neighbors in \( C(x, w) \cup C(w, y) \) and \( C(w, y) = \emptyset \), then \( x \) is apex in \( G \). Similarly, if \( a \) has no neighbors in \( C(y, z) \cup C(z, x) \) and \( C(z, x) = \emptyset \), then \( y \) is apex in \( G \), a contradiction. Therefore, either \( N(a) \cap C(x, w) \cup C(w, y) \) \( \neq \emptyset \) or \( C(w, y) \neq \emptyset \); and either \( N(a) \cap (C(y, z) \cup C(z, x)) \neq \emptyset \) or \( C(z, x) \neq \emptyset \).

It can easily be seen that any one of the four combination yields a \( Q_2 \)-minor in \( G \), a contradiction.

**Subcase 2.2.** There are chords within \( C[x, y] \) and within \( C[y, x] \), and all chords of \( G \) have an endpoint at \( y \).

Let \( c_1y \) and \( d_1y \) be innermost chords within \( C[x, y] \) and \( C[y, x] \), respectively. By 12, \( a \) has a neighbor \( w \in C(c_1, y) \), and a neighbor \( z \in C(y, d_1) \), and edges \( aw \) and \( az \) are not subdivided.

Note that \( a \) has a neighbor \( u \neq y \) such that \( au \) is subdivided, for otherwise \( y \) is apex in \( G \), a contradiction. Then, by 12, \( u \notin C(c_1, y) \cup C(y, d_1) \), hence \( u \in C(x, c_1) \cup C(d_1, x) \). By symmetry, we only need to consider \( u \in C(x, c_1) \). First, if \( u = x \), then since \( G \not\prec_m Q_2 \), it follows that \( N(a) \cap (C(x, w) \cup C(w, y) \cup C(y, z) \cup C(z, x)) = \emptyset \), \( C(w, y) \cap C(y, z) = \emptyset \); and if \( y \in N(a) \), then \( ay \) is not subdivided. Therefore, \( x \) is apex in \( G \), a contradiction. Second, if \( u \in C(x, c_1) \), then since \( G \not\prec_m Q_3 \), it follows that \( N(a) \cap C(z, u) = \emptyset \). Also, since \( G \not\prec_m Q_2 \), it follows that \( N(a) \cap C(u, w) = \emptyset \), and \( C(y, z) = \emptyset \). And, since \( G \not\prec_m Q_2 \), it follows that \( C(y, z) = \emptyset \), and if \( y \in N(a) \), then \( ay \) is not subdivided. Therefore, \( u \) is apex in \( G \), a contradiction.

**Subcase 2.3.** All the chords of \( G \) lie within \( C[x, y] \) and they all have an endpoint at \( y \).

Let \( c_1y \) be an innermost chord within \( C[x, y] \). By 12, \( a \) has a neighbor \( w \in C(c_1, y) \), and edge \( aw \) is not subdivided.

Note that \( a \) has a neighbor \( u \neq y \) such that \( au \) is subdivided, for otherwise \( y \) is apex in \( G \), a contradiction. Then, by 12, \( u \notin C(c_1, y) \). Let \( z \in C(y, x) \) be the neighbor of \( a \) closest to \( y \), in the sense that \( yz \) is an edge of \( G \) or an edge subdivided once. Then \( u \in C(z, c_1) \), for otherwise \( y \) is apex in \( G \). First, if \( u \in C(z, x) \), then \( N(a) \cap C(u, x) = \emptyset \), for otherwise \( G - \{x, y\} \) contains a \( K_{2,3} \)-subdivision. Also, since \( G \not\prec_m Q_2 \), it follows that \( N(a) \cap C(u, x) = \emptyset \), \( C(y, z) = \emptyset \), and if \( y \in N(a) \), then \( ay \) is not subdivided. Therefore, \( x \) is apex in \( G \), a contradiction. Second, if \( u \in C(x, c_1) \), then, by 12, there are no chords with an endpoint in \( C(x, u) \). Also, since \( G \not\prec_m Q_3 \), it follows that \( N(a) \cap C(z, u) = \emptyset \), and since \( G \not\prec_m Q_5 \), we have that \( C(y, z) = \emptyset \). Also, since \( G \not\prec_m Q_2 \), it follows that \( N(a) \cap C(u, w) = \emptyset \), and \( C(y, z) = \emptyset \), and if \( y \in N(a) \), then \( ay \) is not subdivided. Therefore, \( u \) is apex in \( G \), a contradiction. Therefore, we must have \( u = c_1 \). Hence, by 12, \( c_1y \) is the only chord in \( G \). Again, since \( G \not\prec_m Q_3 \), it follows that \( N(a) \cap C(z, u) = \emptyset \). Hence, \( zx \) and \( xc_1 \) are either edges of \( G \) or edges subdivided once. Also, since \( G \not\prec_m Q_2 \), it follows that if \( y \in N(a) \), then \( ay \) is not subdivided. Hence, \( C(y, z) \neq \emptyset \), for otherwise \( u \) is apex in \( G \). Finally, since \( G \not\prec_m J_1 \), it follows that \( C(u, w) = \emptyset \) and \( N(a) \cap C(w, y) = \emptyset \), and hence \( z \) is apex in \( G \), a contradiction.

This concludes the proof of Case 2 in (2), and the entire proof of (2) of Proposition 5.1. \( \square \)
6. Connectivity 3

In this section, we focus on the case that \( G \in \text{ob}(O^*) - \mathcal{S} \) has connectivity three (recall from Lemma 2.2 that \( G \in \text{ob}(O^*) - \mathcal{S} \) is not 4-connected, and thus \( K_5 \) and \( \text{Oct} \) are the only 4-connected members of \( \text{ob}(O^*) \)). Here, we rely on the existence of contractible edges in 3-connected graphs (Lemma 6.2) and the minor-minimality of \( G \) to prove the following proposition, which says that such a \( G \) does not exist.

**Proposition 6.1.** There are no 3-connected graphs in \( \text{ob}(O^*) - \{K_5, K_{3,3}, \text{Oct}, Q\} \). In other words, the only graphs of connectivity 3 in \( \text{ob}(O^*) \) are \( K_{3,3} \) and \( Q \).

**Lemma 6.2** (see [3]). If \( G \) is 3-connected and \( |V(G)| \geq 5 \), then \( G \) has an edge \( e \) such that \( G/e \) is also 3-connected.

Such an edge is called contractible. We denote by \( v_{xy} \) the new vertex obtained by contracting edge \( xy \) in a graph.

The proof of Proposition 6.1 follows from Lemma 6.2 and two lemmas which are stated and proved below.

**Lemma 6.3.** There is no 3-connected graph \( G \) in \( \text{ob}(O^*) - \{K_5, K_{3,3}, \text{Oct}, Q\} \) that has a contractible edge \( xy \) such that \( v_{xy} \) is not apex vertex in \( G/xy \).

**Proof.** Suppose otherwise that there exists a 3-connected graph \( G \) in \( \text{ob}(O^*) - \{K_5, K_{3,3}, \text{Oct}, Q\} \) that has a contractible edge \( xy \) such that \( v_{xy} \) is not apex vertex in \( G/xy \), and hence there is an apex vertex \( a \neq v_{xy} \) in \( G/xy \). Then, \( (G/xy) - a \in \mathcal{O} \) is 2-connected. Since \( G \) is 3-connected (and simple and planar), it has a unique planar embedding by the well-known theorem of Whitney from 1933 (see [2]). Since \( (G/xy) - a \in \mathcal{O} \) is 2-connected, it follows that restricting this embedding to \( (G/xy) - a \), we have that all the vertices of \( (G/xy) - a \) lie on a cycle \( C' \) and are incident with the outer face. This is so because, by Whitney’s theorem, it follows that every simple 2-connected outerplanar graph has a unique outerplanar embedding. Since \( G - a \not\in \mathcal{O} \), it follows that \( x \) or \( y \), say \( x \), is embedded in the interior of the disk bounded by \( C \), where \( C \subseteq G \) is the cycle isomorphic to \( C' \), and the corresponding isomorphism \( \phi : V(C') \rightarrow V(C) \) is the identity map on \( V(C') - v_{xy} \) and \( \phi(v_{xy}) = y \).

Let \( u_1, u_2, \ldots, u_n \in V(C) \) \( (n \geq 3) \) be the neighbors of \( x \) in the clockwise order around \( C \). For \( i = 1, \ldots, n \), let \( S_i := C[u_i, u_{i+1}] \), where \( S_n \) is understood to be \( C[u_n, u_1] \). We call the \( S_i \)’s the segments of \( C \). We call \( u_i \)’s the endpoint vertices of the segments and the vertices in \( C(u_i, u_{i+1}) \) for \( i = 1, \ldots, n \), the interior vertices of the segments. Two segments of \( S_i \) and \( S_j \) are said to be consecutive if \( |i - j| = 1 \), or \( \{i, j\} = \{1, n\} \). We observe the following facts.

1. The edges of \( G \) are:
   - edges of \( C \);
   - edges \( xu_i \) for \( i = 1, \ldots, n \);
   - chords of \( C \), that is, edges not in \( E(C) \) with both endpoints in a single segment of \( C \) (note that such edges are embedded in the interior of the disk bounded by \( C \));
   - edges with one endpoint in \( C \) and the other at \( a \).
It follows by the above that:

2. Interior vertices of the segments are either endpoints of chords or neighbors of $a$.

3. For every chord $c_1c_2$ in $G$ with $c_1 < c_2$ (in the clockwise order of $C$ restricted to the segment containing $c_1c_2$), $a$ has a neighbor (in the usual sense, as opposed to the one from Section 5) in $C(c_1, c_2)$ (by 3-connectedness of $G$).

Let $N(a) := N_G(a)$. We now prove:

4. $N(a)$ is covered by exactly two consecutive segments of $C$.

**Pf.** First, we show that $N(a)$ is covered by exactly two segments of $C$. If there are four internally disjoint paths from $a$ to $x$, then the subgraph of $G$ formed from the union of those paths and $C$ contains an $Oct$-minor, a contradiction.

Therefore, by Menger’s theorem and the fact that $G$ is 3-connected, it follows that $G$ has a 3-cut separating $a$ and $x$. By 1 above, it follows that this 3-cut is a subset of $V(C)$, and therefore at least one of $a$ or $x$ has degree 3. Let $u \in \{a, x\}$ be such that $deg_G(u) = 3$, and let $v \in \{a, x\} - \{u\}$. The three neighbors of $u$ divide $C$ into three segments. If all three segments contain interior vertices that are in $N(v)$, then $G$ contains a $Q$-minor, a contradiction.

Hence, one segment does not contain any interior vertices that are in $N(v)$. Then, if $u = x$ then we are done. And similarly, if $u = a$ then we are done. Hence, we have shown that $N(a)$ is covered by exactly two segments of $C$.

Furthermore, the two segments that cover $N(a)$ are consecutive. Suppose not, and let $S_i$ and $S_j$ be the two segments that cover $N(a)$ with $|i - j| > 1$. If both of them contain at least two neighbors of $a$, then two of those neighbors in each segment can be contracted to four distinct endpoint vertices and thus $G \succeq_m Oct$, a contradiction. Hence, one of them, say $S_i$, contains only one neighbor of $a$, call it $n_1$. Since $deg(a) \geq 3$, $S_j$ must contain at least two neighbors of $a$: let $n_2$ be the closest one to $u_j$, and $n_3$ be the closest one to $u_{j+1}$.

Suppose $n_1$ is an endpoint vertex, so that $n_1 = u_i$ or $u_{i+1}$. Note that in this case $deg(x) \geq 5$, for otherwise two consecutive segments cover $N(a)$. Then, since $G \notin O^*$, it follows that $C(n_2, n_3) \neq \emptyset$ (for otherwise $n_1$ is an apex vertex). But then, $G \succeq_m Q_1$, a contradiction (by deleting edge $n_1x$ and contracting $n_2$ to $u_j$, and $n_3$ to $u_{j+1}$).
Therefore, $n_1$ must be an interior vertex, so $n_1 \in C(u_i, u_{i+1})$. Again, since $G \notin O^*$, there is a vertex in $C(n_2, n_3)$, or there is a chord with one endpoint in $C[u_i, n_1]$ and the other in $C(n_1, u_{i+1}]$ (for otherwise $n_1$ is an apex vertex). In the first case, $G \geq m Q_1$ (just like above) while in the second, $G \geq m Oct$ (by contracting edge $n_1a$), a contradiction. This proves 4.

We now show that $C$ actually has exactly three segments.

5. $C$ has exactly three segments, or equivalently \( \deg(x) = 3 \), or equivalently $n = 3$.

\textbf{Pf.} By 4, we may assume that $N(a)$ is covered by $S_1$ and $S_2$. Since interior vertices are either endpoints of chords or neighbors of $a$, it follows by 2 and 3 that $C(u_i, u_{i+1}) = \emptyset$ for $i = 3, 4, \ldots, n$ (where $u_{n+1} = u_1$).

Suppose that $n \geq 4$. By 4, it follows that $a$ has neighbors in $C[u_1, u_2]$ and in $C(u_2, u_3]$. Therefore, in the graph $G \setminus xu_4$, none of the vertices $a$, $u_2$, $x$, $u_4$ can be apex (since the deletion of any one of them still leaves a $K_{2,3}$-subdivision as a subgraph). Let $s$ be an apex vertex in $G \setminus xu_4$. Then $s \in V(C)$. Therefore, the unique embedding of $G$ restricted to the graph $(G \setminus xu_4) - s \in O$ is an embedding in which all the vertices (including $x$) are incident with the outer face. By adding edge $xu_4$ to this embedding, we obtain an embedding of $G - s$ in which all the vertices are incident with the outer face, a contradiction.

We have shown that for $i = 3, 4, \ldots, n$ $xu_i \notin E(G)$ which, by 3-connectivity of $G$, implies that $C$ has exactly three segments and proves 5.

By 5, $G$ has the following general structure:

Therefore, let $S_1$ and $S_2$ cover $N(a)$. It follows by 2, and 3 that $C(u_3, u_1) = \emptyset$ (that is $u_3u_1 \in E(G)$). Also, similarly to 4 from the proof of Proposition 5.1, since $G \not\geq m Q_1$, we have:

6. Within a single segment $S_1$ or $S_2$, there are no non-overlapping chords (or equivalently, all the chords are nested).
We say that segment \( S_1 \) (respectively \( S_2 \)) is of type-one, if \( \{z\} := N(a) \cap C[u_1, u_2] \) with \( z \neq u_1 \), and \( C(z, u_2) \neq \emptyset \) (respectively, \( \{w\} := N(a) \cap C(u_2, u_3) \) with \( w \neq u_3 \), and \( C(u_2, w) \neq \emptyset \)). And we say that \( S_1 \) (respectively \( S_2 \)) is of type-two, if \( |N(a) \cap C[u_1, u_2]| \geq 2 \) (respectively \( |N(a) \cap C(u_2, u_3)| \geq 2 \)). Note that if \( S_1 \) (respectively \( S_2 \)) is not of type-one nor type-two, then \( \{z\} := N(a) \cap C[u_1, u_2] \) and \( zu_2 \in E(C) \) (respectively \( \{w\} := N(a) \cap C(u_2, u_3) \) and \( uw \in E(C) \)). Finally, note that at least one of \( S_1 \) or \( S_2 \) is of type-one or type-two, for otherwise \( u_2 \) is apex in \( G \).

There are two cases to consider.

**Case 1.** Each of \( S_1 \) and \( S_2 \) is of type-one or type-two.

Suppose that one of the segments, say \( S_2 \), is of type-one. Then, \( \{w\} := N(a) \cap C(u_2, u_3) \) with \( w \neq u_3 \), and \( C(u_2, w) \neq \emptyset \). Hence, it follows by 2, that there is a chord with one endpoint \( c_1 \in C(u_2, w) \), and the other \( c_2 \in C(w, u_3) \). Choose \( c_1 \) and \( c_2 \) so that the chord \( c_1c_2 \) is innermost. Then by 6, all other chords in \( S_2 \) have one endpoint in \( C[u_2, c_1] \) and the other in \( C[c_2, u_3] \). However, since \( S_1 \) is of type-one or type-two, we either have \( \{z\} := N(a) \cap C[u_1, u_2] \) with \( z \neq u_1 \), and \( C(z, u_2) \neq \emptyset \) (which by 2 implies that there is a chord with one endpoint in \( C[u_1, z] \) and the other in \( C(z, u_2) \)), or \( |N(a) \cap C[u_1, u_2]| \geq 2 \). This implies that the only other chords in \( S_2 \) that do not have an endpoint at \( u_2 \) (that is, those that do have an endpoint in \( C(u_2, c_1) \)) have an endpoint at \( c_2 \), for otherwise \( G \geq_m Q_1 \) (by contracting \( wa \) and \( za \) if necessary, see figure below).

Therefore, \( u_2 \) is apex in \( G \), a contradiction.

Similarly, suppose that for one of the segments, say \( S_1 \), is of type-two. Then, \( |N(a) \cap C[u_1, u_2]| \geq 2 \). If there are chords with endpoints distinct from \( u_2 \) in \( S_1 \), then let \( d_1d_2 \), with \( d_1 < d_2 \) in the cyclic order of \( C \), be an innermost chord of \( S_1 \) with \( d_2 \neq u_2 \), and let \( z \in N(a) \cap C(d_1, d_2) \). Then again, since \( S_2 \) is of type-one or type-two, we either have \( \{w\} := N(a) \cap C(u_2, u_3) \) with \( w \neq u_3 \), and \( C(u_2, w) \neq \emptyset \) (which by 2 implies that there is a chord with one endpoint in \( C(u_2, w) \) and the other in \( C(w, u_3) \)), or \( |N(a) \cap C(u_2, u_3)| \geq 2 \). This implies that the only other chords in \( S_1 \) that do not have an endpoint at \( u_2 \) (that is, those that do have an endpoint in \( C[d_2, u_2] \)) have an endpoint at \( d_1 \), for otherwise \( G \geq_m Q_1 \) as above. Furthermore, \( N(a) \cap (C(d_1, z) \cup C(z, u_2)) = \emptyset \), for otherwise \( G \geq_m Q_1 \) as above. Therefore again, \( u_2 \) is apex in \( G \), a contradiction.

**Case 2.** Exactly one of the segments \( S_1 \) or \( S_2 \) is of type-one or type-two.

By symmetry, suppose that \( S_2 \) is not of type-one nor type-two, and that \( S_1 \) is. Then, \( \{w\} := N(a) \cap C(u_2, u_3) \) and \( uw \in E(C) \). We divide this case into two subcases depending on whether \( u_1u_2 \) is an edge of \( G \).

**Subcase 2.1.** \( u_1u_2 \notin E(G) \).

Let \( s \) be an apex vertex in \( G \setminus xu_3 \), and we assume that the graph \( (G \setminus xu_3) - s \in \mathcal{O} \) is embedded in the plane with all of its vertices incident with the outer face. Clearly, \( s \neq x \) and \( s \neq u_3 \), for otherwise \( x \) or \( u_3 \) is apex in \( G \), a contradiction. Also, \( s \neq a \), since \( (G \setminus xu_3) - a \) contains a \( K_{2,3} \)-subdivision (because \( C(u_1, u_2) \neq \emptyset \), since \( S_1 \) is of type-one or type-two).
First, suppose that \( w = u_3 \). Then \( u_2u_3 \in E(C) \) (that is, \( C(u_2, u_3) = \emptyset \)). If \( s = u_2 \) (or by symmetry, if \( s = u_1 \)), then in \( (G \setminus xu_3) - s \), \( \deg(u_3) = 2 \) and \( \deg(x) = 1 \) hence edges \( u_3u_1, u_3a, \) and \( xu_1 \) are also incident with the outer face. Since \( u_3u_1 \) is a simple edge, by putting the edge \( xu_3 \) back in, we can embed \( G - s \) so that all the vertices are still incident with the outer face, hence \( G - s \) is outerplanar, a contradiction (see figure below).

Therefore, \( s \notin \{u_1, u_2, u_3, x, a\} \), so that \( s \in C(u_1, u_2) \). Then, in \( (G \setminus xu_3) - s \), \( \deg(x) = 2 \), and so \( (G \setminus xu_3) - s \) has an outerplanar embedding such that edges \( xu_1 \) and \( xu_2 \) are incident with the outer face. Also, note that \( x, u_1, u_3, u_2 \) is a 4-cycle in \( (G \setminus xu_3) - s \). Therefore, since \( u_1u_2 \notin E(G) \), we can put the edge \( xu_3 \) back in to obtain an embedding of \( G - s \) in which all the vertices are still incident with the outer face, hence \( G - s \) is outerplanar, a contradiction (see figure below).

Therefore, \( w \neq u_3 \) and so \( w \in C(u_2, u_3) \). Since \( u_2w \in E(C) \), the only possible chords in \( S_2 \) have one endpoint at \( u_2 \) and the other in \( C(w, u_3) \). Note that by Case 2 hypothesis, \( u_3a \notin E(G) \).

If \( s = u_2 \), then in \( (G \setminus xu_3) - s \), \( \deg(u_3) = 2 \) and \( \deg(x) = 1 \), hence edge \( u_3u_1 \) and the other edge incident with \( u_3 \), as well as the pendant edge \( xu_1 \) are all incident with the outer face. Since \( u_3u_1 \) is a simple edge, by putting the edge \( xu_3 \) back in, we obtain an embedding of \( G - s \) in which all the vertices are still incident with the outer face, hence \( G - s \) is outerplanar, a contradiction.

Now suppose \( s = u_1 \). If \( u_2u_3 \) is a chord of \( C \), then in \( (G \setminus xu_3) - s \), \( \deg(u_3) = 2 \) and \( \deg(x) = 1 \), hence edges \( u_3u_2, u_3w, \) and \( xu_2 \) are incident with the outer face. Since \( u_3u_2 \) is a simple edge, by putting the edge \( xu_3 \) back in, we can embed \( G - s \) so that all the vertices are still incident with the outer face, hence \( G - s \) is outerplanar, a contradiction. Hence \( u_2u_3 \) is not a chord of \( C \). If \( G \) has a chord \( c = u_2c_1 \) with \( c_1 \in C(w, u_3) \), then choose \( c_1 \) closest to \( u_3 \), so that \( c_1u_3 \in E(C) \). And if there is no such chord, then let \( c_1 := w \). Then, in \( (G \setminus xu_3) - s \), \( \deg(x) = 1 \), and \( \deg(c_1) = 3 \), but \( c_1 \) is adjacent to \( u_3 \) with \( \deg(u_3) = 1 \), hence edge \( u_2c_1 \) and the pendant edges \( xu_2 \) and \( c_1u_3 \) are all incident with the outer face. Since \( u_2c_1 \) is a simple edge (even if \( c_1 = w \), by putting the edge \( xu_3 \) back in, we can embed \( G - s \) so that all the vertices are still incident with the outer face, hence \( G - s \) is outerplanar, a contradiction (see figure below).
Similarly, if \( s = w \), then \( G \) has no chords with one endpoint at \( u_2 \) and the other in \( C(w, u_3) \), for otherwise \((G \setminus w) - w \) contains a \( K_{2,3} \)-subdivision (because \( C(u_1, w) \neq \emptyset \), since \( S_1 \) is of type-one or type-two). Hence, \( C(w, u_3) = \emptyset \). Therefore, in \((G \setminus w) - w\), \( \deg (x) = 2 \) and \( \deg (u_3) = 1 \), hence edges \( xu_1 \) and \( u_1u_3 \) are incident with the outer face. Since \( xu_1 \) is a simple edge, by putting the edge \( xu_3 \) back in, we can embed \( G - w \) so that all the vertices are still incident with the outer face, hence \( G - w \) is outerplanar, a contradiction.

Therefore, \( s \notin \{u_1, u_2, u_3, x, a, w\} \), and so \( s \in C(u_1, u_2) \) (by (1)). Again, if \( G \) has a chord \( c = u_2c_1 \) with \( c_1 \in C(w, u_3) \), then choose \( c_1 \) closest to \( u_3 \), so that \( c_1u_3 \in E(G) \). And if there is no such chord, then let \( c_1 := w \). Then, in \((G \setminus w) - w\), \( \deg (x) = 2 \), hence edges \( xu_2 \) and \( xu_1 \) are incident with the outer face. Also, note that \( x, u_1, u_3, c_1, u_2 \) is a 5-cycle in \((G \setminus w) - w\). Therefore, since \( u_1u_2 \notin E(G) \) (the Subcase 2.1 hypothesis) and \( u_1c_1 \notin E(G) \) (by (1)), we can put the edge \( xu_3 \) back in (even if \( u_2u_3 \in E(G) \)) to obtain an embedding of \( G - w \) in which all the vertices are still incident with the outer face, hence \( G - w \) is outerplanar, a contradiction (see figure below).

**Subcase 2.2.** \( u_1u_2 \in E(G) \).

If all chords within \( S_1 \) have an endpoint at \( u_1 \) or all have an endpoint at \( u_2 \), then \( u_1 \) or \( u_2 \) respectively, is apex in \( G \), a contradiction. Hence, there is a chord with both endpoints in \( C(u_1, u_2) \).

Let \( c_1c_2 \in E(G) \) be the innermost chord with \( c_1, c_2 \in C(u_1, u_2) \) (in the sense that there are no other chords with both endpoints in \( C[c_1, c_2] \)), and let \( a_1 \in N(a) \cap C(c_1, c_2) \).

Suppose that \( a_2 \neq a_1 \) is another neighbor of \( a \) in \( C(c_1, c_2) \). Then by choice of \( c_1c_2 \), we have that \( \deg (a_1) = 3 = \deg (a_2) \). Note that \( a \) has no other neighbors in \( C(u_1, u_2) \), for otherwise \( G \) contains two disjoint \( K \)-graphs, a contradiction. Let \( s \) be an apex vertex in \( G \setminus a_1a_2 \), and we assume that the graph \((G \setminus a_1a_2) - s \in O \) is embedded in the plane with all of its vertices incident with the outer face.

It is easy to see that \( s = w \) (regardless of whether \( w = u_3 \)), for otherwise: if \( s \notin \{a\} \cup C(u_1, u_2) \), then \((G \setminus a_1a_2) - s \) contains a \( K_{1,1} \)-subdivision; and if \( u \notin \{u_1, u_2\} \cup C(w, u_3) \), then \((G \setminus a_1a_2) - s \) contains a \( K_{2,3} \)-subdivision. Therefore \( s = w \), and hence the only neighbors of \( a \) are \( a_1, a_2 \) and \( w \) (because if \( u_1 \) or \( u_2 \) is a neighbor of \( a \) then \((G \setminus a_1a_2) - s \) contains a \( K_{2,3} \)-subdivision). Then, in \((G \setminus a_1a_2) - s \), \( \deg (a) = 2 \), hence edges \( aa_1 \) and \( aa_2 \) are incident with the outer face, and by putting the edge \( a_1a_2 \) back in, we obtain an embedding of \( G - s \) in which all the vertices are still incident with the outer face, hence \( G - s \) is outerplanar, a contradiction.

Hence, we have shown that \( a_1 \) is the only neighbor of \( a \) in \( C(c_1, c_2) \).

We now show furthermore that \( N(a) \cap (C(u_1, c_1) \cup C(c_2, u_2)) = \emptyset \). For suppose otherwise, and let \( c_3 \in C(u_1, c_1) \) (the argument for \( c_3 \in C(c_2, u_2) \) is similar). Then \( N(a) \cap C(c_2, u_2) = \emptyset \) (for otherwise \( G \geq 2K_4 \)). Let \( s \) be an apex vertex in \( G \setminus c_1a_1 \). Then clearly \( s \in \{u_2, w\} \). If \( s = w \), then since \( w \) is apex in \( G \setminus c_1a_1 \) we have that: \( c_3 = c_1; N(a) \cap (C[u_1, c_1] \cup \{u_2\}) = \emptyset \); and \( G \) does not have any chords with one endpoint at \( u_2 \) and the other in \( C(w, u) \) (in the case that \( w \neq u_3 \)).
Therefore \( w \) is apex in \( G \), a contradiction. If \( s = u_2 \), then since \( u_2 \) is apex in \( G \setminus c_1 a_1 \), it follows that \( G \) has no chords with one endpoint in \( C[u_1, c_1] \) and the other in \( C[c_2, u_2] \). Hence all chords of \( G \) have one endpoint at \( c_1 \) or at \( u_2 \). Therefore \( u_2 \) is apex in \( G \), a contradiction.

Hence, we have shown that \( N(a) \cap (C(u_1, a_1) \cup C(a_1, u_2)) = \emptyset \). Thus the only possible neighbors of \( a \) (other than \( a_1 \) and \( w \)) are \( u_1 \) and \( u_2 \). In fact, at least one of them is a neighbor of \( a \) since \( \deg_{(G/xy)} a \geq 3 \). Let \( s \) be an apex vertex in \( G/aa_1 \). Then clearly \( s \in \{u_1, u_2\} \). Suppose that \( s = u_2 \) (the argument for \( s = u_1 \) is similar). Since \( u_2 \) is apex in \( G/aa_1 \), it follows that \( G \) has no chords with one endpoint in \( C[u_1, c_1] \) and the other in \( C[c_2, u_2] \). Hence all chords of \( G \) have one endpoint at \( c_1 \) or at \( u_2 \). Therefore \( u_2 \) is apex in \( G \), a contradiction. This concludes the proof of Lemma 6.3. \( \square \)

**Lemma 6.4.** There is no 3-connected graph \( G \) in \( ob(\mathcal{O}^*) - \{K_5, K_{3,3}, Oct, Q\} \) with the property that for every contractible edge \( xy \) in \( G \) the vertex \( v_{xy} \) is apex in \( G/xy \).

**Proof.** Suppose otherwise that there exists a 3-connected graph \( G \) in \( ob(\mathcal{O}^*) - \{K_5, K_{3,3}, Oct, Q\} \) with the property that for any contractible edge \( xy \) in \( G \) the vertex \( v_{xy} \) is apex in \( G/xy \).

The following claim provides a way of testing whether an edge in a 3-connected graph is contractible.

1. Let \( G \) be a 3-connected graph with edge \( xy \). Then, \( G/xy \) is 3-connected if and only if \( G - \{x, y\} \) \( (= (G/xy) - v_{xy}) \) is 2-connected.

**Pf.** If \( G/xy \) is 3-connected, then clearly \( (G/xy) - v_{xy} \) is 2-connected. Now, suppose that \( G - \{x, y\} \) \( (= (G/xy) - v_{xy}) \) is 2-connected and that \( G/xy \) is not 3-connected, so that \( G/xy \) has a 2-cut. Since \( G \) is 3-connected, it follows that \( v_{xy} \) is one of the vertices in that 2-cut (for otherwise, this 2-cut would also be a 2-cut in \( G \)). Therefore, \( (G/xy) - v_{xy} \) has a cut-vertex, a contradiction which proves 1.

Let \( xy \) be a contractible edge in \( G \). Then, by 1, \( (G/xy) - v_{xy} \in \mathcal{O} \) is 2-connected. Since \( G \) is 3-connected it has a unique planar embedding. Restricting this embedding to \( (G/xy) - v_{xy} \), we have that all the vertices of \( (G/xy) - v_{xy} \) lie on a cycle \( C \) and are incident with the outer face.

Let \( x_1, x_2, \ldots, x_m \in V(C) \) \( (m \geq 2) \) be the neighbors of \( x \) in the clockwise order around \( C \). And let \( y_1, y_2, \ldots, y_n \in V(C) \) \( (n \geq 2) \) be the neighbors of \( y \) in the clockwise order around \( C \). Note that \( x_i \notin C(y_1, y_n) \) for all \( i \) and \( y_j \notin C(x_1, x_m) \) for all \( j \), for otherwise \( G \) would contain a \( K_{3,3} \)-minor. Also, note that possibly \( x_m = y_1 \) or \( y_n = x_1 \).

2. The edges of \( G \) are:
   - edges of \( C \);
   - edges \( xx_i \) for \( i = 1, \ldots, m \), and \( yy_j \) for \( j = 1, \ldots, n \);
   - chords of \( C \), that is, edges not in \( E(C) \) with both endpoints in \( C \) (note that such edges are embedded in the interior of the disk bounded by \( C \));
   - edge \( xy \).

Just as in the proof of Lemma 6.3, it follows from 2 that:

3. The vertices of \( C \) are either endpoints of chords or neighbors of \( x \) or \( y \).

4. For every chord \( c_1c_2 \) in \( G \) with \( c_1 < c_2 \) (in the clockwise order restricted to the segment containing \( c_1c_2 \)), there is a neighbor of \( x \) or \( y \) in \( C(c_1, c_2) \).

Also, since neither \( y \) nor \( x \) is apex in \( G \), it follows, respectively, that:
5. $C(x_1, x_m) \neq \emptyset$ and $C(y_1, y_n) \neq \emptyset$.

Hence $G$ has the following general structure:

Before we proceed, we prove a claim regarding the structure of $G$.

6. $G$ does not have a chord with both endpoints in $C[y_n, y_1]$. And by symmetry, the same statement holds for $C[x_m, x_1]$.

Pf. Let $c_1c_2$ be a chord of $G$ with both endpoints in $C[y_n, y_1]$. Without loss of generality, we may assume that $c_1c_2$ is the innermost such chord, in the sense that there are no other chords with both endpoints in $C[c_1, c_2]$. By 4, it follows that $x$ has a neighbor $s$ in $C(c_1, c_2)$. Note that $x$ does not have another such neighbor $t$ in $C(c_1, c_2)$, for otherwise edge $st$ is contractible (because $G - \{s, t\}$ is 2-connected), but $(G/st) - v_{st} \notin \mathcal{O}$ (because it contains a $K_{2,3}$-subdivision, since $C(y_1, y_n) \neq \emptyset$), a contradiction because $v_{st}$ by the assumption of the proof is an apex vertex in $G/st$. So, the only vertex in $C(c_1, c_2)$ is $s$. But then, edge $xs$ is contractible (because $G - \{x, s\}$ is 2-connected), and $(G/xs) - v_{xs} \notin \mathcal{O}$ (because it contains a $K_{2,3}$-subdivision, since $C(y_1, y_n) \neq \emptyset$), a contradiction. This proves 6.

By 6, we have:

7. The only chords in $G$ have one endpoint in $C(x_1, x_m)$ and the other in $C(y_1, y_n)$.

The following claim further tightens up the structure of $G$.

8. There is exactly one vertex in $C(x_1, x_m)$ and exactly one in $C(y_1, y_n)$.

Pf. Suppose that $C(x_1, x_m)$ has two vertices $s$ and $t$. Then, by 3 it follows that both $s$ and $t$ are neighbors of $x$, or endpoints of chords whose other endpoints lie in $C(y_1, y_n)$ by 7, or both. Note that $st$ is contractible (by 1, because $G - \{s, t\}$ is 2-connected), and $(G/st) - v_{st} \notin \mathcal{O}$ (because it contains a $K_{1,4}$-subdivision, consisting of the cycle formed by edge $xx_m$, the clockwise path along $C$ from $x_m$ to $x_1$, and edge $x_1x$; and the three spokes from $y$ to this cycle), violating the hypothesis of Lemma 6.4. This proves 8.

With the structure of $G$ restricted by 6 and 8, we are ready to finish the proof of the lemma. Let $s$ and $t$ be the unique vertices in $C(x_1, x_m)$ and $C(y_1, y_n)$, respectively. Note that $st \in E(G)$, for otherwise any one of $x_1$, $x_m$, $y_1$, $y_n$ is apex in $G$, a contradiction. Also, it follows by 2 and 7 that $C(x_m, y_1) = \emptyset$ and $C(y_n, x_1) = \emptyset$.

If $x_m \neq y_1$ and $y_n \neq x_1$, then $G \ni Q$, a contradiction (see figure below).
Hence, by symmetry, we have either the case that $x_m \neq y_1$ and $y_n = x_1$, or that $x_m = y_1$ and $y_n = x_1$. In either case, we cannot have that both $sx, ty \in E(G)$, for otherwise $G \geq_m \text{Oct}$ (see figure below).

Hence, by symmetry, $sx \notin E(G)$, and it follows that $x_m$ is apex, a contradiction. This concludes the proof of the lemma. □

REFERENCES


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