Ranking Tournaments with No Errors

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Abstract

This paper examines the classical problem of ranking a set of players on the basis of a set of pairwise comparisons arising from a sports tournament; the objective is to minimize the total number of upsets, where an *upset* occurs if a higher ranked player was actually defeated by a lower ranked player. This problem can be rephrased as the so-called minimum feedback arc set problem on tournaments, which arises in a rich variety of applications and has been a subject of extensive research. In this paper we study this NP-hard problem using polyhedral and linear programming approaches. Let $T = (V, A)$ be a tournament with a nonnegative integral weight $w(e)$ on each arc e. A subset F of arcs is called a *feedback arc set* if $T \backslash F$ contains no cycles (directed). A collection $\mathcal{C}$ of cycles (with repetition allowed) is called a *cycle packing* if each arc e is used at most $w(e)$ times by members of $\mathcal{C}$. We call T *cycle Mengerian* (CM) if, for any nonnegative integral function $\boldsymbol{w}$ defined on A, the minimum total weight of a feedback arc set is equal to the maximum size of a cycle packing. The purpose of this paper is to present a structural characterization of all CM tournaments, which yields a polynomial-time algorithm for the minimum-weight feedback arc set problem on such tournaments.

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1 Introduction

Consider a sports tournament in which each of n players is required to play precisely one game with each other player, and assume that each game ends in a win or a loss. After completion of the tournament, it is desirable to find a ranking of all n players that minimizes the number of upsets, where an *upset* occurs if a higher ranked player was actually defeated by a lower ranked player. This problem can be rephrased as the so-called minimum feedback arc set problem on tournaments, and will be investigated in the more general weighted setting in this paper.

Let $G = (V, A)$ be a digraph with a nonnegative integral weight $w(e)$ on each arc e. A subset F of arcs is called a *feedback arc set* (FAS) of G if $G \backslash F$ contains no cycles (directed). The *minimum-weight FAS problem* (or simply *FAS problem*) is to find an FAS in G with minimum total weight. Digraph G is called a *tournament* if there is precisely one arc between any two vertices in G. The FAS problem on tournaments, abbreviated FAST, dates back to as early as the 1780s when Borda [7] and Condorcet [12] each proposed voting systems for elections with more than two candidates. Since the FAST arises in a rich variety of applications in sports, databases, and statistics, where it is necessary to effectively combine rankings from different sources, FAS's in tournaments have been studied extensively from the combinatorial [19, 20, 34, 38], statistical [33], and algorithmic [1, 2, 13, 28, 37, 36] points of view, and thus have produced a vast body of literature. In [1], Ailon, Charikar, and Newman proved that the FAST is NP-hard under randomized reductions even in the unweighted case. In [3], Alon showed that this unweighted version is in fact NP-hard; in [10], Charbit, Thomassé, and Yeo established this result independently. In [28], Mathieu and Schudy devised a polynomial time approximation scheme (PTAS) for the FAST. Given these results, it is natural to ask the following question: When can the FAST be solved exactly in polynomial time? Inspired by the title of Mathieu and Schudy's paper [28], this is equivalent to asking: Which tournaments can be ranked with no errors? The purpose of this paper is to resolve this problem using polyhedral and linear programming approaches.

We introduce some terminology before proceeding. Let $C\boldsymbol{x} \geq \boldsymbol{d}$, $\boldsymbol{x} \geq \boldsymbol{0}$ be a rational linear system and let P denote the polyhedron $\{\boldsymbol{x} : C\boldsymbol{x} \geq \boldsymbol{d}, \boldsymbol{x} \geq \boldsymbol{0}\}$. We call P *integral* if it is the convex hull of all integral vectors contained in P. As shown by Edmonds and Giles [18], P is integral iff the minimum in the LP-duality equation

$$\min\{\boldsymbol{w}^T\boldsymbol{x} : C\boldsymbol{x} \geq \boldsymbol{d},\ \boldsymbol{x} \geq \boldsymbol{0}\} = \max\{\boldsymbol{y}^T\boldsymbol{d} : \boldsymbol{y}^T C \leq \boldsymbol{w}^T,\ \boldsymbol{y} \geq \boldsymbol{0}\}$$

has an integral optimal solution, for every integral vector $\boldsymbol{w}$ for which the optimum is finite. If, instead, the maximum in the equation enjoys this property, then the system $C\boldsymbol{x} \geq \boldsymbol{d}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is called *totally dual integral* (TDI). It is well known that many combinatorial optimization problems can be naturally formulated as integer programs of the form $\min\{\boldsymbol{w}^T\boldsymbol{x} : \boldsymbol{x} \in P,\ \text{integral}\}$; if P is integral, then such a problem reduces to its LP-relaxation. Edmonds and Giles [18] proved that total dual integrality implies primal integrality: if $C\boldsymbol{x} \geq \boldsymbol{d}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is TDI and $\boldsymbol{d}$ is integer-valued, then P is integral. Thus the model of TDI systems serves as a general framework for establishing many combinatorial min-max theorems. Over the past six decades, these two integrality properties have been the subjects of extensive research and the major concern of polyhedral combinatorics (see Schrijver [29, 30] for comprehensive accounts).

Let us return to the FAS problem. Let M be the cycle-arc incidence matrix of the input digraph G, and let $\pi(G)$ denote the linear system $M\boldsymbol{x} \geq \mathbf{1}$, $\boldsymbol{x} \geq \mathbf{0}$. We call G *cycle ideal* (CI) if $\pi(G)$ defines an integral polyhedron, and call G *cycle Mengerian* (CM) if $\pi(G)$ is a TDI system. To facilitate better understanding, we give an intuitive interpretation of these concepts. A collection $\mathcal{C}$ of cycles (with repetition allowed) in G is called a *cycle packing* of G if each arc e is used at most $w(e)$ times by members of $\mathcal{C}$. The *cycle packing problem* consists in finding a cycle packing with maximum size, which can be viewed as the dual version of the FAS problem. To see this, let $\mathbb{P}(G, \boldsymbol{w})$ stand for the linear program

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{w}^T\boldsymbol{x} \\ \text{Subject to} & M\boldsymbol{x} \geq \mathbf{1} \\ & \quad \boldsymbol{x} \geq \mathbf{0}, \end{array}$$

and let $\mathbb{D}(G, \boldsymbol{w})$ denote its dual

$$\begin{array}{ll} \text{Maximize} & \boldsymbol{y}^T\mathbf{1} \\ \text{Subject to} & \boldsymbol{y}^T M \leq \boldsymbol{w}^T \\ & \boldsymbol{y} \geq \mathbf{0}, \end{array}$$

where $\boldsymbol{w} = (w(e) : e \in A)$. Then $\mathbb{P}(G, \boldsymbol{w})$ (resp. $\mathbb{D}(G, \boldsymbol{w})$) is exactly the LP-relaxation of the FAS problem (resp. cycle packing problem), and thus is called the *fractional FAS problem* (resp. *fractional cycle packing problem*). Let $\tau_w(G)$ be the minimum total weight of an FAS, let $\nu_w(G)$ be the maximum size of a cycle packing, let $\tau_w^*(G)$ be the optimal value of $\mathbb{P}(G, \boldsymbol{w})$, and let $\nu_w^*(G)$ be the optimal value of $\mathbb{D}(G, \boldsymbol{w})$. Clearly,

$$\nu_w(G) \leq \nu_w^*(G) = \tau_w^*(G) \leq \tau_w(G);$$

these two inequalities, however, need not hold with equalities in general (as we shall see in Section 2). The aforementioned Edmonds-Giles theorems give rise to the following two observations:

• G is CI iff $\mathbb{P}(G, \boldsymbol{w})$ has an integral optimal solution for any nonnegative integral $\boldsymbol{w}$ iff $\tau_w^*(G) = \tau_w(G)$ for any nonnegative integral $\boldsymbol{w}$. Since the separation problem of $\mathbb{P}(G, \boldsymbol{w})$ is the minimum-weight cycle problem, which admits a polynomial-time algorithm, it follows from a theorem of Grötschel, Lovász, and Schrijver [22] that $\mathbb{P}(G, \boldsymbol{w})$ is always solvable in polynomial time. Therefore, the FAS problem can be solved in polynomial time for any nonnegative integral $\boldsymbol{w}$, provided its input digraph G is CI; and

• G is CM iff $\mathbb{D}(G, \boldsymbol{w})$ has an integral optimal solution for any nonnegative integral $\boldsymbol{w}$ iff $\nu_w^*(G) = \nu_w(G)$ for any nonnegative integral $\boldsymbol{w}$ iff the beautiful min-max relation $\nu_w(G) = \tau_w(G)$ holds for any nonnegative integral $\boldsymbol{w}$. (This gives an equivalent definition of CM digraphs.)

So the study of CI and CM digraphs has both great theoretical interest and practical value. Initiated in the early 1960s [14, 38], it has inspired many min-max theorems in combinatorial optimization, such as Lucchesi and Younger [27], Seymour [31, 32], Geelen and Guenin [21], Guenin [23, 24], Guenin and Thomas [25], Cai *et al.* [8, 9], and Ding *et al.* [16, 17]. Despite tremendous research efforts, only some special classes of CI and CM digraphs [27, 23, 25] have been identified to date, and a complete characterization seems extremely hard to obtain.

Let D_5 be the digraph obtained from K_5 (the complete graph with five vertices) by replacing each edge ij with a pair of opposite arcs (i, j) and (j, i). Applegate, Cook, and McCormick [4] and

Barahona, Fonlupt, and Mahjoub [5] independently proved that D_5 is CM, thereby confirming a conjecture posed in both Barahona and Mahjoub [6] and Jünger [26]. This theorem is equivalent to saying that every tournament with five vertices is cycle Mengerian.

The purpose of this paper is to give a complete characterization of all CI and CM tournaments. We say that a tournament is *Möbius-free* if it contains none of $K_{3,3}$, $K'_{3,3}$, M_5, and M_5^* depicted in Figure 1 as a subgraph. This class of tournaments is so named because the forbidden structures are all Möbius ladders. Observe that M_5^* is obtained from M_5 by reversing the direction of each arc.

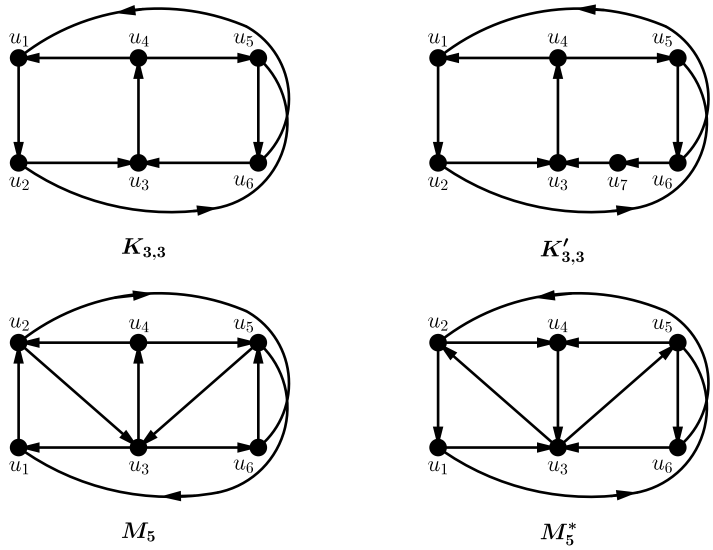


Figure 1. Forbidden Structures

Theorem 1.1. *For a tournament $T = (V, A)$, the following statements are equivalent:*

(i) T is Möbius-free;

(ii) T is cycle ideal; and

(iii) T is cycle Mengerian.

Throughout this paper we shall repeatedly use the following notations and terminology.

As usual, $\mathbb{R}_+$ and $\mathbb{Z}_+$ stand for the sets of nonnegative real numbers and nonnegative integers, respectively. For any two sets Ω and K, where Ω is always a set of numbers and K is always finite, we use Ω^K to denote the set of vectors $\boldsymbol{x} = (x(k) : k \in K)$ whose coordinates are members of Ω. If f is a function defined on a finite set S and $R \subseteq S$, then $f(R)$ denotes $\sum_{s \in R} f(s)$.

Digraphs considered in this paper contain no parallel arcs nor loops unless otherwise stated, but they may contain opposite arcs. Let G be a digraph. We use $V(G)$ and $A(G)$ to denote its vertex set and arc set, respectively, if they are not specified. For each $v \in V(G)$, we use $d_G^+(v)$ and $d_G^-(v)$ to denote the out-degree and in-degree of v, respectively. We call v a *near-sink* of G if its out-degree is one, and call v a *near-source* if its in-degree is one. For simplicity, an arc $e = (u, v)$ of G is also denoted by uv. Arc e is called *special* if either u is a near-sink or v is a near-source of G. For each $U \subseteq V(G)$, we use $\delta^+(U)$ (resp. $\delta^-(U)$) to denote the set of all arcs from U to $V(G)\backslash U$ (resp. from $V(G)\backslash U$ to U), and write $\delta^+(U) = \delta^+(u)$ and $\delta^-(U) = \delta^-(u)$ if $U = \{u\}$. We also use G/U to denote the digraph obtained from G by first deleting arcs between any two vertices in U, then identifying all vertices in U, and finally deleting the parallel arcs except one from each vertex to each other vertex; we say that G/U is obtained from G by *contracting* U. We say that U is a *homogeneous set* of G if $|U| \geq 2$ and the arcs between U and any vertex v outside U are either all directed to U or all directed to v. For each arc $e = (u, v)$ of G, the digraph obtained from G by *contracting* e, denoted by G/e, is exactly $G/\{u, v\}$. A *dicut* of G is a partition (X, Y) of $V(G)$ such that all arcs between X and Y are directed to Y. A dicut (X, Y) is *trivial* if $|X| = 1$ or $|Y| = 1$. Recall that G is called *weakly connected* if its underlying undirected graph is connected, and is called *strongly connected* or *strong* if each vertex is reachable from each other vertex. Clearly, a weakly connected digraph G is strong iff G has no dicut. Furthermore, a weakly connected digraph G is called *internally strong* if every dicut of G is trivial, and is called *internally 2-strong* (*i2s*) if G is strong and $G\backslash v$ is internally strong for every vertex v. A *strong component* of G is a maximal strong subgraph, where the adjective maximal is meant with respect to set-inclusion rather than size. Note that each vertex of G belongs to exactly one strong component. Thus the strong components of G can be ordered as $A_1, A_2, \ldots, A_p$, such that the arcs between A_i and A_j are all directed from A_i to A_j for any $1 \leq i < j \leq p$; we refer to $(A_1, A_2, \ldots, A_p)$ as a *strong partition* of G. The *reverse* of G, denoted by G^*, is obtained from G be reversing the direction of each arc.

By a cycle or a path in a digraph we always mean a directed one. By a *triangle* we mean a directed cycle of length three. Let P be a directed path from a to b and let c and d be two vertices on P such that a, b, c, d (not necessarily distinct) occur on P in order as we traverse P in its direction from a. Then $P[c, d]$ denotes the subpath of P from c to d, and $P(c, d) = P[c, d]\backslash\{c, d\}$. Let C be a directed cycle. For each vertex a on C, we use a^- (resp. a^+) to denote the vertex precedes (resp. succeeds) a as we traverse C in its direction. For each pair of vertices a and b on C, we use $C[a, b]$ to denote the segment of C from a to b.

The remainder of this paper is organized as follows. In Section 2, we first show that every cycle ideal tournament is Möbius-free. We then introduce a summing operation, which plays an important role in the structural description of Möbius-free tournaments. In Section 3, we prove that every *i2s* Möbius-free tournament comes from a finite list. In Section 4, we give a structural decomposition of Möbius-free tournaments that are not *i2s*, and exhibit some basic properties satisfied by the optimal solutions to the fractional cycle packing and FAS problems. In Sections 5 (resp. 6), we carry out a series of basic (resp. composite) reduction operations involved in the reduction step. In Section 7, we accomplish the last step of our proof. In Section 8, we conclude this paper with some remarks.

2 Preliminaries

In this section, we first show that each digraph displayed in Figure 1 is a forbidden structure of cycle ideal (CI) tournaments. We then introduce a summing operation on tournaments, which will be used to lift the connectivity of the Möbius-free tournament involved in Theorem 1.1. Finally, we prove that being Möbius-free is preserved under this summing operation and under contracting special arcs.

Lemma 2.1. *Every cycle ideal tournament is Möbius-free.*

Proof. Assume the contrary: Some CI tournament $T = (V, A)$ contains a member D of $\{K_{3,3}, K'_{3,3}, M_5, M_5^*\}$. Let B be the arc set of D and let $\mathcal{C}$ be the family of all cycles in T. Define $w(e) = 1$ if $e \in B$ and $w(e) = 0$ if $e \in A \backslash B$. We propose to show that, for this weight function $\boldsymbol{w}$, the optimal value of $\mathbb{P}(T, \boldsymbol{w})$, denoted by $\tau_w^*(T)$, is not integral. Depending on the structure of D, we consider four cases.

Case 1. $D = K_{3,3}$.

Define $\boldsymbol{x} \in \mathbb{R}_+^A$ and $\boldsymbol{y} \in \mathbb{R}_+^{\mathcal{C}}$ as follows:

- $x(e) = 1$ if $e \in A \backslash B$, $x(e) = 1/2$ if $e \in \{u_1u_2, u_3u_4, u_5u_6\}$, and $x(e) = 0$ otherwise; and
- $y(C) = 1/2$ if $C \in \{u_1u_2u_3u_4u_1, u_3u_4u_5u_6u_3, u_1u_2u_5u_6u_1\}$ and $y(C) = 0$ otherwise.

It is easy to see that $\boldsymbol{x}$ and $\boldsymbol{y}$ are feasible solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively. Since both of their objective values are $3/2$, by the LP-duality theorem, $\boldsymbol{x}$ and $\boldsymbol{y}$ are actually optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively. Thus $\tau_w^*(T) = 3/2$.

Case 2. $D = K'_{3,3}$.

Define $\boldsymbol{x} \in \mathbb{R}_+^A$ and $\boldsymbol{y} \in \mathbb{R}_+^{\mathcal{C}}$ as follows:

- $x(e) = 1$ if $e \in A \backslash B$, $x(e) = 1/2$ if $e \in \{u_1u_2, u_3u_4, u_5u_6\}$, and $x(e) = 0$ otherwise; and
- $y(C) = 1/2$ if $C \in \{u_1u_2u_3u_4u_1, u_3u_4u_5u_6u_7u_3, u_1u_2u_5u_6u_1\}$ and $y(C) = 0$ otherwise.

Similar to Case 1, we can show that $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively, and $\tau_w^*(T) = 3/2$.

Case 3. $D = M_5$.

Define $\boldsymbol{x} \in \mathbb{R}_+^A$ and $\boldsymbol{y} \in \mathbb{R}_+^{\mathcal{C}}$ as follows:

- $x(e) = 1$ if $e \in A \backslash B$, $x(e) = 1/2$ if $e \in \{u_1u_2, u_2u_3, u_3u_4, u_5u_3, u_6u_5\}$, and $x(e) = 0$ otherwise; and
- $y(C) = 1/2$ if $C \in \{u_1u_2u_3u_1, u_2u_3u_4u_2, u_3u_4u_5u_3, u_3u_6u_5u_3, u_1u_2u_6u_5u_1\}$ and $y(C) = 0$ otherwise.

Similar to Case 1, we can show that $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively, and $\tau_w^*(T) = 5/2$.

Case 4. $D = M_5^*$.

Consider the reverse T^*. In view of the $1-1$ correspondence between cycles in T and those in T^* and using the statement established in Case 3, we obtain $\tau_w^*(T) = 5/2$ in this case as well.

Combining the above cases, we conclude that $\tau_w^*(T)$ is not integral. So $\mathbb{P}(T, \boldsymbol{w})$ has no integral optimal solution and hence T is not CI, a contradiction. ∎

As an important endeavor towards a proof of Theorem 1.1, we shall demonstrate that all Möbius-free tournaments can be constructed from some prime tournaments using the following summing operation: Let $T_1 = (V_1, A_1)$ and $T_2 = (V_2, A_2)$ be two strong tournaments, with $|V_i| \geq 3$ for $i = 1, 2$. Suppose (a_1, b_1) is a special arc of T_1 with $d_{T_1}^+(a_1) = 1$ and (b_2, a_2) is a special arc of T_2 with $d_{T_2}^-(a_2) = 1$. The *1-sum* of T_1 and T_2 over (a_1, b_1) and (b_2, a_2) is the tournament arising from the disjoint union of $T_1 \backslash a_1$ and $T_2 \backslash a_2$ by identifying b_1 with b_2 (the resulting vertex is denoted by b) and adding all arcs from $T_1 \backslash \{a_1, b_1\}$ to $T_2 \backslash \{a_2, b_2\}$. We call b the *hub* of the 1-sum. See Figure 2 for an illustration. Note that if $|V_i| = 3$ for $i = 1$ or 2, then T_i is a triangle, and thus $T = T_{3-i}$. We say that T_1 is *smaller* than T_2 if $|V_1| < |V_2|$.

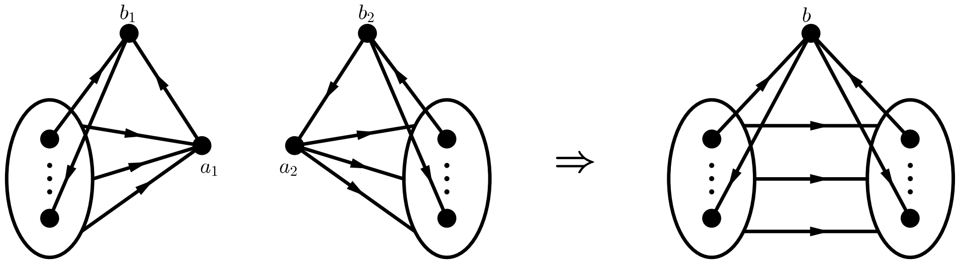


Figure 2. 1-sum of T_1 and T_2.

Lemma 2.2. *Let $T = (V, A)$ be a strong tournament. If T is not i2s, then T is the 1-sum of two smaller strong tournaments.*

Proof. Since T is not *i2s*, it contains a vertex b such that $T\backslash b$ has a nontrivial dicut (X, Y). As T is strong, there exist $a_1 \in Y$ and $a_2 \in X$ such that $\{(a_1, b), (b, a_2)\} \subseteq A$. Set $T_1 = T\backslash(Y\backslash a_1)$, $T_2 = T\backslash(X\backslash a_2)$, and rename b as b_i in T_i for $i = 1, 2$. Clearly, a_1 has out-degree one in T_1 and a_2 has in-degree one in T_2. From the definition we see that T is the 1-sum of T_1 and T_2 over (a_1, b_1) and (b_2, a_2). Furthermore, T_i is strong and has fewer vertices than T for $i = 1, 2$. ∎

Let us show that being Möbius-free is maintained under the 1-sum operation.

Lemma 2.3. *Let $T = (V, A)$ be the 1-sum of two tournaments T_1 and T_2. Then T is Möbius-free iff both T_1 and T_2 are Möbius-free.*

Proof. Since both T_1 and T_2 are sub-tournaments of T, the "only if" part holds trivially. To establish the "if" part, assume the contrary: T contains a member D of $\{K_{3,3}, K'_{3,3}, M_5, M_5^*\}$; subject to this, the number of vertices in D is minimum. Let b be the hub of the 1-sum. Then b is contained in D. Observe that

(1) if $D = K'_{3,3}$, then $(u_3, u_6) \in A$ (see the labeling in Figure 1), for otherwise T would contain $K_{3,3}$, contradicting the minimality assumption on D.

Set $D' = D \cup \{(u_3, u_6)\}$ if $D = K'_{3,3}$ and set $D' = D$ otherwise. It it a routine matter to check that D' is *i2s* (while $K'_{3,3}$ is not). Since T is the 1-sum of T_1 and T_2 and since T contains D' by (1), either $T_1\backslash b$ or $T_2\backslash b$ contains precisely one vertex from $D'\backslash b$. Therefore, either T_1 or T_2 contains a subgraph isomorphic to D' and hence is not Möbius-free. ∎

In the remainder of this section, we show that being Möbius-free is also preserved under the operation of contracting a special arc. Note that the resulting digraph may contain opposite arcs.

Lemma 2.4. *Let $T = (V, A)$ be a Möbius-free tournament with a special arc $a = (x, y)$. Then T/a is also Möbius-free.*

Proof. Replacing T by its reverse T^* if necessary, we may assume that x is a near-sink of T. Thus y is the only out-neighbor of x. Let z be the vertex obtained by identifying x and y in T/a and let $\mathcal{F} = \{K_{3,3}, K'_{3,3}, M_5, M_5^*\}$. Assume the contrary: T/a contains a subdigraph $D \in \mathcal{F}$.

Then z is in D. We use D' to denote the digraph obtained from $D\backslash z$ by adding two vertices x and y and adding all arcs in $\{(x,y)\} \cup \{(y,u) : (z,u) \in A(D)\} \cup \{(u,x) : u \in V(D)\backslash z\}$. Clearly, D' is a subgraph of T. We propose to prove that

(1) T contains a member of $\mathcal{F}$.

Let us label the vertices of D as in Figure 1. Depending on the structure of D, we distinguish among four cases.

Case 1. $D = K_{3,3}$. In this case, symmetry allows us to assume that $z = u_4$ or u_5.

• $z = u_4$. Then u_1 and u_5 are the only out-neighbors of y in D'. Thus the union of the three cycles $u_1u_2u_5u_6u_1$, xyu_1u_2x, and xyu_5u_6x forms a $K_{3,3}$ in T.

• $z = u_5$. Then u_6 is the only out-neighbor of y in D'. If $(u_4, y) \in A$, then the union of the three cycles $u_1u_2u_3u_4u_1$, $u_4yu_6u_3u_4$, and $u_1u_2xyu_6u_1$ forms a $K'_{3,3}$ in T. Similarly, if $(u_2, y) \in A$, then the union of the three cycles $u_1u_2u_3u_4u_1$, $u_1u_2yu_6u_1$, and $u_4xyu_6u_3u_4$ also forms a $K'_{3,3}$ in T. So we assume that $\{(y, u_4), (y, u_2)\} \subseteq A$. Thus the union of the three cycles $u_4u_1xyu_4$, $u_1u_2u_3u_4u_1$, and $u_2u_3xyu_2$ forms a $K_{3,3}$ in T.

Case 2. $D = K'_{3,3}$. In this case, we may assume that $(u_3, u_6) \in A$, for otherwise the present case reduces to Case 1.

• $z = u_2$. Then u_5 and u_3 are the only out-neighbors of y in D'. It follows that the union of the three cycles $u_3u_4u_5u_6u_7u_3$, xyu_3u_4x, and xyu_5u_6x forms a $K'_{3,3}$ in T.

• $z = u_3$. Then u_4 is the only out-neighbor of y in D'. If $(u_6, y) \in A$, then the union of the three cycles $u_1u_2u_5u_6u_1$, $yu_4u_5u_6y$, and $xyu_4u_1u_2x$ forms a $K'_{3,3}$ in T; if $(u_2, y) \in A$, then the union of the three cycles $u_1u_2u_5u_6u_1$, $yu_4u_1u_2y$, and $xyu_4u_5u_6x$ also forms a $K'_{3,3}$ in T. So we assume that $\{(y, u_6), (y, u_2)\} \subseteq A$. It follows that a $K_{3,3}$ is formed in T by the three cycles xyu_6u_1x, xyu_2u_5x, and $u_1u_2u_5u_6u_1$.

• $z = u_4$. Then u_1 and u_5 are the only out-neighbors of y in D'. Thus the union of the three cycles $u_1u_2u_5u_6u_1$, xyu_5u_6x, and xyu_1u_2x forms a $K_{3,3}$ in T.

• $z = u_6$. Then u_1 and u_7 are the only out-neighbors of y in D'. It follows that the union of the three cycles $u_1u_2u_3u_4u_1$, $xyu_7u_3u_4x$, and xyu_1u_2x forms a $K'_{3,3}$ in T.

• $z = u_1$. Then u_2 is the only out-neighbor of y in D'. If $\{(u_4, y), (u_6, y)\} \subseteq A$, then the union of the three cycles $yu_2u_3u_4y$, $yu_2u_5u_6y$, and $u_3u_4u_5u_6u_7u_3$ forms a $K'_{3,3}$ in T. So we assume that at least one of (y, u_4) and (y, u_6) is in A.

Consider the first subcase when $(y, u_4) \in A$. If $(u_6, u_2) \in A$, then the union of the three cycles xyu_2u_3x, xyu_4u_5x, and $u_2u_3u_4u_5u_6u_2$ forms a $K'_{3,3}$ in T; if $(y, u_7) \in A$, then the union of the three cycles xyu_7u_3x, xyu_4u_5x, and $u_3u_4u_5u_6u_7u_3$ forms a $K'_{3,3}$ in T. So we assume that $\{(u_2, u_6), (u_7, y)\} \subseteq A$. If $(u_4, u_6) \in A$, then a $K'_{3,3}$ is formed by the three cycles $yu_2u_6u_7y$, $u_3u_4u_6u_7u_3$, and $xyu_2u_3u_4x$; if $(u_3, u_5) \in A$, then a $K'_{3,3}$ is formed by the three cycles $yu_2u_6u_7y$, $u_3u_5u_6u_7u_3$, and $xyu_2u_3u_5x$. So we further assume that $\{(u_6, u_4), (u_5, u_3)\} \subseteq A$. It follows that the union of the three cycles $u_3u_6u_4u_5u_3$, xyu_4u_5x, and $xyu_2u_3u_6x$ forms a $K'_{3,3}$.

Consider the second subcase when $(y, u_6) \in A$. If $(u_7, u_2) \in A$, then the union of the three cycles xyu_2u_3x, $u_2u_3u_6u_7u_2$, and xyu_6u_7x forms a $K_{3,3}$; if $(y, u_3) \in A$, then a $K'_{3,3}$ is formed by the three cycles xyu_6u_7x, xyu_3u_4x, and $u_3u_4u_5u_6u_7u_3$; if $(u_4, u_6) \in A$, then a $K'_{3,3}$ is formed by the three cycles xyu_6u_7x, $u_3u_4u_6u_7$, and $xyu_2u_3u_4x$. So we assume that $\{(u_2, u_7), (u_3, y), (u_6, u_4)\} \subseteq A$. If $(u_5, u_3) \in A$, then a $K'_{3,3}$ is formed by the three cycles $u_3u_6u_4u_5u_3$, xyu_6u_4x, and $xyu_2u_5u_3x$; if $(u_4, u_2) \in A$, then a $K_{3,3}$ is formed by the three cycles

xyu_2u_3x, xyu_6u_4x, and $u_2u_3u_6u_4u_2$. So we further assume that $\{(u_3, u_5), (u_2, u_4)\} \subseteq A$. Now if $(y, u_5) \in A$, then the union of the three cycles $u_3u_5u_6u_7u_3$, xyu_5u_6x, and $xyu_2u_7u_3x$ forms a $K'_{3,3}$; if $(u_5, y) \in A$, then the union of the three cycles $yu_2u_7u_3y$, $yu_2u_4u_5y$, and $u_3u_4u_5u_6u_7u_3$ also forms a $K'_{3,3}$.

• $z = u_5$. Then u_6 is the only out-neighbor of y in D'. If $(y, u_2) \in A$, then the union of the three cycles $u_1u_2u_3u_6u_1$, xyu_2u_3x, and xyu_6u_1x forms a $K_{3,3}$ in T. So we assume that $(u_2, y) \in A$. If $(u_4, y) \in A$, then the union of the three cycles $yu_6u_1u_2y$, $u_1u_2u_3u_4u_1$, and $yu_6u_7u_3u_4y$ forms a $K'_{3,3}$ in T. So we also assume that $(y, u_4) \in A$. If $(u_1, u_7) \in A$, then a $K'_{3,3}$ is formed by the three cycles $u_1u_7u_3u_4u_1$, xyu_4u_1x, and $xyu_6u_7u_3x$. So we further assume that $(u_7, u_1) \in A$. If $(y, u_7) \in A$, then a $K'_{3,3}$ is formed by the three cycles $u_1u_2u_3u_4u_1$, $u_1u_2yu_7u_1$, and $xyu_7u_3u_4x$. Similarly, if $(y, u_3) \in A$, then a $K'_{3,3}$ is formed by the three cycles $u_1u_2u_3u_4u_1$, xyu_3u_4x, and $xyu_6u_1u_2x$; if $(y, u_1) \in A$, then a $K'_{3,3}$ is formed by the three cycles xyu_1u_2x, xyu_6u_7x, and $u_1u_2u_3u_6u_7u_1$. Thus it remains to consider the subcase when $\{(u_7, y), (u_3, y), (u_1, y)\} \subseteq A$. If $(u_2, u_7) \in A$, then a $K'_{3,3}$ is formed by the three cycles $u_1u_2yu_6u_1$, $yu_6u_7u_3y$, and $u_1u_2u_7u_3u_4u_1$. So we assume that $(u_7, u_2) \in A$. If $(u_6, u_4) \in A$, then a $K'_{3,3}$ is formed by the three cycles $yu_6u_4u_1y$, $u_1u_2u_3u_4u_1$, and $yu_6u_7u_2u_3y$. So we also assume that $(u_4, u_6) \in A$. If $(u_4, u_7) \in A$, then a $K'_{3,3}$ is formed by the three cycles $yu_4u_6u_1y$, $u_1u_2u_3u_6u_1$, and $yu_4u_7u_2u_3y$. So we further assume that $(u_7, u_4) \in A$.

From the above observations, we conclude that u_7 has a unique in-neighbor u_6 in the subtournament T' of T induced by $V(D')$. If $\{(u_6, u_2), (u_2, u_4)\} \subseteq A$, then an M_5^* is formed by the five cycles yu_4u_1y, $u_1u_2u_4u_1$, $u_2u_4u_6u_2$, $u_4u_6u_7u_4$, and $yu_6u_7u_1y$. If $(u_2, u_6) \in A$, then a $K_{3,3}$ is formed by $u_1u_2u_3u_4u_1$, $u_3u_4u_6u_7u_3$, and $u_1u_2u_6u_7u_1$; if $(u_4, u_2) \in A$, then the union of the three cycles $u_1u_2u_3u_6u_1$, $yu_4u_2u_3y$, and $yu_4u_6u_1y$ also forms a $K_{3,3}$ in T.

• $z = u_7$. Then u_3 is the only out-neighbor of y in D'. If $(u_6, y) \in A$, then the union of the three cycles $u_1u_2u_5u_6u_1$, $u_1u_2u_3u_4u_1$, and $yu_3u_4u_5u_6y$ forms a $K'_{3,3}$ in T; if $(y, u_1) \in A$, then a $K_{3,3}$ is formed in T by the three cycles $u_1u_2u_3u_4u_1$, xyu_1u_2x, and xyu_3u_4x. So we assume that $\{(y, u_6), (u_1, y)\} \subseteq A$. If $(u_1, u_3) \in A$, then a $K'_{3,3}$ is formed by xyu_6u_1x, xyu_3u_4x, and $u_1u_3u_4u_5u_6u_1$; if $(u_4, u_6) \in A$, then the union of the three cycles xyu_6u_1x, xyu_3u_4x, and $u_1u_2u_3u_4u_6u_1$ forms a $K'_{3,3}$ in T; if $(y, u_2) \in A$, then the union of the three cycles $u_1u_2u_5u_6u_1$, xyu_6u_1x, and xyu_2u_5x forms a $K_{3,3}$ in T. So we further assume that $\{(u_3, u_1), (u_6, u_4), (u_2, y)\} \subseteq A$. Depending on whether $(u_5, y) \in A$, we distinguish between two subcases.

Consider the first subcase when $(u_5, y) \in A$. If $(u_4, u_2) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles $u_1u_2u_5u_6u_1$, $yu_3u_6u_1y$, and $yu_3u_4u_2u_5y$. So we assume that $(u_2, u_4) \in A$. If $(u_6, u_2) \in A$, then a $K_{3,3}$ is formed by the three cycles $u_2u_4u_5u_6u_2$, $yu_3u_6u_2y$, and $yu_3u_4u_5y$. So we further assume that $(u_2, u_6) \in A$. If $(u_4, y) \in A$, then a $K_{3,3}$ is formed in T by the three cycles $u_1u_2u_6u_4u_1$, $yu_3u_6u_4y$, and $yu_3u_1u_2y$ in T; if $(y, u_4) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_6u_1x, xyu_4u_5x, and $u_1u_2u_4u_5u_6u_1$.

Consider the second subcase when $(y, u_5) \in A$. If $(y, u_4) \in A$, then a $K'_{3,3}$ is formed by the three cycles xyu_5u_6x, xyu_4u_1x, and $u_1u_2u_5u_6u_4u_1$. So we assume that $(u_4, y) \in A$. If $(u_2, u_6) \in A$, then a $K_{3,3}$ is formed by the three cycles $u_1u_2u_6u_4u_1$, $yu_3u_6u_4y$, and $yu_3u_1u_2y$. So we also assume that $(u_6, u_2) \in A$. If $(u_2, u_4) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles $u_2u_4u_5u_6u_2$, xyu_6u_2x, and $xyu_3u_4u_5x$; if $(u_5, u_3) \in A$, then a $K_{3,3}$ is formed in T by the three cycles xyu_6u_2x, xyu_5u_3x, and $u_2u_5u_3u_6u_2$. Thus we further assume that $\{(u_4, u_2), (u_3, u_5)\} \subseteq A$. It follows that a $K'_{3,3}$ is formed in T by the three cycles $u_2u_5u_6u_4u_2$, $yu_3u_4u_2y$, and $yu_3u_5u_6u_1y$.

Case 3. $D = M_5$. In this case, u_1 and u_6 are symmetric, so are u_2 and u_5.

• $z = u_4$. Then vertices u_2 and u_5 are the only out-neighbors of y in D'. If $(u_3, y) \in A$, then an M_5 is formed in T by the five cycles $u_3u_6u_5u_3$, yu_5u_3y, yu_2u_3y, $u_1u_2u_3u_1$, and $u_1u_2u_6u_5u_1$. If $(y, u_3) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_5u_1x, xyu_3u_6x, and $u_1u_2u_3u_6u_5u_1$.

• $z = u_5$. Then u_1 and u_3 are the only out-neighbors of y in D'. If $(u_4, u_1) \in A$, then a $K_{3,3}$ is formed by the three cycles $u_1u_2u_3u_4u_1$, xyu_1u_2x, and xyu_3u_4x. So we assume that $(u_1, u_4) \in A$. If $(y, u_4) \in A$, then a $K_{3,3}$ is formed by the three cycles $u_1u_4u_2u_3u_1$, xyu_4u_2x, and xyu_3u_1x. Thus we further assume that $(u_4, y) \in A$. It follows that an M_5 is formed in T by the five cycles $u_1u_2u_3u_1$, $u_2u_3u_4u_2$, yu_3u_4y, xyu_3x, and xyu_1u_2x.

• $z = u_6$. Then u_5 is the only out-neighbor of y in D'. If $\{(u_2, y), (u_3, y)\} \subseteq A$, then an M_5 is formed in T by the five cycles $u_3u_4u_5u_3$, yu_5u_3y, $u_2u_3u_4u_2$, $u_1u_2u_3u_1$, and $yu_5u_1u_2y$. Otherwise, if $(y, u_2) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_5u_1x, xyu_2u_3x, and $u_1u_2u_3u_4u_5u_1$; if $(y, u_3) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_5u_1x, xyu_3u_4x, and $u_1u_2u_3u_4u_5u_1$.

• $z = u_3$. Then u_1, u_4, and u_6 are the only out-neighbors of y in D'. If $\{(u_5, y), (u_2, y)\} \subseteq A$, then an M_5 is formed in T by the five cycles yu_6u_5y, yu_4u_5y, yu_4u_2y, yu_1u_2y, and $u_1u_2u_6u_5u_1$. Suppose at least one of (y, u_5) and (y, u_2) is in T. If both (y, u_5) and (y, u_2) are in T, then a $K_{3,3}$ is formed by the three cycles $u_1u_2u_6u_5u_1$, xyu_5u_1x, and xyu_2u_6x. So we assume that either $\{(y, u_5)(u_2, y)\} \subseteq A$ or $\{(y, u_2), (u_5, y)\} \subseteq A$.

Consider the first subcase when $\{(y, u_5), (u_2, y)\} \subseteq A$. If $(u_6, u_4) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_5u_1x, xyu_6u_4x, and $u_1u_2u_6u_4u_5u_1$. So we may assume that $(u_4, u_6) \in A$. If $(u_1, u_4) \in A$, then a $K_{3,3}$ is formed in T by the three cycles $u_1u_4u_6u_5u_1$, xyu_5u_1x, and xyu_4u_6x. If $(u_4, u_1) \in A$, then a $K'_{3,3}$ is formed in T by the the three cycles $u_1u_2u_6u_5u_1$, $yu_4u_1u_2y$, and $xyu_4u_6u_5x$.

Consider the second subcase when $\{(y, u_2), (u_5, y)\} \subseteq A$. If $(u_6, u_4) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_4u_5x, xyu_2u_6x, and $u_1u_2u_6u_4u_5u_1$. If $(u_1, u_4) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_2u_6x, xyu_1u_4x, and $u_1u_4u_2u_6u_5u_1$. So we assume that $\{(u_4, u_6), (u_4, u_1)\} \subseteq A$. Then a $K'_{3,3}$ is formed in T by the three cycles $u_1u_2u_6u_5u_1$, $yu_4u_6u_5y$, and $xyu_4u_1u_2x$.

Case 4. $D = M_5^*$. In this case, u_1 and u_6 are symmetric, so are u_2 and u_5.

• $z = u_3$. Then u_5 and u_2 are the only out-neighbors of y in D'. Thus a $K_{3,3}$ is formed in T by the three cycle $u_1u_5u_6u_2u_1$, xyu_2u_1x, and xyu_5u_6x.

• $z = u_4$. Then u_3 is the only out-neighbor of y in D'. If both $\{(u_2, y), (u_5, y)\} \subseteq A$, then an M_5^* is formed by the five cycles $u_1u_3u_2u_1$, yu_3u_2y, yu_3u_5y, $u_3u_5u_6u_3$, and $u_1u_5u_6u_2u_1$. So we assume that at most one of (u_2, y) and (u_5, y) is in T. If $(y, u_2) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles $u_1u_3u_5u_6u_2u_1$, xyu_2u_1x, and xyu_3u_5x; if $(y, u_5) \in T$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_3u_2x, xyu_5u_6x, and $u_1u_5u_6u_3u_2u_1$.

• $z = u_6$. Then u_2 and u_3 are the only out-neighbors of y in D'. If $(u_5, y) \in A$, then an M_5^* is formed in T by the five cycles $u_1u_3u_2u_1$, $u_2u_4u_3u_2$, $u_3u_5u_4u_3$, yu_3u_5y, and $yu_2u_1u_5y$. If $(y, u_5) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_5u_4x, xyu_2u_1x, and $u_1u_5u_4u_3u_2u_1$.

• $z = u_5$. Then u_4 and u_6 are the only out-neighbors of y in D'. Observe that if both (u_3, y) and (u_1, y) are arcs in T, then an M_5^* is formed in T by the five cycles $u_1u_3u_2u_1$, $u_2u_4u_3u_2$, yu_4u_3y, yu_6u_3y, and $yu_6u_2u_1y$. So we assume that at least one of (y, u_3) and (y, u_1) is in T.

Suppose $(u_4, u_1) \in A$. If $(u_1, u_6) \in A$, then a $K_{3,3}$ is formed in T by the three cycles $u_1u_6u_2u_4u_1$, xyu_4u_1x, and xyu_6u_2x; if $(y, u_3) \in A$, then a $K_{3,3}$ is formed in T by the three cycles $u_1u_3u_2u_4u_1$, xyu_4u_1x, and xyu_3u_2x. So we assume that $\{(u_6, u_1), (u_3, y)\} \subseteq A$. Then a $K'_{3,3}$ is formed in T by the three cycles $u_1u_3u_2u_4u_1$, $yu_6u_1u_3y$, and $xyu_6u_2u_4x$.

Suppose $(u_1, u_4) \in A$. If $(y, u_2) \in A$, then a $K_{3,3}$ is formed by the three cycles $u_1u_4u_3u_2u_1$, xyu_2u_1x, and xyu_4u_3x. So we assume that $(u_2, y) \in A$. Consider the subcase when $(u_1, y) \in A$. Now $(y, u_3) \in A$. If $(u_4, u_6) \in A$, then the union of the three cycles $u_2u_4u_6u_3u_2$, xyu_4u_6x, and xyu_3u_2x forms a $K_{3,3}$ in T; if $(u_6, u_4) \in A$, then the union of the three cycles $u_1u_4u_3u_2u_1$, $yu_6u_2u_1y$, and $xyu_6u_4u_3x$ forms a $K'_{3,3}$ in T. Next, consider the subcase when $(y, u_1) \in A$. If $(y, u_3) \in A$, then the union of the three cycles $u_1u_4u_3u_2u_1$, xyu_1u_4x, and xyu_3u_2x forms a $K_{3,3}$ in T; if $(u_4, u_6) \in A$, then the union of the three cycles $u_2u_4u_6u_3u_2$, $yu_1u_3u_2y$, and $xyu_1u_4u_6x$ forms a $K'_{3,3}$ in T. Suppose $\{(u_3, y), (u_6, u_4)\} \subseteq A$. Then a $K'_{3,3}$ is formed in T by the three cycles $u_1u_4u_3u_2u_1$, $yu_6u_4u_3y$, and $xyu_6u_2u_1x$.

Combining the above four cases, we establish (1). Therefore T is not Möbius-free, a contradiction. ∎

3 Structural Descriptions

In this section we show that every Möbius-free tournament can be constructed from some prime tournaments using 1-sum operations. Our proof relies on the following chain theorem, which asserts that every *i2s* tournament $T = (V, A)$ with $|V| \geq 5$ can be constructed from $\{F_1, F_2, F_3, F_4, F_5\}$ (see Figure 3) by repeatedly adding vertices such that all the intermediate tournaments are also *i2s*.

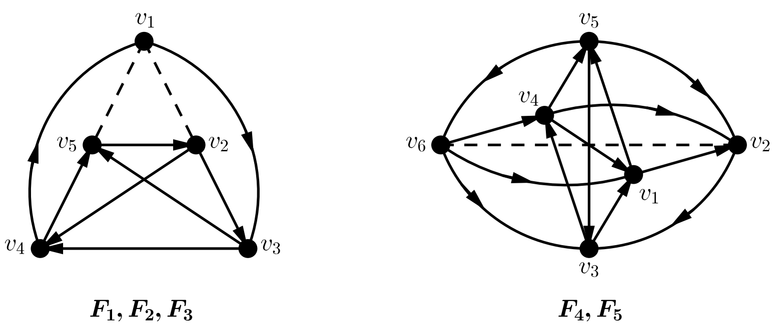


Figure 3. $v_1v_2, v_5v_1 \in F_1$; $v_2v_1, v_1v_5 \in F_2$; $v_2v_1, v_5v_1 \in F_3$; $v_6v_2 \in F_4$; $v_2v_6 \in F_5$.

Theorem 3.1. *Let $T = (V, A)$ be an i2s tournament with $|V| \geq 5$. Then the following statements hold:*

(i) If $|V| = 5$, then $T \in \{F_1, F_2, F_3\}$;

(ii) If $|V| = 6$, then either T has a vertex z with $T \backslash z \in \{F_1, F_2, F_3\}$ or $T \in \{F_4, F_5\}$;

(iii) If $|V| \geq 7$, then T has a vertex z such that $T \backslash z$ remains to be i2s.

Our next theorem states that every *i2s* Möbius-free tournament with at least six vertices comes from a finite family of sporadic tournaments, which, together with Lemmas 3.4 and 3.5, gives a structural description of all *i2s* Möbius-free tournaments.

Theorem 3.2. *Let $T = (V, A)$ be an i2s tournament with $|V| \geq 6$. Then T is Möbius-free iff T is one of G_1, G_2, G_3, and F_4 (see Figure 4).*

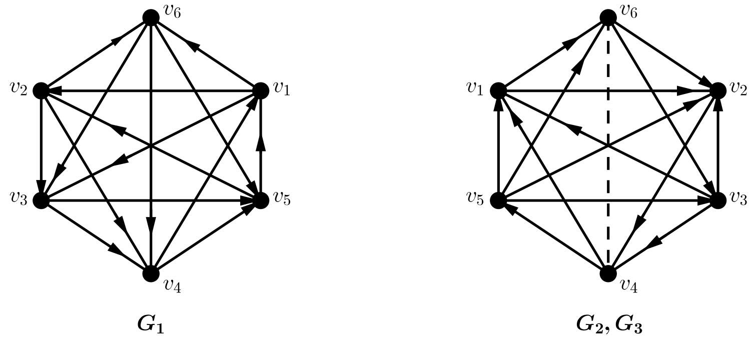


Figure 4. $v_6v_4 \in G_2$ and $v_4v_6 \in G_3$.

Let C_3 and F_0 be the two tournaments depicted in Figure 5, and let

$$\mathcal{T}_1 = \{C_3, F_0, F_2, F_3, F_4, G_2, G_3\}.$$

Our third theorem gives a structural description of all strong Möbius-free tournaments.

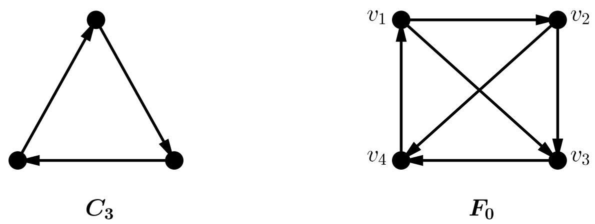


Figure 5. Strong tournaments with three or four vertices.

Theorem 3.3. *Let $T = (V, A)$ be a strong Möbius-free tournament with $|V| \geq 3$. Then T can be obtained by repeatedly taking 1-sums starting from tournaments in $\mathcal{T}_1$, unless $T \in \{F_1, G_1\}$.*

We break the proofs of these theorems into a series of lemmas.

Lemma 3.4. *Let $T = (V, A)$ be a strong tournament. If $|V| = 3$, then T is C_3; if $|V| = 4$, then T is F_0. (So T is strong iff it is i2s when $|V| = 3$ or 4.)*

Proof. Since every strong tournament has a Hamilton cycle, it is clear that $T = C_3$ if $|V| = 3$ and $T = F_0$ if $|V| = 4$. Note that both C_3 and F_0 are *i2s*, so T is strong iff it is *i2s* when $|V| = 3$ or 4. ∎

Lemma 3.5. *Let $T = (V, A)$ be an i2s tournament. If $|V| = 5$, then $T \in \{F_1, F_2, F_3\}$.*

Proof. If $T\backslash u$ is strong for each $u \in V$, then both the in-degree and out-degree of each vertex equal two, and hence T is isomorphic to F_1.

So we assume that $T\backslash u$ has a trivial dicut (X, Y) for some $u \in V$. Since each F_i is isomorphic to its reverse for $i = 1, 2, 3$, replacing T by its reverse if necessary, we may assume that $|X| = 1$ and $|Y| = 3$. Let $X = \{x\}$ and $Y = \{y_1, y_2, y_3\}$. Since $T\backslash u$ is internally strong, Y induces a C_3. Since T is strong, $(u, x) \in A$, and u has at most two out-neighbors in Y. If u has exactly two out-neighbors in Y, say y_1 and y_2 (by symmetry), then $(\{u, x\}, \{y_1, y_2\})$ would be a nontrivial dicut of $T\backslash y_3$, a contradiction. So u has at most one out-neighbor in Y. If u has no out-neighbors in Y, then all arcs between Y and u are directed to u, so T is isomorphic to F_2. If u has only one out-neighbor in Y, then T is isomorphic to F_3.

Combining the above observations, we conclude that $T \in \{F_1, F_2, F_3\}$. ∎

Lemma 3.6. *Let $T = (V, A)$ be a strong tournament and let x and y be two distinct vertices of T. Then T has a third vertex z such that $T\backslash z$ is still strong, unless T has a Hamilton path between x and y such that the remaining arcs are all backward.*

Proof. Since T is strong, it has a Hamilton cycle C. Let us first consider the case when

(1) T has a strong subgraph S containing both x and y with $|V(S)| < |V|$.

For notational simplicity, we assume that, subject to (1), S is chosen so that $|V(S)|$ is as large as possible. Then the vertices of S are consecutive on C. Let $P = C\backslash V(S)$. If P has only one vertex, then we are done. So we assume that P has two or more vertices. Let s and t be the initial and terminal vertices of P, respectively. Using the maximality assumption on S, we see that $\{(v, s), (t, v)\} \subseteq A$ for any vertex v in S. We claim that P contains no vertex other than s and t, for otherwise, let z be an internal vertex of P and let v be a vertex in S. Then either $S \cup C[s^-, z] \cup \{(z, v)\}$ or $S \cup C[z, t^+] \cup \{(v, z)\}$ would be a strong subgraph of T properly containing S; this contradiction to (1) justifies the claim. Since $\{(v, s), (t, v)\} \subseteq A$ for all vertices v in S, we deduce that $T\backslash z$ is strong for any vertex z in $S\backslash\{x, y\}$.

Next, let us consider the case when (1) does not occur. Renaming x and y if necessary, we may assume that $(x, y) \in A$. From the hypothesis of the present case, we deduce that (x, y) is an arc on C, $\{(x, y^+), (x^-, y)\} \subseteq A$, and $\{(x, v), (v, y)\} \subseteq A$ for any $v \in V\backslash\{x, y, x^-, y^+\}$. Thus $C\backslash(x, y)$ is a Hamilton path from y to x such that the remaining arcs are all backward. ∎

Corollary 3.7. *Let $T = (V, A)$ be a strong tournament with $|V| \geq 4$ and let x be a vertex in T. Then there exists a vertex $z \neq x$ such that $T\backslash z$ is strong.*

Proof. Let y be a vertex of T with $y \neq x$. By Lemma 2.3,

- either T has a vertex $z \neq x, y$ such that $T\backslash z$ is strong
- or T has a Hamilton path between x and y such that the remaining arcs are all backward.

In the former case z is a desired vertex, and in the latter case y is as desired. ∎

A digraph is called *trivial* if it contains only one vertex. The following lemma on strong partitions of tournaments (see Section 1) is straightforward, so we omit its proof here.

Lemma 3.8. *Let $T = (V, A)$ be an internally strong tournament and let $(A_1, A_2, \ldots, A_p)$ be the strong partition of T. If $|V| \geq 3$, then one of the following statements holds:*

(i) $p = 1$; A_1 is nontrivial;

(ii) $p = 2$*; exactly one of* A_1 *and* A_2 *is nontrivial;*

(iii) $p = 3$*; both* A_1 *and* A_3 *are trivial.*

The lemma below follows instantly from the preceding one.

Lemma 3.9. *Let* $T = (V, A)$ *be an i2s tournament, let* x *be a vertex in* T*, and let* $(A_1, A_2, \ldots, A_p)$ *be the strong partition of* $T\backslash x$*. Then* $1 \leq p \leq 3$*. (The value of* p *is called the* type *of* x *in* T*).*

For convenience, we shall not distinguish each A_i from its vertex set $V(A_i)$ in subsequent proofs, if there is no risk of confusion. Thus $|A_i| = |V(A_i)|$.

The following two lemmas guarantee the existence of a vertex z in an $i2s$ tournament T with at least six vertices such that $T\backslash z$ remains to be $i2s$.

Lemma 3.10. *Let* $T = (V, A)$ *be an i2s tournament with* $|V| \geq 6$*. If* T *contains a vertex* x *of type 3 (see Lemma 3.9), then it contains a vertex* z *such that* $T\backslash z$ *remains to be i2s.*

Proof. Let (A_1, A_2, A_3) be the strong partition of $T\backslash x$. Since x is of type 3, $|A_1| = |A_3| = 1$ by Lemma 3.8. So $|A_2| \geq 3$. Let z_i be the only vertex in A_i for $i = 1, 3$. Since T is $i2s$, both (x, z_1) and (z_3, x) are arcs in T. Furthermore, x has at least one in-neighbor x_1 and at least one out-neighbor x_2 in A_2. If there exists $z \in A_2\backslash\{x_1, x_2\}$ such that $A_2\backslash z$ is strong, then $T\backslash z$ is $i2s$. Otherwise, by Lemma 3.6, A_2 has a Hamilton path between x_1 and x_2 such that the remaining arcs of A_2 are all backward. Let $z = x_2$ if x_1 is the only in-neighbor of x in A_2 and let $z = x_1$ otherwise. Then $A_2\backslash z$ is strong and has at least one in-neighbor and at least one out-neighbor of x. Therefore $T\backslash z$ is $i2s$. ∎

Lemma 3.11. *Let* $T = (V, A)$ *be an i2s tournament with* $|V| \geq 6$ *and* $T \notin \{F_4, F_5\}$ *(see Figure 3). Then* T *contains a vertex* z *such that* $T\backslash z$ *remains to be i2s.*

Proof. We proceed by contradiction. By a *triple* $(T; x, y)$ we mean an $i2s$ tournament $T = (V, A)$ with $|V| \geq 6$ and $T \notin \{F_4, F_5\}$ such that $T\backslash z$ is not $i2s$ for any vertex z, together with two distinguished vertices x and y in T. Let us choose a triple $(T; x, y)$ such that

(1) $T\backslash x$ is strong while $T\backslash\{x, y\}$ is not internally strong;

(2) subject to (1), letting $(A_1, A_2, \ldots, A_p)$ be the strong partition of $T\backslash\{x, y\}$, A_1 contains an out-neighbor x' of x; and

(3) subject to (1) and (2), the tuple $(|A_1|, |A_2|, ..., |A_p|)$ is minimized lexicographically.

By Corollary 3.7, there exists a triple $(T; x, y)$ satisfying (1). To verify the existence of a triple $(T; x, y)$ satisfying both (1) and (2), note that if x has no out-neighbor in A_1, then it must have an in-neighbor in A_p, for otherwise, y would be of type 3, and hence $T\backslash z$ would be $i2s$ for some vertex z by Lemma 3.10, a contradiction. Since each of F_4 and F_5 is isomorphic to its reverse, replacing T by T^* if necessary, we see that the triple $(T; x, y)$ is available.

Let us make some simple observations about the triple $(T; x, y)$. Since $|V| \geq 6$, by (1) we have

(4) $p \geq 2$, and y has an out-neighbor y' in A_1 and an in-neighbor y'' in A_p.

(5) If $p = 2$, then x has an in-neighbor in A_p.

Otherwise, since $|V| \geq 6$ and $T\backslash y$ is internally strong, $|A_2| = 1$ and $|A_1| \geq 3$, which implies that $T\backslash\{x, y\}$ is internally strong, this contradiction justifies (5).

Once again, since $T\backslash y$ is internally strong, the statement below follows instantly from Lemma 3.8.

(6) If $p \geq 3$ and x has no in-neighbor in A_p, then $|A_p| = 1$ and x has an in-neighbor in A_{p-1}.

Since A_i is strong, either $|A_i| = 1$ or $|A_i| \geq 3$ for $1 \leq i \leq p$. Let $A_i = \{a_i\}$ for each i with $|A_i| = 1$ hereafter. We divide the remainder of the proof into a series of claims.

Claim 1. $|A_1| = 1$.

Assume the contrary: $|A_1| \geq 3$. Replacing x' (resp. y') by a second out-neighbor of x (resp. y) in A_1 if necessary, we may assume that $x' \neq y'$, for otherwise, $x' = y'$ is the unique out-neighbor of both x and y in A_1. Since $T\backslash x'$ is internally strong and $A_1\backslash x'$ has no incoming arcs, $|A_1\backslash x'| \leq 1$ and thus $|A_1| \leq 2$, contradicting the assumption on $|A_1|$. By Lemma 3.6, one of (7), (8), and (9) holds:

(7) $A_1\backslash\{x', y'\}$ has a vertex z such that $A_1\backslash z$ is strong.

(8) $|A_1| = 3$. Renaming the vertices in A_1 as x', y', z if necessary, we assume that both (x, x') and (y, y') are arcs in T, and that if three vertices in A_1 are all out-neighbors of x, then (y', x') is an arc in T; otherwise, if three vertices in A_1 are all out-neighbors of y, then (x', y') is an arc in T.

(9) $|A_1| \geq 4$ and A_1 has a Hamilton path P between x' and y' such that the remaining arcs in A_1 are all backward. Furthermore, we may assume that both (v, x) and (v, y) are arcs in T for any $v \in A_1\backslash\{x', y'\}$, for otherwise, (7) holds true by replacing x' or y' (which is z) with v.

Let z be as specified in (7) or (8), whichever holds, and let z be the terminal vertex of $P\backslash\{x', y'\}$ if (9) holds. Clearly, $T\backslash z$ is strong. We propose to prove that $T\backslash z$ is *i2s*, which amounts to saying that

(10) $T\backslash\{w, z\}$ is internally strong for each $w \in V\backslash z$.

From (5), (6), and the definition of z, we see that (10) holds trivially for any $w \in \cup_{i=2}^{p-1} A_i \cup \{x, y\}$. It remains to consider the following two cases.

Case 1.1. $w \in A_p$.

Depending on whether $w = y''$ (see (4)), we distinguish between two subcases.

• $w \neq y''$. In this subcase, $|A_p| \geq 2$. Thus x has at least one in-neighbor in A_p by (5) and (6). Let $(B_1, B_2, \ldots, B_q)$ be the strong partition of $A_p\backslash w$, let r be the largest subscript such that B_r contains an in-neighbor of x or y, and let $B = \cup_{i=r+1}^{q} B_i$. Since B has no outgoing arcs in $T\backslash w$ (which is internally strong), $|B| \leq 1$. Let us show that $T\backslash\{w, z\}$ is internally strong, for otherwise, x is a source and x' is a near-source of $T\backslash\{w, z\}$; in particular, $(x', y') \in A$. From the descriptions of (7)-(9), we deduce that $|A_1| = 3$ and $(z, x) \in A$. Consider the triple $(T; z, w)$. Let $(A'_1, A'_2, \ldots, A'_t)$ be the strong partition of $T\backslash\{z, w\}$. Then $A'_1 = \{x\}$. Since $T\backslash z$ is strong while $T\backslash\{z, w\}$ is not internally strong, and $|A'_1| < |A_1|$, the existence of the triple $(T; z, w)$ contradicts the minimality assumption on $(|A_1|, |A_2|, ..., |A_p|)$ in the choice of $(T; x, y)$ (see (1)-(3)).

• $w = y''$. In this subcase, we may assume that y'' is the only in-neighbor of y in A_p, for otherwise, replacing y'' by a second in-neighbor of y in A_p, we reduce the present subcase to the preceding one. If x has an in-neighbor in $A_p\backslash w$, then $T\backslash y$ is strong. Interchanging the roles of x and y, we reduce the present subcase to the preceding one as well. Thus we further assume that $A_p\backslash w$ contains no in-neighbors of x. Since $T\backslash w$ is internally strong, $A_p = \{w\}$. If w is an in-neighbor of x, then the existence of the triple $(T^*; x, y)$ contradicts the minimality assumption on $(|A_1|, |A_2|, ..., |A_p|)$ in the choice of $(T; x, y)$ (see (1)-(3)). So w is an out-neighbor of x. By (5) and (6), A_{p-1} contains an in-neighbor of x. Let us show that $T\backslash\{w,z\}$ is internally strong, for otherwise, y is a source and y' is a near-source of $T\backslash\{w,z\}$; in particular, both (y',x') and (y',x) are arcs in T. From the descriptions of (7)-(9), we deduce that $|A_1|=3$ and $(z,y)\in A$. Thus the existence of the triple $(T;z,w)$ contradicts the minimality assumption on $(|A_1|,|A_2|,...,|A_p|)$ in the choice of $(T;x,y)$ (see (1)-(3)).

Case 1.2. $w\in A_1\backslash z$.

Depending on whether (7), (8), or (9) holds, we distinguish between two subcases.

• (7) holds. In this subcase, let $(B_1,B_2,...,B_q)$ be the strong partition of $A_1\backslash\{w,z\}$, let r be the smallest subscript such that B_r contains an out-neighbor of x or y, and let $B=\cup_{i=1}^{r-1}B_i$. Then $(T\backslash\{w,z\})\backslash B$ is strong. If $|B|\le 1$, then $T\backslash\{w,z\}$ is internally strong. So we assume that $|B|\ge 2$. Since $T\backslash w$ is internally strong and since B has no incoming arcs in $T\backslash\{w,z\}$, $T\backslash w$ contains at least one arc from z to B. Thus the triple $(T;z,w)$ is a better choice than $(T;x,y)$ (see (1)-(3)) because $|B|<|A_1|$, a contradiction.

• (8) or (9) holds. In this subcase, if $w=x'$, then $T\backslash\{w,x,z\}$ is strong, so $T\backslash\{w,z\}$ is internally strong. If $w=y'$ and x has an in-neighbor contained in A_p, then $T\backslash\{w,y,z\}$ is strong, so $T\backslash\{w,z\}$ is also internally strong; if $w=y'$ and x has no in-neighbor contained in A_p, then x has an in-neighbor x'' contained in A_{p-1} by (5) and (6), and y has an out-neighbor contained in $\{x\}\cup(A_1\backslash y')\cup(\cup_{i=2}^{p-1}A_i)$ (as $T\backslash y'$ is internally strong), and hence $T\backslash\{w,z\}$ is strong. Suppose $w\notin\{x',y'\}$. In view of (5) and (6), it is clear that $T\backslash\{w,z\}$ is strong.

Combining the above two cases, we establish (10) for all $w\in A_p\cup(A_1\backslash z)$ and hence for all $w\in V\backslash z$. So $T\backslash z$ is *i2s*; this contradiction justifies Claim 1.

Claim 2. $|A_2|=1$.

Assume the contrary: $|A_2|\ge 3$. Since $T\backslash a_1$ is internally strong, A_2 contains a vertex a_2 which is an out-neighbor of x or y. If $|A_2|\ge 4$, let z be a vertex in $A_2\backslash a_2$ such that $A_2\backslash z$ is strong (see Corollary 3.7); if $|A_2|=3$, let z be the vertex in A_2 with $(z,a_2)\in A$. Since T is *i2s* and since x has an in-neighbor in $A_{p-1}\cup A_p$ by (5) and (6), $T\backslash z$ is strong. We propose to show that $T\backslash z$ is *i2s*, which amounts to saying that

(11) $T\backslash\{w,z\}$ is internally strong for each $w\in V\backslash z$.

From (5), (6), and the definition of z, we see that (11) holds trivially for any $w\in\{x,y\}\cup(A_2\backslash z)\cup(\cup_{i=3}^{p-1}A_i)$. It remains to consider the following two cases.

Case 2.1. $w=a_1$.

In this case, if a_2 is an out-neighbor of y, then $T\backslash\{a_1,x,z\}$ is strong and hence $T\backslash\{a_1,z\}$ is internally strong. So we assume that a_2 is an out-neighbor of x. If x has an in-neighbor in A_p, then $T\backslash\{a_1,y,z\}$ is strong and hence $T\backslash\{a_1,z\}$ is internally strong. So we further assume that x has no in-neighbor in A_p. Then $|A_p|=1$ and x has an in-neighbor in A_{p-1} by (5) and (6). We claim that y has an out-neighbor in $\{x\}\cup(A_2\backslash z)\cup(\cup_{i=3}^{p-1}A_i)$, for otherwise, let $B=\{y,y''\}$ and $\bar{B}=V\backslash\{a_1,y,y'',z\}$. Then $(\bar{B},B)$ is a nontrivial dicut in $T\backslash\{a_1,z\}$, so $T\backslash\{a_1,z\}$ is not internally strong. Therefore the existence of the triple $(T^*;z,a_1)$ contradicts the minimality assumption on $(|A_1|,|A_2|,...,|A_p|)$ in the choice of $(T;x,y)$ (see (1)-(3)). It follows instantly from the claim that $T\backslash\{a_1,z\}$ is strong.

Case 2.2. $w\in A_p$.

Depending on whether $w=y''$, we distinguish between two subcases.

• $w\neq y''$. In this subcase, $|A_p|\ge 2$. So x has an in-neighbor in A_p by (5) and (6). Let

$(B_1, B_2, ..., B_q)$ be the strong partition of $A_p \backslash w$, let r be the largest subscript such that B_r contains an in-neighbor of x or y, and let $B = \cup_{i=r+1}^q B_i$. Since B has no outgoing arcs in $T \backslash w$ (which is internally strong), $|B| \leq 1$. Clearly, $(T \backslash \{w, z\}) \backslash B$ is strong, so $T \backslash \{w, z\}$ is internally strong.

• $w = y''$. In this subcase, we may assume that y'' is the only in-neighbor of y in A_p, for otherwise, replacing y'' by a second in-neighbor of y in A_p, we reduce the present subcase to the preceding one. If x has an in-neighbor in $A_p \backslash w$, then $T \backslash y$ is strong. Interchanging the roles of x and y, we reduce the present subcase to the preceding one as well. So we assume that $A_p \backslash w$ contains no in-neighbors of x. Since $T \backslash w$ is internally strong, $|A_p \backslash w| \leq 1$, so $|A_p| \leq 2$. Since A_p is strong, we have $A_p = \{w\}$. If w is an out-neighbor of x, then x has an in-neighbor in A_{p-1} by (5) and (6). Thus $T \backslash \{w, y, z\}$ is strong and hence $T \backslash \{w, z\}$ is internally strong. So we further assume that w is an in-neighbor of x. If A_{p-1} contains an in-neighbor of x or y, then $T \backslash \{w, z\}$ is also internally strong; if A_{p-1} contains no in-neighbor of x or y, then $(\cup_{i=1}^{p-2} A_i \cup \{x, y\}, A_{p-1})$ is a dicut in $T \backslash w$. Since $T \backslash w$ is internally strong, $|A_{p-1}| = 1$. Thus the existence of the triple $(T^*; x, y)$ contradicts the minimality assumption on $(|A_1|, |A_2|, ..., |A_p|)$ in the choice of $(T; x, y)$ (see (1)-(3)).

Combining the above two cases, we establish (11) for all $w \in \{a_1\} \cup A_p$ and hence for all $w \in V \backslash z$. So $T \backslash z$ is $i2s$; this contradiction justifies Claim 2.

Claim 3. At least one of (x, a_2) and (y, a_2) is an arc in T.

Assume the contrary: both (a_2, x) and (a_2, y) are arcs in T. By (5) and (6), x has an in-neighbor in $A_{p-1} \cup A_p$, so $T \backslash a_2$ is strong. We propose to show that $T \backslash a_2$ is $i2s$, which amounts to saying that

(12) $T \backslash \{w, a_2\}$ is internally strong for each $w \in V \backslash a_2$.

Clearly, (12) holds for $w \in \cup_{i=3}^{p-1} A_i \cup \{x, y\}$. It remains to consider the following two cases.

Case 3.1. $w = a_1$.

Since $T \backslash a_1$ is internally strong, A_3 contains an out-neighbor of x or y. If A_3 contains an out-neighbor of y, then $T \backslash \{a_1, x, a_2\}$ is strong, and hence $T \backslash \{a_1, a_2\}$ is internally strong. So we assume that A_3 contains an out-neighbor of x. If A_p contains an in-neighbor of x, then $T \backslash \{a_1, y, a_2\}$ is strong, so $T \backslash \{a_1, a_2\}$ is internally strong. If A_p contains no in-neighbor of x, then $|A_p| = 1$ and x has an in-neighbor in A_{p-1} by (5) and (6). Thus $\{x\} \cup A_3 \cup \ldots \cup A_{p-1}$ induces a strong sub-tournament. Since $T \backslash a_1$ is internally strong, y has an out-neighbor in $\{x\} \cup A_4 \cup \ldots \cup A_{p-1}$. It follows that $T \backslash \{a_1, a_2\}$ is strong.

Case 3.2. $w \in A_p$.

Depending on whether $w = y''$, we distinguish between two subcases.

• $w \neq y''$. In this subcase, the argument is exactly the same as the one employed in Case 2.2 when $w \neq y''$.

• $w = y''$. In this subcase, we may assume that $A_p = \{w\}$ and w is an in-neighbor of x (see the proof in Case 2.2 when $w = y''$). If A_{p-1} contains an in-neighbor of x or y, then $T \backslash \{w, a_2\}$ is internally strong; otherwise, $(\cup_{i=1}^{p-2} A_i \cup \{x, y\}, A_{p-1})$ is a dicut in $T \backslash w$, so $|A_{p-1}| = 1$. If $p = 4$, then T is isomorphic to F_4 (see its labeling in Figure 3) under the mapping

$$(a_1, a_2, a_3, a_4, \{x, y\}) \to (v_5, v_6, v_2, v_3, \{v_1, v_4\}),$$

contradicting the hypothesis. So $p \geq 5$. Thus A_{p-2} contains an in-neighbor of x or y, for otherwise $(\cup_{i=1}^{p-3} A_i \cup \{x,y\}, A_{p-1} \cup A_{p-2})$ would a nontrivial dicut in $T\backslash w$, contradicting the fact that $T\backslash w$ is internally strong. It follows that $T\backslash\{w, a_2\}$ is internally strong, in which a_{p-1} is a sink and possible y is a source.

Combining the above two cases, we establish (12) for all $w \in \{a_1\} \cup A_p$ and hence for all $w \in V\backslash a_2$. So $T\backslash a_2$ is *i2s*; this contradiction justifies Claim 3.

Claim 4. Let k be the largest subscript such that A_k contains an in-neighbor of x. Then $k = 3$.

Assume the contrary: $k \neq 3$. Since $|V| \geq 6$ and $|A_1| = |A_2| = 1$ by Claims 1 and 2, we have $p \geq 3$. If $p = 3$, then $|A_p| \geq 2$, so x has an in-neighbor in A_p by (5) and (6) and hence $k = 3$, this contradiction implies that $p \geq 4$. We propose to show that

(13) $T\backslash z$ is *i2s* for some vertex z of T.

Depending on the size of A_p and value of p, we distinguish among three cases.

Case 4.1. $|A_p| \geq 3$.

In this case, x has an in-neighbor x'' in A_p by (5) and (6). Let z be an arbitrary vertex in A_3. Clearly, $T\backslash z$ is strong. We aim to show that (13) holds for this z. By Claim 3, at least one of (x, a_2) and (y, a_2) is in T. Thus $T\backslash\{w, z\}$ is internally strong for $w \in \cup_{i=3}^{p-1} A_i \cup \{x, y, a_1, a_2\}$. To establish this statement for $w \in A_p$, we consider two subcases.

• $w \neq y''$. In this subcase, the argument is exactly the same as that employed in Case 2.2 when $w \neq y''$.

• $w = y''$. In this subcase, we may assume that w is the only in-neighbor of y in A_p. Observe that x has an in-neighbor in $A_p\backslash w$, for otherwise, since $T\backslash w$ is internally strong, $|A_p\backslash w| \leq 1$, so $|A_p| \leq 2$, a contradiction. Interchanging the roles of x and y, we reduce the present subcase to the preceding one.

Case 4.2. $|A_p| = 1$ and $p \geq 5$.

In this case, x has an in-neighbor in $A_{p-1} \cup A_p$ by (5) and (6). To prove (13), we proceed by considering two subcases.

• A_3 contains an out-neighbor of y. In this subcase, let us show that $T\backslash a_2$ is *i2s*. Clearly, $T\backslash a_2$ is strong, and $T\backslash\{a_2, w\}$ is internally strong for any $w \in \cup_{i=3}^{p-1} A_i \cup \{a_1, x, y\}$. If A_{p-1} contains an in-neighbor of x or y, then $T\backslash\{a_2, a_p\}$ is internally strong. So we assume that A_{p-1} contains no in-neighbor of x or y. Note that $(\cup_{i=1}^{p-2} A_i \cup \{x, y\}, A_{p-1})$ is a dicut in $T\backslash a_p$. Since $T\backslash a_p$ is internally strong, $|A_{p-1}| = 1$. Since $p \geq 5$, A_{p-2} contains an in-neighbor of x or y, for otherwise $(\cup_{i=1}^{p-3} A_i \cup \{x, y\}, A_{p-1} \cup A_{p-2})$ would be a nontrivial dicut in $T\backslash a_p$, a contradiction. It follows that $T\backslash\{a_2, a_p\}$ is internally strong, in which a_{p-1} is a sink and possible one of x and y is a source.

• All vertices in A_3 are in-neighbors of y. In this subcase, let z be an arbitrary vertex in A_3; let us show that $T\backslash z$ is *i2s*. Clearly, $T\backslash z$ is strong. Observe that if a_p is an out-neighbor of x, then $\cup_{i=4}^{p-1} A_i \cup \{a_2, x\}$ contains an out-neighbor of y, for otherwise $(\cup_{i=3}^{p-1} A_i \cup \{a_2, x\}, \{y, a_p\})$ would be an nontrivial dicut in $T\backslash a_1$, a contradiction. It follows that $T\backslash\{w, z\}$ is internally strong for any $w \in \cup_{i=3}^{p-1} A_i \cup \{x, y, a_2\}$ no matter whether (x, a_p) is an arc in T. Let us make two more observations.

(14) $T\backslash\{a_1, z\}$ is internally strong. To justify this, note that if (y, a_2) is an arc in T, then $T\backslash\{a_1, x, z\}$ is strong, so $T\backslash\{a_1, z\}$ is internally strong. Thus we may assume that (a_2, y) is an arc in T. By Claim 3, (x, a_2) is also in T. If a_p is an in-neighbor of x, then $T\backslash\{a_1, y, z\}$ is

strong and hence $T\backslash\{a_1, z\}$ is internally strong; if a_p is an out-neighbor of x, then x contains an in-neighbor in A_{p-1} by (6), and $\cup_{i=4}^{p-1} A_i \cup \{a_2, x\}$ contains an out-neighbor of y as observed in the preceding paragraph. Thus (14) follows.

(15) $T\backslash\{a_p, z\}$ is internally strong. To justify this, note that if A_{p-1} contains an in-neighbor of x or y, then $T\backslash\{a_p, z\}$ is internally strong. If A_{p-1} contains no in-neighbor of x or y, then $(\cup_{i=1}^{p-2} A_i \cup \{x, y\}, A_{p-1})$ is a dicut in $T\backslash a_p$, which implies that $|A_{p-1}| = 1$. Since $p \geq 5$, A_{p-2} contains an in-neighbor of x or y, for otherwise $(\cup_{i=1}^{p-3} A_i \cup \{x, y\}, A_{p-1} \cup A_{p-2})$ is a nontrivial dicut in $T\backslash a_p$, a contradiction. Thus (15) holds.

Case 4.3. $|A_p| = 1$ and $p = 4$.

In this case, (a_4, x) is an arc in T by (5), (6), and the assumption $k \neq 3$. Depending on the size of A_3, we consider two subcases.

• $|A_3| \geq 3$. In this subcase, A_3 contains a vertex a_3 which is an in-neighbor of x or y, because $T\backslash a_4$ is internally strong. If $|A_3| = 3$, let z be the vertex such that $(a_3, z) \in A_3$; if $|A_3| \geq 4$, Corollary 3.7 guarantees the existence of a vertex $z \in A_3\backslash a_3$ such that $A_3\backslash z$ is strong. Let us show that $T\backslash z$ is *i2s*. Clearly, $T\backslash z$ is strong, and $T\backslash\{w, z\}$ is internally strong for any $w \neq a_1$. If (y, a_2) is an arc in T, then $T\backslash\{a_1, x, z\}$ is strong and hence $T\backslash\{a_1, z\}$ is internally strong. If (a_2, y) is an arc in T, then so is (x, a_2) by Claim 3. Since $T\backslash\{a_1, y, z\}$ is strong, $T\backslash\{a_1, z\}$ is internally strong.

• $|A_3| = 1$. In this subcase, if exactly one of (y, a_3) and (x, a_3) is an arc in T, then $T\backslash a_2$ is *i2s*. So we assume that either both (a_3, y) and (a_3, x) are in T or both (y, a_3) and (x, a_3) are in T. If exactly one of (x, a_2) and (y, a_2) is in T, then $T\backslash a_3$ is *i2s*. So we further assume that both (x, a_2) and (y, a_2) are in T by Claim 3. Thus both (a_3, y) and (a_3, x) are in T, for otherwise, $(\{x, y\}, \{a_1, a_2, a_3\})$ would be a dicut in $T\backslash a_4$, a contradiction. Now we can see that T is isomorphic to F_5 (see its labeling in Figure 3) under the mapping

$$(a_1, a_2, a_3, a_4, \{x, y\}) \rightarrow (v_5, v_2, v_6, v_3, \{v_1, v_4\}),$$

contradicting the hypothesis of the present lemma.

Combining the above three cases, we have proved (13); this contradiction justifies Claim 4.

Claim 5. $p = 4$.

Assume the contrary: $p \neq 4$. Since $|V| \geq 6$ and $|A_i| = 1$ for $i = 1, 2$, we have $p \geq 3$. By Claim 4, (5), and (6), we also have $p \leq 4$. So $p = 3 = k$. Let x'' be an in-neighbor of x in A_3. Replacing x'' (resp. y'') by a second in-neighbor of x (resp. y) in A_3 if necessary, we may assume that $x'' \neq y''$, for otherwise, x'' is the only in-neighbor of x and y in A_3. Since $T\backslash x''$ is internally strong, $|A_3\backslash x''| \leq 1$, so $|A_3| \leq 2$ and hence $|A_3| = 1$, contradicting the hypothesis that $|V| \geq 6$. If all vertices in A_3 are in-neighbors of both x and y, then $T\backslash z$ is *i2s* for any $z \in A_3$ by Claim 3.

So we assume that A_3 contains an out-neighbor of x or y. We propose to show that $T\backslash a_2$ is *i2s*. Clearly, $T\backslash a_2$, $T\backslash\{x, a_2\}$, and $T\backslash\{y, a_2\}$ are all strong. By the hypothesis of the present case, $A_3 \cup \{x\}$ or $A_3 \cup \{y\}$ induces a strong sub-tournament of T, so $T\backslash\{a_1, a_2\}$ is internally strong. Let w be an arbitrary vertex in A_3. Since $x'' \neq y''$, symmetry allows us to assume that $w \neq x''$. If $A_3\backslash w$ is strong, then $T\backslash\{w, a_2\}$ is internally strong; otherwise, let $(B_1, B_2, ..., B_q)$ be the strong partition of $A_3\backslash w$. Then $q \geq 2$. Let r be the largest subscript such that B_r contains an in-neighbor of x or y and let $B = \cup_{i=r+1}^{q} B_i$. Since $(\cup_{i=1}^{r} B_i \cup \{a_1, a_2, x, y\}, B)$ is a dicut in $T\backslash w$, we have $|B| \leq 1$. If $T\backslash(B \cup \{a_2, w\}) = \cup_{i=1}^{r} B_i \cup \{a_1, x, y\}$ is strong, then $T\backslash\{w, a_2\}$ is internally strong; otherwise, w is the only in-neighbor of y in $A_3 \cup \{x\}$. Since $\cup_{i=1}^{r} B_i \cup \{a_1, x\}$ is strong, $T\backslash\{w, a_2\}$ is also internally strong.

Combining the above observations, we see that $T\backslash z$ is $i2s$ for some vertex z of T; this contradiction justifies Claim 5.

From (6) and Claims 4 and 5, we deduce that $|A_4| = 1$ and (x, a_4) is an arc in T. Depending on the size of A_3, we distinguish between two cases.

• $|A_3| \geq 3$. In this case, let x'' be an in-neighbor of x in A_3 (see Claim 4). If $|A_3| = 3$, let z be the vertex in A_3 such that (x'', z) is an arc; otherwise, let z be a vertex in $A_3\backslash x''$ such that $A_3\backslash z$ is strong (see Corollary 3.7). Clearly, $T\backslash z$ is $i2s$. Let us show it is actually $i2s$; that is, $T\backslash\{w, z\}$ is internally strong for any $w \in V\backslash z$. This statement holds trivially when $w \neq a_1$. So we assume that $w = a_1$. If (y, a_2) is an arc in T, then $T\backslash\{a_1, z\}$ is strong; otherwise, by Claim 3, (x, a_2) is an arc in T. So $(A_3\backslash z) \cup \{a_2, x\}$ induces a strong sub-tournament of T. Since $T\backslash a_1$ is internally, $(A_3\backslash z) \cup \{a_2, x\}$ contains an out-neighbor of y. Thus $T\backslash\{a_1, z\}$ is strong.

• $|A_3| = 1$. In this subcase, (a_3, x) is an arc in T by Claim 4. If (y, a_3) or (y, x) is an arc in T, then $T\backslash a_2$ is $i2s$. So we assume that both (a_3, y) and (x, y) are arcs in T. Since $T\backslash a_1$ is internally strong, (y, a_2) is an arc in T. Note that (a_2, x) is an arc of T, for otherwise T would be isomorphic to F_4 (see its labeling in Figure 3) under the mapping

$$(a_1, a_2, a_3, a_4, x, y) \to (v_4, v_1, v_5, v_2, v_6, v_3),$$

contradicting the hypothesis of the present lemma. It follows that $T\backslash a_3$ is $i2s$.

Combining the above two cases, we conclude that T contains a vertex z such that $T\backslash z$ remains to be $i2s$; this contradiction proves the lemma. ∎

With the above preparations, we are ready to establish the main results of this section.

Proof of Theorem 3.1. The desired statements follow instantly from Lemmas 3.5 and 3.11. ∎

Proof of Theorem 3.2. For convenience, we say that an $i2s$ Möbius-free tournament T' is an *extension* of T if $T'\backslash v$ is isomorphic to T for some vertex v of T'.

Claim 1. G_1 is the only extension of F_1.

To justify this, let T be an extension of F_1, let v_6 be a vertex of T such that $T\backslash v_6$ is isomorphic to F_1, and label the vertices of $T\backslash v_6$ as in Figure 3 for F_1. We propose to show that T is isomorphic to G_1. Since the in-degree and out-degree of each vertex in F_1 are two, F_1 enjoys a high degree of symmetry in which all vertices behave likewise.

Since T is strong, symmetry allows us to assume that v_1 is an in-neighbor of v_6. Then at most one of (v_6, v_2) and (v_6, v_5) is in T, for otherwise, the union of the five cycles $v_1v_6v_5v_1$, $v_1v_3v_5v_1$, $v_2v_3v_5v_2$, $v_2v_4v_5v_2$, and $v_1v_6v_2v_4v_1$ would form an M_5^* in T, a contradiction. Thus we may proceed by considering the following three cases.

• Both (v_2, v_6) and (v_5, v_6) are in T. In this case, since T is strong, at most one of (v_3, v_6) and (v_4, v_6) is contained in T. If both (v_6, v_3) and (v_4, v_6) are in T, then the five cycles $v_1v_2v_4v_1$, $v_5v_2v_4v_5$, $v_1v_3v_4v_1$, $v_6v_3v_4v_6$, and $v_2v_6v_3v_5v_2$ would form an M_5. Similarly, if both (v_3, v_6) and (v_6, v_4) are in T, then the five cycles $v_1v_3v_5v_1$, $v_2v_3v_5v_2$, $v_2v_4v_5v_2$, $v_6v_4v_5v_6$, and $v_1v_3v_6v_4v_1$

would form an M_5 in T as well. So both (v_6, v_3) and (v_6, v_4) are in T. Thus T is isomorphic to G_1, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_2, v_6, v_4, v_5, v_1, v_3)$ in G_1 as labeled in Figure 4.

• Both (v_6, v_2) and (v_5, v_6) are in T. In this case, (v_6, v_3) is in T, for otherwise, the five cycles $v_1v_3v_4v_1$, $v_1v_3v_5v_1$, $v_2v_3v_5v_2$, $v_2v_3v_6v_2$, and $v_1v_6v_2v_4v_1$ would form an M_5, a contradiction. If (v_6, v_4) is in T, then T is isomorphic to G_1, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_2, v_3, v_4, v_5, v_1, v_6)$ in G_1 as labeled in Figure 3. If (v_4, v_6) is in T, then T is also isomorphic to G_1, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_6, v_4, v_5, v_1, v_2, v_3)$ in G_1 as labeled in Figure 4.

• Both (v_2, v_6) and (v_6, v_5) are in T. In this case, (v_6, v_4) is in T, for otherwise, the five cycles $v_1v_2v_4v_1$, $v_1v_3v_4v_1$, $v_1v_3v_5v_1$, $v_1v_6v_5v_1$, and $v_2v_4v_6v_5v_2$ would form an M_5, a contradiction. If (v_6, v_3) is in T, then T is isomorphic to G_1, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_1, v_2, v_3, v_4, v_5, v_6)$ in G_1 as labeled in Figure 3. If (v_3, v_6) is in T, then T is also isomorphic to G_1, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_1, v_2, v_6, v_4, v_5, v_3)$ in G_1 as labeled in Figure 4.

Combining the above observations, we see that G_1 is the only extension of F_1.

Claim 2. F_2 has no extension.

Assume the contrary: T is an extension of F_2 such that $T\backslash v_6$ is isomorphic to F_2 for some vertex v_6 of T. Let us label the vertices of $T\backslash v_6$ as in Figure 3 for F_2. Since T is $i2s$, v_6 has an in-neighbor in $\{v_1, v_3, v_4\}$, for otherwise, $(\{v_2, v_6\}, \{v_1, v_3, v_4\})$ would be a nontrivial dicut in $T\backslash v_5$, a contradiction. By symmetry, we may assume that (v_1, v_6) is an arc in T. Next, v_3 or v_4 is an out-neighbor of v_6, for otherwise $(\{v_1, v_3, v_4\}, \{v_5, v_6\})$ would be a nontrivial dicut in $T\backslash v_2$. Depending on the direction of the arc between v_6 and v_3, we consider two cases.

• (v_6, v_3) is in T. In this case, if (v_6, v_5) is an arc in T, then the union of the three cycles $v_1v_6v_3v_4v_1$, $v_2v_3v_4v_5v_2$, and $v_1v_6v_5v_2v_1$ is a $K_{3,3}$. So (v_5, v_6) is an arc in T. If (v_4, v_6) is an arc in T, then the union of the five cycles $v_2v_3v_5v_2$, $v_6v_3v_5v_6$, $v_6v_3v_4v_6$, $v_1v_3v_4v_1$, and $v_1v_5v_2v_4v_1$ would form an M_5^*; if (v_6, v_4) is an arc in T, then the union of the five cycles $v_2v_4v_5v_2$, $v_6v_4v_5v_6$, $v_6v_4v_1v_6$, $v_3v_4v_1v_3$, and $v_1v_3v_5v_2v_1$ would form an M_5^* as well. Thus we reach a contradiction in either subcase.

• (v_3, v_6) is in T. In this case, (v_6, v_4) is in T. If (v_6, v_5) is in T, then the union of the three cycles $v_1v_3v_6v_4v_1$, $v_2v_3v_6v_5v_2$, and $v_1v_5v_2v_4v_1$ would form a $K_{3,3}$. Thus (v_5, v_6) is in T. But then the union of the five cycles $v_2v_4v_5v_2$, $v_6v_4v_5v_6$, $v_6v_4v_1v_6$, $v_3v_4v_1v_3$, and $v_1v_3v_5v_2v_1$ would form an M_5^*, a contradiction.

Combining the above observations, we see that F_2 has no extension.

Claim 3. G_2 and G_3 are the only extensions of F_3.

To justify this, let T be an extension of F_3 such that $T\backslash v_6$ is isomorphic to F_3 for some vertex v_6 of T. Let us label the vertices of $T\backslash v_6$ as in Figure 3 for F_3. Since T is $i2s$, v_6 has at least one in-neighbor in $\{v_1, v_3, v_4\}$, for otherwise $(\{v_2, v_6\}, \{v_1, v_3, v_4\})$ would be a nontrivial dicut in $T\backslash v_5$, a contradiction.

• (v_6, v_1) is in T. In this case, at most one of (v_6, v_3) and (v_6, v_4) is in T. Let us first consider the subcase when (v_4, v_6) is in T. Now at most one of (v_2, v_6) and (v_5, v_6) is in T, for otherwise, $(\{v_2, v_4, v_5\}, \{v_1, v_6\})$ would be a nontrivial dicut in $T\backslash v_3$. Next, (v_5, v_6) is in T, for otherwise, the three cycles $v_1v_3v_4v_6v_1$, $v_2v_4v_6v_5v_2$, and $v_1v_3v_5v_2v_1$ would form a $K_{3,3}$. It follows that (v_6, v_2)

is also in T. If (v_6, v_3) is in T, then the five cycles $v_1v_3v_4v_1$, $v_6v_3v_4v_6$, $v_2v_4v_6v_2$, $v_2v_4v_5v_2$, and $v_1v_3v_5v_2v_1$ would form an M_5; if (v_3, v_6) is in T, then the three cycles $v_1v_3v_6v_2v_1$, $v_1v_3v_4v_5v_1$, and $v_6v_2v_4v_5v_6$ would form a $K_{3,3}$, a contradiction. It remains to consider the subcase when (v_6, v_4) is in T. Thus (v_3, v_6) is also in T. If (v_5, v_6) is in T, then the five cycles $v_1v_3v_5v_1$, $v_2v_3v_5v_2$, $v_2v_4v_5v_2$, $v_6v_4v_5v_6$, and $v_1v_3v_6v_4v_1$ would form an M_5, this contradiction implies that (v_6, v_5) is an arc of T. If (v_2, v_6) is in T, then the three cycles $v_1v_3v_6v_4v_1$, $v_2v_6v_4v_5v_2$, and $v_1v_3v_5v_2v_1$ would form a $K_{3,3}$. So (v_6, v_2) is in T and thus T is isomorphic to G_3, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_2, v_3, v_4, v_1, v_6, v_5)$ in G_3 as labeled in Figure 4.

• (v_1, v_6) is in T. Let us first consider the subcase when (v_6, v_3) is in T. Now (v_5, v_6) is in T, for otherwise, the three cycles $v_1v_6v_3v_4v_1$, $v_1v_6v_5v_2v_1$, and $v_2v_3v_4v_5v_2$ would form a $K_{3,3}$. It follows that (v_4, v_6) is in T, for otherwise the five cycles $v_2v_4v_5v_2$, $v_6v_4v_5v_6$, $v_1v_6v_4v_1$, $v_1v_3v_4v_1$, and $v_1v_3v_5v_2v_1$ would form an M_5^*. Thus (v_4, v_6) is in T, which in turn implies that (v_6, v_2) is in T, for otherwise $(\{v_2, v_4, v_5\}, \{v_1, v_6\})$ would be a nontrivial dicut in $T\backslash v_3$. But then the five cycles $v_2v_3v_5v_2$, $v_2v_4v_5v_2$, $v_2v_4v_6v_2$, $v_2v_1v_6v_2$, and $v_1v_6v_3v_5v_1$ would form an M_5, a contradiction. It remains to consider the subcase when (v_3, v_6) is in T. If (v_6, v_2) is in T, then the five cycles $v_1v_3v_4v_1$, $v_1v_3v_5v_1$, $v_2v_3v_5v_2$, $v_2v_3v_6v_2$, and $v_1v_6v_2v_4v_1$ would form an M_5. Thus (v_2, v_6) is in T. If (v_6, v_4) is in T, then the three cycles $v_1v_3v_6v_4v_1$, $v_1v_3v_5v_2v_1$, and $v_2v_6v_4v_5v_2$ would form a $K_{3,3}$. Thus (v_4, v_6) is in T. Since T is strong, (v_6, v_5) must be in T. Therefore, T is isomorphic to G_2, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_1, v_5, v_6, v_3, v_4, v_2)$ in G_2 as labeled in Figure 4.

Combining the above observations, we see that G_2 and G_3 are the only extensions of F_3.

Claim 4. F_4 is Möbius-free while F_5 is not.

It is routine to check that F_4 contains none of the digraphs displayed in Figure 1, so F_4 is Möbius-free. Let us label F_5 as in Figure 3. Then the union of the three cycles $v_1v_5v_3v_4v_1$, $v_2v_6v_3v_4v_2$, and $v_1v_5v_2v_6v_1$ forms a $K_{3,3}$. Thus F_5 is not Möbius-free.

Claim 5. G_1 has no extension.

Assume the contrary: T is an extension of G_1 such that $T\backslash v_7$ is isomorphic to G_1 for some vertex v_7 of T. Let us label the vertices of $T\backslash v_7$ as in Figure 4 for G_1. Depending on the direction of the arc between v_7 and v_1, we distinguish between two cases.

• (v_1, v_7) is in T. In this case, (v_5, v_7) is in T, for otherwise, the union of the three cycles $v_1v_7v_5v_2v_4v_1$, $v_2v_6v_3v_5v_2$, and $v_1v_6v_3v_4v_1$ would form a $K'_{3,3}$. If (v_7, v_6) is in T, then the union of the three cycles $v_1v_7v_6v_4v_1$, $v_7v_6v_3v_5v_7$, and $v_1v_3v_5v_2v_4v_1$ would form a $K'_{3,3}$. Thus (v_6, v_7) is in T, which in turn implies that (v_2, v_7) is in T, for otherwise, the union of the three cycles $v_3v_5v_7v_2v_3$, $v_1v_6v_7v_2v_4v_1$, and $v_1v_6v_3v_5v_1$ would form a $K'_{3,3}$. If (v_7, v_4) is in T, then the union of the three cycles $v_2v_6v_3v_5v_2$, $v_1v_3v_5v_7v_4v_1$, and $v_1v_2v_6v_4v_1$ would form a $K'_{3,3}$. Thus (v_4, v_7) is in T. Since T is strong, (v_7, v_3) is in T. It follows that the union of the five cycles $v_1v_2v_4v_1$, $v_5v_2v_4v_5$, $v_1v_3v_4v_1$, $v_7v_3v_4v_7$, and $v_2v_7v_3v_5v_2$ would form an M_5, a contradiction. Therefore G_1 has no extension.

• (v_7, v_1) is in T. Note that G_1 is isomorphic to its reverse under the mapping

$$(v_1, v_2, v_3, v_4, v_5, v_6) \to (v_5, v_4, v_6, v_2, v_1, v_3).$$

So if T is an extension of G_1, then T^* is also an extension of G_1. If (v_7, v_5) appears in T, then (v_1, v_7) is in T^* and hence the present case reduces to the preceding one. So we may assume that

(v_5, v_7) is in T, which implies that (v_7, v_3) is in T, for otherwise the union of the three cycles $v_2v_3v_4v_5v_2$, $v_1v_2v_3v_7v_1$, and $v_1v_6v_4v_5v_7v_1$ would form a $K'_{3,3}$. Thus (v_7, v_2) is in T, for otherwise, the union of the five cycles $v_1v_2v_7v_1$, $v_1v_2v_4v_1$, $v_1v_3v_4v_1$, $v_1v_3v_5v_1$, and $v_2v_7v_3v_5v_2$ would form an M_5^*. But then the union of the three cycles $v_2v_6v_5v_7v_2$, $v_3v_4v_5v_7v_3$, and $v_1v_2v_6v_3v_4v_1$ would form a $K'_{3,3}$, a contradiction.

Combining the above observations, we see that G_1 has no extension.

Claim 6. Neither G_2 nor G_3 has an extension.

To justify this, observe that G_3 is isomorphic to G_2^* under the mapping

$$(v_1, v_2, v_3, v_4, v_5, v_6) \to (v_3, v_5, v_1, v_4, v_2, v_6).$$

So if T is an extension of G_2, then T^* is an extension of G_3. Hence it suffices to show that G_2 has no extension. Assume the contrary: T is an extension of G_2 such that $T\backslash v_7$ is isomorphic to G_2 for some vertex v_7 of T. Let us label the vertices of $T\backslash v_7$ as in Figure 4 for G_2. Depending the direction of the arc between v_7 and v_1, we distinguish between two cases.

• (v_7, v_1) is in T. Let us first consider the subcase when (v_3, v_7) is in T. Now (v_4, v_7) is in T, for otherwise, the union of the three cycles $v_1v_6v_4v_5v_1$, $v_1v_6v_3v_7v_1$, and $v_3v_7v_4v_5v_3$ would form a $K_{3,3}$. Next, (v_7, v_5) is in T, for otherwise, the union of the three cycles $v_1v_6v_3v_7v_1$, $v_3v_4v_5v_6v_3$, and $v_1v_2v_4v_5v_7v_1$ would form a $K'_{3,3}$. If (v_7, v_6) is in T, then the union of the five cycles $v_1v_6v_3v_1$, $v_7v_6v_3v_7$, $v_3v_7v_5v_3$, $v_3v_4v_5v_3$, and $v_1v_6v_4v_5v_1$ would form an M_5; if (v_6, v_7) is in T, the the union of the three cycles $v_1v_6v_7v_5v_1$, $v_1v_6v_3v_4v_1$, and $v_7v_5v_3v_4v_7$ would form a $K_{3,3}$, a contradiction. It remains to consider the subcase when (v_7, v_3) is in T. Now (v_7, v_2) is in T, for otherwise, the union of the three cycles $v_1v_6v_3v_4v_1$, $v_1v_6v_2v_7v_1$, and $v_2v_7v_3v_4v_5v_2$ would form a $K'_{3,3}$. Since T is $i2s$, (v_6, v_7) must be in T, for otherwise $(\{v_5, v_7\}, \{v_1, v_6, v_3, v_2\})$ would be a nontrivial dicut in $T\backslash v_4$. Thus (v_7, v_4) is in T, for otherwise the union of the three cycles $v_1v_6v_7v_3v_1$, $v_2v_4v_7v_3v_2$, and $v_1v_6v_2v_4v_5v_1$ would form a $K'_{3,3}$. But then the union of the three cycles $v_1v_6v_7v_3v_1$, $v_4v_5v_6v_7v_4$, and $v_1v_2v_4v_5v_3v_1$ also forms a $K'_{3,3}$, a contradiction.

• (v_1, v_7) is in T. Let us first consider the subcase when (v_7, v_6) is in T. If (v_7, v_4) is in T, then the union of the three cycles $v_1v_7v_6v_3v_1$, $v_1v_7v_4v_5v_1$, and $v_3v_4v_5v_6v_3$ would form a $K_{3,3}$; if (v_4, v_7) is in T, then the union of the three cycles $v_1v_7v_6v_3v_1$, $v_2v_4v_7v_6v_2$, and $v_1v_2v_4v_5v_3v_1$ would form a $K'_{3,3}$, a contradiction. It remains to consider the subcase when (v_6, v_7) is in T. Now (v_5, v_7) is in T, for otherwise the union of the five cycles $v_1v_6v_3v_1$, $v_1v_6v_4v_1$, $v_4v_5v_6v_4$, $v_7v_5v_6v_7$, and $v_1v_7v_5v_3v_1$ would form an M_5. Since T is $i2s$, (v_7, v_3) is in T, for otherwise $(\{v_1, v_3, v_5, v_6\}, \{v_2, v_7\})$ would be a nontrivial dicut in $T\backslash v_4$. But then the union of the three cycles $v_1v_6v_7v_3v_1$, $v_1v_6v_4v_5v_1$, and $v_3v_4v_5v_7v_3$ forms a $K_{3,3}$, a contradiction.

Combining the above observations, we see that G_2 has no extension.

Claim 7. F_4 has no extension.

Assume the contrary: T is an extension of F_4 such that $T\backslash v_7$ is isomorphic to F_4 for some vertex v_7 of T. Let us label the vertices of $T\backslash v_7$ as in Figure 3 for F_4. Depending on the direction of the arc between v_2 and v_7, we distinguish between two cases.

• (v_2, v_7) is in T. In this case, (v_5, v_7) appears in T, for otherwise, the union of the three cycles $v_1v_2v_3v_4v_1$, $v_3v_4v_5v_6v_3$, and $v_1v_2v_7v_5v_6v_1$ would form a $K'_{3,3}$. Next, (v_6, v_7) is in T, for otherwise, if (v_3, v_7) is in T, then the union of the three cycles $v_1v_5v_2v_3v_1$, $v_1v_5v_6v_4v_1$, and $v_2v_3v_7v_6v_4v_2$ would form a $K'_{3,3}$; if (v_7, v_3) is in T, then the union of the three cycles $v_1v_5v_3v_4$,

$v_2v_7v_3v_4v_2$, and $v_1v_5v_2v_7v_6v_1$ would also form a $K'_{3,3}$, a contradiction. Since T is $i2s$, at least one of (v_7, v_1) and (v_7, v_4) is in T, for otherwise $(\{v_6, v_5, v_1, v_4\}, \{v_2, v_7\})$ would be a nontrivial dicut in $T\backslash v_3$. Assume that (v_7, v_1) is in T. If (v_7, v_3) is in T, then the union of the three cycles $v_1v_5v_3v_4v_1$, $v_2v_7v_3v_4v_2$, and $v_1v_5v_2v_7v_1$ would form a $K_{3,3}$; if (v_3, v_7) is in T, then the union of the three cycles $v_1v_5v_6v_7v_1$, $v_1v_2v_3v_7v_1$, and $v_2v_3v_4v_5v_6v_2$ would form a $K'_{3,3}$, a contradiction. Thus (v_1, v_7) is in T and hence so is (v_7, v_4). Consequently, the union of the three cycles $v_1v_5v_6v_3v_1$, $v_7v_4v_5v_6v_7$, and $v_1v_7v_4v_2v_3v_1$ forms a $K'_{3,3}$, a contradiction.

• (v_7, v_2) is in T. Observe that F_4 is isomorphic to its reverse under the mapping

$$(v_1, v_2, v_3, v_4, v_5, v_6) \to (v_4, v_6, v_5, v_1, v_3, v_2).$$

If T is an extension of F_4, then T^* is also an extension of F_4. If (v_7, v_6) occurs in T, then (v_2, v_7) occurs in T^*, and hence the present case reduces to the preceding case. So we may assume (v_6, v_7) is in T.

Let us first consider the subcase when (v_3, v_7) is in T. Then (v_5, v_7) is in T, for otherwise, the union of the five cycles $v_1v_5v_6v_4v_1$, $v_1v_2v_3v_4v_1$, and $v_2v_3v_7v_5v_6v_2$ would form a $K'_{3,3}$, a contradiction. If (v_7, v_4) is in T, then the union of the three cycles $v_1v_5v_7v_4v_1$, $v_2v_3v_7v_4v_2$, and $v_1v_5v_2v_3v_1$ would form a $K_{3,3}$. So (v_4, v_7) is in T. Since T is $i2s$, (v_7, v_1) is in T, for otherwise $(\{v_6, v_5, v_1, v_4\}, \{v_2, v_7\})$ would be a nontrivial dicut in $T\backslash v_3$. Thus the union of the three cycles $v_1v_2v_3v_7v_1$, $v_1v_5v_6v_7v_1$, and $v_2v_3v_4v_5v_6v_2$ would form a $K'_{3,3}$, a contradiction.

It remains to consider the second subcase when (v_7, v_3) is in T. Assume that (v_1, v_7) is in T. Then (v_4, v_7) is in T, for otherwise, the union of the three cycles $v_1v_5v_6v_3v_1$, $v_7v_4v_5v_6v_7$, and $v_1v_7v_4v_2v_3v_1$ would form a $K'_{3,3}$, a contradiction. Since T is $i2s$, (v_7, v_5) is in T, for otherwise $(\{v_6, v_5, v_1, v_4\}, \{v_2, v_7\})$ would be a nontrivial dicut in $T\backslash v_3$. But then the union of the three cycles $v_2v_3v_4v_7v_2$, $v_4v_7v_5v_6v_4$, and $v_1v_5v_6v_2v_3v_1$ would form a $K'_{3,3}$, a contradiction. So (v_7, v_1) must appear in T. Since T is $i2s$, (v_4, v_7) is in T, for otherwise $(\{v_6, v_7\}, \{v_1, v_2, v_3, v_4\})$ would be a nontrivial dicut in $T\backslash v_5$. But then the union of the three cycles $v_1v_5v_2v_3v_1$, $v_7v_2v_3v_4v_7$, and $v_1v_5v_6v_4v_7v_1$ would form a $K'_{3,3}$, a contradiction again. So Claim 7 is justified.

From Claims 1-4, we conclude that G_1, G_2, G_3, and F_4 are the only $i2s$ Möbius-free tournaments on six vertices. By Claims 5-7 and Theorem 3.1(iii), there is no $i2s$ Möbius-free tournament on seven or more vertices. This completes the proof of Theorem 3.2. ∎

Proof of Theorem 3.3. We apply induction on $|V|$. By Lemma 3.4, $T = C_3$ if $|V| = 3$ and $T = F_0$ if $|V| = 4$, so $T \in \mathcal{T}_1$ if $|V| \leq 4$. Let us proceed to the induction step.

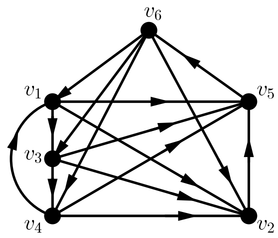
If T is $i2s$, then $T \in \mathcal{T}_1$ by Theorem 3.2 and Lemmas 3.4 and 3.5. So we assume that T is not $i2s$. Thus T can be expressed as the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 by Lemmas 2.2 and 2.3. Note that $T_i \notin \{F_1, G_1\}$ because neither F_1 nor G_1 contains a special arc for $i = 1, 2$. By induction hypothesis, both T_1 and T_2 can be constructed by repeatedly taking 1-sums starting from tournaments in $\mathcal{T}_1$, and hence so can T. ∎

So far we have demonstrated that every $i2s$ Möbius-free tournament comes from a finite set. Let us proceed to consider a strong tournament T that is not $i2s$; in this case, it is hard to give a clear description of T. Nevertheless, by Lemma 2.2, T can be expressed as the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2. We can gain enough structural information about T_2 if we impose minimality constraint on $|V(T_2)|$.

Let F_6 be the tournament depicted in Figure 6. Observe that it is not *i2s* because $F_6 \backslash v_6$ has a nontrivial dicut. Let

$$\mathcal{T}_2 = \{F_0, F_2, F_3, F_4, F_6, G_2, G_3\}.$$

Then $\mathcal{T}_2 = (\mathcal{T}_1 \backslash \{C_3\}) \cup \{F_6\}$.



***F*₆**

Figure 6. A minimal tournament involved in 1-sum

Lemma 3.12. *Let $T = (V, A)$ be a strong Möbius-free tournament. Suppose T is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 such that $|V(T_2)|$ is as small as possible. Then $T_2 \in \mathcal{T}_2$.*

Proof. Since T is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2, we have $|V(T_i)| \geq 4$ for $i = 1, 2$. If T_2 is *i2s*, then $T_2 \in \mathcal{T}_1 \backslash \{C_3\}$ by Theorem 3.3, and hence $T_2 \in \mathcal{T}_2$. It remains to consider the case when T_2 is not *i2s*.

Recall the definition of the 1-sum operation in Section 2. There exist a special arc (a_1, b_1) in T_1 and a special arc (b_2, a_2) in T_2, with $d^+_{T_1}(a_1) = d^-_{T_2}(a_2) = 1$, such that T is obtained from the disjoint union of $T_1 \backslash a_1$ and $T_2 \backslash a_2$ by identifying b_1 with b_2 (the resulting vertex is denoted by b) and adding all arcs from $T_1 \backslash \{a_1, b_1\}$ to $T_2 \backslash \{a_2, b_2\}$. We propose to show that

(1) $T_2 \backslash v$ is internally strong for any $v \in V(T_2) \backslash a_2$.

Assume the contrary: $T_2 \backslash v$ has a nontrivial dicut (X, Y) for some $v \in V(T_2) \backslash a_2$. Since a_2 is a near-source in T_2, we have $a_2 \in X$ and $Y \subseteq V(T_2) \backslash \{a_2, b_2\}$. Let x be a vertex in X and y be a vertex in Y such that both (v, x) and (y, v) are arcs in T_2. Set $T'_1 = T \backslash (Y \backslash y)$ and $T'_2 = T_2 \backslash (X \backslash x)$. Then T is the 1-sum of T'_1 and T'_2 over (v, x) and (y, v), with $3 < |V(T'_2)| < |V(T_2)|$, contradicting the minimality hypothesis on T_2. So (1) is justified.

Since T_2 is not *i2s*, $T \backslash a_2$ has a nontrivial dicut (X, Y) by (1). Since T_2 is strong, $b_2 \in Y$. Observe that

(2) $|Y| = 2$, for otherwise, $(X \cup \{a_2\}, Y \backslash b_2)$ would be a nontrivial dicut in $T_2 \backslash b_2$, contradicting (1).

Let c_2 be the vertex in $Y \backslash b_2$. Since T_2 contains no sink, (c_2, b_2) is an arc in T_2. Let S be the sub-tournament induced by X. Then

(3) S is strong, for otherwise, let $(A_1, A_2, ..., A_p)$ be the strong partition of S. Then $p \geq 2$. Thus $(A_1 \cup \{a_2\}, \cup_{i=2}^p A_i \cup \{c_2\})$ would be a nontrivial dicut in $T \backslash b_2$, contradicting (1).

(4) $|X| = 3$.

Suppose not. Then $|X| \geq 4$. Since S is strong by (3), it has a 4-cycle $d_1d_2d_3d_4d_1$. Note that both (a_2, d_i) and (d_i, b_2) are arcs in T_2 for $1 \leq i \leq 4$. Thus the cycle $d_1d_2d_3d_4d_1$ together with the five arcs (b_2, a_2), (a_2, d_2), (a_2, d_4), (d_1, b_2), and (d_3, b_2) would form a $K_{3,3}$ in T_2, a contradiction.

Combining (1)-(4), we see that T_2 is isomorphic to F_6. ∎

Lemma 3.13. *Let $T = (V, A)$ be the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 over the special arcs (a_1, b_1) and (b_2, a_2) such that $T_2 \in \mathcal{T}_2$, and let T' be the digraph obtained from T by contracting two vertices x and y in $T_2\backslash\{a_2, b_2\}$. Then T' is also Möbius-free.*

Proof. Let T_2' be the digraph obtained from T_2 by contracting x and y. Notice that T_2' may contain opposite arcs. Since $T_2 \in \mathcal{T}$, we have $|V(T_2')| \leq 5$, so T_2' is Möbius-free. Let T_2'' be an arbitrary spanning tournament contained in T_2', and let T'' be the 1-sum of T_1 and T_2''. Then T'' is a spanning tournament contained in T'. By Lemma 2.3, T'' is Möbius-free. It follows that T' is also Möbius-free, because none of $K_{3,3}$, $K_{3,3}'$, M_5, and M_5^* contains a pair of opposite arcs. ∎

4 Reductions: Getting Started

Throughout this paper, an *instance* $(T, \boldsymbol{w})$ consists of a Möbius-free tournament $T = (V, A)$ with a nonnegative integral weight $w(e)$ on each arc e. We say that another instance $(T', \boldsymbol{w}')$ is *smaller* than $(T, \boldsymbol{w})$ if $|V'| < |V|$ or if $|V'| = |V|$ but $w(A') < w(A)$, where $T' = (V', A')$. Recall the fractional FAS problem $\mathbb{P}(T, \boldsymbol{w})$ and the fractional cycle packing problem $\mathbb{D}(T, \boldsymbol{w})$ introduced in Section 1. We shall prove Theorem 1.1 using reduction methods; the objective of the reduction step is given below.

Theorem 4.1. *Let $(T, \boldsymbol{w})$ be an instance, such that $\mathbb{D}(T', \boldsymbol{w}')$ has an integral optimal solution for any smaller instance $(T', \boldsymbol{w}')$ than $(T, \boldsymbol{w})$. Then $\mathbb{D}(T, \boldsymbol{w})$ also has an integral optimal solution.*

Our proof relies heavily on a structural description of T. Clearly, we may assume that T is strong. As shown in Section 3, if T is *i2s*, then it comes from a finite list. The following lemma asserts that if T is not *i2s*, then it can be expressed as the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2, such that the structure of T_2 is relatively simple. Thus our proof may proceed by merely performing reduction on T_2. In our lemma, s^* is the vertex arising from contracting S in T/S, the tournaments G_4, G_5, G_6 are shown in Figure 7, and

$$\mathcal{T}_3 = \{F_0, F_3, F_4, F_6, G_2, G_3, G_4, G_5, G_6\} = (\mathcal{T}_2\backslash F_2) \cup \{G_4, G_5, G_6\}.$$

Moreover, we say that a cycle C in T is *positive* if $w(e) > 0$ for each arc e on C, and say that C crosses b (the hub of the 1-sum) if it contains an arc between $T_1\backslash\{b, a_1\}$ and $T_2\backslash\{b, a_2\}$.

Lemma 4.2. *Let $T = (V, A)$ be a strong Möbius-free tournament with a nonnegative integral weight $w(e)$ on each arc e. Suppose $\tau_w(T) > 0$ and T is not i2s. Then T is the* 1*-sum of two smaller strong Möbius-free tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2), such that one of the following three cases occurs:*

(i) $\tau_w(T_2\backslash a_2) > 0$ *and* $T_2 \in \mathcal{T}_2$*;*

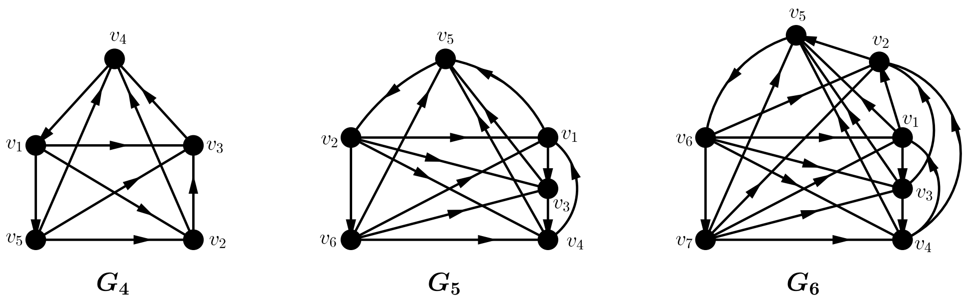


Figure 7. Three more tournaments involved in structural description

(ii) $\tau_w(T_2\backslash a_2) > 0$ *and there exists a vertex subset* S *of* $T_2\backslash\{a_2, b_2\}$ *with* $|S| \geq 2$*, such that* $T[S]$ *is acyclic,* $T_2/S \in \mathcal{T}_3$*, and* s^* *is a near-sink in* T/S*. Furthermore,*

- $(b_2, a_2) = (v_1, v_5)$ *and* $s^* = v_4$ *if* $T_2/S = G_4$*;*
- $(b_2, a_2) = (v_2, v_6)$ *and* $s^* = v_5$ *if* $T_2/S = G_5$*;*
- $(b_2, a_2) = (v_6, v_7)$ *and* $s^* = v_5$ *if* $T_2/S = G_6$*; and*

(iii) every positive cycle in T *crosses* b*.*

Proof. To establish the statement, we shall construct a sequence of 1-sums of T until one of the three desired cases occurs.

By Lemmas 2.2 and 2.3, T can be expressed as the 1-sum of two smaller strong Möbius-free tournaments T_{11} and T_{12}; subject to this, $|V(T_{12})|$ is as small as possible. Let (a_{11}, b_{11}) in T_{11} and (b_{12}, a_{12}) in T_{12} be the two special arcs involved in the definition of the 1-sum, and let b_1 denote the hub of the 1-sum. By Lemma 3.12, we have $T_{12} \in \mathcal{T}_2$. If $\tau_w(T_{12}\backslash a_{12}) > 0$, then (i) occurs, with $T_1 = T_{11}$ and $T_2 = T_{12}$. So we may assume that $\tau_w(T_{12}\backslash a_{12}) = 0$. Furthermore,

(1) $T_{12}\backslash a_{12}$ is an acyclic tournament in which b_1 is the sink.

To justify this, let K be an MFAS in $T_{12}\backslash a_{12}$. Then $w(K) = 0$ and $T_{12}\backslash K$ is acyclic. Let J be the set of all arcs leaving b_1 in $T_{12}\backslash a_{12}$. Note that no arc in J is contained in any positive cycle in T that crosses b_1. Let T'_{12} be obtained from T_{12} by reversing the directions of all arcs in J and some arcs in K so that $T'_{12}\backslash a_{12}$ is acyclic, and define the weight of each reversed arc in T'_{12} to be zero. Let $T' = (V, A')$ denote the resulting tournament and let $\boldsymbol{w}'$ denote the resulting weight function defined on A'. Then T' remains strong. Since no arc in $K \cup J$ is contained in any positive cycle in T, it is clear that every optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ corresponds to a feasible solution to $\mathbb{D}(T', \boldsymbol{w}')$ with the same objective value, and vice versa. So we may assume that T is T' and that $\boldsymbol{w}$ is $\boldsymbol{w}'$. Thus (1) holds.

At a general step i, suppose T is the 1-sum of two smaller strong Möbius-free tournaments T_{i1} and T_{i2} over two special arcs (a_{i1}, b_{i1}) and (b_{i2}, a_{i2}), such that $T_{i2}\backslash a_{i2}$ is an acyclic tournament in which b_i (the hub of the 1-sum) is the sink. Let S_i be the vertex set of $T_{i2}\backslash\{a_{i2}, b_i\}$, and let T_i be the tournament obtained from T by contracting S_i into a single vertex s_i^*. Clearly, T_i is isomorphic to T_{i1}, in which s_i^* corresponds to a_{i1} and is a near-sink. If $\tau_w(T_i\backslash s_i^*) = 0$, then every positive cycle in T crosses b_i. So (iii) occurs, with $T_1 = T_{i1}$, $T_2 = T_{i2}$, and $b = b_i$. Thus we may assume that $\tau_w(T_i\backslash s_i^*) > 0$. We construct a new 1-sum of T as follows.

Assume first that T_i is $i2s$. In this case, T_i and hence T_{i1} is a member of $\mathcal{T}_2$ by Lemma 3.12. Furthermore, $T_{i1} \neq F_6$. Let T', T'_{i1}, and T'_{i2} be the reverses of T, T_{i1}, and T_{i2}, respectively. Then T' is the 1-sum of two smaller strong Möbius-free tournaments T'_{i2} and T'_{i1}, with $T'_{i1} \in \mathcal{T}_2 \backslash F_6$. Since there is a one-to-one correspondence between cycles in T and those in T', $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution iff so does $\mathbb{D}(T', \boldsymbol{w}')$. Thus we may assume that T is T' and hence (i) occurs.

Assume next that T_i is not $i2s$. By Lemmas 2.2 and 2.3, T_i can be expressed as the 1-sum of two smaller strong Möbius-free tournaments T'_{i1} and T'_{i2}; subject to this, $|V(T'_{i2})|$ is as small as possible. By Lemma 3.12, we have $T'_{i2} \in \mathcal{T}_2$. Let (a'_{i1}, b'_{i1}) in T'_{i1} and (b'_{i2}, a'_{i2}) in T'_{i2} be the two special arcs involved in the definition of the 1-sum, and let b'_i denote the hub of this 1-sum. We proceed by considering two subcases.

• $b'_i \neq s^*_i$. In this subcase, s^*_i is contained in $T'_{i2} \backslash \{a'_{i2}, b'_i\}$, because it is a near-sink in T_i. Hence b_i is contained in $T'_{i2} \backslash a'_{i2}$. Observe that T is the 1-sum of two smaller strong Möbius-free tournaments $T_{(i+1)1}$ and $T_{(i+1)2}$, such that the hub b_{i+1} of this 1-sum is exactly b'_i and that $T_{(i+1)1} = T'_{i1}$. Let $(a_{(i+1)1}, b_{(i+1)1})$ in $T_{(i+1)1}$ and $(b_{(i+1)2}, a_{(i+1)2})$ in $T_{(i+1)2}$ be the two special arcs involved in the definition of this 1-sum. Then $T_{i2} \backslash a_{i2}$ is a proper subtournament of $T_{(i+1)2} \backslash a_{(i+1)2}$. If $\tau_w(T_{(i+1)2} \backslash a_{(i+1)2}) > 0$, then (ii) occurs, with $T_1 = T_{(i+1)1}$, $T_2 = T_{(i+1)2}$, $S = S_i$, and $s^* = s^*_i$. Furthermore, $T_2/S \neq F_2$, because no vertex in $\{v_1, v_3, v_4\}$ (see the labeling in Figure 3) is a near-sink in $F_2 \backslash v_2$ and hence corresponds to s^*. So we assume that $\tau_w(T_{(i+1)2} \backslash a_{(i+1)2}) = 0$. Furthermore,

(2) $T_{(i+1)2} \backslash a_{(i+2)2}$ is an acyclic tournament in which b_{i+1} is the sink.

Since the proof goes along the same line as that of (1), the details are omitted here. In view of (2), we can repeat the construction process by replacing i with $i+1$.

• $b'_i = s^*_i$. In this subcase, b_i is contained in $T'_{i1} \backslash \{a'_{i1}, b'_i\}$, because it is the only out-neighbor of s^*_i in T_i. Since $T'_{i2} \in \mathcal{T}_2$ (see the labeling in Figures 3-6) and since (b'_{i2}, a'_{i2}) is a special arc of T'_{i2}, and $b'_{i2} = b'_i = s^*_i$ is a sink of $T'_{i2} \backslash a'_{i2}$, it is routine to check that one of (3)-(5) occurs:

(3) $T'_{i2} = F_0$, $(b'_{i2}, a'_{i2}) = (v_4, v_1)$, and $s^*_i = v_4$;

(4) $T'_{i2} = F_2$, $(b'_{i2}, a'_{i2}) = (v_5, v_2)$, and $s^*_i = v_5$; and

(5) $T'_{i2} = F_6$, $(b'_{i2}, a'_{i2}) = (v_5, v_6)$, and $s^*_i = v_5$.

Observe that T is the 1-sum of two smaller strong Möbius-free tournaments $T_{(i+1)1}$ and $T_{(i+1)2}$ along two special arcs $(a_{(i+1)1}, b_{(i+1)1})$ in $T_{(i+1)1}$ and $(b_{(i+1)2}, a_{(i+1)2})$, such that the hub b_{i+1} of this 1-sum is exactly b_i and that $T_{(i+1)1} \backslash a_{(i+1)1} = T'_{i1} \backslash \{s^*_i, a'_{i1}\}$. Clearly, $T_{i2} \backslash a_{i2}$ is a proper subtournament of $T_{(i+1)2} \backslash a_{(i+1)2}$. It is a simple matter to check that $T_{(i+1)2}/S_i$ is isomorphic to G_{t+1} when (t) holds for $t = 3, 4, 5$. If $\tau_w(T_{(i+1)2} \backslash a_{(i+1)2}) > 0$, then (ii) occurs, with $T_1 = T_{(i+1)1}$, $T_2 = T_{(i+1)2}$, $S = S_i$, and $s^* = s^*_i$. So we assume that $\tau_w(T_{(i+1)2} \backslash a_{(i+1)2}) = 0$. Furthermore, $T_{(i+1)2} \backslash a_{(i+2)2}$ is an acyclic tournament in which b_{i+1} is the sink. Since the proof is exactly the same as that of (2), we omit the details here. Thus we can repeat the construction process by replacing i with $i+1$.

Since $T_{i2} \backslash a_{i2}$ is a proper subtournament of $T_{(i+1)2} \backslash a_{(i+1)2}$ for each step i, the construction process terminates in a finite number of steps. Therefore one of (i)-(iii) holds. ∎

In the remainder of this section, we assume that $(T, \boldsymbol{w})$ is an instance as described in Theorem 4.1, and that $T = (V, A)$ is the 1-sum of two strong Möbius-free tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2).

Let $\mathcal{C}$ be the set of all cycles in T, let $\mathcal{C}_i$ be the set of all cycles in $T_i \backslash a_i$ for $i = 1, 2$, and let $\mathcal{C}_0 = \mathcal{C} \backslash (\mathcal{C}_1 \cup \mathcal{C}_2)$. Note that each cycle in $\mathcal{C}_0$ crosses b, the hub of the 1-sum. For each arc e of T, let $\mathcal{C}(e) = \{C \in \mathcal{C} : e \in C\}$ and $\mathcal{C}_i(e) = \{C \in \mathcal{C}_i : e \in C\}$ for $i = 0, 1, 2$.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, and let $\nu_w^*(T)$ denote the optimal value of $\mathbb{D}(T, \boldsymbol{w})$. Then $\nu_w^*(T) = \boldsymbol{y}^T\mathbf{1}$. Set $\mathcal{C}^y = \{C \in \mathcal{C} : y(C) > 0\}$ and $\mathcal{C}_i^y = \{C \in \mathcal{C}_i : y(C) > 0\}$ for $i = 0, 1, 2$. For each arc e of T, set $z(e) = y(\mathcal{C}(e))$. We say that e is *saturated* by $\boldsymbol{y}$ if $w(e) = z(e)$ and *unsaturated* otherwise, and say that e is *saturated* by $\boldsymbol{y}$ in T_i if $w(e) = y(\mathcal{C}_i(e))$ for $i = 1, 2$. For each $\mathcal{D} \subseteq \mathcal{C}^y$, we say that arc e is *outside* $\mathcal{D}$ if e is not contained in any cycle in $\mathcal{D}$.

Let us exhibit some properties enjoyed by optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, and make further technical preparations for the proof of Theorem 4.1.

Lemma 4.3. *Let $T = (V, A)$ be a tournament with a nonnegative integral weight $w(e)$ on each arc, and let $\boldsymbol{x}$ (resp. $\boldsymbol{y}$) be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$ (resp. $\mathbb{D}(T, \boldsymbol{w})$). Then the following statements hold:*

- *(i) $x(C) = 1$ for any cycle C of T with $y(C) > 0$;*
- *(ii) $x(e) = 0$ for all $e \in A$ with $z(e) < w(e)$;*
- *(iii) $w(e) = z(e)$ for all $e \in A$ with $x(e) > 0$; and*
- *(iv) Let C_1 and C_2 be two cycles of T with $y(C_i) > 0$ for $i = 1, 2$. Suppose a and b are two common vertices of C_1 and C_2 such that $C_i(a, b)$ is vertex-disjoint from $C_{3-i}(b, a)$ for $i = 1, 2$. Then $\sum_{e \in C_1[a,b]} x(e) = \sum_{e \in C_2[a,b]} x(e)$.*

Proof. Statements (i)-(iii) follow directly from the complementary slackness conditions. To justify (iv), let $\theta = \min\{y(C_1), y(C_2)\}$, let $C_i' = C_{3-i}[a, b] \cup C_i[b, a]$ for $i = 1, 2$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C_i)$ with $y(C_i) - \theta$ and replacing $y(C_i')$ with $y(C_i') + \theta$ for $i = 1, 2$. Clearly, $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Using (i), we obtain $x(C_i) = x(C_i') = 1$ for $i = 1, 2$, which implies $\sum_{e \in C_1[a,b]} x(e) = \sum_{e \in C_2[a,b]} x(e)$. ∎

Lemma 4.4. *Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution if one of the following conditions is satisfied:*

- *(i) $w(e) > \lceil z(e) \rceil$ for some $e \in A$;*
- *(ii) $\mathcal{C}_0^y = \emptyset$; and*
- *(iii) $y(C)$ is integral for some $C \in \mathcal{C}^y$.*

Proof. (i) Define $\boldsymbol{w}' \in \mathbb{Z}_+^A$ by $w'(e) = \lceil z(e) \rceil$ and $w'(a) = w(a)$ for all $a \in A \backslash e$. Then $w(A) > w'(A)$. By the hypothesis of Theorem 4.1, $\mathbb{D}(T, \boldsymbol{w}')$ has an integral optimal solution $\boldsymbol{y}'$. Since $\boldsymbol{y}$ is also a feasible solution to $\mathbb{D}(T, \boldsymbol{w}')$, we have $(\boldsymbol{y}')^T\mathbf{1} \geq \boldsymbol{y}^T\mathbf{1}$. So $\boldsymbol{y}'$ is an integral optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ as well.

(ii) Since $\mathcal{C}_0^y = \emptyset$, each cycle in $\mathcal{C}^y$ is contained in $T_i \backslash a_i$ for $i = 1$ or 2. Let $\boldsymbol{w}_i$ be the restriction of $\boldsymbol{w}$ to $T_i \backslash a_i$. Then the hypothesis of Theorem 4.1 guarantees the existence of an integral optimal solution $\boldsymbol{y}_i$ to $\mathbb{D}(T_i \backslash a_i, \boldsymbol{w}_i)$. Clearly, the union of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ yields an integral optimal solution to $\mathbb{D}(T, \boldsymbol{w})$.

(iii) Define $\boldsymbol{w}' \in \mathbb{Z}_+^A$ by $w'(e) = w(e) - y(C)$ for each arc e on C and $w'(a) = w(a)$ for all other arcs a. Then $w(A) > w'(A)$. By the hypothesis of Theorem 4.1, $\mathbb{D}(T, \boldsymbol{w}')$ has an integral optimal solution $\boldsymbol{y}'$. Clearly, $\boldsymbol{y}$ yields a feasible solution to $\mathbb{D}(T, \boldsymbol{w}')$ with value $\boldsymbol{y}^T\mathbf{1} - y(C)$.

So $(\boldsymbol{y}')^T\mathbf{1} \geq \boldsymbol{y}^T\mathbf{1} - y(C)$. Let $\boldsymbol{y}^* \in \mathbb{Z}_+^{\mathcal{C}}$ be defined by $y^*(C) = y(C) + y'(C)$ and $y^*(D) = y'(D)$ for all $D \in \mathcal{C}\backslash C$. Then $\boldsymbol{y}^*$ is an integral feasible solution to $\mathbb{D}(T, \boldsymbol{w})$ with value at least $(\boldsymbol{y}')^T\mathbf{1} + y(C) \geq \boldsymbol{y}^T\mathbf{1}$. Hence $\boldsymbol{y}^*$ is an integral optimal solution to to $\mathbb{D}(T, \boldsymbol{w})$. ∎

Lemma 4.5. *Let $G = (U, E)$ be a Möbius-free digraph obtained from a tournament by adding some arcs, and let $c(e)$ be a nonnegative integral weight associated with each arc $e \in E$. If $|U| < |V|$ or if $|U| = |V|$ but $c(E) < w(A)$, where V and $w(A)$ are as defined in Theorem 4.1, then $\mathbb{D}(G, \mathbf{c})$ has an integral optimal solution.*

Proof. The proof technique employed below is due to Barahona and Mahjoub [6].

Let us repeatedly apply the following operations on G whenever possible: For each pair of opposite arcs e and f, replace $c(g)$ by $c(g) - \theta$ for $g = e, f$, where $\theta = \min\{c(e), c(f)\}$, and delete exactly one arc $g \in \{e, f\}$ with $c(g) = 0$ from G. Let $G' = (V', A')$ be the resulting digraph and let $\boldsymbol{c}'$ be the resulting weight function. Clearly, G' is a tournament. Hence, by the hypothesis of Theorem 4.1, G' is CM. Let F' be a minimum FAS of G' and let $\boldsymbol{y}'$ be a maximum cycle packing in G'. Then $c'(F') = (\boldsymbol{y}')^T\mathbf{1}$.

Define $y(C) = y'(C)$ for all cycles C in G'. For each 2-cycle C formed by arcs e and f in G, define $y(C) = \theta$, where $\theta = \min\{c(e), c(f)\}$, and place g and all arcs in F' into F, where g is the arc in $\{e, f\}\backslash A'$. Repeat the process until all 2-cycles in G are exhausted. Clearly, F is an FAS of G, $\boldsymbol{y}$ is a cycle packing of G, and $c(F) = \boldsymbol{y}^T\mathbf{1}$. By the LP-duality theorem, $\boldsymbol{y}$ is an integral optimal solution to $\mathbb{D}(G, \mathbf{c})$. ∎

Lemma 4.6. *Suppose $a = (s, t)$ is a special arc of $T = (V, A)$, where s is a near-sink. Then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution if one of the following conditions is satisfied:*

(i) $w(e) = z(e)$ for all arcs $e \in \delta^-(s)$;

(ii) $\nu_w^*(T)$ is an integer;

(iii) $x(a) = 0$ for some optimal solution $\boldsymbol{x}$ of $\mathbb{P}(T, \boldsymbol{w})$;

(iv) a is unsaturated by $\boldsymbol{y}$; that is, $z(a) < w(a)$.

Proof. (i) By Lemma 4.4(i), we may assume that $w(a) = \lceil z(a) \rceil$. Since $w(e) = z(e)$ for all $e \in \delta^-(s)$ and $z(a) = \sum_{e \in \delta^-(s)} z(e)$, we obtain $w(a) = \sum_{e \in \delta^-(s)} w(e)$. Let $T' = (V', A')$ be the digraph obtained from T by contracting the arc a; we still use t to denote the resulting vertex. By Lemma 2.4, T' is also Möbius-free. Define $\boldsymbol{w}' \in \mathbb{Z}_+^{A'}$ as follows: $w'(e) = w(e)$ if e is not directed to t, $w'(e) = w(f) + w(e)$ if $f = (r, s)$ and $e = (r, t)$ are both in A, and $w'(e) = w(f)$ if $f = (r, s)$ is in A while $e = (r, t)$ is not. It is easy to see that every integral feasible solution of $\mathbb{D}(T, \boldsymbol{w})$ yields an integral feasible solution to $\mathbb{D}(T', \boldsymbol{w}')$ with the same objective value, and vice versa. As $\mathbb{D}(T', \boldsymbol{w}')$ has an integral optimal solution by Lemma 4.5, so does $\mathbb{D}(T, \boldsymbol{w})$.

(ii) By (i), we may assume that $w(e) \neq z(e)$ for some arc $e = (r, s)$ in A. By Lemma 4.4(i), we may assume that $w(e) = \lceil z(e) \rceil$. So $\lceil z(e) \rceil \neq z(e)$. Set $\theta = z(e) - \lfloor z(e) \rfloor$. Then $0 < \theta < 1$. Let $\boldsymbol{w}'$ be obtained from $\boldsymbol{w}$ by replacing $w(e)$ with $w(e) - 1$. Then any optimal solution $\boldsymbol{y}$ of $\mathbb{D}(T, \boldsymbol{w})$ yields a feasible solution of $\mathbb{D}(T, \boldsymbol{w}')$ with value at least $\nu_w^*(T) - \theta$. By the hypothesis of Theorem 4.1, $\mathbb{D}(T, \boldsymbol{w}')$ has an integral optimal solution $\boldsymbol{y}'$ with value at least $\nu_w^*(T) - \theta$ and hence at least $\nu_w^*(T)$. So $\boldsymbol{y}'$ is also an integral optimal solution to $\mathbb{D}(T, \boldsymbol{w})$.

(iii) For each $r \in V\backslash\{s, t\}$ with $e = (r, t) \in A$, we claim that $x(e) = x(f)$, where $f = (r, s)$. If $w(e) = 0$ or $w(f) = 0$, clearly we may assume that $x(e) = x(f)$ (modifying one of them if necessary, the resulting solution remains optimal). Next, consider the case when $w(e) > 0$ and $w(f) > 0$. Let C_1 and C_2 be two cycles passing through e and f, respectively, with $y(C_i) > 0$ for $i = 1, 2$. By Lemma 4.3(iv), $x(e) = x(a) + x(f) = x(f)$. So the claim is justified.

Let $T' = (V', A')$ be the digraph obtained from T by contracting the arc a. By Lemma 2.4, T' is also Möbius-free. Define $\boldsymbol{w'} \in \mathbb{Z}_+^{A'}$ as follows: $w'(e) = w(e)$ if e is not directed to t, $w'(e) = w(f) + w(e)$ if $f = (r, s)$ and $e = (r, t)$ are both in A, and $w'(e) = w(f)$ if $f = (r, s)$ is in A while $e = (r, t)$ is not. Let $\boldsymbol{x'} \in \mathbb{R}_+^{A'}$ be the projection of $\boldsymbol{x}$, and let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through (r, s) in T with $y(C) > 0$, let C' be the cycle in T' arising from C by replacing the path rst with (r, t) and set $y'(C') = y(C) + y(C')$. By the LP-duality theorem, $\boldsymbol{x'}$ and $\boldsymbol{y'}$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w'})$ and $\mathbb{D}(T', \boldsymbol{w'})$, respectively, with the same objective value as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\mathbb{D}(T', \boldsymbol{w'})$ has an integral optimal solution. So $\nu_w^*(T)$ is an integer. Thus (iii) follows from (ii).

(iv) Since $z(a) < w(a)$, we have $x(a) = 0$ by Lemma 4.3(ii). Therefore (iv) can be deduced from (iii). ∎

Recall that $\mathcal{C}_2$ is the set of all cycles in $T_2 \backslash a_2$. In the following lemma, $\mathcal{D}_k$ is the set of all cycles of length k in $T_2 \backslash a_2$, and q is the length of a longest cycle in $T_2 \backslash a_2$. Thus $\mathcal{C}_2 = \cup_{k=3}^{q} \mathcal{D}_k$. Let $H_i = (V_i, E_i)$ be a digraph for $i = 1, 2, \ldots, k$. A digraph $H = (V, E)$ is called a *multiset sum* of these k digraphs if $V = \cup_{i=1}^{k} V_i$ and E is the multiset sum of all these E_i's; that is, if an arc (u, v) is contained in t of these H_i's, then there are precisely t parallel arcs from u to v in H.

Lemma 4.7. *Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that $y(\mathcal{C}_2)$ is maximized and, subject to this, $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically. Then the following statements hold:*

(i) Every $C \in \mathcal{C}$ contains an arc e that is saturated by $\boldsymbol{y}$;

(ii) Every $C \in \mathcal{C}_2$ contains an arc that is outside $\mathcal{C}_0^y$;

(iii) If $C_1 \in \mathcal{C}_0^y$ and $C_2 \in \mathcal{C}_2$ share arcs, then some arc on C_2 but outside C_1 is saturated by $\boldsymbol{y}$;

(iv) If exactly one arc on $C \in \mathcal{C}_2$ is outside $\mathcal{C}_0^y$, then it is saturated by $\boldsymbol{y}$ in T_2;

(v) Every chord of $C \in \mathcal{C}_2^y$ is saturated by $\boldsymbol{y}$ in T_2;

(vi) If the multiset sum of $C_1 \in \mathcal{C}_0$, $C_2 \in \mathcal{C}_2$, and unsaturated arcs in $T_2 \backslash a_2$ contains two arc-disjoint cycles in $T_2 \backslash a_2$, then $y(C_1)$ or $y(C_2)$ is 0;

(vii) Every triangle $C \in \mathcal{C}_2$ contains an arc that is saturated by $\boldsymbol{y}$ in T_2;

(viii) If the multiset sum of $C_1 \in \mathcal{C}_0$ and $C_2 \in \mathcal{C}_2$ contains two arc-disjoint cycles $C_1' \in \mathcal{C}_0$ and $C_2' \in \mathcal{C}_2$, with $|C_2'| < |C_2|$, then $y(C_1)$ or $y(C_2)$ is 0.

Proof. (i) Assume the contrary: $w(e) > z(e)$ for each arc e on C. Set $\theta = \min\{w(e) - z(e) : e \in C\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C)$ with $y(C) + \theta$. Then $\boldsymbol{y'}$ is a feasible solution to $\mathbb{D}(T, \boldsymbol{w})$, with $(\boldsymbol{y'})^T \mathbf{1} = \boldsymbol{y}^T \mathbf{1} + \theta > \boldsymbol{y}^T \mathbf{1}$, contradicting the optimality on $\boldsymbol{y}$.

(ii) Assume the contrary: each arc e_i on C is contained in some $C_i \in \mathcal{C}_0^y$. Observe that b, the hub of the 1-sum, is not on C, for otherwise, let e_j be the arc on C that leaves b. From the definition of the 1-sum, we see that e_j is contained in no cycle in $\mathcal{C}_0$, contradicting the definition of C_j. Let $k = |C|$ and let H be the multiset sum of $C_1, C_2, ..., C_k$. Then H is an even digraph and $d_H^+(b) = d_H^-(b) = k$. Let H' be obtained from H by deleting all arcs on

C. Then H' remains even and $d^+_{H'}(b) = d^-_{H'}(b) = k$ because b is outside C. So H' contains k arc-disjoint cycles $C'_1, C'_2, ..., C'_k$ passing through b and hence in $\mathcal{C}_0$. Set $\theta = \min_{1 \le i \le k} \; y(C_i)$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C)$ with $y(C) + \theta$, replacing $y(C_i)$ with $y(C_i) - \theta$, and replacing $y(C'_i)$ with $y(C'_i) + \theta$ for $1 \le i \le k$. Clearly, $\boldsymbol{y}'$ is a feasible solution to $\mathbb{D}(T, \boldsymbol{w})$ with $(\boldsymbol{y}')^T \mathbf{1} = \boldsymbol{y}^T \mathbf{1} + \theta > \boldsymbol{y}^T \mathbf{1}$, a contradiction.

(iii) Assume the contrary: $w(e) > z(e)$ for each arc e in B, the set of all arcs on C_2 but outside C_1. Set $\theta = \min\{y(C_1),\, w(e) - z(e) : e \in B\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C_1)$ with $y(C_1) - \theta$ and replacing $y(C_2)$ with $y(C_2) + \theta$. Then $\boldsymbol{y}'$ is also optimal, with $y'(\mathcal{C}_2) = y(\mathcal{C}_2) + \theta$, so the existence of $\boldsymbol{y}'$ contradicts the maximality assumption on $y(\mathcal{C}_2)$ in the choice of $\boldsymbol{y}$.

(iv) Assume the contrary: the only arc $e_0 = (u, v)$ on C outside $\mathcal{C}^y_0$ is not unsaturated by $\boldsymbol{y}$ in T_2. Then $w(e_0) > z(e_0)$. Let C_i a cycle in $\mathcal{C}^y_0$ that passes through each e_i on $C \backslash e_0$. Let $k = |C| - 1$ and let H be the multiset sum of $C_0, C_1, C_2, ..., C_k$, where C_0 is the 2-cycle formed by (u, v) and (v, u). Then H is an even digraph, and $d^+_H(b) = d^-_H(b) = k$ if $b \ne u$ and $d^+_H(b) = d^-_H(b) = k + 1$ otherwise, where b is the hub of the 1-sum. Let H' be obtained from H by deleting all arcs on C. Then H' remains even and contains k arc-disjoint cycles $C'_1, C'_2, ..., C'_k$ passing through b (and hence in $\mathcal{C}_0$). Clearly, at most one of $C'_1, C'_2, ..., C'_k$, say C'_k if any, contains the arc (v, u). Then $C'_1, C'_2, ..., C'_{k-1}$ are all in $\mathcal{C}_0$. Set $\theta = \min\{w(e_0) - z(e_0),\, y(C_i) : 1 \le i \le k\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C)$ with $y(C) + \theta$, replacing $y(C_i)$ with $y(C_i) - \theta$ for $1 \le i \le k$, and replacing $y(C'_j)$ with $y(C'_j) + \theta$ for $1 \le j \le k - 1$. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y(C) < y'(C)$, the existence of $\boldsymbol{y}'$ contradicts the maximality assumption on $y(\mathcal{C}_2)$ in the choice of $\boldsymbol{y}$.

(v) Assume the contrary: some chord $e = (u, v)$ of C is not saturated by $\boldsymbol{y}$ in T_2. Let $C' = C[v, u] \cup \{(u, v)\}$. Note that $C' \in \mathcal{C}_2$ and $|C'| < |C|$.

We first consider the case when e is outside $\mathcal{C}^y_0$. Then $w(e) - z(e) > 0$. Set $\theta = \min\{y(C), w(e) - z(e)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C)$ with $y(C) - \theta$ and replacing $y(C')$ with $y(C') + \theta$. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(C) < y(C)$, the existence of $\boldsymbol{y}'$ contradicts the minimality assumption on $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ in the choice of $\boldsymbol{y}$.

We next consider the case when e is contained in some cycle D in $\mathcal{C}^y_0$. Then the multiset sum of C and D contains a cycle D' in $\mathcal{C}_0$ that is disjoint from C'. Set $\sigma = \min\{y(C), y(D)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C)$, $y(D)$, $y(C')$, and $y(D')$ with $y(C) - \sigma$, $y(D) - \sigma$, $y(C') + \sigma$, and $y(D') + \sigma$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(C) < y(C)$, the existence of $\boldsymbol{y}'$ contradicts the minimality assumption on $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ in the choice of $\boldsymbol{y}$.

(vi) Assume the contrary: $y(C_1)y(C_2) > 0$. Let B be the set of unsaturated arcs in $T_2 \backslash a_2$, and let C'_1 and C'_2 be two arc-disjoint cycles in $\mathcal{C}_2$ that are contained in the multiset sum of C_1, C_2, and B. Set $\theta = \min\{y(C_1), y(C_2), w(e) - z(e) : e \in B\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C_1)$, $y(C_2)$, $y(C'_1)$, and $y(C'_2)$ with $y(C_1) - \theta$, $y(C_2) - \theta$, $y(C'_1) + \theta$, and $y(C'_2) + \theta$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(\mathcal{C}_2) = y(\mathcal{C}_2) + \theta$, the existence of $\boldsymbol{y}'$ contradicts the maximality assumption on $\boldsymbol{y}$.

(vii) Let $C = ijki$ be a triangle in $T_2 \backslash u_2$. By (ii), at least one arc on C is outside $\mathcal{C}^y_0$, say (i, j). If all arcs on C are outside $\mathcal{C}^y_0$, then by (i) one of the three arcs is saturated by $\boldsymbol{y}$ in T and hence in T_2. If (i, j) is the only arc on C that is outside $\mathcal{C}^y_0$, then (i, j) is saturated by $\boldsymbol{y}$ in T_2 by (iv). If exactly one arc on C, say (j, k), is contained in some cycle in $\mathcal{C}^y_0$, then by (iii) one of (i, j) and (k, i) is saturated by $\boldsymbol{y}$ in T and hence in T_2.

(viii) Assume the contrary: $y(C_1)y(C_2) > 0$. Set $\theta = \min\{y(C_1), y(C_2)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C_1)$, $y(C_2)$, $y(C'_1)$, and $y(C'_2)$ with $y(C_1) - \theta$, $y(C_2) - \theta$, $y(C'_1) + \theta$, and $y(C'_2)+\theta$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $|C'_2| < |C_2|$ and $y'(C_2) < y(C_2)$, the existence of $\boldsymbol{y}'$ contradicts the minimality assumption on $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \dots, y(\mathcal{D}_3))$ in the choice of $\boldsymbol{y}$. ∎

Lemma 4.8. *Suppose $T_2 \backslash a_2$ contains a unique cycle C, which is a triangle. If $w(a) > 0$ for each arc a on C, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that $y(C)$ is maximized. By Lemma 4.7(vii), some arc e on C is saturated by $\boldsymbol{y}$ in T_2. Since C is the unique cycle in $T_2 \backslash a_2$, we have $y(C) = w(e)$. Thus $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 4.4(iii). ∎

5 Basic Reductions

Throughout this section, we assume that $(T, \boldsymbol{w})$ is an instance as described in Theorem 4.1, and that $T = (V, A)$ is the 1-sum of two strong Möbius-free tournaments T_1 and T_2 over the two special arcs (a_1, b_1) and (b_2, a_2), with $\tau_w(T_2 \backslash a_2) > 0$ and $T_2 \in \mathcal{T}_2$. (Possibly T_1 is a triangle and thus $T = T_2$.) Let us label T_2 as in Figures 3-6. Since (b_2, a_2) is a special arc and a_2 is a near-source of T_2,

- $(b_2, a_2) = (v_1, v_2)$ or (v_4, v_1) if $T_2 = F_0$;
- $(b_2, a_2) = (v_5, v_2)$ if $T_2 = F_2$ or F_3;
- $(b_2, a_2) = (v_5, v_6)$ if $T_2 = F_4$;
- $(b_2, a_2) = (v_5, v_6)$ if $T_2 = F_6$; and
- $(b_2, a_2) = (v_4, v_5)$ if $T_2 = G_2$ or G_3.

Note that $T_2 \backslash a_2$ is a transitive triangle when $T_2 = F_0$ and $(b_2, a_2) = (v_4, v_1)$; in this case, unfortunately, no reduction on $T_2 \backslash a_2$ is available, and the information on $T_2 \backslash a_2$ alone does not lead to a proof of the desired statement; that is, $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. In fact, the same problem occurs when $\tau_w(T_2 \backslash a_2) = 0$, no matter what T_2 is. That may partly explain why the assumption of this section is so made and Lemma 4.2 is so stated.

Theorem 5.1. *For the above instance $(T, \boldsymbol{w})$, problem $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

We shall carry out a proof by performing reduction on $T_2 \backslash a_2$. We employ the same notations as introduced before. In particular, $\nu^*_w(T)$ stands for the common optimal value of $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, and $\tau_w(T)$ stands for the minimum total weight of an FAS in T. An FAS K of T is called *minimal* if no proper subset of K is an FAS of T. A minimum-weight FAS is denoted by MFAS. We use $\mathcal{F}_2$ to denote the family of all minimal FAS's in $T_2 \backslash a_2$. Recall that $\mathcal{C}_2$ stands for the set of all cycles in $T_2 \backslash a_2$, and $\mathcal{D}_k$ is the set of all cycles of length k in $T_2 \backslash a_2$. For every real number r, set $[r] = r - \lfloor r \rfloor$.

We break the proof of Theorem 5.1 into a series of lemmas.

Lemma 5.2. *If $T_2 \in \{F_0, F_2, F_6\}$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. By the hypothesis of Theorem 5.1, $\tau_w(T_2 \backslash a_2) > 0$. So if $T_2 = F_0$, then $(b_2, a_2) = (v_1, v_2)$ and hence $T_2 \backslash a_2$ is a triangle. It is then routine to check that, for each $T_2 \in \{F_0, F_2, F_6\}$, there is a unique cycle contained in $T_2\backslash a_2$, which is a triangle. Therefore $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 4.8. ∎

Lemma 5.3. *If $T_2 = F_3$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. It is routine to check that

• $\mathcal{C}_2 = \{v_1v_3v_4v_1, v_1v_3v_5v_1, v_1v_3v_4v_5v_1\}$ and

• $\mathcal{F}_2 = \{\{v_1v_3\}, \{v_3v_4, v_3v_5\}, \{v_3v_4, v_5v_1\}, \{v_4v_1, v_5v_1\}, \{v_3v_5, v_4v_1, v_4v_5\}\}$.

We also have a computer verification of these results. So $|\mathcal{C}_2| = 3$ and $|\mathcal{F}_2| = 5$. Recall that $(b_2, a_2) = (v_5, v_2)$.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(1) $y(\mathcal{C}_2)$ is maximized;

(2) subject to (1), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically; and

(3) subject to (1) and (2), $y(v_1v_3v_5v_1)$ is minimized.

Observe that

(4) if $K \in \mathcal{F}_2$ satisfies $y(\mathcal{C}_2) = w(K)$, then K is an MFAS.

Indeed, since $y(\mathcal{C}_2) = \nu_w^*(F_3\backslash v_2)$, we have $w(K) = \nu_w^*(F_3\backslash v_2) \le \tau_w(F_3\backslash v_2) \le w(K)$. So $w(K) = \tau_w(F_3\backslash v_2)$.

Claim 1. $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$.

To justify this, observe that v_1v_3 is a special arc of T and v_1 is a near-sink. By Lemma 4.6(iv), we may assume that v_1v_3 is saturated by $\boldsymbol{y}$ in T. If v_1v_3 is outside $\mathcal{C}_0^y$, then v_1v_3 is saturated by $\boldsymbol{y}$ in F_3. Thus $y(\mathcal{C}_2) = w(v_1v_3)$. By (4), $\{v_1v_3\}$ is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$. So we assume that v_1v_3 is contained in some cycle $C \in \mathcal{C}_0^y$; subject to this, C is chosen to have the maximum number of arcs in $F_3\backslash v_2$. Depending on whether C passes through v_4v_1, we consider two cases.

• C contains v_4v_1. In this case, C contains the path $v_4v_1v_3v_5$. Applying Lemma 4.7(ii) to the triangles $v_1v_3v_4v_1$ and $v_1v_3v_5v_1$ respectively, we see that both v_3v_4 and v_5v_1 are outside $\mathcal{C}_0^y$. By Lemma 4.7(iv), both v_3v_4 and v_5v_1 are saturated by $\boldsymbol{y}$ in F_3. Moreover, $y(v_1v_3v_4v_5v_1) = 0$, for otherwise, by Lemma 4.7(v), v_3v_5 is saturated by $\boldsymbol{y}$ in F_3, contradicting the fact that $v_3v_5 \in C$. So $y(v_1v_3v_4v_1) = w(v_3v_4)$, $y(v_1v_3v_5v_1) = w(v_5v_1)$, and $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_4, v_5v_1\}$. By (4), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$.

• C does not contain v_4v_1. In this case, we may assume that v_4v_1 is outside $\mathcal{C}_0^y$, for otherwise, let D be a cycle in $\mathcal{C}_0^y$ passing through v_4v_1. Then D contains the path $v_1v_3v_5$. Replacing C by D, we see that the previous case occurs. Since C contains v_1v_3, it also contains v_3v_4 or v_3v_5. If C contains v_3v_4, then it contains the path $v_1v_3v_4v_5$. Using Lemma 4.7(ii) and (iv) and the cycles $v_1v_3v_4v_1$ and $v_1v_3v_4v_5v_1$, we see that both v_4v_1 and v_5v_1 are saturated by $\boldsymbol{y}$ in F_3. So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_5v_1\}$. Using (4), we obtain $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$. If C contains v_3v_5, then v_5v_1 is saturated by $\boldsymbol{y}$ in F_3 by Lemma 4.7(ii) and (iv). Thus we may assume that v_4v_1 is not saturated by $\boldsymbol{y}$ in F_3, otherwise we are done (as shown above). It follows from Lemma 4.7(v) that $y(v_1v_3v_4v_5v_1) = 0$, and from Lemma 4.7(ii) and (iv) (using the triangle $v_1v_3v_4v_1$) that v_3v_4 is outside $\mathcal{C}_0^y$. So, by Lemma 4.7(iii), v_3v_4 is saturated by $\boldsymbol{y}$ in F_3. Since $y(\mathcal{C}_2) = w(J)$, where $J = \{v_3v_4, v_5v_1\}$, Claim 1 is justified by (4).

Claim 2. $y(C)$ is an integer for each $C \in \mathcal{C}_2$.

To justify this, observe that $y(v_1v_3v_4v_5v_1) = 0$, for otherwise, by Lemma 4.7(v), both v_4v_1 and v_3v_5 are saturated by $\boldsymbol{y}$ in F_3. So $y(v_1v_3v_4v_1) = w(v_4v_1)$ and $y(v_1v_3v_5v_1) = w(v_3v_5)$; both of them are integers. By Claim 1, $y(v_1v_3v_4v_5v_1)$ is also integral, as desired.

From the proof of Claim 1, we see that one of the following three cases occurs:

- $y(v_1v_3v_4v_1) + y(v_1v_3v_5v_1) = w(v_1v_3)$;
- $y(v_1v_3v_4v_1) = w(v_3v_4)$ and $y(v_1v_3v_5v_1) = w(v_5v_1)$; and
- $y(v_1v_3v_4v_1) = w(v_4v_1)$ and $y(v_1v_3v_5v_1) = w(v_5v_1)$.

Thus the desired statement holds trivially in the second and third cases. It remains to consider the first case.

Suppose on the contrary that neither $y(v_1v_3v_4v_1)$ nor $y(v_1v_3v_5v_1)$ is an integer. Then $[y(v_1v_3v_4v_1)] + [y(v_1v_3v_5v_1)] = 1$. By the hypothesis of the present case, v_1v_3 is saturated by $\boldsymbol{y}$ in F_3, so v_4v_1 is outside $\mathcal{C}_0^y$. Thus

$$w(v_4v_1) \geq \lceil y(v_1v_3v_4v_1) \rceil = \lfloor y(v_1v_3v_4v_1) \rfloor + 1 = y(v_1v_3v_4v_1) + [y(v_1v_3v_5v_1)].$$

We propose to show that

(5) v_3v_4 is saturated by $\boldsymbol{y}$ in F_3.

Suppose not. If v_3v_4 is unsaturated in T, set $\theta = \min\{w(v_3v_4) - z(v_3v_4), [y(v_1v_3v_5v_1)]\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_3v_4v_1)$ and $y(v_1v_3v_5v_1)$ with $y(v_1v_3v_4v_1) + \theta$ and $y(v_1v_3v_5v_1) - \theta$, respectively; if v_3v_4 is saturated in T and contained in some $C \in \mathcal{C}_0^y$, set $\theta = \min\{y(C), [y(v_1v_3v_5v_1)]\}$ and $C' = C[v_5, v_3] \cup \{v_3v_5\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_3v_4v_1)$, $y(v_1v_3v_5v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_3v_4v_1)+\theta$, $y(v_1v_3v_5v_1)-\theta$, $y(C)-\theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_3v_5v_1) < y(v_1v_3v_5v_1)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (3) on $\boldsymbol{y}$. So (5) is established.

By (5), we have $y(v_1v_3v_4v_1) = w(v_3v_4)$ and $y(v_1v_3v_5v_1) = w(v_1v_3) - w(v_3v_4)$; both of them are integers. This contradiction proves Claim 2.

Since $\tau_w(F_3 \backslash v_2) > 0$, by Claims 1 and 2, $y(C)$ is a positive integer for some $C \in \mathcal{C}_2$. Thus, by Lemma 4.4(iii), $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. ∎

Lemma 5.4. *If $T_2 = F_4$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. It is routine to check that

- $\mathcal{C}_2 = \{v_1v_2v_3v_1, v_2v_3v_4v_2, v_1v_5v_3v_1, v_3v_4v_5v_3, v_1v_2v_3v_4v_1, v_1v_5v_2v_3v_1, v_1v_5v_3v_4v_1, v_2v_3v_4v_5v_2,$ $v_1v_5v_2v_3v_4v_1\}$ and
- $\mathcal{F}_2 = \{\{v_2v_3, v_5v_3\}, \{v_3v_1, v_3v_4\}, \{v_1v_2, v_1v_5, v_3v_4\}, \{v_1v_5, v_2v_3, v_3v_4\}, \{v_1v_5, v_2v_3, v_4v_5\},$ $\{v_1v_2, v_1v_5, v_4v_2, v_4v_5\}, \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}, \{v_1v_2, v_4v_2, v_5v_2, v_5v_3\},$ $\{v_2v_3, v_3v_1, v_4v_1, v_4v_5\}, \{v_3v_1, v_4v_1, v_4v_2, v_4v_5\}, \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}\}$.

We also have a computer verification of these results. So $|\mathcal{C}_2| = 9$ and $|\mathcal{F}_2| = 11$. Recall that $(b_2, a_2) = (v_5, v_6)$.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(1) $y(\mathcal{C}_2)$ is maximized;

(2) subject to (1), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically;

(3) subject to (1) and (2), $y(v_1v_5v_2v_3v_1) + y(v_1v_5v_3v_4v_1)$ is minimized;

(4) subject to (1)-(3), $y(v_2v_3v_4v_5v_2)$ is minimized;

(5) subject to (1)-(4), $y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3)$ is minimized; and

(6) subject to (1)-(5), $y(v_1v_5v_3v_1)$ is minimized.

Let us make some simple observations about $\boldsymbol{y}$.

(7) If $K \in \mathcal{F}_2$ satisfies $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 5.3.)

(8) If $y(v_1v_5v_2v_3v_4v_1) > 0$, then each arc in the set $\{v_1v_2, v_3v_1, v_4v_2, v_4v_5, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. Furthermore, $y(v_1v_2v_3v_1) = y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$.

To justify this, note that each arc in the given set is a chord of the cycle $v_1v_5v_2v_3v_4v_1$. So the first half follows instantly from Lemma 4.7(v). Let $\uplus$ stand for the multiset sum. Then $v_1v_5v_2v_3v_4v_1 \uplus v_1v_2v_3v_1 = v_1v_5v_2v_3v_1 \uplus v_1v_2v_3v_4v_1$, $v_1v_5v_2v_3v_4v_1 \uplus v_1v_5v_3v_1 = v_1v_5v_2v_3v_1 \uplus v_1v_5v_3v_4v_1$, and $v_1v_5v_2v_3v_4v_1 \uplus v_3v_4v_5v_3 = v_1v_5v_3v_4v_1 \uplus v_2v_3v_4v_5v_2$. Suppose on the contrary that $y(v_1v_2v_3v_1) > 0$. Let $\theta = \min\{y(v_1v_5v_2v_3v_4v_1), y(v_1v_2v_3v_1)\}$ and let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_2v_3v_1)$, $y(v_1v_5v_2v_3v_1)$, and $y(v_1v_2v_3v_4v_1)$ with $y(v_1v_5v_2v_3v_4v_1)-\theta$, $y(v_1v_2v_3v_1)-\theta$, $y(v_1v_5v_2v_3v_1)+\theta$, and $y(v_1v_2v_3v_4v_1)+\theta$. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_5v_2v_3v_4v_1) < y(v_1v_5v_2v_3v_4v_1)$, the existence of $\boldsymbol{y'}$ contradicts the assumption (2) on $\boldsymbol{y}$. So $y(v_1v_2v_3v_1) = 0$. Similarly, $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$.

(9) If $y(v_1v_5v_2v_3v_1) > 0$, then v_1v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4; so is v_4v_5 provided $y(v_1v_2v_3v_4v_1) > 0$. Furthermore, $y(v_3v_4v_5v_3) = 0$.

To justify this, note that both v_1v_2 and v_5v_3 are chords of the cycle $v_1v_5v_2v_3v_1$, so they are saturated by $\boldsymbol{y}$ in F_4 by Lemma 4.7(v). Since $v_1v_5v_2v_3v_1 \uplus v_3v_4v_5v_3 = v_1v_5v_3v_1 \uplus v_2v_3v_4v_5v_2$, from (3) we deduce that $y(v_3v_4v_5v_3) = 0$ (for a proof, see that of (8)).

Consider the case when $y(v_1v_2v_3v_4v_1) > 0$. If v_4v_5 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles $v_1v_5v_2v_3v_1$, $v_1v_2v_3v_4v_1$, and the arc v_4v_5 contains two arc-disjoint cycles $v_1v_2v_3v_1$ and $v_2v_3v_4v_5v_2$; if v_4v_5 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C \in \mathcal{C}_0^y$, then the multiset sum of $v_1v_5v_2v_3v_1$, $v_1v_2v_3v_4v_1$, and C contains three arc-disjoint cycles $v_1v_2v_3v_1$, $v_2v_3v_4v_5v_2$, and $C' = C[v_5, v_4] \cup \{v_4v_1, v_1v_5\}$. In either subcase we can obtain from $\boldsymbol{y}$ an optimal solution $\boldsymbol{y'}$ to $\mathbb{D}(T, \boldsymbol{w})$ that is better than $\boldsymbol{y}$ by (2). So v_4v_5 is saturated by $\boldsymbol{y}$ in F_4.

(10) If $y(v_1v_5v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4; so is v_4v_2 provided $y(v_1v_5v_2v_3v_1) > 0$, and so is v_1v_2 provided $y(v_2v_3v_4v_5v_2) > 0$. Furthermore, $y(v_1v_2v_3v_1) = 0$.

To justify this, note that both v_3v_1 and v_4v_5 are chords of the cycle $v_1v_5v_3v_4v_1$, so they are saturated by $\boldsymbol{y}$ in F_4 by Lemma 4.7(v). Since $v_1v_5v_3v_4v_1 \uplus v_1v_2v_3v_1 = v_1v_5v_3v_1 \uplus v_1v_2v_3v_4v_1$, from (3) we deduce that $y(v_1v_2v_3v_1) = 0$ (for a proof, see that of (8)).

Consider the case when $y(v_1v_5v_2v_3v_1) > 0$. If v_4v_2 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles $v_1v_5v_2v_3v_1$, $v_1v_5v_3v_4v_1$, and the arc v_4v_2 contains arc-disjoint cycles $v_1v_5v_3v_1$ and $v_2v_3v_4v_2$; if v_4v_2 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C_1 \in \mathcal{C}_0^y$, then the multiset sum of C_1, $v_1v_5v_2v_3v_1$, and $v_1v_5v_3v_4v_1$ contains three arc-disjoint cycles $v_1v_5v_3v_1$, $v_2v_3v_4v_2$, and $C_1' = C_1[v_5, v_4] \cup \{v_4v_1, v_1v_5\}$. In either subcase we can obtain from $\boldsymbol{y}$ an optimal solution $\boldsymbol{y'}$ to $\mathbb{D}(T, \boldsymbol{w})$ that is better than $\boldsymbol{y}$ by (2). So v_4v_5 is saturated by $\boldsymbol{y}$ in F_4.

Next, consider the case when $y(v_2v_3v_4v_5v_2) > 0$. If v_1v_2 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles $v_1v_5v_3v_4v_1$, $v_2v_3v_4v_5v_2$, and the arc v_1v_2 contains arc-disjoint cycles $v_3v_4v_5v_3$ and $v_1v_2v_3v_4v_1$; if v_1v_2 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C_2 \in \mathcal{C}_0^y$, then the multiset sum of C_2, $v_2v_3v_4v_5v_2$, and $v_1v_5v_3v_4v_1$ contains three arc-disjoint cycles $v_3v_4v_5v_3$, $v_1v_2v_3v_4v_1$, and $C_2' = C_2[v_5, v_1] \cup \{v_1v_5\}$. In either subcase we can obtain from $\boldsymbol{y}$ an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ that is better than $\boldsymbol{y}$ by (2). So v_1v_2 is saturated by $\boldsymbol{y}$ in F_4.

(11) If $y(v_1v_2v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_2 are saturated by $\boldsymbol{y}$ in F_4; so is v_4v_5 provided $y(v_1v_5v_3v_1) > 0$.

The first half follows instantly from Lemma 4.7(v). Suppose $y(v_1v_5v_3v_1) > 0$. If v_4v_5 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles $v_1v_5v_3v_1$, $v_1v_2v_3v_4v_1$, and the arc v_4v_5 contains arc-disjoint cycles $v_1v_2v_3v_1$ and $v_3v_4v_5v_3$; if v_4v_5 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C \in \mathcal{C}_0^y$, then the multiset sum of $v_1v_2v_3v_4v_1$, $v_1v_5v_3v_1$, and C contains three arc-disjoint cycles $v_1v_2v_3v_1$, $v_3v_4v_5v_3$, and $C' = C[v_5, v_4] \cup \{v_4v_1, v_1v_5\}$. In either subcase we can obtain from $\boldsymbol{y}$ an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ that is better than $\boldsymbol{y}$ by (2). So v_4v_5 is saturated by $\boldsymbol{y}$ in F_4.

(12) If $y(v_2v_3v_4v_5v_2) > 0$, then both v_4v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4; so is v_1v_2 provided $y(v_1v_5v_3v_1) > 0$.

The first half follows instantly from Lemma 4.7(v). Suppose $y(v_1v_5v_3v_1) > 0$. If v_1v_2 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles $v_1v_5v_3v_1$, $v_2v_3v_4v_5v_2$, and the arc v_1v_2 contains arc-disjoint cycles $v_1v_2v_3v_1$ and $v_3v_4v_5v_3$; if v_1v_2 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C \in \mathcal{C}_0^y$, then the multiset sum of C, $v_2v_3v_4v_5v_2$, and $v_1v_5v_3v_1$ contains three arc-disjoint cycles $v_3v_4v_5v_3$, $v_1v_2v_3v_1$, and $C' = C[v_5, v_1] \cup \{v_1v_5\}$. In either subcase we can obtain from $\boldsymbol{y}$ an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ that is better than $\boldsymbol{y}$ by (2). So v_1v_2 is saturated by $\boldsymbol{y}$ in F_4.

Claim 1. $y(\mathcal{C}_2) = \tau_w(F_4 \backslash v_6)$.

To justify this, observe that v_2v_3 is a special arc of T and v_2 is a near-sink. By Lemma 4.6(iv), we may assume that v_2v_3 is saturated by $\boldsymbol{y}$ in T. Depending on whether v_2v_3 is outside $\mathcal{C}_0^y$, we distinguish between two cases.

Case 1.1. v_2v_3 is contained in some cycle in $\mathcal{C}_0^y$.

Choose $C \in \mathcal{C}_0^y$ that contains v_2v_3 and, subject to this, has the maximum number of arcs in $F_4 \backslash v_6$. We proceed by considering three subcases.

• C contains v_1v_2. In this subcase, C contains the path $P = v_1v_2v_3v_4v_5$. By Lemma 4.7(ii) and (iv), each arc in the set $K = \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. Since no arc on C (and hence on P) is saturated by $\boldsymbol{y}$ in F_4, we have $y(v_1v_5v_2v_3v_4v_1) = y(v_1v_5v_2v_3v_1) = y(v_1v_5v_3v_4v_1) = 0$ by (8) $-$ (10). Since the multiset sum of $v_1v_5v_3v_1$ and C contains three arc-disjoint cycles $v_1v_2v_3v_1$, $v_3v_4v_5v_3$, and $C' = C[v_5, v_1] \cup \{v_1v_5\}$, from the optimality of $\boldsymbol{y}$, we deduce that $y(v_1v_5v_3v_1) = 0$. So $y(\mathcal{C}_2) = w(K)$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$.

• C contains v_4v_2. In this subcase, C contains the path $P = v_4v_2v_3v_1v_5$. By Lemma 4.7(ii) and (iv), each arc in the set $K = \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. Since no arc on C (and hence on P) is saturated by $\boldsymbol{y}$ in F_4, $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_5v_3v_4v_1)$, $y(v_1v_2v_3v_4v_1)$, and $y(v_2v_3v_4v_5v_2)$ are all 0 by (8) and (10)-(12). Since the multiset sum of $v_3v_4v_5v_3$ and C contains three arc-disjoint cycles $v_1v_5v_3v_1$, $v_2v_3v_4v_2$, and $C' = C[v_5, v_4] \cup \{v_4v_5\}$, from the optimality of $\boldsymbol{y}$, we deduce that $y(v_3v_4v_5v_3) = 0$. So $y(\mathcal{C}_2) = w(K)$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3 \backslash v_2)$.

• C contains neither v_1v_2 nor v_4v_2. In this subcase, we may assume that both v_1v_2 and v_4v_2 are outside $\mathcal{C}_0^y$, for otherwise, each cycle containing v_1v_2 or v_4v_2 passes through v_2v_3, and thus one of the preceding subcases occurs. Clearly, C contains v_3v_4 or v_3v_1.

Assume first that C contains v_3v_4. If C contains v_4v_1, then it also contains v_1v_5. By Lemma 4.7(ii) and (iv), each arc in the set $K = \{v_1v_2, v_4v_2, v_5v_2, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. So $y(\mathcal{C}_2) = w(K)$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$. If C does not contain v_4v_1, then C contains v_4v_5. By Lemma 4.7(ii) and (iv), each arc in the set $\{v_4v_2, v_5v_2, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. If v_1v_2 is also saturated by $\boldsymbol{y}$ in F_4, then $y(\mathcal{C}_2) = w(K)$, where K is as defined above. Again, K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$. So we assume that v_1v_2 is not saturated by $\boldsymbol{y}$ in T. Since v_1v_2 is outside $\mathcal{C}_0^y$, so are v_4v_1 and v_3v_1. By Lemma 4.7(iii), both v_4v_1 and v_3v_1 are saturated by $\boldsymbol{y}$ in T and hence in F_4. Moreover, by (8)-(10), $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_5v_2v_3v_1)$, and $y(v_1v_5v_3v_4v_1)$ are all 0. Since the multiset sum of the cycles $v_1v_5v_3v_1$, C, and the unsaturated arc v_1v_2 contains two arc-disjoint cycles $v_1v_2v_3v_1$ and $v_3v_4v_5v_3$. By Lemma 4.7(vi), we have $y(v_1v_5v_3v_1) = 0$. So $y(\mathcal{C}_2) = w(J)$, where $J = \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}$. By (7), J is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$.

Assume next that C contains v_3v_1. Then C contains v_1v_5. By Lemma 4.7(ii) and (iv), each arc in the set $\{v_1v_2, v_5v_2, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. If v_4v_2 is also saturated by $\boldsymbol{y}$ in F_4, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_4v_2, v_5v_2, v_5v_3\}$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$. So we assume that v_4v_2 is not saturated by $\boldsymbol{y}$ in F_4 and hence in T (recall that v_4v_2 is outside $\mathcal{C}_0^y$). By Lemma 4.7(iv), v_3v_4 is outside $\mathcal{C}_0^y$. By Lemma 4.7(iii), v_3v_4 is saturated by $\boldsymbol{y}$ in T and hence in F_4. By (8) and (10)-(12), $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_5v_3v_4v_1)$, $y(v_1v_2v_3v_4v_1)$, and $y(v_2v_3v_4v_5v_2)$ are all 0. Since the multiset sum of the cycles $v_3v_4v_5v_3$, C, and the unsaturated arc v_4v_2 contains two arc-disjoint cycles $v_1v_5v_3v_1$ and $v_2v_3v_4v_2$, we have $y(v_3v_4v_5v_3) = 0$ by Lemma 4.7(vi). So $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$.

Case 1.2. v_2v_3 is outside $\mathcal{C}_0^y$.

By the previous observation, v_2v_3 is saturated by $\boldsymbol{y}$ in F_4 now. Note also that v_5v_3 is outside $\mathcal{C}_0$. If v_5v_3 is saturated by $\boldsymbol{y}$ in T, so is it in F_4, and hence $y(\mathcal{C}_2) = w(K)$, where $K = \{v_2v_3, v_5v_3\}$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$. So we assume that v_5v_3 is unsaturated. By (8), (9), and (12), $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_5v_2v_3v_1)$, and $y(v_2v_3v_4v_5v_2)$ are all 0. Observe that both v_3v_1 and v_3v_4 are outside $\mathcal{C}_0^y$, for otherwise, since each cycle passing through v_3v_1 or v_3v_4 contains v_1v_5 or v_4v_5, from Lemma 4.7(iv) we deduce that v_5v_3 is saturated, a contradiction. If both v_3v_1 and v_3v_4 are saturated by $\boldsymbol{y}$ in F_4, then $y(\mathcal{C}_2) = w(J)$, where $J = \{v_3v_1, v_3v_4\}$. By (7), J is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$. So we assume that

(13) at most one of v_3v_1 and v_3v_4 is saturated by $\boldsymbol{y}$ in F_4.

Since $\mathcal{C}_0^y \neq \emptyset$, there is a cycle $C \in \mathcal{C}_0^y$ passing through v_4v_1, or v_1v_5, or v_4v_5; subject to this, let C be chosen to have the maximum number of arcs in $F_4\backslash v_6$. We proceed by considering three subcases.

- C contains both v_4v_1 and v_1v_5. In this subcase, since v_5v_3 is unsaturated, by Lemma 4.7(iii), v_3v_1 and v_3v_4 are both saturated by $\boldsymbol{y}$ in F_4, a contradiction.
- C contains v_1v_5 but not v_4v_1. In this subcase, from the choice of C, we see that v_4v_1 is outside $\mathcal{C}_0^y$, because every cycle containing v_4v_1 passes through v_1v_5. Since v_5v_3 is unsaturated, Lemma 4.7(iii) implies that v_3v_1 is saturated by $\boldsymbol{y}$ in F_4, and thus v_3v_4 is not saturated by $\boldsymbol{y}$ in F_4 and hence in T by (13). Once again, by Lemma 4.7(iii), v_4v_1 is saturated by $\boldsymbol{y}$ in F_4, and v_4v_5 is outside $\mathcal{C}_0^y$. Since both v_5v_3 and v_3v_4 are unsaturated, it follows from Lemma 4.7(i) that v_4v_5 is saturated by $\boldsymbol{y}$ in F_4. If v_4v_2 is also saturated by $\boldsymbol{y}$ in F_4, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_2, v_4v_5\}$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_4\backslash v_6)$. If v_4v_2 is not

saturated by $\boldsymbol{y}$ in F_4, then $y(v_1v_2v_3v_4v_1) = 0$ by (11). Moreover, since the multiset sum of the cycles $v_1v_2v_3v_1$, C, and the unsaturated arcs v_5v_3, v_3v_4, and v_4v_2 contains two arc-disjoint cycles $v_2v_3v_4v_2$ and $v_1v_5v_3v_1$, we have $y(v_1v_2v_3v_1) = 0$ by Lemma 4.7(vi). Therefore, $y(\mathcal{C}_2) = w(J)$, where $J = \{v_2v_3, v_3v_1, v_4v_1, v_4v_5\}$. By (7), J is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$.

• C contains v_4v_5. In this subcase, we may assume that both v_4v_1 and v_1v_5 are outside $\mathcal{C}_0^y$, otherwise one of the preceding subcases occurs. By Lemma 4.7(iii), v_3v_4 is saturated by $\boldsymbol{y}$ in T and hence in F_4, which together with (13) implies that v_3v_1 is not saturated by $\boldsymbol{y}$ in F_4. Using (10) and (11), we deduce that $y(v_1v_5v_3v_4v_1) = y(v_1v_2v_3v_4v_1) = 0$. Using Lemma 4.7(iii) and the triangle $v_1v_5v_3v_1$, we see that v_1v_5 is outside $\mathcal{C}_0^y$. Using Lemma 4.7(i) and the triangle $v_1v_5v_3v_1$, we also deduce that v_1v_5 is saturated by $\boldsymbol{y}$ in T and hence in F_4. If v_1v_2 is also saturated by $\boldsymbol{y}$ in F_4, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_1v_5, v_3v_4\}$. By (7), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_4\backslash v_6)$. So we assume that v_1v_2 is not saturated by $\boldsymbol{y}$ in F_4 and hence in T, because v_1v_2 is outside $\mathcal{C}_0^y$, by the hypothesis of the present case. Since the multiset sum of the cycles C, $v_2v_3v_4v_2$, and unsaturated arcs v_5v_3, v_3v_1, and v_1v_2 contains two arc-disjoint cycles $v_1v_2v_3v_1$ and $v_3v_4v_5v_3$, we have $y(v_2v_3v_4v_2) = 0$ by Lemma 4.7(vi). It follows that $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1v_5, v_2v_3, v_3v_4\}$. By (7), J is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_4\backslash v_6)$. This completes the proof of Claim 1.

Claim 2. $y(C)$ is integral for all $C \in \mathcal{C}_2$ or $\nu_w^*(T)$ is an integer.

To justify this, let $\mathcal{G}_2 = \mathcal{F}_2\backslash\{\{v_1v_5, v_2v_3, v_4v_5\}, \{v_1v_2, v_1v_5, v_4v_2, v_4v_5\}\}$. From the proof of Claim 1, we see that $y(\mathcal{C}_2) = w(K)$ for some $K \in \mathcal{G}_2$. Observe that if $y(\mathcal{C}_2) = w(J)$ for $J = \{v_1v_5, v_2v_3, v_4v_5\}$ or $\{v_1v_2, v_1v_5, v_4v_2, v_4v_5\}$, then both v_1v_5 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4, so $\mathcal{C}_0^y = \emptyset$ in this case, which has been excluded by Lemma 4.4(ii).

Let us make some further observations about $\boldsymbol{y}$.

(14) $y(v_1v_5v_2v_3v_4v_1) = 0$.

Suppose on the contrary that $y(v_1v_5v_2v_3v_4v_1) > 0$. By (8), we have $y(v_1v_2v_3v_1) = y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$, and each arc in the set $\{v_1v_2, v_3v_1, v_4v_2, v_4v_5, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in F_4. So $y(\mathcal{C}_2(v_1v_2)) = w(v_1v_2)$, $y(\mathcal{C}_2(v_3v_1)) = w(v_3v_1)$, $y(\mathcal{C}_2(v_4v_2)) = w(v_4v_2)$, $y(\mathcal{C}_2(v_4v_5)) = w(v_4v_5)$, and $y(\mathcal{C}_2(v_5v_3)) = w(v_5v_3)$. It follows that $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$, $y(v_1v_5v_2v_3v_1) = w(v_3v_1)$, $y(v_2v_3v_4v_2) = w(v_4v_2)$, $y(v_2v_3v_4v_5v_2) = w(v_4v_5)$, and $y(v_1v_5v_3v_4v_1) = w(v_5v_3)$. From Claim 1 we deduce that $y(v_1v_5v_2v_3v_4v_1)$ is also integral, and hence $\nu_w^*(T)$ is an integer by Lemma 4.4(iii).

(15) $y(v_1v_5v_2v_3v_1)$ or $y(v_1v_5v_3v_4v_1)$ is 0.

Assume the contrary: both $y(v_1v_5v_2v_3v_1)$ and $y(v_1v_5v_3v_4v_1)$ are positive. By (9) and (10), we have $y(v_1v_2v_3v_1) = y(v_3v_4v_5v_3) = 0$, and each arc in the set $\{v_1v_2, v_5v_3, v_3v_1, v_4v_2, v_4v_5\}$ is saturated by $\boldsymbol{y}$ in F_4. So $y(\mathcal{C}_2(v_1v_2)) = w(v_1v_2)$, $y(\mathcal{C}_2(v_5v_3)) = w(v_5v_3)$, $y(\mathcal{C}_2(v_3v_1)) = w(v_3v_1)$, $y(\mathcal{C}_2(v_4v_2)) = w(v_4v_2)$, and $y(\mathcal{C}_2(v_4v_5)) = w(v_4v_5)$. It follows that $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$, $y(v_2v_3v_4v_2) = w(v_4v_2)$, $y(v_2v_3v_4v_5v_2) = w(v_4v_5)$, $y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1) = w(v_3v_1)$, and $y(v_1v_5v_3v_1) + y(v_1v_5v_3v_4v_1) = w(v_5v_3)$. Given the above equations and (14), to prove that $y(C)$ is integral for all $C \in \mathcal{C}_2$, it suffices to show that one of $y(v_1v_5v_3v_4v_1)$, $y(v_1v_5v_2v_3v_1)$, and $y(v_1v_5v_3v_1)$ is integral.

By Lemma 4.3 and Claim 1, each arc $e \in K$ satisfies $w(e) = z(e) = y(\mathcal{C}_2(e))$. Let us proceed by considering four subcases.

If $v_2v_3 \in K$, then $w(v_2v_3) = y(\mathcal{C}_2(v_2v_3)) = y(v_2v_3v_4v_2) + y(v_1v_2v_3v_4v_1) + y(v_1v_5v_2v_3v_1)$ $+ y(v_2v_3v_4v_5v_2)$, which implies that $y(v_1v_5v_2v_3v_1)$ is integral.

If $v_3v_4 \in K$, then $w(v_3v_4) = y(\mathcal{C}_2(v_3v_4)) = y(v_2v_3v_4v_2) + y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_5v_2)$ $+ y(v_1v_5v_3v_4v_1)$, which implies that $y(v_1v_5v_3v_4v_1)$ is integral.

If $v_4v_1 \in K$, then $w(v_4v_1) = y(\mathcal{C}_2(v_4v_1)) = y(v_1v_2v_3v_4v_1) + y(v_1v_5v_3v_4v_1)$, which implies that $y(v_1v_5v_3v_4v_1)$ is integral.

If $v_5v_2 \in K$, then $w(v_5v_2) = y(\mathcal{C}_2(v_5v_2)) = y(v_1v_5v_2v_3v_1) + y(v_2v_3v_4v_5v_2)$, which implies that $y(v_1v_5v_2v_3v_1)$ is integral.

Since each $K \in \mathcal{G}_2$ contains at least one arc in the set $\{v_2v_3, v_3v_4, v_4v_1, v_5v_2\}$, it follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$. So $y(v_1v_5v_2v_3v_1)$ is a positive integer, and hence $\nu_w^*(T)$ is an integer by Lemma 4.4(iii). Therefore we may assume that (15) holds.

Depending on what $K \in \mathcal{G}_2$ is, we distinguish among nine cases.

Case 2.1. $K = \{v_1v_5, v_2v_3, v_3v_4\}$.

In this case, by Lemma 4.3(i) and (iii), we have $y(v_2v_3v_4v_2) = y(v_1v_2v_3v_4v_1) = y(v_2v_3v_4v_5v_2) = y(v_1v_5v_2v_3v_1) = y(v_1v_5v_3v_4v_1) = 0$ and $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields $w(v_1v_5) = y(\mathcal{C}_2(v_1v_5)) = y(v_1v_5v_3v_1)$, $w(v_2v_3) = y(\mathcal{C}_2(v_2v_3)) = y(v_1v_2v_3v_1)$, and $w(v_3v_4) = y(\mathcal{C}_2(v_3v_4)) = y(v_3v_4v_5v_3)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.2. $K = \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}$.

In this case, by Lemma 4.3(i) and (iii), we have $y(v_1v_5v_3v_4v_1) = y(v_3v_4v_5v_3) = y(v_1v_2v_3v_4v_1) = y(v_2v_3v_4v_5v_2) = 0$, which together with (14) yields $w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_1)$, $w(v_3v_4) = y(\mathcal{C}_2(v_3v_4)) = y(v_2v_3v_4v_2)$, $w(v_5v_2) = y(\mathcal{C}_2(v_5v_2)) = y(v_1v_5v_2v_3v_1)$, and $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.3. $K = \{v_2v_3, v_3v_1, v_4v_1, v_4v_5\}$.

In this case, by Lemma 4.3(i) and (iii), we have $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = y(v_1v_2v_3v_4v_1)$ $= y(v_2v_3v_4v_5v_2) = 0$, which together with (14) yields $w(v_2v_3) = y(\mathcal{C}_2(v_2v_3)) = y(v_2v_3v_4v_2)$, $w(v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = y(v_1v_5v_3v_1)$, $w(v_4v_1) = y(\mathcal{C}_2(v_4v_1)) = y(v_1v_5v_3v_4v_1)$, and $w(v_4v_5) = y(\mathcal{C}_2(v_4v_5)) = y(v_3v_4v_5v_3)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.4. $K = \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}$.

In this case, by Lemma 4.3(i) and (iii), we have $y(v_1v_5v_3v_1) = y(v_1v_5v_2v_3v_1) = y(v_1v_5v_3v_4v_1) = 0$, which together with (14) yields $w(v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = y(v_1v_2v_3v_1)$, $w(v_4v_1) = y(\mathcal{C}_2(v_4v_1)) = y(v_1v_2v_3v_4v_1)$, $w(v_4v_2) = y(\mathcal{C}_2(v_4v_2)) = y(v_2v_3v_4v_2)$, $w(v_5v_2) = y(\mathcal{C}_2(v_5v_2)) = y(v_2v_3v_4v_5v_2)$, and $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_3v_4v_5v_3)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.5. $K = \{v_1v_2, v_1v_5, v_3v_4\}$.

In this case, by Lemma 4.3(i) and (iii), we have $y(v_1v_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$ and $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields the following three equations:

$w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_1)$;

$w(v_1v_5) = y(\mathcal{C}_2(v_1v_5)) = y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1)$; and

$w(v_3v_4) = y(\mathcal{C}_2(v_3v_4)) = y(v_2v_3v_4v_2) + y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2)$.

Depending on the value of $y(v_1v_5v_2v_3v_1)$, we consider two subcases.

- $y(v_1v_5v_2v_3v_1) = 0$. In this subcase, $y(v_1v_5v_3v_1) = w(v_1v_5)$. If $y(v_2v_3v_4v_5v_2) > 0$, then $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3)$ and $w(v_4v_2) = y(\mathcal{C}_2(v_4v_2)) = y(v_2v_3v_4v_2)$ by (12). Thus both $y(v_3v_4v_5v_3)$ and $y(v_2v_3v_4v_5v_2)$ are integral, and hence $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_2v_3v_4v_5v_2) = 0$. Then $w(v_3v_4) = y(v_2v_3v_4v_2) + y(v_3v_4v_5v_3)$. If $y(v_2v_3v_4v_2)$ is an integer, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we further assume that $y(v_2v_3v_4v_2)$ is not integral. Thus $[y(v_2v_3v_4v_2)] + [y(v_3v_4v_5v_3)] = 1$. Since each arc in K is saturated by $\boldsymbol{y}$ in F_4, both v_2v_3 and v_4v_2 are outside $\mathcal{C}_0^y$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing

$y(v_2v_3v_4v_2)$ and $y(v_3v_4v_5v_3)$ with $y(v_2v_3v_4v_2) + \lceil y(v_3v_4v_5v_3)\rceil$ and $\lfloor y(v_3v_4v_5v_3)\rfloor$ respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_3v_4v_5v_3) < y(v_3v_4v_5v_3)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (5) on $\boldsymbol{y}$.

• $y(v_1v_5v_2v_3v_1) > 0$. In this subcase, $y(v_3v_4v_5v_3) = 0$ and v_5v_3 is saturated by $\boldsymbol{y}$ in F_4 by (9). So $w(v_3v_4) = y(v_2v_3v_4v_2) + y(v_2v_3v_4v_5v_2)$ and $w(v_5v_3) = y(v_1v_5v_3v_1)$. It follows that $y(v_1v_5v_2v_3v_1) = w(v_1v_5) - w(v_5v_3)$. If $y(v_2v_3v_4v_5v_2) = 0$, then $y(v_2v_3v_4v_2) = w(v_3v_4)$; otherwise, by (12), both v_1v_5 and v_4v_2 are saturated by $\boldsymbol{y}$ in F_4. Thus $y(v_2v_3v_4) = w(v_4v_2)$ and $y(v_2v_3v_4v_5v_2) = w(v_3v_4) - w(v_4v_2)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.6. $K = \{v_3v_1, v_4v_1, v_4v_2, v_4v_5\}$.

In this case, by Lemma 4.3 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields the following four equations:

$w(v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1)$;
$w(v_4v_1) = y(\mathcal{C}_2(v_4v_1)) = y(v_1v_2v_3v_4v_1) + y(v_1v_5v_3v_4v_1)$;
$w(v_4v_2) = y(\mathcal{C}_2(v_4v_2)) = y(v_2v_3v_4v_2)$; and
$w(v_4v_5) = y(\mathcal{C}_2(v_4v_5)) = y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2)$.

Depending on the values of $y(v_1v_5v_3v_4v_1)$ and $y(v_1v_5v_2v_3v_1)$, we consider three subcases.

• $y(v_1v_5v_3v_4v_1) > 0$. In this subcase, by (10) and (15), we have $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = 0$. So $y(v_1v_5v_3v_1) = w(v_3v_1)$. If $y(v_2v_3v_4v_5v_2) > 0$, then both v_1v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4 by (10) and (12). So $w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_4v_1)$ and $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1)$. Since $y(v_1v_5v_3v_4v_1) = w(v_4v_1) - y(v_1v_2v_3v_4v_1)$ and $y(v_2v_3v_4v_5v_2) = w(v_4v_5) - y(v_3v_4v_5v_3)$, it follows that $y(v_1v_5v_3v_4v_1)$, $y(v_3v_4v_5v_3)$, and $y(v_2v_3v_4v_5v_2)$ are all integral. So we assume that $y(v_2v_3v_4v_5v_2) = 0$. Then $y(v_3v_4v_5v_3) = w(v_4v_5)$. Since each arc in K is saturated by $\boldsymbol{y}$ in F_4, both v_1v_2 and v_2v_3 are outside $\mathcal{C}_0^y$. By Lemma 4.4(i), we may assume that $w(e) = \lceil z(e)\rceil$ for all arcs e in T. Thus, from (3) we deduce that $y(v_1v_2v_3v_4v_1) = \min\{w(v_1v_2), w(v_2v_3) - w(v_4v_2)\}$ and $y(v_1v_5v_3v_4v_1) = w(v_4v_1) - y(v_1v_2v_3v_4v_1)$. Therefore $y(C)$ is integral for all $C \in \mathcal{C}_2$.

• $y(v_1v_5v_2v_3v_1) > 0$. In this subcase, from (9) and (15), we deduce that $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_4v_1) = 0$, and that both v_1v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4. So $y(v_1v_2v_3v_4v_1) = w(v_4v_1)$ $y(v_2v_3v_4v_5v_2) = w(v_4v_5)$, $w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1)$, and $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1)$. Thus $y(v_1v_2v_3v_1) = w(v_1v_2) - w(v_4v_1)$ is integral, so is $y(v_1v_5v_2v_3v_1)$. Therefore $y(C)$ is integral for all $C \in \mathcal{C}_2$.

• $y(v_1v_5v_3v_4v_1) = y(v_1v_5v_2v_3v_1) = 0$. In this subcase, $y(v_1v_2v_3v_4v_1) = w(v_4v_1)$. Suppose $y(v_2v_3v_4v_5v_2) > 0$. Then v_5v_3 is saturated by $\boldsymbol{y}$ in F_4 by (12). So $w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3)$. If $y(v_1v_5v_3v_1) > 0$, then v_1v_2 is saturated by $\boldsymbol{y}$ in F_4 by (12). So $w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1)$, It follows that $y(v_1v_2v_3v_1)$ and hence $y(C)$ is integral for any $C \in \mathcal{C}_2$. If $y(v_1v_5v_3v_1) = 0$, then $y(v_1v_2v_3v_1) = w(v_3v_1)$, which implies that $y(C)$ is integral for any $C \in \mathcal{C}_2$. So we assume that $y(v_2v_3v_4v_5v_2) = 0$. Then $y(v_3v_4v_5v_3) = w(v_4v_5)$. Observe that $y(v_1v_2v_3v_1)$ is integral, for otherwise, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_2v_3v_1)$ and $y(v_1v_5v_3v_1)$ with $y(v_1v_2v_3v_1) + \lceil y(v_1v_5v_3v_1)\rceil$ and $\lfloor y(v_1v_5v_3v_1)\rfloor$, respectively. Since v_1v_2 and v_2v_3 are outside $\mathcal{C}_0^y$, we see $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_5v_3v_1) < y(v_1v_5v_3v_1)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (5) on $\boldsymbol{y}$. From the above observation, it is easy to see that $y(C)$ is integral for any $C \in \mathcal{C}_2$.

Case 2.7. $K = \{v_1v_2, v_4v_2, v_5v_2, v_5v_3\}$.

In this case, by Lemma 4.3(iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields the following four equations:

$w(v_1v_2) = y(\mathcal{C}_2(v_1v_2)) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1)$;

$w(v_4v_2) = y(\mathcal{C}_2(v_4v_2)) = y(v_2v_3v_4v_2)$;

$w(v_5v_2) = y(\mathcal{C}_2(v_5v_2)) = y(v_1v_5v_2v_3v_1) + y(v_2v_3v_4v_5v_2)$; and

$w(v_5v_3) = y(\mathcal{C}_2(v_5v_3)) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1)$.

Depending on the values of $y(v_1v_5v_3v_4v_1)$ and $y(v_1v_5v_2v_3v_1)$, we consider three subcases.

• $y(v_1v_5v_3v_4v_1) > 0$. In this subcase, by (10) and (15), $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = 0$ and both v_3v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4. So $y(v_2v_3v_4v_2) = w(v_4v_2)$, $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$, $y(v_2v_3v_4v_5v_2) = y(\mathcal{C}_2(v_5v_2)) = w(v_5v_2)$, and $y(v_1v_5v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = w(v_3v_1)$. Thus $y(v_3v_4v_5v_3)$ and $y(v_1v_5v_3v_4v_1)$ are also integral.

• $y(v_1v_5v_2v_3v_1) > 0$. In this subcase, by (9) and (15), we have $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_4v_1) = 0$. So $y(v_1v_5v_3v_1) = w(v_5v_3)$. If $y(v_1v_2v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4 by (9) and (11). So $w(v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1)$ and $w(v_4v_5) = y(\mathcal{C}_2(v_4v_5)) = y(v_2v_3v_4v_5v_2)$. It follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_1)$, $y(v_2v_3v_4v_2)$, and $y(v_1v_5v_3v_1)$ are integral, and $y(v_1v_5v_2v_3v_1)+y(v_2v_3v_4v_5v_2) = w(v_5v_2)$. If $y(v_2v_3v_4v_5v_2)$ is an integer, then $y(C)$ is integral for any $C \in \mathcal{C}_2$. So we assume that $y(v_2v_3v_4v_5v_2)$ is not integral. We propose to show that

(16) $\nu^*_w(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. By Lemma 4.4(iii), we may assume that $w(v_1v_2) = w(v_4v_2) = w(v_5v_3) = 0$. Thus $y(C) = 0$ for all $C \in \mathcal{C}_2 \backslash \{v_1v_5v_2v_3v_1, v_2v_3v_4v_5v_2\}$. Observe that v_3v_4 is outside $\mathcal{C}^y_0$, for otherwise, let D be a cycle in $\mathcal{C}^y_0$ that contains v_3v_4. It is then easy to see that an optimal solution $\boldsymbol{y'}$ to $\mathbb{D}(T, \boldsymbol{w})$ can be obtained from $\boldsymbol{y}$ by modifying $y(D)$, $y(v_1v_5v_2v_3v_1)$, and $y(v_2v_3v_4v_5v_2)$ and by possibly rerouting D, so that $y'(v_1v_5v_2v_3v_1) < y(v_1v_5v_2v_3v_1)$, contradicting (3). Since $y(v_2v_3v_4v_5v_2) < w(v_3v_4)$, we have $x(v_3v_4) = 0$ by Lemma 4.3(ii). Since both $y(v_1v_5v_2v_3v_1)$ and $y(v_2v_3v_4v_5v_2)$ are positive, $x(v_3v_1) + x(v_1v_5) = x(v_3v_4) + x(v_4v_5)$ by Lemma 4.3(i). So $x(v_4v_5) = x(v_3v_1) + x(v_1v_5)$.

Let us show that if $w(v_4v_1) > 0$, then $x(v_4v_1) = x(v_3v_1)$. For this purpose, note that both v_4v_1 and v_4v_5 are contained in some cycles in $\mathcal{C}^y_0$, for otherwise, we can obtain a new optimal solution $\boldsymbol{y'}$ from $\boldsymbol{y}$ satisfying (1) and (2), but $y'(v_1v_5v_2v_3v_1) = \lfloor y(v_1v_5v_2v_3v_1) \rfloor$ and $y'(v_2v_3v_4v_5v_2) = y(v_2v_3v_4v_5v_2) + [y(v_1v_5v_2v_3v_1)]$, which again contradicts (3). Thus $x(v_4v_5) = x(v_1v_5)+x(v_4v_1)$ by Lemma 4.3(iii). Combining it with the equality established in the preceding paragraph, we obtain the $x(v_4v_1) = x(v_3v_1)$. If $w(v_4v_1) = 0$, then we may assume that $x(v_4v_1) = x(v_3v_1)$ (replacing the smaller of these two with the larger if necessary).

Similarly, we can prove that $x(uv_3) = x(uv_4)$ for each $u \in V(T_1)\backslash\{b, a_1\}$, where b is the hub of the 1-sum. Let T' be the the digraph obtained from T by identifying v_3 and v_4; the resulting vertex is still denoted by v_4. Let $\boldsymbol{w'}$ be obtained from the restriction of $\boldsymbol{w}$ to $A(T')$ by replacing $w(uv_4)$ with $w(uv_3)+w(uv_4)$ for each $u \in V(T_1)\backslash\{b, a_1\}$. Note that T' is Möbius-free by Lemma 3.13, $\boldsymbol{x}$ corresponds to a feasible solution $\boldsymbol{x'}$ to $\mathbb{P}(T', \boldsymbol{w'})$, and $\boldsymbol{y}$ corresponds to a feasible solution $\boldsymbol{y'}$ to $\mathbb{P}(T', \boldsymbol{w'})$ with $y'(v_4v_5v_4) = y'(v_4v_2v_4) = 0$, both having the same objective value $\nu^*_w(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. So $\boldsymbol{x'}$ and $\boldsymbol{y'}$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively. By Lemma 4.5, the optimal value $\nu^*_w(T)$ of $\mathbb{P}(T', \boldsymbol{w'})$ is integral. So (16) is established.

• $y(v_1v_5v_3v_4v_1) = y(v_1v_5v_2v_3v_1) = 0$. In this subcase, $y(v_2v_3v_4v_2)$ and $y(v_2v_3v_4v_5v_2)$ are integral. Assume first that $y(v_1v_2v_3v_4v_1) > 0$. Then, by (11), the arc v_3v_1 is saturated by $\boldsymbol{y}$ in F_4.

So $w(v_3v_1) = y(\mathcal{C}_2(v_3v_1)) = y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1)$. If $y(v_1v_5v_3v_1) = 0$, then $y(v_3v_4v_5v_3) = w(v_5v_3)$. So $y(C)$ is integral for any $C \in \mathcal{C}_2$. If $y(v_1v_5v_3v_1) > 0$, then v_4v_5 is is saturated by $\boldsymbol{y}$ in F_4 by (11). Thus $w(v_4v_5) = y(\mathcal{C}_2(v_4v_5)) = y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2)$, which is integral. It follows that $y(v_3v_4v_5v_3) = w(v_4v_5) - w(v_5v_2)$. So $y(C)$ is integral for any $C \in \mathcal{C}_2$. Assume next that $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_1)$ is integral and $y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) = w(v_5v_3)$. Clearly, we may assume that neither $y(v_1v_5v_3v_1)$ nor $y(v_3v_4v_5v_3)$ is integral, otherwise we are done. Similar to (16), we can show that

(17) $\nu_w^*(T)$ is an integer.

The proof goes along the same line as that of (16). In fact, we only need to replace $y(v_1v_5v_2v_3v_1)$ and $y(v_2v_3v_4v_5v_2)$ with $y(v_1v_5v_3)$ and $y(v_3v_4v_5v_3)$, respectively. So we omit the details here.

Case 2.8. $K = \{v_2v_3, v_5v_3\}$.

In this case, by Lemma 4.3(iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields the following two equations:

$w(v_2v_3) = y(v_1v_2v_3v_1) + y(v_2v_3v_4v_2) + y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_5v_2) + y(v_1v_5v_2v_3v_1)$; and

$w(v_5v_3) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1)$.

Since v_2v_3 is saturated by $\boldsymbol{y}$ in F_4, we have $w(uv_2) = z(uv_2) = 0$ for any $u \in V(T_1)\backslash\{b, a_1\}$ in this case. Depending on the values of $y(v_1v_5v_3v_4v_1)$ and $y(v_1v_5v_2v_3v_1)$, we consider three subcases.

- $y(v_1v_5v_3v_4v_1) > 0$. In this subcase, from (10) and (15) we deduce that $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = 0$ and that both v_3v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4. So $y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2) = w(v_4v_5)$ and $y(v_1v_5v_3v_1) = w(v_3v_1)$. If $y(v_2v_3v_4v_5v_2) > 0$, then both v_1v_2 and v_4v_2 are saturated by $\boldsymbol{y}$ in F_4 by (10) and (12). Thus $y(v_2v_3v_4v_2) = w(v_4v_2)$ and $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$. It follows that $y(v_3v_4v_5v_3)$, $y(v_2v_3v_4v_5v_2)$, and $y(v_1v_5v_3v_4v_1)$ are all integral. So we assume that $y(v_2v_3v_4v_5v_2) = 0$. Then $y(v_3v_4v_5v_3) = w(v_4v_5)$, and $y(v_1v_5v_3v_4v_1) = w(v_5v_3) - w(v_3v_1) - w(v_4v_5)$. Moreover, $y(v_2v_3v_4v_2) = w(v_4v_2)$ and $y(v_1v_2v_3v_4v_1) = w(v_2v_3) - w(v_4v_2)$ if $y(v_1v_2v_3v_4v_1) > 0$, and $y(v_2v_3v_4v_2) = w(v_2v_3)$ otherwise. Therefore $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether if $y(v_2v_3v_4v_5v_2) > 0$.
- $y(v_1v_5v_2v_3v_1) > 0$. In this subcase, by (9) and (15) we deduce that $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_4v_1) = 0$ and that v_1v_2 is saturated by $\boldsymbol{y}$ in F_4. So $y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1) = w(v_1v_2)$. If $y(v_1v_2v_3v_4v_1) > 0$, then v_3v_1, v_4v_2, and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4 by (9) and (11). So $y(v_2v_3v_4v_2) = w(v_4v_2)$, $y(v_2v_3v_4v_5v_2) = w(v_4v_5)$, and $y(v_2v_3v_4v_2) + y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1) = w(v_3v_1)$. It follows that $y(v_1v_2v_3v_1)$, $y(v_1v_2v_3v_4v_1)$, and $y(v_1v_5v_2v_3v_1)$ are all integral. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_1) = w(v_1v_2)$. If $y(v_2v_3v_4v_5v_2) = 0$, then $y(v_2v_3v_4v_2) + y(v_1v_5v_2v_3v_1) = w(v_2v_3) - w(v_1v_2)$. Since $y(v_1v_5v_2v_3v_1) > 0$, we see that v_3v_4 is outside $\mathcal{C}_0^y$, for otherwise, we can obtain an optimal solution $\boldsymbol{y'}$ to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_5v_2v_3v_1) < y(v_1v_5v_2v_3v_1)$, contradicting (3). It follows that $y(v_2v_3v_4v_2) = \min\{w(v_4v_2), w(v_3v_4)\}$ and $y(v_1v_5v_2v_3v_1) = w(v_2v_3) - w(v_1v_2) - y(v_2v_3v_4v_2)$. If $y(v_2v_3v_4v_5v_2) > 0$, then $y(v_2v_3v_4v_2) = w(v_4v_2)$ by (12) and $y(v_1v_5v_2v_3v_1) + y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_1v_2) - w(v_4v_2)$. Thus we always have $w(v_iv_2) = \lceil z(v_iv_2) \rceil = z(v_iv_2)$ for $i = 1, 4, 5$. Since v_2 is a near-sink, $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 4.6(i).
- $y(v_1v_5v_2v_3v_1) = y(v_1v_5v_3v_4v_1) = 0$. In this subcase, depending on whether $y(v_2v_3v_4v_5v_2) > 0$, we distinguish between two subsubcases.

(a) We first assume that $y(v_2v_3v_4v_5v_2) > 0$. Now, in view of (12), v_4v_2 is saturated by

$\boldsymbol{y}$ in F_4, which yields $w(v_4v_2) = y(v_2v_3v_4v_2)$. If $y(v_1v_5v_3v_1) > 0$, then v_1v_2 is saturated by $\boldsymbol{y}$ in F_4. So $y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1) = w(v_1v_2)$ and $y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_1v_2) - w(v_4v_2)$. Thus $w(v_iv_2) = \lceil z(v_iv_2) \rceil = z(v_iv_2)$ for $i = 1, 4, 5$. By Lemma 4.6(i), $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. So we assume that $y(v_1v_5v_3v_1) = 0$. If $y(v_1v_2v_3v_4v_1) = 0$, then $y(v_1v_2v_3v_1) + y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_4v_2)$. Since $\boldsymbol{y}$ satisfies (1), we have $y(v_1v_2v_3v_1) = \min\{w(v_1v_2), w(v_3v_1)\}$ and $y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_4v_2) - y(v_1v_2v_3v_1)$. If $y(v_1v_2v_3v_4v_1) > 0$, then $y(v_1v_2v_3v_1) = w(v_3v_1)$ by (11) and $y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_3v_1) - w(v_4v_2)$. Assume $y(v_1v_2v_3v_4v_1)$ is not integral. Then $[y(v_1v_2v_3v_4v_1)] + [y(v_2v_3v_4v_5v_2)] = 1$. We propose to show that

(18) v_4v_1 is saturated by $\boldsymbol{y}$ in F_4.

Suppose the contrary. If v_4v_1 is not saturated by $\boldsymbol{y}$ in T, we set $\theta = \min\{w(v_4v_1) - z(v_4v_1), [y(v_2v_3v_4v_5v_2)]\}$, and let $\boldsymbol{y'}$ arise from $\boldsymbol{y}$ by replacing $y(v_1v_2v_3v_4v_1)$ and $y(v_2v_3v_4v_5v_2)$ with $y(v_1v_2v_3v_4v_1) + \theta$ and $y(v_2v_3v_4v_5v_2) - \theta$, respectively. Since v_1v_2 is outside $\mathcal{C}_0^y$, $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, contradicting (4). If v_4v_1 is saturated by $\boldsymbol{y}$ in T but contained in a cycle $C \in \mathcal{C}_0^y$, let $C' = C[v_5, v_4] \cup \{v_4v_5\}$ and $\sigma = \min\{y(C), [y(v_2v_3v_4v_5v_2)]\}$, and let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_2v_3v_4v_1)$, $y(v_2v_3v_4v_5v_2)$, $y(C)$, and $y(C')$ with $y(v_1v_2v_3v_4v_1) + \sigma$, $y(v_2v_3v_4v_5v_2) - \sigma$, $y(C) - \sigma$, and $y(C') + \sigma$, respectively. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, contradicting (4) again. So (18) is established.

By (18), we have $y(v_1v_2v_3v_4v_1) = w(v_4v_1)$. It follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$.

(b) We next assume that $y(v_2v_3v_4v_5v_2) = 0$. If $y(v_1v_2v_3v_4v_1) > 0$, then v_4v_2 is saturated by $\boldsymbol{y}$ in F_4 by (11). So $y(v_2v_3v_4v_2) = w(v_4v_2)$ and $y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1) = w(v_2v_3) - w(v_4v_2)$. Thus $w(v_iv_2) = \lceil z(v_iv_2) \rceil = z(v_iv_2)$ for $i = 1, 4, 5$. By Lemma 4.6(i), $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. So we assume that $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_1) + y(v_2v_3v_4v_2) = w(v_2v_3)$ and $y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) = w(v_5v_3)$. If $y(v_1v_2v_3v_1)$ is integral, then $w(v_iv_2) = \lceil z(v_iv_2) \rceil = z(v_iv_2)$ for $i = 1, 4, 5$. Hence, by Lemma 4.6(i), $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. So we assume that $y(v_1v_2v_3v_1)$ is not integral. We propose to show that

(19) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since $0 < y(v_1v_2v_3v_1) < w(v_1v_2)$ and $0 < y(v_2v_3v_4v_2) < w(v_4v_2)$, by Lemma 4.3(i) and (ii), we have $x(v_1v_2) = x(v_4v_2) = 0$ and $x(v_3v_1) = x(v_3v_4)$.

Let us show that $x(v_1v_5) = x(v_4v_5)$. If both $y(v_1v_5v_3v_1)$ and $y(v_3v_4v_5v_3)$ are positive, then, by Lemma 4.3(i), we have $x(v_1v_5v_3v_1) = x(v_3v_4v_5v_3) = 1$, which implies $x(v_1v_5) = x(v_4v_5)$, as desired. If one of $y(v_1v_5v_3v_1)$ and $y(v_3v_4v_5v_3)$ is zero, then the other equals $w(v_5v_3)$. By Lemma 4.4(iii), we may assume that $w(v_5v_3) = 0$. Since v_2v_3 is saturated by $\boldsymbol{y}$ in F_4, both v_1v_2 and v_4v_2 are outside $\mathcal{C}_0^y$. If v_3v_4 is also outside $\mathcal{C}_0^y$, let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_5v_3)$ and $y(v_1v_5v_3v_1)$ with $y(v_3v_4v_5v_3) + [y(v_1v_5v_3v_1)]$ and $\lfloor y(v_1v_5v_3v_1) \rfloor$, respectively, then $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_3v_4v_5v_3)$ is a positive integer, $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 4.4(iii). So we may assume that v_3v_4 is contained in some cycle in $\mathcal{C}_0^y$; the same holds for v_3v_1. Let C_1 and C_2 be two cycles in $\mathcal{C}_0^y$ passing through v_3v_1 and v_3v_4, respectively. By Lemma 4.3(iii), we have $x(v_3v_1) + x(v_1v_5) = x(v_3v_4) + x(v_4v_5)$. Thus $x(v_1v_5) = x(v_4v_5)$ also holds.

Similarly, we can prove that $x(uv_1) = x(uv_4)$ for each vertex $u \in V(T_1) \backslash \{b, a_1\}$, where b is the hub of the 1-sum. Let $T' = (V', A')$ be the digraph obtained from T by identifying v_1 and v_4; the resulting vertex is still denoted by v_1. Let $\boldsymbol{w'}$ be the restriction of $\boldsymbol{w}$ to A'. Then $\boldsymbol{x}$

corresponds to a feasible solution $\boldsymbol{x}'$ to $\mathbb{P}(T', \boldsymbol{w}')$ with $x'(v_1v_5) = x(v_4v_1) + x(v_1v_5) = x(v_4v_5)$ by Lemma 4.3(iii), and $\boldsymbol{y}$ corresponds to a feasible solution $\boldsymbol{y}'$ to $\mathbb{D}(T', \boldsymbol{w}')$; both having the same objective value $\nu^*_w(T)$ as $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$. By the LP-duality theorem, $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively. By Lemma 4.5, $\mathbb{D}(T', \boldsymbol{w}')$ has an integral optimal solution. So $\nu^*_w(T)$ is an integer. This proves (19).

Case 2.9. $K = \{v_3v_1, v_3v_4\}$.

In this case, by Lemma 4.3(iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (14) yields the following two equations:

$w(v_3v_1) = y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1)$; and

$w(v_3v_4) = y(v_2v_3v_4v_2) + y(v_3v_4v_5v_3) + y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_5v_2) + y(v_1v_5v_3v_4v_1)$.

Since each $e \in K$ is saturated by $\boldsymbol{y}$ in F_4, we have $w(uv_i) = z(uv_i) = 0$ for $i = 2, 3$ and all $u \in V(T_1)\backslash\{b, a_1\}$, where b is the hub of the 1-sum. Depending on the values of $y(v_1v_5v_3v_4v_1)$ and $y(v_1v_5v_2v_3v_1)$, we consider three subcases.

• $y(v_1v_5v_2v_3v_1) > 0$. In this subcase, from (9) and (15) we deduce that $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_4v_1) = 0$ and that v_1v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4. So $w(v_1v_2) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1)$ and $w(v_5v_3) = y(v_1v_5v_3v_1)$. If $y(v_1v_2v_3v_4v_1) > 0$, then both v_4v_2 and v_4v_5 are saturated by $\boldsymbol{y}$ in F_4 by (9) and (11). Thus $y(v_2v_3v_4v_2) = w(v_4v_2)$ and $y(v_2v_3v_4v_5v_2) = w(v_4v_5)$. It follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_3v_4v_1) = 0$. If $y(v_2v_3v_4v_5v_2) > 0$, then v_4v_2 is saturated by $\boldsymbol{y}$ in F_4 by (12), which implies that $y(v_2v_3v_4v_2) = w(v_4v_2)$; if $y(v_2v_3v_4v_5v_2) = 0$, then $y(v_2v_3v_4v_2) = w(v_3v_4)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$, regardless of the value of $y(v_2v_3v_4v_5v_2)$.

• $y(v_1v_5v_3v_4v_1) > 0$. In this subcase, from (10) and (15) we deduce that $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = 0$ and that v_4v_5 is saturated by $\boldsymbol{y}$ in F_4. So $w(v_3v_1) = y(v_1v_5v_3v_1)$ and $w(v_4v_5) = y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2)$. If $y(v_2v_3v_4v_5v_2) > 0$, then v_1v_2, v_4v_2, and v_5v_3 are all saturated by $\boldsymbol{y}$ in F_4 by (10) and (12). So $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$, $y(v_2v_3v_4v_2) = w(v_4v_2)$, and $y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1) = w(v_5v_3) - y(v_1v_5v_3v_1)$. It follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_2v_3v_4v_5v_2) = 0$. Then $y(v_3v_4v_5v_3) = w(v_4v_5)$. If $y(v_1v_2v_3v_4v_1) > 0$, then v_4v_2 is saturated by $\boldsymbol{y}$ in F_4 by (11). So $y(v_2v_3v_4v_2) = w(v_4v_2)$ and hence $y(v_1v_2v_3v_4v_1) + y(v_1v_5v_3v_4v_1) = w(v_3v_4) - w(v_4v_5) - w(v_4v_2)$; if $y(v_1v_2v_3v_4v_1) = 0$, then $y(v_2v_3v_4v_2) + y(v_1v_5v_3v_4v_1) = w(v_3v_4) - w(v_4v_5)$. Since all arcs in $F_4\backslash v_6$ except $\{v_1v_5, v_4v_1, v_4v_5\}$ are outside $\mathcal{C}^y_0$ and $y(v_1v_5v_3v_4v_1) > 0$, by (ii) we have $y(v_1v_2v_3v_4v_1) = \min\{w(v_1v_2), w(v_2v_3) - w(v_4v_2)\}$ if $y(v_1v_2v_3v_4v_1) > 0$ and $y(v_2v_3v_4v_2) = \min\{w(v_4v_2), w(v_2v_3)\}$ otherwise. So $y(v_1v_5v_3v_4v_1)$ is integral, and hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, regardless of the value of $y(v_1v_2v_3v_4v_1)$.

• $y(v_1v_5v_2v_3v_1) = y(v_1v_5v_3v_4v_1) = 0$. In this subcase, depending on whether $y(v_2v_3v_4v_5v_2) > 0$, we distinguish between two subsubcases.

(a) We first assume that $y(v_2v_3v_4v_5v_2) > 0$. By (12), both v_4v_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in F_4, which implies $w(v_4v_2) = y(v_2v_3v_4v_2)$ and $w(v_5v_3) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3)$. If $y(v_1v_5v_3v_1) > 0$, then v_1v_2 is saturated by $\boldsymbol{y}$ in F_4 by (12). So $w(v_1v_2) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1)$. Moreover, if $y(v_1v_2v_3v_4v_1) > 0$, then v_4v_5 is saturated by $\boldsymbol{y}$ in F_4 by (11), which yields one more equation $w(v_4v_5) = y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_2v_3v_4v_1) = 0$. So we assume that $y(v_1v_5v_3v_1) = 0$. Then $y(v_1v_2v_3v_1) = w(v_3v_1)$, $y(v_3v_4v_5v_3) = w(v_5v_3)$ and $y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_5v_2) = w(v_3v_4) - w(v_4v_2) - w(v_5v_3)$. If $y(v_1v_2v_3v_4v_1)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_3v_4v_1)$ is integral. Similar to (18), we can prove that v_4v_1 is saturated by $\boldsymbol{y}$

in F_4. Then $y(v_1v_2v_3v_4v_1) = w(v_4v_1)$, a contradiction.

(b) We next assume that $y(v_2v_3v_4v_5v_2) = 0$. Suppose $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) = w(v_3v_1)$ and $y(v_2v_3v_4v_2) + y(v_3v_4v_5v_3) = w(v_3v_4)$. If neither $y(v_1v_5v_3v_1)$ nor $y(v_3v_4v_5v_3)$ is integral, then neither $y(v_1v_2v_3v_1)$ nor $y(v_2v_3v_4v_2)$ is integral. Similar to (19), we can show that $\nu_w^*(T)$ is an integer. So we may assume that $y(v_1v_5v_3v_1)$ or $y(v_3v_4v_5v_3)$ is integral. Observe that both of them are integral, for otherwise, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_2v_3v_1)$ and $y(v_1v_5v_3v_1)$ with $y(v_1v_2v_3v_1) + [y(v_1v_5v_3v_1)]$ and $\lfloor y(v_1v_5v_3v_1) \rfloor$, respectively. Since v_1v_2, v_2v_3, and v_4v_2 are all outside $\mathcal{C}_0^y$, $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, with $y'(v_1v_5v_3v_1) < y(v_1v_5v_3v_1)$, contradicting (5).

Suppose $y(v_1v_2v_3v_4v_1) > 0$. Then $y(v_2v_3v_4v_2) = w(v_4v_2)$. If $y(v_1v_5v_3v_1) > 0$, then v_4v_5 is saturated by $\boldsymbol{y}$ in F_4 by (11), which implies $y(v_3v_4v_5v_3) = w(v_4v_5)$, $y(v_1v_2v_3v_4v_1) = w(v_3v_4) - w(v_4v_2) - w(v_4v_5)$, and $y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) = w(v_3v_1)$. If $y(v_1v_5v_3v_1)$ is not integral, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_2v_3v_1)$ and $y(v_1v_5v_3v_1)$ with $y(v_1v_2v_3v_1) + [y(v_1v_5v_3v_1)]$ and $\lfloor y(v_1v_5v_3v_1) \rfloor$, respectively. Since both v_1v_2 and v_2v_3 are outside $\mathcal{C}_0$, $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, with $y'(v_1v_5v_3v_1) < y(v_1v_5v_3v_1)$, contradicting (5). So $y(v_1v_5v_3v_1)$ is integral and hence is zero by Lemma 4.4(iii). It follows that $y(v_1v_2v_3v_1) = w(v_3v_1)$ and $y(v_1v_2v_3v_4v_1) + y(v_3v_4v_5v_3) = w(v_3v_4) - w(v_4v_2)$. If $y(v_3v_4v_5v_3)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_3v_4v_5v_3)$ is not integral. Let us show that

(20) $\nu_w^*(T)$ is an integer.

By Lemma 4.4(iii), we may assume that $w(v_3v_1) = w(v_4v_2) = 0$. Recall that $w(v_5v_2) = z(v_5v_2) = 0$ and $w(uv_i) = z(uv_i) = 0$ for $i = 2, 3$ and all $u \in V(T_1) \backslash \{b, a_1\}$. So we may assume that $x(uv_2) = x(uv_3)$. Let $T' = (V', A')$ be the digraph obtained from T by identifying v_2 and v_3; the resulting vertex is still denoted by v_3, and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to A'. Then $\boldsymbol{x}$ corresponds to a feasible solution $\boldsymbol{x}'$ to $\mathbb{P}(T', \boldsymbol{w}')$, and $\boldsymbol{y}$ corresponds to a feasible solution $\boldsymbol{y}'$ to $\mathbb{D}(T', \boldsymbol{w}')$; both having the same objective value $\nu_w^*(T)$ as $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$. By the LP-duality theorem, $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively. By Lemma 4.5, $\mathbb{D}(T', \boldsymbol{w}')$ has an integral optimal solution. So $\nu_w^*(T)$ is an integer. This proves (20) and hence Claim 2.

Since $\tau_{\boldsymbol{w}}(F_4 \backslash v_6) > 0$, from Claim 2, Lemma 4.4(iii) and Lemma 4.6(ii) we deduce that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. This completes the proof of Lemma 5.4. ∎

Lemma 5.5. *If $T_2 = G_2$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. It is routine to check that

- $\mathcal{C}_2 = \{v_1v_2v_4v_1, v_1v_6v_3v_1, v_1v_6v_4v_1, v_1v_6v_2v_4v_1, v_1v_6v_3v_4v_1, v_1v_6v_3v_2v_4v_1\}$ and
- $\mathcal{F}_2 = \{\{v_1v_6, v_1v_2\}, \{v_1v_6, v_2v_4\}, \{v_1v_6, v_4v_1\}, \{v_3v_1, v_4v_1\}, \{v_4v_1, v_6v_3\}, \{v_2v_4, v_6v_3, v_6v_4\}, \{v_2v_4, v_3v_1, v_3v_4, v_6v_4\}, \{v_1v_2, v_6v_2, v_6v_3, v_6v_4\}, \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2, v_6v_4\}\}$.

We also have a computer verification of these results. So $|\mathcal{C}_2| = 6$ and $|\mathcal{F}_2| = 9$. Recall that $(b_2, a_2) = (v_4, v_5)$.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(1) $y(\mathcal{C}_2)$ is maximized;

(2) subject to (1), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically;

(3) subject to (1) and (2), $y(v_1v_6v_3v_4v_1)$ is minimized; and

(4) subject to (1)-(3), $y(v_1v_6v_4v_1)$ is minimized;

Let us make some simple observations about $\boldsymbol{y}$.

(5) If $K \in \mathcal{F}_2$ satisfies $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 5.3.)

The three statements below follow instantly from Lemma 4.7(v).

(6) If $y(v_1v_6v_3v_2v_4v_1) > 0$, then each arc in the set $\{v_1v_2, v_3v_1, v_3v_4, v_6v_2, v_6v_4\}$ is saturated by $\boldsymbol{y}$ in G_2.

(7) If $y(v_1v_6v_3v_4v_1) > 0$, then both v_3v_1 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2.

(8) If $y(v_1v_6v_2v_4v_1) > 0$, then both v_1v_2 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2.

Claim 1. $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$.

To justify this, observe that if both v_1v_2 and v_1v_6 are saturated by $\boldsymbol{y}$ in G_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_1v_6\}$; if both v_3v_1 and v_4v_1 are saturated by $\boldsymbol{y}$ in G_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1\}$. By (5), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$ in either case. So we assume that

(9) at most one of v_1v_2 and v_1v_6 is saturated by $\boldsymbol{y}$ in G_2. The same holds for v_3v_1 and v_4v_1.

As v_2v_4 is a special arc of T and v_2 is a near-sink, by Lemma 4.6(iv), we may assume that v_2v_4 is saturated by $\boldsymbol{y}$ in T. Depending on whether v_2v_4 is outside $\mathcal{C}_0^y$, we distinguish between two cases.

Case 1.1. v_2v_4 is contained by some cycle in $\mathcal{C}_0^y$.

In this case, we proceed by considering two subcases.

• v_3v_1 is saturated by $\boldsymbol{y}$ in G_2. In this subcase, by (9), v_4v_1 is not saturated by $\boldsymbol{y}$ in G_2 and hence in T, because v_4v_1 is outside $\mathcal{C}_0$. By the hypothesis of the present case and Lemma 4.7(iii), v_1v_2 is saturated by $\boldsymbol{y}$ in T. Observe that v_1v_2 is outside $\mathcal{C}_0^y$, for otherwise, a cycle $C \in \mathcal{C}_0^y$ containing v_1v_2 must pass through v_2v_4. Thus, by Lemma 4.7(iv), v_4v_1 is saturated by $\boldsymbol{y}$ in G_2, a contradiction. It follows that v_1v_2 is saturated by $\boldsymbol{y}$ in G_2. So, by (9), v_1v_6 is not saturated by $\boldsymbol{y}$ in G_2. If v_1v_6 is contained in some cycle $C \in \mathcal{C}_0^y$, applying Lemma 4.7(iv) to the cycle $C[v_1, v_4] \cup \{v_4v_1\}$ in $\mathcal{C}_2$, we see that v_4v_1 is saturated by $\boldsymbol{y}$ in T, a contradiction. So v_1v_6 is outside $C \in \mathcal{C}_0^y$. By Lemma 4.7(iii), v_6v_2 is saturated by $\boldsymbol{y}$ in G_2 and v_6v_4 is outside $\mathcal{C}_0^y$. Using Lemma 4.7(i), we further deduce that v_6v_4 is saturated by $\boldsymbol{y}$ in G_2. If v_6v_3 is also saturated by $\boldsymbol{y}$ in G_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_6v_2, v_6v_3, v_6v_4\}$. By (5), K is an MFAS and thus $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$. If v_6v_3 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C \in \mathcal{C}_0^y$, applying Lemma 4.7(iii) to the cycle $C[v_6, v_4] \cup \{v_4v_1, v_1v_6\} \in \mathcal{C}_2$, we see that v_4v_1 or v_1v_6 is saturated, a contradiction. If v_6v_3 is not saturated by $\boldsymbol{y}$ in T then, by Lemma 4.7(iii), v_3v_2 is saturated by $\boldsymbol{y}$ in G_2 and and v_3v_4 is outside $\mathcal{C}_0^y$. Using Lemma 4.7(i), we further deduce that v_3v_4 is saturated by $\boldsymbol{y}$ in G_2. Thus $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2, v_6v_4\}$. By (5), J is an MFAS and thus $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$.

• v_3v_1 is not saturated by $\boldsymbol{y}$ in G_2. In this subcase, we have $y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$ by (6) and (7). Assume first that v_1v_2 is saturated by $\boldsymbol{y}$ in G_2. Then v_1v_6 is not saturated by $\boldsymbol{y}$ in G_2 by (9). Thus v_6v_3 is saturated by $\boldsymbol{y}$ in G_2 by Lemma 4.7(iii) and (iv). If v_4v_1 is also saturated by $\boldsymbol{y}$ in G_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3\}$; otherwise, both v_6v_2 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2 by Lemma 4.7(iii) and (iv). So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_6v_2, v_6v_3, v_6v_4\}$. By (5), K is an MFAS in either subsubcase, and thus $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$.

Assume next that v_1v_2 is not saturated by $\boldsymbol{y}$ in G_2. By (8), we have $y(v_1v_6v_2v_4v_1) = 0$. By the hypothesis of the present case and by Lemma 4.7(iii) and (iv), v_4v_1 is saturated by $\boldsymbol{y}$ in G_2.

If v_6v_3 is also saturated by $\boldsymbol{y}$ in G_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3\}$. By (5), K is an MFAS and thus $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$. So we assume that v_6v_3 is not saturated by $\boldsymbol{y}$ in G_2. Thus v_1v_6 is saturated by $\boldsymbol{y}$ in G_2 by Lemma 4.7(iii) and (iv). We propose to show that

(10) $y(v_1v_6v_4v_1) = 0$.

Assume the contrary: $y(v_1v_6v_4v_1) > 0$. Observe that v_1v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_1v_2. Then the multiset sum of $v_1v_6v_4v_1$ and C contains two arc-disjoint cycles $v_1v_2v_4v_1$ and $C[v_4, v_1] \cup \{v_1v_6, v_6v_4\}$. Set $\theta = \min\{y(v_1v_6v_4v_1), y(C)\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_4v_1)$, $y(v_1v_2v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_4v_1) - \theta$, $y(v_1v_2v_4v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$, the existence of $\boldsymbol{y'}$ contradicts the assumption (4) on $\boldsymbol{y}$. It follows that v_3v_1 is also outside $\mathcal{C}_0^y$, because every cycle containing v_3v_1 in $\mathcal{C}_0^y$ must pass through v_1v_2. So neither v_1v_2 nor v_3v_1 is saturated by $\boldsymbol{y}$ in T.

Let us show that v_6v_3 is outside $\mathcal{C}_0^y$, for otherwise, let $C \in \mathcal{C}_0^y$ contain v_6v_3. Then the multiset sum of $v_1v_6v_4v_1$, C, and the unsaturated arc v_3v_1 contains arc-disjoint cycles $v_1v_6v_3v_1$ and $C' = C[v_4, v_6] \cup \{v_6v_4\}$. Set $\theta = \min\{y(v_1v_6v_4v_1), y(C), w(v_3v_1) - z(v_3v_1)\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_4v_1)$, $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_4v_1) - \theta$, $y(v_1v_6v_3v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$, the existence of $\boldsymbol{y'}$ contradicts the assumption (4) on $\boldsymbol{y}$. Let $D \in \mathcal{C}_0^y$ be a cycle containing v_2v_4. Then the multiset sum of D, $v_1v_6v_4v_1$, and the unsaturated arcs v_6v_3, v_3v_1, and v_1v_2 contains two arc-disjoint cycles $v_1v_2v_4v_1$ and $v_1v_6v_3v_1$. Thus, by Lemma 4.7(vi), we obtain $y(v_1v_6v_4v_1) = 0$; this contradiction proves (10).

From (10), we deduce that $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_1\}$. So, by (5), K is an MFAS and thus $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$.

Case 1.2. v_2v_4 is outside $\mathcal{C}_0^y$.

In this case, v_2v_4 is saturated by $\boldsymbol{y}$ in G_2. So v_1v_2, v_3v_2, and v_6v_2 are all outside $\mathcal{C}_0^y$. Assume first that v_1v_6 is saturated by $\boldsymbol{y}$ in G_2. Then v_1v_2 is not saturated by $\boldsymbol{y}$ by (9). By (6) and (8), we have $y(v_1v_6v_3v_2v_4v_1) = y(v_1v_6v_2v_4v_1) = 0$ and hence $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_2v_4\}$. It follows from (5) that K is an MFAS and thus $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$. Assume next that v_1v_6 is not saturated by $\boldsymbol{y}$ in G_2. If v_4v_1 is not saturated by $\boldsymbol{y}$ in T, then v_6v_4 is outside $\mathcal{C}_0^y$ by Lemma 4.7(iii). So v_3v_4 is contained in some cycle in $\mathcal{C}_0^y$ because $\mathcal{C}_0^y \neq \emptyset$. Using Lemma 4.7(iii), we deduce that both v_6v_3 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2. Using (6), we obtain $y(v_1v_6v_3v_2v_4v_1) = 0$. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_2v_4, v_6v_3, v_6v_4\}$. If v_4v_1 is saturated by $\boldsymbol{y}$ in T, then so is it in G_2 because v_4v_1 is outside $\mathcal{C}_0^y$. By (9), v_3v_1 is not saturated by $\boldsymbol{y}$ in G_2. By Lemma 4.7(iii), v_6v_3 is saturated by $\boldsymbol{y}$ in G_2. By (6) and (7), we have $y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$. Hence $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3\}$. In either subsubcase, K is an MFAS by (5) and thus $y(\mathcal{C}_2) = \tau_w(G_2\backslash v_5)$. This proves Claim 1.

Claim 2. $y(C)$ is integral for all $C \in \mathcal{C}_2$ or $\nu_w^*(T)$ is an integer.

To justify this, we may assume that

(11) $y(v_1v_6v_3v_2v_4v_1) = 0$.

Otherwise, by (6), we have $w(e) = y(\mathcal{C}_2(e))$ for each e in the set $\{v_1v_2, v_3v_1, v_3v_4, v_6v_2, v_6v_4\}$. So $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$, and $y(v_1v_6v_4v_1) = w(v_6v_4)$. By Claim 1, $y(\mathcal{C}_2)$ is an integer, so is $y(v_1v_6v_3v_2v_4v_1)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$.

By Claim 1, $y(\mathcal{C}_2) = w(K)$ for some $K \in \mathcal{F}_2$. Depending on what K is, we distinguish among nine cases.

Case 2.1. $K = \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2, v_6v_4\}$.

In this case, by Lemma 4.3 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$. It follows instantly that $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.2. $K = \{v_1v_6, v_4v_1\}$.

In this case, by Lemma 4.3 (i), we have $y(v_1v_6v_4v_1) = y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$. By Lemma 4.3 (iii), we further obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$. It follows that $y(v_1v_2v_4v_1) = w(v_4v_1)$ and $y(v_1v_6v_3v_1) = w(v_1v_6)$. Therefore $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.3. $K = \{v_1v_2, v_6v_2, v_6v_3, v_6v_4\}$.

In this case, by Lemma 4.3 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$, $y(v_1v_6v_4v_1) = w(v_6v_4)$, and $y(v_1v_6v_3v_1)+y(v_1v_6v_3v_4v_1) = w(v_6v_3)$. Note that if $y(v_1v_6v_3v_4v_1) > 0$, we have one more equation $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (7). Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_3v_4v_1) = 0$.

Case 2.4. $K = \{v_2v_4, v_6v_3, v_6v_4\}$.

In this case, by Lemma 4.3 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_2v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_2v_4)$, $y(v_1v_6v_3v_1) + y(v_1v_6v_3v_4v_1) = w(v_6v_3)$, and $y(v_1v_6v_4v_1) = w(v_6v_4)$. Note that if $y(v_1v_6v_2v_4v_1) > 0$, we have one more equation $y(v_1v_2v_4v_1) = w(v_1v_2)$ by (8); if $y(v_1v_6v_3v_4v_1) > 0$, we have one more equation $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (7). Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$ in any subcase.

Case 2.5. $K = \{v_2v_4, v_3v_1, v_3v_4, v_6v_4\}$.

In this case, by Lemma 4.3 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_2v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_2v_4)$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, and $y(v_1v_6v_4v_1) = w(v_6v_4)$. Note that if $y(v_1v_6v_2v_4v_1) > 0$, we have one more equation $y(v_1v_2v_4v_1) = w(v_1v_2)$ by (8). Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1) = 0$.

Case 2.6. $K = \{v_1v_6, v_2v_4\}$.

In this case, by Lemma 4.3 (i), we have $y(v_1v_6v_2v_4v_1) = 0$. By Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_2v_4v_1) = w(v_2v_4)$ and $y(v_1v_6v_3v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$. Moreover, in this case v_1v_2, v_3v_2, and v_6v_2 are all outside $\mathcal{C}_0^y$, and $w(uv_2) = z(uv_2) = 0$ for any $u \in V(T_1)\backslash\{b, a_1\}$, where b is the hub of the 1-sum. Examining the cycles in $\mathcal{C}_2$, we see that $z(v_3v_2) = z(v_6v_2) = 0$ and so $w(v_iv_2) = \lceil z(v_iv_2) \rceil = z(v_iv_2)$ for $i = 1, 3, 6$. Thus $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 4.6(i).

Case 2.7. $K = \{v_4v_1, v_6v_3\}$.

In this case, by Lemma 4.3 (i) and (iii), we have $y(v_1v_6v_3v_4v_1) = 0$, $y(v_1v_6v_3v_1) = w(v_6v_3)$, and $y(v_1v_2v_4v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_4v_1)$. Lemma 4.4(iii) allows us to assume that $w(v_6v_3) = 0$. If $y(v_1v_6v_2v_4v_1) > 0$, then both v_1v_2 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2 by (8). So $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_1v_6v_4v_1) = w(v_6v_4)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$; the same holds if $y(v_1v_6v_2v_4v_1) = 0$ and $y(v_1v_2v_4v_1)$ is integral. So we assume that $y(v_1v_6v_2v_4v_1) = 0$ and $y(v_1v_2v_4v_1)$ is not integral. Observe that v_1v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_1v_2, let $C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_4\}$, and set $\theta = \min\{y(C), y(v_1v_6v_4v_1)\}$. Let $\boldsymbol{y}'$

be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_4v_1)$, $y(v_1v_2v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_4v_1)-\theta$, $y(v_1v_2v_4v_1)+\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$. Since $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (4) on $\boldsymbol{y}$. Similarly, we can prove that v_6v_2 is outside $\mathcal{C}_0^y$. Examining cycles in $\mathcal{C}_2$, we see that $w(v_6v_2)=z(v_6v_2)=0$. Now we propose to show that

(12) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T,\boldsymbol{w})$. Since both $y(v_1v_2v_4v_1)$ and $y(v_1v_6v_4v_1)$ are positive, we have $x(v_1v_2)+x(v_2v_4)=x(v_1v_6)+x(v_6v_4)$ by Lemma 4.3(i). Since $y(v_1v_2v_4v_1)<w(v_1v_2)$, we have $x(v_1v_2)=0$ by Lemma 4.3(ii). So $x(v_2v_4)=x(v_1v_6)+x(v_6v_4)$. If each of v_3v_1 and v_3v_2 is contained in some cycle in $\mathcal{C}_0^y$, then $x(v_3v_1)=x(v_3v_2)$ by Lemma 4.3(iv). If one of v_3v_1 and v_3v_2 is outside $\mathcal{C}_0^y$, say v_3v_1, then we may assume that $w(v_3v_1)=0$ and $x(v_3v_1)=x(v_3v_2)$. Similarly, we can prove that $x(uv_1)=x(uv_2)$ for each $u\in V(T_1)\backslash\{a_1,b\}$.

Let $T'=(V',A')$ be obtained from T by deleting vertex v_2, let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by defining $w'(uv_1)=w(uv_1)+w(uv_2)$ for $u=v_3$ or $u\in V(T_1)\backslash\{b,a_1\}$ and $w'(v_iv_j)=w(v_iv_j)+w(v_2v_4)$ for $(i,j)=(1,6)$ or $(6,4)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through the path uv_2v_4 with $u\in(V(T_1)\backslash\{a_1,b\})\cup\{v_3\}$, let C' be the cycle arising from C by replacing uv_2v_4 with $uv_1v_6v_4$, and set $y'(C')=y(C)+y(C')$ and $y'(v_1v_6v_4v_1)=y(v_1v_6v_4v_1)+y(v_1v_2v_4v_1)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T',\boldsymbol{w}')$ and $\mathbb{D}(T',\boldsymbol{w}')$ respectively, both having the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence $\nu_w^*(T)$ is an integer by the hypothesis of Theorem 4.1.

Case 2.8. $K=\{v_1v_6,v_1v_2\}$.

In this case, by Lemma 4.3 (iii), we have $w(e)=y(\mathcal{C}_2(e))$ for each $e\in K$, which together with (11) yields the following equations: $y(v_1v_2v_4v_1)=w(v_1v_2)$ and $y(v_1v_6v_3v_1)+y(v_1v_6v_4v_1)+y(v_1v_6v_2v_4v_1)+y(v_1v_6v_3v_4v_1)=w(v_1v_6)$. Moreover, v_3v_1 is outside $\mathcal{C}_0^y$. Depending on whether $y(v_1v_6v_3v_4v_1)=0$, we consider two subcases.

• $y(v_1v_6v_3v_4v_1)=0$. In this subcase, we first assume that $y(v_1v_6v_2v_4v_1)>0$. Then $y(v_1v_6v_4v_1)=w(v_6v_4)$ by (8). Thus $y(v_1v_6v_3v_1)+y(v_1v_6v_2v_4v_2)=w(v_1v_6)-w(v_6v_4)$. Let us show that $y(v_1v_6v_3v_1)$ is integral. Suppose not. If v_6v_3 is outside $\mathcal{C}_0^y$, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$ and $y(v_1v_6v_2v_4v_1)$ with $y(v_1v_6v_3v_1)+[y(v_1v_6v_2v_4v_1)]$ and $\lfloor y(v_1v_6v_2v_4v_1)\rfloor$, respectively; if v_6v_3 is contained in some cycle C in $\mathcal{C}_0^y$, set $\theta=\min\{y(C),[y(v_1v_6v_2v_4v_1)]\}$ and $C'=C[v_4,v_6]\cup\{v_6v_2,v_2v_4\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$, $y(v_1v_6v_2v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_1)+\theta$, $y(v_1v_6v_2v_4v_1)-\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. In both subsubcases, $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$ with $y'(v_1v_6v_2v_4v_1)<y(v_1v_6v_2v_4v_1)$, contradicting (2). We next assume that $y(v_1v_6v_2v_4v_1)=0$. The proof of this subsubcase is similar to that in the preceding one (with $y(v_1v_6v_4v_1)$ in place of $y(v_1v_6v_2v_4v_1)$). Thus we reach a contradiction to (4).

• $y(v_1v_6v_3v_4v_1)>0$. In this subcase, by (7), both v_3v_1 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2. So $y(v_1v_6v_3v_1)=w(v_3v_1)$, $y(v_1v_6v_4v_1)=w(v_6v_4)$, and $y(v_1v_6v_2v_4v_1)+y(v_1v_6v_3v_4v_1)=w(v_1v_6)-w(v_3v_1)-w(v_6v_4)$. If $y(v_1v_6v_2v_4v_1)$ is integral, then $y(C)$ is integral for all $C\in\mathcal{C}_2$. So we assume that $y(v_1v_6v_2v_4v_1)$ is not integral. Then $[y(v_1v_6v_2v_4v_1)]+[y(v_1v_6v_3v_4v_1)]=1$. Observe that v_6v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_6v_2, let $C'=C[v_4,v_6]\cup\{v_6v_3,v_3v_4\}$, let $\theta=\min\{y(C),[y(v_1v_6v_3v_4v_1]\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_4v_1)$, $y(v_1v_6v_2v_4v_1)$, $y(C)$, $y(C')$ with $y(v_1v_6v_3v_4v_1)-\theta$, $y(v_1v_6v_2v_4v_1)+\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (2). Similarly, we can show that v_3v_2 is also outside $\mathcal{C}_0^y$. Thus $w(v_3v_2) = z(v_3v_2) = 0$. By Lemma 4.4(iii), we may assume that $w(v_1v_2)$, $w(v_3v_1)$, and $w(v_6v_4)$ are all 0. We propose to show that

(13) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since $y(v_1v_6v_2v_4v_1) > 0$ and $y(v_1v_6v_3v_4v_1) > 0$, from Lemma 4.3(i) we deduce that $x(v_6v_2) + x(v_2v_4) = x(v_6v_3) + x(v_3v_4)$. Since $y(v_1v_6v_2v_4v_1) < w(v_6v_2)$, we have $x(v_6v_2) = 0$ by Lemma 4.3(ii). It follows that $x(v_2v_4) = x(v_6v_3) + x(v_3v_4)$. Since $w(v_6v_4) = 0$ and v_6v_2 is outside $\mathcal{C}_0^y$, $x(uv_6) = x(uv_2)$ for each $u \in V(T_1)\backslash\{b, a_1\}$. Let $T' = (V', A')$ be the tournament obtained from T by deleting vertex v_2, let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(uv_6)$ with $w(uv_6)+w(uv_2)$ for each $u \in V(T_1)\backslash\{b, a_1\}$ and replacing $w(v_iv_j)$ with $w(v_iv_j) + w(v_2v_4)$ for $(i, j) = (6, 3)$ or $(3, 4)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A', and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through uv_2v_4 with $u \in V(T_1)\backslash\{b, a_1\}$, let C' be the cycle arising from C by replacing uv_2v_4 with $uv_6v_3v_4$, and set $y'(C') = y(C')+y(C)$ and $y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1)+y(v_1v_6v_2v_4v_1)$. From the LP-duality theorem, we deduce that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, both having the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence $\nu_w^*(T)$ is an integer by the hypothesis of Theorem 4.1.

Case 2.9. $K = \{v_3v_1, v_4v_1\}$.

In this case, by Lemma 4.3 (iii), we have $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_6v_3v_1) = w(v_3v_1)$ and $y(v_1v_2v_4v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1)$. Assume first that $y(v_1v_6v_2v_4v_1) = 0$. If $y(v_1v_6v_3v_4v_1) > 0$, then v_6v_4 is saturated by $\boldsymbol{y}$ in G_2. So $y(v_1v_6v_4v_1) = w(v_6v_4)$ and hence $y(v_1v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1) - w(v_6v_4)$; if $y(v_1v_6v_3v_4v_1) = 0$, then $y(v_1v_2v_4v_1) + y(v_1v_6v_4v_1) = w(v_4v_1)$. If $y(v_1v_2v_4v_1)$ is an integer, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_2v_4v_1)$ is not integral. Then we can prove that both v_6v_2 and v_1v_2 are outside $\mathcal{C}_0^y$ and that $\nu_w^*(T)$ is an integer. The proof is the same as that of (12) (with $y(v_1v_6v_4v_1)$ in place of $y(v_1v_6v_3v_4v_1)$ when $y(v_1v_6v_3v_4v_1) > 0$), so we omit the details here .

Assume next that $y(v_1v_6v_2v_4v_1) > 0$. Then both v_1v_2 and v_6v_4 are saturated by $\boldsymbol{y}$ in G_2. So $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_4v_1) = w(v_6v_4)$, and $y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1) - w(v_1v_2) - w(v_6v_4)$. If $y(v_1v_6v_3v_4v_1)$ is an integer, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_6v_3v_4v_1)$ is not integral. Then we can prove that both v_6v_2 and v_3v_2 are outside $\mathcal{C}_0^y$ and that $\nu_w^*(T)$ is an integer. The proof is the same as that of (13), so we omit the details here. Thus Claim 2 is established.

Since $\tau_{\boldsymbol{w}}(F_4\backslash v_6) > 0$, from Claim 2, Lemma 4.4(iii) and Lemma 4.6(ii) we deduce that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. This completes the proof of Lemma 5.5. ∎

Lemma 5.6. *If $T_2 = G_3$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. It is routine to check that

- $\mathcal{C}_2 = \{v_1v_2v_4v_1, v_1v_6v_3v_1, v_2v_4v_6v_2, v_3v_4v_6v_3, v_1v_6v_2v_4v_1, v_1v_6v_3v_4v_1, v_2v_4v_6v_3v_2, v_1v_6v_3v_2v_4v_1, v_1v_2v_4v_6v_3v_1\}$ and
- $\mathcal{F}_2 = \{\{v_2v_4, v_6v_3\}, \{v_1v_2, v_1v_6, v_4v_6\}, \{v_1v_2, v_6v_2, v_6v_3\}, \{v_1v_6, v_2v_4, v_3v_4\}, \{v_1v_6, v_2v_4, v_4v_6\}, \{v_1v_6, v_4v_1, v_4v_6\}, \{v_2v_4, v_3v_1, v_3v_4\}, \{v_3v_1, v_4v_1, v_4v_6\}, \{v_4v_1, v_4v_6, v_6v_3\}, \{v_4v_1, v_6v_2, v_6v_3\}, \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2\}, \{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\},$

$\{v_3v_1, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}\}$.

We also have a computer verification of these results. So $|\mathcal{C}_2| = 9$ and $|\mathcal{F}_2| = 13$. Recall that $(b_2, a_2) = (v_4, v_5)$.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(1) $y(\mathcal{C}_2)$ is maximized;

(2) subject to (1), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically;

(3) subject to (1) and (2), $y(v_1v_6v_3v_4v_1)$ is minimized; and

(4) subject to (1)-(3), $y(v_1v_2v_4v_1) + y(v_3v_4v_6v_3)$ is minimized;

Let us make some simple observations about $\boldsymbol{y}$.

(5) If $K \in \mathcal{F}_2$ satisfies $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 5.3.)

(6) If $y(v_1v_2v_4v_6v_3v_1) > 0$, then each arc in the set $\{v_1v_6, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}$ is saturated by $\boldsymbol{y}$ in G_3. Furthermore, $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$.

To justify this, note that each arc in the given set is a chord of the cycle $v_1v_2v_4v_6v_3v_1$. So the first half follows instantly from Lemma 4.7(v). Once again let $\uplus$ stand for the multiset sum. Then $v_1v_2v_4v_6v_3v_1 \uplus v_1v_6v_2v_4v_1 = v_1v_2v_4v_1 \uplus v_1v_6v_3v_1 \uplus v_2v_4v_6v_2$, $v_1v_2v_4v_6v_3v_1 \uplus v_1v_6v_3v_4v_1 = v_1v_2v_4v_1 \uplus v_1v_6v_3v_1 \uplus v_3v_4v_6v_3$, and $v_1v_2v_4v_6v_3v_1 \uplus v_1v_6v_3v_2v_4v_1 = v_1v_2v_4v_1 \uplus v_1v_6v_3v_1 \uplus v_2v_4v_6v_3v_2$. It follows from the optimality of $\boldsymbol{y}$ that $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$.

(7) If $y(v_1v_6v_3v_2v_4v_1) > 0$, then each arc in the set $\{v_1v_2, v_3v_1, v_3v_4, v_4v_6, v_6v_2\}$ is saturated by $\boldsymbol{y}$ in G_3. Furthermore, $y(v_2v_4v_6v_2) = y(v_3v_4v_6v_3) = 0$.

To justify this, note that each arc in the given set is a chord of the cycle $v_1v_2v_4v_6v_3v_1$. So the first half follows instantly from Lemma 4.7(v). Observe that $v_1v_6v_3v_2v_4v_1 \uplus v_3v_4v_6v_3 = v_1v_6v_3v_4v_1 \uplus v_2v_4v_6v_3v_2$ and $v_1v_6v_3v_2v_4v_1 \uplus v_2v_4v_6v_2 = v_1v_6v_2v_4v_1 \uplus v_2v_4v_6v_3v_2$. Since $\boldsymbol{y}$ satisfies (2), it is clear that $y(v_2v_4v_6v_2) = y(v_3v_4v_6v_3) = 0$.

(8) If $y(v_1v_6v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_6 are saturated by $\boldsymbol{y}$ in G_3; so is v_1v_2 if $y(v_2v_4v_6v_3v_2) > 0$. Furthermore, $y(v_2v_4v_6v_2) = 0$.

To justify this, note that both v_3v_1 and v_4v_6 are chords of the cycle $v_1v_2v_4v_6v_3v_1$, so they are saturated by $\boldsymbol{y}$ in G_3 by Lemma 4.7(v). Suppose $y(v_2v_4v_6v_3v_2) > 0$. If v_1v_2 is not saturated by $\boldsymbol{y}$ in T, then $v_1v_6v_3v_4v_1 \uplus v_2v_4v_6v_3v_2 \uplus \{v_1v_2\} = v_1v_2v_4v_1 \uplus v_3v_4v_6v_3$; if v_1v_2 is saturated by $\boldsymbol{y}$ in T but contained in some cycle $C \in \mathcal{C}_0^y$, then the multiset sum of C, $v_1v_6v_3v_4v_1$, and $v_2v_4v_6v_3v_2$ contains arc-disjoint cycles $v_1v_2v_4v_1$, $v_3v_4v_6v_3$, and $C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_3, v_3v_2, v_2v_4\}$. Thus we can obtain an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ that contradicts the assumption (3) on $\boldsymbol{y}$. Moreover, since $v_1v_6v_3v_4v_1 \uplus v_2v_4v_6v_2 = v_3v_4v_6v_3 \uplus v_1v_6v_2v_4v_1$, it follows from (3) that $y(v_2v_4v_6v_2) = 0$.

(9) If $y(v_1v_6v_2v_4v_1) > 0$, then both v_1v_2 and v_4v_6 are saturated by $\boldsymbol{y}$ in G_3; so is v_3v_1 if $y(v_3v_4v_6v_3) > 0$ or $y(v_2v_4v_6v_3v_2) > 0$.

The first half follows instantly from Lemma 4.7(v). To prove the second half, assume the contrary. If v_3v_1 is not saturated by $\boldsymbol{y}$ in T, then $v_3v_4v_6v_3 \uplus v_1v_6v_2v_4v_1 \uplus \{v_3v_1\} = v_2v_4v_6v_2 \uplus v_1v_6v_3v_1$, and $v_2v_4v_6v_3v_2 \uplus v_1v_6v_2v_4v_1 \uplus \{v_3v_1\} = v_2v_4v_6v_2 \uplus v_1v_6v_3v_1$; if v_3v_1 is saturated by $\boldsymbol{y}$ in T but contained in some cycle C in $\mathcal{C}_0^y$, then the multiset sum of C, $v_1v_6v_2v_4v_1$, and $v_3v_4v_6v_3$ (resp. $v_2v_4v_6v_3v_2$) contains arc-disjoint cycles $v_2v_4v_6v_2$, $v_1v_6v_3v_1$, and $C' = C[v_4, v_3] \cup \{v_3v_4\}$ (resp. $C' = C[v_4, v_3] \cup \{v_3v_2, v_2v_4\}$). Since $\boldsymbol{y}$ satisfies (2), we have $y(v_3v_4v_6v_3) = y(v_2v_4v_6v_3v_2) = 0$, a contradiction.

(10) If $y(v_2v_4v_6v_3v_2) > 0$, then both v_3v_4 and v_6v_2 are saturated by $\boldsymbol{y}$ in G_3 by Lemma 4.7(v).

(11) If v_1v_6 is contained in a cycle in $\mathcal{C}_0^y$, then both v_4v_1 and v_4v_6 are saturated by $\boldsymbol{y}$ in G_3.

Since both $C[v_1, v_4] \cup \{v_4v_1\}$ and $C[v_6, v_4] \cup \{v_4v_6\}$ are cycles in $\mathcal{C}_2$, the statement follows instantly from Lemma 4.7(iv).

(12) If v_6v_3 is contained in a cycle in $\mathcal{C}_0^y$, then v_4v_6 is saturated by $\boldsymbol{y}$ in G_3; so is v_1v_6 or v_4v_1.

The first half follows instantly from Lemma 4.7(iv). To prove the second half, we may assume, by (11), that v_1v_6 is outside $\mathcal{C}_0^y$. Let C be a cycle in $\mathcal{C}_0^y$ containing v_6v_3. Then both $C[v_6, v_4] \cup \{v_4v_6\}$ and $C[v_6, v_4] \cup \{v_4v_1, v_1v_6\}$ are cycles in $\mathcal{C}_2$. Thus, by Lemma 4.7(iv), v_4v_6 and at least one of v_1v_6 and v_4v_1 are saturated by $\boldsymbol{y}$ in G_3.

Claim 1. $y(\mathcal{C}_2) = \tau_w(G_3 \backslash v_5)$.

To justify this, observe that v_2v_4 is a special arc of T and v_2 is a near-sink. By Lemma 4.6(iv), we may assume that v_2v_4 is saturated by $\boldsymbol{y}$ in T. Let $\mathcal{G}_2 = \{\{v_1v_2, v_1v_6, v_4v_6\}, \{v_1v_2, v_6v_2, v_6v_3\}, \{v_2v_4, v_3v_1, v_3v_4\}, \{v_3v_1, v_4v_1, v_4v_6\}\}$. Then $\mathcal{G}_2 \subset \mathcal{F}_2$. Observe that

(13) if $y(v_1v_2v_4v_6v_3v_1) = 0$, then for each $K \in \mathcal{G}_2$, not all arcs in K are saturated by $\boldsymbol{y}$ in G_3.

Suppose the contrary: all arcs in K are saturated by $\boldsymbol{y}$ in G_3. Examining cycles in $\mathcal{C}_2$, we see that $y(\mathcal{C}_2) = w(K)$. By (5), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(G_3 \backslash v_5)$. So we may assume that (13) holds.

Depending on whether v_2v_4 is outside $\mathcal{C}_0^y$, we distinguish between two cases.

Case 1.1. v_2v_4 is contained in some cycle in $\mathcal{C}_0^y$.

We proceed by considering four subcases.

• Neither v_4v_1 nor v_4v_6 is saturated by $\boldsymbol{y}$ in G_3. In this subcase, by Lemma 4.7(iii) and (iv), both v_1v_2 and v_6v_2 are saturated by $\boldsymbol{y}$ in G_3. By (6)-(9), $y(v_1v_2v_4v_6v_3v_1)$, $y(v_1v_6v_3v_2v_4v_1)$, $y(v_1v_6v_3v_4v_1)$, and $y(v_1v_6v_2v_4v_1)$ are all zero. By (12) and (13), v_6v_3 is outside $\mathcal{C}_0^y$ and not saturated by $\boldsymbol{y}$. By Lemma 4.7(iii), both v_3v_2 and v_3v_4 are saturated by $\boldsymbol{y}$ in G_3. By Lemma 4.7(i) and (iii), at least one of v_1v_6 and v_3v_1 is saturated by $\boldsymbol{y}$ in G_3. Thus $y(\mathcal{C}_2) = w(K)$, where K is $\{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\}$ or $\{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2\}$. By (5), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(G_3 \backslash v_5)$.

• v_4v_6 is saturated by $\boldsymbol{y}$ in G_3 while v_4v_1 is not. In this subcase, by Lemma 4.7(iii), v_1v_2 is saturated by $\boldsymbol{y}$ in G_3. By (6), we have $y(v_1v_2v_4v_6v_3v_1) = 0$. By (11) and (13), v_1v_6 is outside $\mathcal{C}_0^y$ and not saturated by $\boldsymbol{y}$. By Lemma 4.7(i) and (iii), v_6v_2 is saturated by $\boldsymbol{y}$ in G_3. So, by (12) and (13), v_6v_3 is outside $\mathcal{C}_0^y$ and not saturated by $\boldsymbol{y}$. It follows from Lemma 4.7(i) and (iii) that v_3v_1, v_3v_2, and v_3v_4 are all saturated by $\boldsymbol{y}$ in G_3. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2\}$. By (5), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(G_3 \backslash v_5)$.

• v_4v_1 is saturated by $\boldsymbol{y}$ in G_3 while v_4v_6 is not. In this subcase, by Lemma 4.7(iii), v_6v_2 is saturated by $\boldsymbol{y}$ in G_3. By (7)-(9), $y(v_1v_6v_3v_2v_4v_1)$, $y(v_1v_6v_3v_4v_1)$, and $y(v_1v_6v_2v_4v_1)$ are all zero. By (12), v_6v_3 is outside $\mathcal{C}_0^y$. Furthermore, we may assume that v_6v_3 is not saturated by $\boldsymbol{y}$, for otherwise $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_2, v_6v_3\}$. Then, by Lemma 4.7(iii) and (iv), both v_3v_2 and v_3v_4 are saturated by $\boldsymbol{y}$ in G_3. If v_3v_1 is also saturated by $\boldsymbol{y}$ in G_3, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}$; otherwise, by Lemma 4.7(i) and (iii), both v_1v_2 and v_1v_6 are saturated by $\boldsymbol{y}$ in G_3. So $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\}$.

• Both v_4v_1 and v_4v_6 are saturated by $\boldsymbol{y}$ in G_3. In this subcase, if $y(v_1v_2v_4v_6v_3v_1) > 0$, then v_1v_6 is saturated by $\boldsymbol{y}$ in G_3 and $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$ by (6).

Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_1v_6\}$. So we assume that $y(v_1v_2v_4v_6v_3v_1) = 0$. Then v_3v_1 is not saturated by $\boldsymbol{y}$ in G_3 by (13). Thus $y(v_1v_6v_3v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$ by (7) and (8). If $y(v_1v_6v_2v_4v_1) > 0$, then v_1v_2 is saturated by $\boldsymbol{y}$ in G_3 and $y(v_3v_4v_6v_3) = y(v_2v_4v_6v_3v_2) = 0$ by (9). By (13), v_1v_6 is not saturated by $\boldsymbol{y}$ in G_3. Hence, by Lemma 4.7(iii), v_6v_3 is saturated by $\boldsymbol{y}$ in G_3. Therefore, $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_6v_3\}$. So we may assume that $y(v_1v_6v_2v_4v_1) = 0$ and that v_1v_6 is not saturated by $\boldsymbol{y}$ in G_3, for otherwise $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_1v_6\}$. Thus, by Lemma 4.7(iii) and (iv), v_6v_3 is saturated by $\boldsymbol{y}$ in G_3. We may further assume that v_6v_2 is not saturated by $\boldsymbol{y}$ in G_3, for otherwise, $y(\mathcal{C}_2) = w(J)$, where $J = \{v_4v_1, v_6v_2, v_6v_3\}$. Then $y(v_2v_4v_6v_3v_2) = 0$ by (10). We propose to show that

(14) $y(v_3v_4v_6v_3) = 0$.

Assume the contrary: $y(v_3v_4v_6v_3) > 0$. Since neither v_1v_6 nor v_3v_1 is saturated by $\boldsymbol{y}$ in G_3, we distinguish among four subsubcases.

(a) Neither v_1v_6 nor v_3v_1 is saturated by $\boldsymbol{y}$ in T. In this subsubcase, set $\theta = \min\{w(v_1v_6) - z(v_1v_6), w(v_3v_1) - z(v_3v_1), y(v_3v_4v_6v_3)\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_6v_3)$ and $y(v_1v_6v_3v_1)$ with $y(v_3v_4v_6v_3) - \theta$ and $y(v_1v_6v_3v_1) + \theta$, respectively. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (4).

(b) v_3v_1 is not saturated by $\boldsymbol{y}$ in T and v_1v_6 is contained in some cycle $C_1 \in \mathcal{C}_0^y$. In this subsubcase, since v_6v_3 is saturated by $\boldsymbol{y}$ in G_3, cycle C_1 contains the path $v_6v_2v_4$. Thus the multiset sum of C_1, $v_3v_4v_6v_3$, and v_3v_1 contains two arc-disjoint cycles $v_2v_4v_6v_2$ and $v_1v_6v_3v_1$. By Lemma 4.7(iv), we have $y(v_3v_4v_6v_3) = 0$, a contradiction.

(c) v_1v_6 is not saturated by $\boldsymbol{y}$ in T and v_3v_1 is contained in some cycle $C_2 \in \mathcal{C}_0^y$. In this subsubcase, it is clear that C_2 contains the path $v_1v_2v_4$. Observe that the multiset sum of C_2, $v_3v_4v_6v_3$, and the unsaturated v_1v_6 contains two arc-disjoint cycles $v_1v_6v_3v_1$ and $C_2' = C_2[v_4, v_3] \cup \{v_3v_4\}$. Set $\theta = \min\{y(C_2), y(v_3v_4v_6v_3), w(v_1v_6) - z(v_1v_6)\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C_2)$, $y(v_3v_4v_6v_3)$, $y(v_1v_6v_3v_1)$, and $y(C_2')$ with $y(C_2) - \theta$, $y(v_3v_4v_6v_3) - \theta$, $y(v_1v_6v_3v_1) + \theta$, and $y(C_2') + \theta$, respectively. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (4).

(d) v_1v_6 and v_3v_1 are contained in some cycles C_1 and C_2 in $\mathcal{C}_0^y$, respectively. In this subsubcase, if v_3v_1 is also on C_1, then the multiset sum of C_1 and $v_3v_4v_6v_3$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_2v_4v_6v_2$, and $C_1' = C_1[v_4, v_3] \cup \{v_3v_4\}$. From the optimality of $\boldsymbol{y}$, we deduce that $y(v_3v_4v_6v_3) = 0$. If v_3v_1 is outside C_1, then the multiset sum of C_1, C_2, and $v_3v_4v_6v_3$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_2v_4v_6v_2$, $C_1' = C_1[v_4, v_1] \cup \{v_1v_2, v_2v_4\}$, and $C_2' = C_2[v_4, v_3] \cup \{v_3v_4\}$. From the optimality of $\boldsymbol{y}$, we again deduce that $y(v_3v_4v_6v_3) = 0$.

By (14), we have $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_6v_3\}$. So K is an MFAS by (5) and hence $y(\mathcal{C}_2) = \tau_w(G_3 \backslash v_5)$.

Case 1.2. v_2v_4 is outside $\mathcal{C}_0^y$.

In this case, v_2v_4 is saturated by $\boldsymbol{y}$ in G_3, and v_1v_2, v_3v_2, and v_6v_2 are all outside $\mathcal{C}_0^y$. Since $\mathcal{C}_0^y \neq \emptyset$, there exists a cycle $C \in \mathcal{C}_0^y$ containing v_3v_4. From (6), (7), and (10), we see that $y(v_1v_2v_4v_6v_3v_1)$, $y(v_1v_6v_3v_2v_4v_1)$, and $y(v_2v_4v_6v_3v_2)$ are all zero. If v_6v_3 is also saturated by $\boldsymbol{y}$ in G_3, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_2v_4, v_6v_3\}$. So we assume that v_6v_3 is not saturated by $\boldsymbol{y}$ in G_3. By Lemma 4.7(iii) and (iv), v_4v_6 is saturated by $\boldsymbol{y}$ in G_3.

Assume first that v_4v_1 is not saturated by $\boldsymbol{y}$ in G_3. Then, by Lemma 4.7(iii) and (iv), v_1v_6 is saturated by $\boldsymbol{y}$ in G_3. By (13), v_1v_2 is not saturated by $\boldsymbol{y}$ in G_3 and hence in T. By (9),

$y(v_1v_6v_2v_4v_1) = 0$. If v_6v_3 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of C, $v_2v_4v_6v_2$, and the unsaturated arcs v_6v_3, v_4v_1, and v_1v_2 contains two arc-disjoint cycles $v_1v_2v_4v_1$ and $v_3v_4v_6v_3$; if v_6v_3 is saturated by $\boldsymbol{y}$ in T but contained in some cycle C in $\mathcal{C}_0^y$, then the multiset sum of C, $v_2v_4v_6v_2$, and the unsaturated arcs v_4v_1 and v_1v_2 contains two arc-disjoint cycles $v_1v_2v_4v_1$ and $v_3v_4v_6v_3$. By Lemma 4.7(vi), we have $y(v_2v_4v_6v_2) = 0$ in either subcase. So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_6, v_2v_4\}$.

Assume next that v_4v_1 is saturated by $\boldsymbol{y}$ in G_3. Then, by (13), v_3v_1 and at least one of v_1v_2 and v_1v_6 are not saturated by $\boldsymbol{y}$ in G_3. By Lemma 4.7(iii) and (iv), both v_3v_1 and v_1v_6 are outside $\mathcal{C}_0^y$; using Lemma 4.7(i) and (iii), we further deduce that v_1v_6 is saturated by $\boldsymbol{y}$ in G_3. Thus, by (13), v_1v_2 is not saturated by $\boldsymbol{y}$ in G_3. It follows from (8) and (9) that $y(v_1v_6v_3v_4v_1) = y(v_1v_6v_2v_4v_1) = 0$. Therefore $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_1, v_4v_6\}$. So K is an MFAS by (5) and hence $y(\mathcal{C}_2) = \tau_w(G_3 \backslash v_5)$. This proves Claim 1.

Claim 2. $y(C)$ is integral for all $C \in \mathcal{C}_2$ or $\nu_w^*(T)$ is an integer.

To justify this, we may assume that

(15) $y(v_1v_2v_4v_6v_3v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$.

Assume the contrary: $y(v_1v_2v_4v_6v_3v_1) = 0$. Then, from (6) we deduce that $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$ and that each arc in the set $\{v_1v_6, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}$ is saturated by $\boldsymbol{y}$ in G_3. So $y(v_1v_2v_4v_1) = w(v_4v_1)$, $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_3v_4v_6v_3) = w(v_3v_4)$, $y(v_2v_4v_6v_2) = w(v_6v_2)$, and $y(v_2v_4v_6v_3v_1) = w(v_3v_2)$. By Claim 1, $y(\mathcal{C}_2)$ is an integer; so is $y(v_1v_2v_4v_6v_3v_1)$. Thus Lemma 4.4(iii) allows us to assume that $y(v_1v_2v_4v_6v_3v_1) = 0$.

If $y(v_1v_6v_3v_2v_4v_1) > 0$, then from (7) we deduce that $y(v_2v_4v_6v_2) = y(v_3v_4v_6v_3) = 0$ and that each arc in the set $\{v_1v_2, v_3v_1, v_3v_4, v_4v_6, v_6v_2\}$ is saturated by $\boldsymbol{y}$ in G_3. So $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$, and $y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. By Claim 1, $y(\mathcal{C}_2)$ is an integer; so is $y(v_1v_6v_3v_2v_4v_1)$. Thus Lemma 4.4(iii) allows us to further assume that $y(v_1v_6v_3v_2v_4v_1) = 0$.

By Claim 1, $y(\mathcal{C}_2) = w(K)$ for some $K \in \mathcal{F}_2$. Depending on what K is, we distinguish among 13 cases.

Case 2.1. $K = \{v_1v_6, v_2v_4, v_4v_6\}$.

In this case, by Lemma 4.3 (i), we have $y(v_2v_4v_6v_2) = y(v_1v_6v_2v_4v_1) = y(v_2v_4v_6v_3v_2) = y(v_1v_6v_3v_2v_4v_1) = y(v_1v_2v_4v_6v_3v_1) = 0$. By Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$, $y(v_1v_2v_4v_1) = w(v_2v_4)$, and $y(v_3v_4v_6v_3) = w(v_4v_6)$. If $y(v_1v_6v_3v_4v_1) > 0$, then by (8) we have one more equation $y(v_1v_6v_3v_1) = w(v_3v_1)$. So $y(C)$ is integral for any $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_3v_4v_1) = 0$.

Case 2.2. $K = \{v_4v_1, v_4v_6, v_6v_3\}$.

In this case, by Lemma 4.3 (i), we have $y(v_3v_4v_6v_3) = y(v_1v_6v_3v_4v_1) = y(v_2v_4v_6v_3v_2) = y(v_1v_6v_3v_2v_4v_1) = y(v_1v_2v_4v_6v_3v_1) = 0$. By Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_4v_1)$, $y(v_2v_4v_6v_2) = w(v_4v_6)$, and $y(v_1v_6v_3v_1) = w(v_6v_3)$. If $y(v_1v_6v_2v_4v_1) > 0$, then by (9) we have one more equation $y(v_1v_2v_4v_1) = w(v_1v_2)$. So $y(C)$ is integral for any $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1) = 0$.

Case 2.3. $K = \{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2\}$.

In this case, by Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_2v_4v_6v_3v_2) = w(v_3v_2)$, $y(v_3v_4v_6v_3)+y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, and $y(v_2v_4v_6v_2)+y(v_1v_6v_2v_4v_1) = w(v_6v_2)$. Observe that if $y(v_1v_6v_3v_4v_1) > 0$, then by (8) we have $y(v_3v_4v_6v_3) = w(v_4v_6) - w(v_3v_2)$ and $y(v_2v_4v_6v_2) = 0$; if $y(v_1v_6v_3v_4v_1) = 0$ and $y(v_1v_6v_2v_4v_1) > 0$, then by (9) we have $y(v_2v_4v_6v_2) = w(v_4v_6) - w(v_3v_2) - w(v_3v_4)$. So $y(C)$ is integral for any $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1)$ or $y(v_1v_6v_3v_4v_1)$ is zero.

Case 2.4. $K = \{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\}$.

In this case, by Lemma 4.3 (i), we have $y(v_1v_6v_3v_4v_1) = y(v_1v_6v_2v_4v_1) = 0$. By Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_2v_4v_6v_3v_2) = w(v_3v_2)$, $y(v_3v_4v_6v_3) = w(v_3v_4)$, and $y(v_2v_4v_6v_2) = w(v_6v_2)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.5. $K = \{v_3v_1, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}$.

In this case, by Lemma 4.3 (i), we have $y(v_1v_6v_3v_4v_1) = 0$. By Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_2v_4v_6v_3v_2) = w(v_3v_2)$, $y(v_3v_4v_6v_3) = w(v_3v_4)$, $y(v_1v_2v_4v_1) = w(v_4v_1)$, and $y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) = w(v_6v_2)$. Observe that if $y(v_1v_6v_2v_4v_1) > 0$, then by (9) we have $y(v_2v_4v_6v_2) = w(v_4v_6) - w(v_3v_2) - w(v_3v_4)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1)$ is zero.

Case 2.6. $K = \{v_1v_6, v_2v_4, v_3v_4\}$.

In this case, by Lemma 4.3 (i), we have $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. By Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) + y(v_2v_4v_6v_3v_2) = w(v_2v_4)$, and $y(v_3v_4v_6v_3) = w(v_3v_4)$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_2v_4v_6v_2) = w(v_6v_2)$ by (10). Since v_4v_1 and v_1v_2 are outside $\mathcal{C}_0^y$ and $\boldsymbol{y}$ satisfies (2), it is easy to see that $y(v_1v_2v_4v_1) = \min\{w(v_1v_2), w(v_4v_1)\}$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$. Thus we may assume that $y(v_2v_4v_6v_3v_2) = 0$. Since both v_4v_6 and v_6v_2 are outside $\mathcal{C}_0^y$, by (4) we have $y(v_2v_4v_6v_2) = \min\{w(v_6v_2), w(v_4v_6) - w(v_3v_4)\}$. It follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.7. $K = \{v_2v_4, v_3v_1, v_3v_4\}$.

In this case, by Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) + y(v_2v_4v_6v_3v_2) = w(v_2v_4)$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, and $y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) = w(v_3v_4)$.

Assume first that $y(v_1v_6v_3v_4v_1) > 0$. Then, by (8), we have $y(v_2v_4v_6v_2) = 0$ and $y(v_3v_4v_6v_3)+ y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. If $y(v_2v_4v_6v_3v_2) > 0$, then, by (8) and (10), we obtain $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$; if $y(v_2v_4v_6v_3v_2) = 0$ and $y(v_1v_6v_2v_4v_1) > 0$, then, by (9), we get $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_2v_4v_1) = w(v_2v_4) - w(v_1v_2)$, and $y(v_3v_4v_6v_3) = w(v_4v_6)$; if $y(v_1v_6v_2v_4v_1) = y(v_2v_4v_6v_3v_2) = 0$, then $y(v_1v_2v_4v_1) = w(v_2v_4)$, and $y(v_3v_4v_6v_3) = w(v_4v_6)$. Thus $y(C)$ is integral for all $C \in \mathcal{C}_2$ in any subcase.

Assume next that $y(v_1v_6v_3v_4v_1) = 0$. If $y(v_1v_6v_2v_4v_1) > 0$, then, by (9), we have $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_2v_4v_6v_2) + y(v_2v_4v_6v_3v_2) = w(v_4v_6) - y(v_3v_4v_6v_3) = w(v_4v_6) - w(v_3v_4)$, and so $y(v_1v_6v_2v_4v_1) = w(v_2v_4) + w(v_3v_4) - w(v_1v_2) - w(v_4v_6)$. Observe that if $y(v_2v_4v_6v_3v_2) > 0$, then we have one more equation $y(v_2v_4v_6v_2)+y(v_1v_6v_2v_4v_1) = w(v_6v_2)$ by (10). Thus $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_2v_4v_6v_3v_2) = 0$. So we assume that $y(v_1v_6v_2v_4v_1) = 0$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_2v_4v_6v_2) = w(v_6v_2)$ and $y(v_1v_2v_4v_1) + y(v_2v_4v_6v_3v_2) = w(v_2v_4) - w(v_6v_2)$; if $y(v_2v_4v_6v_3v_2) = 0$, then $y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) = w(v_2v_4)$. Since $\boldsymbol{y}$ satisfies

(2) and (4) and since v_4v_1, v_4v_6, v_1v_2, and v_6v_2 are all outside $\mathcal{C}_0^y$, if $y(v_1v_2v_4v_1) > 0$, then $y(v_1v_2v_4v_1) = \min\{w(v_4v_1), w(v_1v_2)\}$ or $y(v_2v_4v_6v_2) = \min\{w(v_4v_6) - y(v_3v_4v_6v_3), w(v_6v_2)\}$, regardless of the value of $y(v_2v_4v_6v_3v_2)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.8. $K = \{v_1v_2, v_6v_2, v_6v_3\}$.

In this case, by Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) = w(v_6v_2)$, and $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3)$. Depending on the value of $y(v_1v_6v_3v_4v_1)$, we consider two subcases.

• $y(v_1v_6v_3v_4v_1) > 0$. In this subcase, by (8), we have $y(v_2v_4v_6v_2) = 0$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, and $y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. So $y(v_1v_6v_3v_4v_1) = w(v_6v_3) - w(v_3v_1) - w(v_4v_6)$. Observe that if $y(v_2v_4v_6v_3v_2) > 0$, then we have one more equation $y(v_3v_4v_6v_3) = w(v_3v_4) - y(v_1v_6v_3v_4v_1)$ by (10). So $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_2v_4v_6v_3v_2) = 0$.

• $y(v_1v_6v_3v_4v_1) = 0$. In this subcase, assume first that $y(v_1v_6v_2v_4v_1) > 0$. If $y(v_3v_4v_6v_3) > 0$ or $y(v_2v_4v_6v_3v_2) > 0$, then, by (9), we have $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_3v_4v_6v_3)+y(v_2v_4v_6v_3v_2) = w(v_6v_3) - w(v_3v_1)$, and $y(v_2v_4v_6v_2) = w(v_4v_6) + w(v_3v_1) - w(v_6v_3)$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (10). Thus $y(v_2v_4v_6v_3v_2)$ and $y(v_1v_6v_2v_4v_1)$ are integral. If $y(v_3v_4v_6v_3) = y(v_2v_4v_6v_3v_2) = 0$, then $y(v_2v_4v_6v_2) = w(v_4v_6)$ and $y(v_1v_6v_2v_4v_1) = w(v_6v_2) - w(v_4v_6)$. So $y(C)$ is integral for all $C \in \mathcal{C}_2$ in any subsubcase. Assume next that $y(v_1v_6v_2v_4v_1) = 0$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (10) and $y(v_1v_6v_3v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3) - w(v_3v_4)$; if $y(v_2v_4v_6v_3v_2) = 0$, then $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) = w(v_6v_3)$. Note that both v_3v_1 and v_1v_6 are outside $\mathcal{C}_0^y$. As $\boldsymbol{y}$ satisfies (2) and (4), we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_1v_6), w(v_3v_1)\}$, no matter whether $y(v_2v_4v_6v_3v_2) > 0$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.9. $K = \{v_4v_1, v_6v_2, v_6v_3\}$.

In this case, by Lemma 4.3 (i), we have $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. By Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_4v_1)$, $y(v_2v_4v_6v_2) = w(v_6v_2)$, and $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_6v_3)$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (10), so $y(v_1v_6v_3v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3) - w(v_3v_4)$; if $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_1v_6v_3v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3)$. Clearly, v_1v_6 is outside $\mathcal{C}_0^y$. We propose to show that

(16) $y(v_1v_6v_3v_1)$ is integral.

Suppose on the contrary that $y(v_1v_6v_3v_1)$ is not integral. If v_3v_1 is outside $\mathcal{C}_0^y$, then from (2) and (4) we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_1v_6)\}$, a contradiction. So we assume that v_3v_1 is contained in some cycle C in $\mathcal{C}_0^y$. Then C contains the path $v_1v_2v_4$. Set $C' = C[v_4, v_3] \cup \{v_3v_2, v_2v_4\}$ if $y(v_2v_4v_6v_3v_2) > 0$ and $C' = C[v_4, v_3] \cup \{v_3v_4\}$ otherwise, and set $\theta = \min\{[y(v_2v_4v_6v_3v_2)], y(C)\}$ if $y(v_2v_4v_6v_3v_2) > 0$ and $\theta = \min\{[y(v_3v_4v_6v_3)], y(C)\}$ otherwise. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_2v_4v_6v_3v_2)$ (resp. $y(v_3v_4v_6v_3)$), $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_2v_4v_6v_3v_2)-\theta$ (resp. $y(v_3v_4v_6v_3)-\theta$), $y(v_1v_6v_3v_1)+\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. Then $y'(v_2v_4v_6v_3v_2) < y(v_2v_4v_6v_3v_2)$ or $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (2) or (4). So (16) is established.

From (16) it follows that $y(C)$ is integral for all $C \in \mathcal{C}_2$.

Case 2.10. $K = \{v_2v_4, v_6v_3\}$.

In this case, by Lemma 4.3 (i), we have $y(v_2v_4v_6v_3v_2) = 0$. By Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) = w(v_2v_4)$ and $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) = w(v_6v_3)$. It follows that all arcs in $G_3\backslash v_5$ are outside $\mathcal{C}_0^y$ except possibly v_3v_4. If $y(v_1v_6v_3v_4v_1) > 0$, then, by (8), we have $y(v_2v_4v_6v_2) = 0$, $y(v_1v_6v_3v_1) = w(v_3v_1)$, and $y(v_3v_4v_6v_3) = w(v_4v_6)$. Observe that if $y(v_1v_6v_2v_4v_1) > 0$, then we have one more equation $y(v_1v_2v_4v_1) = w(v_1v_2)$. Thus $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1) = 0$. So we assume that $y(v_1v_6v_3v_4v_1) = 0$.

If $y(v_1v_6v_2v_4v_1) > 0$, then, by (9), we obtain $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) = w(v_4v_6)$. Furthermore, $y(v_1v_6v_3v_1) = w(v_3v_1)$ if $y(v_3v_4v_6v_3) > 0$ and $y(v_1v_6v_3v_1) = w(v_6v_3)$ otherwise. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_3v_4v_6v_3) = 0$. So we may assume that $y(v_1v_6v_2v_4v_1) = 0$.

If $y(v_3v_4v_6v_3) = 0$, then $y(v_1v_6v_3v_1) = w(v_6v_3)$. Recall that both v_4v_6 and v_6v_2 are outside $\mathcal{C}_0^y$. If $y(v_1v_2v_4v_1) > 0$, then from (4) we deduce that $y(v_2v_4v_6v_2) = \min\{w(v_4v_6), w(v_6v_2)\}$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_2v_4v_1) > 0$. It remains to consider the subcase when $y(v_3v_4v_6v_3) > 0$. Since both v_3v_1 and v_1v_6 are outside $\mathcal{C}_0^y$, from (4) we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_1v_6)\}$. If $y(v_1v_2v_4v_1) = 0$, then $y(v_2v_4v_6v_2) = w(v_2v_4)$; otherwise, by (4), at least one of v_4v_6 and v_6v_2 is saturated by $\boldsymbol{y}$ in G_3. It follows that $y(v_2v_4v_6v_2) = \min\{w(v_6v_2), w(v_4v_6) - y(v_3v_4v_6v_3)\}$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_2v_4v_1) = 0$.

Case 2.11. $K = \{v_3v_1, v_4v_1, v_4v_6\}$.

In this case, by Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_2v_4v_1) + y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1)$, and $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. Depending on the value of $y(v_1v_6v_3v_4v_1)$, we consider two subcases.

• $y(v_1v_6v_3v_4v_1) > 0$. In this subcase, $y(v_2v_4v_6v_2) = 0$ by (8). If $y(v_2v_4v_6v_3v_2) > 0$, then, by (8) and (10), we have $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$, and $y(v_3v_4v_6v_3) = w(v_3v_4)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_2v_4v_6v_3v_2) = 0$. Then $y(v_3v_4v_6v_3) = w(v_4v_6)$. Depending on the value of $y(v_1v_6v_2v_4v_1)$, we distinguish between two subsubcases.

(a) $y(v_1v_6v_2v_4v_1) > 0$. By (9), $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1) - w(v_1v_2)$. If $y(v_1v_6v_2v_4v_1)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_1v_6v_2v_4v_1)$ is not integral. By Lemma 4.4(iii), we may assume that $w(v_3v_1)$, $w(v_1v_2)$, and $w(v_4v_6)$ are all zero. Observe that v_6v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_6v_2. Then C passes through v_2v_4. Let $C' = C[v_4, v_6] \cup \{v_6v_3, v_3v_4\}$, let $\theta = \min\{y(C), y(v_1v_6v_3v_4v_1)\}$, and let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_4v_1)$, $y(v_1v_6v_2v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_4v_1)-\theta$, $y(v_1v_6v_2v_4v_1)+\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (3). Similarly, we can prove that v_3v_2 is outside $\mathcal{C}_0^y$. Thus $w(v_3v_2) = z(v_3v_2) = 0$. We propose to show that

(17) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since $y(v_1v_6v_2v_4v_1) > 0$ and $y(v_1v_6v_3v_4v_1) > 0$, by Lemma 4.3(i) we have $x(v_6v_2) + x(v_2v_4) = x(v_6v_3) + x(v_3v_4)$. Since $y(v_1v_6v_2v_4v_1) < w(v_6v_2)$, by Lemma 4.3(ii) we obtain $x(v_6v_2) = 0$, which implies $x(v_2v_4) = x(v_6v_3) + x(v_3v_4)$. Since v_6v_2 is outside $\mathcal{C}_0^y$, for each vertex u in $V(T_1)\backslash\{b, a_1\}$, we obtain $x(uv_6) = x(uv_2)$. Let $T' = (V', A')$ be obtained from T by deleting vertex v_2, let $\boldsymbol{w'}$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(uv_6)$ with $w(uv_6)+w(uv_2)$ for each u in $V(T_1)\backslash\{b,a_1\}$ and replacing $w(v_iv_j)$ with $w(v_iv_j)+w(v_2v_4)$ for $(i,j)=(6,3)$ or $(3,4)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be defined from $\boldsymbol{y}$ as follows: for each cycle C passing through uv_2v_4 with $u\in V(T_1)\backslash\{b,a_1\}$, let C' be the cycle arising from C by replacing uv_2v_4 with $uv_6v_3v_4$, and set $y'(C')=y(C)+y(C')$ and $y'(v_1v_6v_3v_4v_1)=y(v_1v_6v_3v_4v_1)+y(v_1v_6v_2v_4v_1)$. Then $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T',\boldsymbol{w}')$ and $\mathbb{D}(T',\boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence $\nu_w^*(T)$ is an integer by the hypothesis of Theorem 4.1. So (17) follows.

(b) $y(v_1v_6v_2v_4v_1)=0$. Then $y(v_1v_2v_4v_1)+y(v_1v_6v_3v_4v_1)=w(v_4v_1)$. If $y(v_1v_2v_4v_1)$ is integral, then $y(C)$ is integral for all $C\in\mathcal{C}_2$. So we assume that $y(v_1v_2v_4v_1)$ is not integral. Observe that v_1v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_1v_2. Since the multiset sum of C and $v_1v_6v_3v_4v_1$ contains arc-disjoint cycles $v_1v_2v_4v_1$ and $C'=C[v_4,v_1]\cup\{v_1v_6,v_6v_3,v_3v_4\}$. By Lemma 4.7(vi), we have $y(C)=0$, a contradiction. Similarly, we can prove that v_6v_2 and v_3v_2 are outside $\mathcal{C}_0^y$ as well. Thus $w(v_iv_2)=z(v_iv_2)=0$ for $i=3,6$. We propose to show that

(18) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T,\boldsymbol{w})$. Since both $y(v_1v_2v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ are positive, by Lemma 4.3(i) we have $x(v_1v_2)+x(v_2v_4)=x(v_1v_6)+x(v_6v_3)+x(v_3v_4)$. Since $y(v_1v_2v_4v_1)<w(v_1v_2)$, by Lemma 4.3(ii) we obtain $x(v_1v_2)=0$, which implies that $x(v_2v_4)=x(v_1v_6)+x(v_6v_3)+x(v_3v_4)$. Since v_1v_2 is outside $\mathcal{C}_0^y$, for each vertex $u\in V(T_1)\backslash\{b,a_1\}$, we obtain $x(uv_1)=x(uv_2)$. Let $T'=(V',A')$ be obtained from T by deleting vertex v_2, and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to A' by replacing $w(uv_1)$ with $w(uv_1)+w(uv_2)$ for each $u\in V(T_1)\backslash\{b,a_1\}$ and replacing $w(v_iv_j)$ with $w(v_iv_j)+w(v_2v_4)$ for $(i,j)=(1,6)$, $(6,3)$, and $(3,4)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be defined from $\boldsymbol{y}$ as follows: for each cycle C passing through uv_2v_4 with $u\in V(T_1)\backslash\{b,a_1\}$, let C' be obtained from C by replacing uv_2v_4 with $uv_1v_6v_3v_4$, and set $y'(C')=y(C)+y(C')$ and $y'(v_1v_6v_3v_4v_1)=y(v_1v_6v_3v_4v_1)+y(v_1v_2v_4v_1)$. Then $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T',\boldsymbol{w}')$ and $\mathbb{D}(T',\boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence $\nu_w^*(T)$ is an integer by the hypothesis of Theorem 4.1. This proves (18).

• $y(v_1v_6v_3v_4v_1)=0$. In this subcase, $y(v_1v_2v_4v_1)=w(v_1v_2)$. By (9), if $y(v_1v_6v_2v_4v_1)>0$, then $y(v_1v_6v_2v_4v_1)=w(v_4v_1)-w(v_1v_2)$; otherwise, $y(v_1v_2v_4v_1)=w(v_4v_1)$. If $y(v_2v_4v_6v_3v_2)>0$, then, by (10), we have $y(v_3v_4v_6v_3)=w(v_3v_4)$, $y(v_2v_4v_6v_2)=w(v_6v_2)-y(v_1v_6v_2v_4v_2)$, and $y(v_2v_4v_6v_3v_2)=w(v_4v_6)-w(v_3v_4)-y(v_2v_4v_6v_2)$. Hence $y(C)$ is integral for all $C\in\mathcal{C}_2$. So we assume that $y(v_2v_4v_6v_3v_2)=0$. Thus $y(v_2v_4v_6v_2)+y(v_3v_4v_6v_3)=w(v_4v_6)$. If $y(v_2v_4v_6v_2)$ is integral, then $y(C)$ is integral for all $C\in\mathcal{C}_2$. So we further assume that $y(v_2v_4v_6v_2)$ is not integral. By Lemma 4.4(iii), we may assume that $w(v_3v_1)=w(v_4v_1)=0$. Observe that v_6v_2 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_6v_2. Then C passes through v_2v_4. Let $C'=C[v_4,v_6]\cup\{v_6v_3,v_3v_4\}$, let $\theta=\min\{y(C),y(v_3v_4v_6v_3)\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_6v_3)$, $y(v_2v_4v_6v_2)$, $y(C)$, and $y(C')$ with $y(v_3v_4v_6v_3)-\theta$, $y(v_2v_4v_6v_2)+\theta$, $y(C)-\theta$, and $y(C')+\theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$ with $y'(v_3v_4v_6v_3)<y(v_3v_4v_6v_3)$, contradicting (4). Similarly, we can show that v_3v_2 is outside $\mathcal{C}_0^y$. So $w(v_3v_2)=z(v_3v_2)=0$. Moreover, $\nu_w^*(T)$ is an integer; the proof is the same as that of (17) (with $y(v_2v_4v_6v_2)$ and $y(v_3v_4v_6v_3)$ in place of $y(v_1v_6v_2v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$, respectively), so we omit the details here.

Case 2.12. $K = \{v_1v_6, v_4v_1, v_4v_6\}$.

In this case, by Lemma 4.3 (i), we have $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. By Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_1v_2v_4v_1) = w(v_4v_1)$, and $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. If $y(v_2v_4v_6v_3v_2) > 0$, then, by (10), we have $y(v_2v_4v_6v_2) = w(v_6v_2)$ and $y(v_3v_4v_6v_3) = w(v_3v_4)$, so $y(v_2v_4v_6v_3v_2) = w(v_4v_6) - w(v_6v_2) - w(v_3v_4)$. Hence $y(C)$ is integral for all $C \in \mathcal{C}_2$, It remains to assume that $y(v_2v_4v_6v_3v_2) = 0$. Then $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) = w(v_4v_6)$. If $y(v_2v_4v_6v_2)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we further assume that $y(v_2v_4v_6v_2)$ is not integral. Then we can prove that $\nu_w^*(T)$ is an integer; the proof is the same as that of (17), so we omit the details here.

Case 2.13. $K = \{v_1v_2, v_1v_6, v_4v_6\}$.

In this case, by Lemma 4.3 (iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2)$, $y(v_1v_6v_3v_1) + y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$, and $y(v_2v_4v_6v_2) + y(v_3v_4v_6v_3) + y(v_2v_4v_6v_3v_2) = w(v_4v_6)$. Clearly, v_3v_1 is outside $\mathcal{C}_0^y$. Depending on the value of $y(v_1v_6v_3v_4v_1)$, we consider two subcases.

• $y(v_1v_6v_3v_4v_1) > 0$. In this subcase, $y(v_2v_4v_6v_2) = 0$ and $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (8). If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$ and $y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) = w(v_3v_4)$ by (10). Thus $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we assume that $y(v_2v_4v_6v_3v_2) = 0$. Then $y(v_3v_4v_6v_3) = w_{46}$ and $y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6) - w(v_3v_1)$. If $y(v_1v_6v_2v_4v_1)$ is integral, then $y(C)$ is integral for all $C \in \mathcal{C}_2$. So we further assume that $y(v_1v_6v_2v_4v_1)$ is not integral. Then we can prove that $\nu_w^*(T)$ is an integer; the proof is the same as that of (17), so we omit the details here.

• $y(v_1v_6v_3v_4v_1) = 0$. In this subcase, $y(v_1v_6v_3v_1)+y(v_1v_6v_2v_4v_1) = w(v_1v_6)$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ and $y(v_2v_4v_6v_2) + y(v_1v_6v_2v_4v_1) = w(v_6v_2)$ by (10). Observe that if $y(v_1v_6v_2v_4v_1) > 0$, then we have one more equation $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (9). So $y(C)$ is integral for all $C \in \mathcal{C}_2$, no matter whether $y(v_1v_6v_2v_4v_1) = 0$. Thus we may assume that $y(v_2v_4v_6v_3v_2) = 0$. We proceed by considering two subsubcases.

(a) Assume first that $y(v_3v_4v_6v_3) = 0$. Then $y(v_2v_4v_6v_2) = w(v_4v_6)$. If $y(v_1v_6v_3v_1)$ is integral, then so is $y(C)$ for all $C \in \mathcal{C}_2$. Thus we assume that $y(v_1v_6v_3v_1)$ is not integral. If v_6v_3 is outside $\mathcal{C}_0^y$, then it follows from (4) that $y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_6v_3)\}$; this contradiction implies that v_6v_3 is contained in a cycle C in $\mathcal{C}_0^y$. Let $C' = C[v_4, v_6] \cup \{v_6v_2, v_2v_4\}$, let $\theta = \min\{[y(v_1v_6v_2v_4v_1)], y(C)\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_2v_4v_1)$, $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_2v_4v_1) - \theta$, $y(v_1v_6v_3v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_2v_4v_1) < y(v_1v_6v_2v_4v_1)$, contradicting (2).

(b) Assume next that $y(v_3v_4v_6v_3) > 0$. If $y(v_1v_6v_2v_4v_1) > 0$, then $y(v_1v_6v_3v_1) = w(v_3v_1)$ and $y(v_1v_6v_2v_4v_1) = w(v_1v_6) - w(v_3v_1)$ by (9); otherwise, $y(v_1v_6v_3v_1) = w(v_1v_6)$. If $y(v_3v_4v_6v_3)$ is integral, then so is $y(C)$ for all $C \in \mathcal{C}_2$. Thus we assume that $y(v_3v_4v_6v_3)$ is not integral. Let us prove that

(19) $\nu_w^*(T)$ is an integer.

By Lemma 4.4(iii), we may assume that $w(v_1v_2) = w(v_1v_6) = 0$. Let $T' = (V', A')$ be obtained from T by deleting v_1, and let $\boldsymbol{w}$ be the restriction of $\boldsymbol{w}$ to A'. It is routine to check that $\mathbb{D}(T', \boldsymbol{w}')$ has the same optimal value $\nu_w^*(T)$ as $\mathbb{D}(T, \boldsymbol{w})$. Hence $\nu_w^*(T)$ is an integer by the hypothesis of Theorem 4.1. This proves (19) and hence Claim 2.

Since $\tau_{\boldsymbol{w}}(G_3 \backslash v_5) > 0$, from Claim 2, Lemma 4.4(iii) and Lemma 4.6(ii) we deduce that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. This completes the proof of Lemma 5.6. ∎

Now we are ready to establish the main result of this section.

Proof of Theorem 5.1. By the hypothesis of this section, T is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2, with $T_2 \in \mathcal{T}_2$. Since $\mathcal{T}_2 = \{F_0, F_2, F_3, F_4, F_6, G_2, G_3\}$, the desired statement follows instantly from Lemmas 5.2-5.6. ∎

6 Composite Reductions

Throughout this section, we assume that $(T, \boldsymbol{w})$ is an instance as described in Theorem 4.1, and that $T = (V, A)$ is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2), such that

(α) $\tau_w(T_2 \backslash a_2) > 0$;

(β) there exists a vertex subset S of $T_2 \backslash \{a_2, b_2\}$ with $|S| \geq 2$ and with the following properties:
- $T[S]$ is acyclic and $T_2/S \in \mathcal{T}_3$; and
- the vertex s^* arising from contracting S is a near-sink in T/S.

From (β) we see that S is actually a homogeneous set of T. The purpose of this section is to establish the following statement.

Theorem 6.1. *For the above instance $(T, \boldsymbol{w})$, problem $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Let us label T_2/S as in Figures 3-7. Since (b_2, a_2) is a special arc, a_2 is a near-source of T_2, and s^* is a near-sink in T/S, we have
- $(b_2, a_2) = (v_1, v_2)$ and $s^* = v_3$ or v_4 if $T_2/S = F_0$;
- $(b_2, a_2) = (v_5, v_2)$ and $s^* = v_1$ if $T_2/S = F_3$;
- $(b_2, a_2) = (v_5, v_6)$ and $s^* = v_2$ if $T_2/S = F_4$;
- $(b_2, a_2) = (v_5, v_6)$ and $s^* = v_2$ if $T_2/S = F_6$;
- $(b_2, a_2) = (v_4, v_5)$ and $s^* = v_2$ if $T_2/S = G_2$ or G_3;
- $(b_2, a_2) = (v_1, v_5)$ and $s^* = v_4$ if $T_2/S = G_4$;
- $(b_2, a_2) = (v_2, v_6)$ and $s^* = v_5$ if $T_2/S = G_5$; and
- $(b_2, a_2) = (v_6, v_7)$ and $s^* = v_5$ if $T_2/S = G_6$,

where the last three follow from Lemma 4.2(ii). Observe that if $T_2/S = F_0$, then $(b_2, a_2) \neq (v_4, v_1)$, for otherwise, $T_2 \backslash v_1$ is acyclic, contradicting (α).

Since $T[S]$ is acyclic, we can label the vertices in S as $s_1, s_2, \ldots, s_r$ such that $s_j s_i$ is an arc in T for any $1 \leq i < j \leq r$, where $r = |S|$. For convenience, we use v_0 to denote the only out-neighbor of S in $T_2 \backslash a_2$ (for example, $v_0 = v_3$ if $T_2/S = F_3$), use f_i to denote the arc $s_i v_0$, and use R to denote the vertex subset $V \backslash (S \cup \{v_0\})$.

In this section, we employ the same notations as introduced in Sections 4 and 5. In particular, given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, we use $\mathcal{C}^y$ to denote $\{C \in \mathcal{C} : y(C) > 0\}$ and use $\mathcal{C}_i^y$ to denote $\{C \in \mathcal{C}_i : y(C) > 0\}$ for $i = 0, 1, 2$. For each arc e of T, we use $z(e)$ to denote $y(\mathcal{C}(e))$. Let G be a digraph with a weight on each arc and let U be a vertex subset of G. By *reorienting $G[U]$ acyclically* we mean the operation of reorienting some arcs of $G[U]$ so that the resulting subgraph induced by U is acyclic, where each new arc is associated with the same weight as its reverse in G.

Lemma 6.2. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively. Then we may assume that the following statements hold:*

(i) $z(s_j s_i) = w(s_j s_i) = 0$ for any $1 \le i < j \le r$ (so if we reorient $T[S]$ acyclically, then the resulting digraph is isomorphic to T, and the optimal value of the resulting $\mathbb{D}(T, \boldsymbol{w})$ remains the same);

(ii) $x(f_i)z(f_i) > 0$ for any $1 \le i \le r$;

(iii) $z(f_i) = w(f_i) > 0$ for any $1 \le i \le r$;

(iv) $x(f_i) \ne x(f_j)$ for any $1 \le i < j \le r$;

(v) Every cycle $C \in \mathcal{C}^y$ contains at most one vertex from S; and

(vi) $z(us_i)z(us_j) = 0$ for any $u \in R$ and $1 \le i < j \le r$.

Proof. (i) Assume the contrary: $z(s_j s_i) > 0$ and, subject to this, $j + i$ is minimized. Then there exists a cycle D passing through $s_j s_i v_0$ with $y(D) > 0$.

Consider first the case when $x(s_j s_i) = 0$. If $z(f_j) > 0$, then $x(f_j) = x(s_j s_i) + x(f_i) = x(f_i)$ by Lemma 4.3(iv). If $z(f_j) = 0$, then $w(f_j) = 0$ by Lemma 4.4(i). Since $x(C) \ge 1$ for any $C \in \mathcal{C}$, we have $x(f_j) \ge x(s_j s_i) + x(f_i)$; replacing $x(f_j)$ by $x(s_j s_i) + x(f_i)$ if necessary, the resulting $\boldsymbol{x}$ is also an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. So we may assume that $x(f_j) = x(s_j s_i) + x(f_i) = x(f_i)$. Similarly, we may assume that $x(us_j) = x(us_i)$ for any $u \in R$. Let $T' = (V', A')$ be obtained from T by deleting s_j. Note that T' also arises from T by identifying s_i with s_j and then deleting some arcs incident with s_j. Let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(f_i)$ with $w(f_i) + w(f_j)$ and replacing $w(us_i)$ with $w(us_i) + w(us_j)$ for every $u \in R$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A', and let $\boldsymbol{y}'$ be the projection of $\boldsymbol{y}$ into the set of all cycles in T'. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively, having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer. It follows from Lemma 4.6(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

Next consider the case when $x(s_j s_i) > 0$. By Lemma 4.3(iii), $w(s_j s_i) = z(s_j s_i)$. Let $\boldsymbol{w}'$ be obtained from $\boldsymbol{w}$ by replacing $w(f_j)$ with $w(f_j) + w(s_j s_i)$ and replacing $w(e)$ with $w(e) - w(s_j s_i)$ for $e = s_j s_i$ and f_i, let $\boldsymbol{x}' = \boldsymbol{x}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through $s_j s_i$ with $y(C) > 0$, let C' be the cycle obtained from C by replacing the path $s_j s_i v_0$ with f_j, and set $y'(C) = 0$ and $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w}')$ and $\mathbb{D}(T, \boldsymbol{w}')$, respectively, having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Since $w'(A) < w(A)$, by the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer. It follows from Lemma 4.6(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

Combining the above two cases, we may assume that $z(s_j s_i) = 0$ and hence $w(z_j z_i) = 0$ by Lemma 4.4(i) for any $1 \le i < j \le r$. From (β) we see that S is a homogeneous set of T, so if we reorient $T[S]$ acyclically, then the resulting digraph is isomorphic to T. Given the weights $w(z_j z_i)$ for all $1 \le i < j \le r$, it is clear that the optimal value of the resulting $\mathbb{D}(T, \boldsymbol{w})$ remains the same.

(ii) Assume the contrary: $x(f_i)z(f_i) = 0$ for some i. Consider first the case $z(f_i) = 0$. Let $T' = (V', A')$ be obtained from T by deleting s_i, and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to A'. Then $\mathbb{D}(T', \boldsymbol{w}')$ has an integral optimal solution by the hypothesis of Theorem 4.1. From (i) and the value of $z(f_i)$, we deduce that s_i is contained in no cycle C with $y(C) > 0$, so $\mathbb{D}(T', \boldsymbol{w}')$ has the same optimal value $\nu_w^*(T)$ as $\mathbb{D}(T', \boldsymbol{w}')$. It follows from Lemma 4.6(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. Thus we may assume that $z(f_j) > 0$ for any $1 \le j \le r$.

Next consider the case when $x(f_i) = 0$. Observe that for any $u \in R$ with $uv_0 \in A$, if $z(uv_0)z(us_i) > 0$, then $x(uv_0) = x(us_i) + x(f_i) = x(us_i)$ by Lemma 4.3(iv), so $x(uv_0) = x(us_i)$; if $z(uv_0)z(us_i) = 0$, modifying $x(uv)$ for $v \in \{v_0, s_i\}$ with $z(uv) = 0$ (thus $w(uv) = 0$) so that the equality $x(uv_0) = x(us_i) + x(f_i) = x(us_i)$ holds, the resulting $\boldsymbol{x}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Hence we may assume that $x(uv_0) = x(us_i)$.

Set $U = \{u \in R : z(us_i) > 0 \text{ and } uv_0 \notin A\}$. Let $T' = (V', A')$ be obtained from $T \backslash s_i$ by adding an arc uv_0 for each $u \in U$, and define $w(uv_0) = w(us_i)$ and $x(uv_0) = x(us_i)$ for each $u \in U$. Let $\boldsymbol{w}'$ be obtained from $\boldsymbol{w}$ by replacing $w(uv_0)$ with $w(uv_0) + w(us_i)$ for each $u \in R$ with $uv_0 \in A$, let $\boldsymbol{x}' = \boldsymbol{x}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through us_i with $y(C) > 0$, let C' be the cycle arising from C by replacing the path us_iv_0 with uv_0, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. In view of (i), we may assume that $i = 1$. So $f_i = f_1$ is a special arc of T. By Lemma 2.4, $T' = T/f_1$ is a Möbius-free digraph and thus, by Lemma 4.5, $\nu_w^*(T)$ is integral. It follows from Lemma 4.6(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

(iii) The statement follows directly from (ii), Lemma 4.4(i), and Lemma 4.3(iii).

(iv) Assume on the contrary that $x(f_i) = x(f_j)$ for some $1 \le i < j \le r$. Observe that for any $u \in R$, if $z(us_i)z(us_j) > 0$, then $x(us_i) + x(f_i) = x(us_j) + x(f_j)$ by Lemma 4.3(iv), so $x(us_i) = x(us_j)$; if $z(us_i)z(us_j) = 0$, letting (k, l) be a permutation of (i, j) with $z(us_k) = 0$, and replacing x_k by x_l if necessary, the resulting $\boldsymbol{x}$ is also an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. So we may assume that $x(us_i) = x(us_j)$. Let $T' = (V', A')$ be obtained from T by deleting s_i, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(us_j)$ with $w(us_j) + w(us_i)$ for any $u \in R$ and replacing $w(f_j)$ with $w(f_j) + w(f_i)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from the restriction of $\boldsymbol{y}$ to cycles in T' as follows: for each cycle C passing through us_i with $y(C) > 0$, let C' be obtained from C by replacing the path us_iv_0 with the path us_jv_0, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer. Thus it follows from Lemma 4.6(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

(v) Suppose on the contrary that C contains two distinct vertices s_i and s_j in S. Let s_k^+ be the vertex succeeding s_k as we traverse C in its direction, for $k = i, j$. Since $y(C) > 0$, from (i) we deduce that s_i^+ and s_j^+ are two distinct vertices outside S. Thus the vertex s^* arising from contracting S would not be a near-sink in T/S, contradicting (β).

(vi) Assume the contrary: $z(us_i)z(us_j) > 0$ for some $u \in R$ and $1 \le i < j \le r$. Consider first the case when $z(us_k) \ge 1$ for $k = i$ or j. In view of (i), we may assume that $z(us_i) \ge 1$. Let T' be obtained from T by adding an arc uv_0 if it is not present in T and define $w(uv_0) = 0$, and let $\boldsymbol{w}'$ be obtained from $\boldsymbol{w}$ by replacing $w(a)$ with $w(a) - \lfloor z(e) \rfloor$ for $a \in \{e, f_i\}$ and replacing $w(uv_0)$ with $w(uv_0) + \lfloor z(e) \rfloor$. Let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$, and let $\boldsymbol{x}'$ be obtained from $\boldsymbol{x}$ by setting $x(uv_0) = x(e) + x(f_i)$. Let $\mathcal{D}$ be the set of all cycles C passing through e with $y(C) > 0$, let $\pi(C)$ be a constant between 0 and $y(C)$ such that $\pi(\mathcal{D}) = \lfloor z(e) \rfloor$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle $C \in \mathcal{D}$, let C' be obtained from C by replacing the path us_iv_0 with uv_0, set $y'(C) = y(C) - \pi(C)$ and $y'(C') = y(C') + \pi(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Let T'' be the tournament obtained from T be adding a new vertex s_0, an arc s_0v_0, and an arc uv_0 for each $u \in V\backslash\{v_0\}$. By Lemma 2.3, T'' is Möbius-free because it is the 1-sum of two smaller Möbius-free tournaments with hub v_0. By Lemma 2.4, the digraph G obtained from T'' by contracting s_0v_0 is also Möbius-free; so is T' because it is a subgraph of G. As $w(A') < w(A)$, from Lemma 4.5 we deduce that $\nu_w^*(T)$ is integral. Therefore, $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 4.6(ii).

So we may assume that $z(us_k) < 1$ for $k = i, j$. Thus $w(us_k) = \lceil z(us_k) \rceil = 1 > z(us_k)$ for $k = i, j$. It follows instantly from Lemma 4.3(ii) that $x(us_k) = 0$ for $k = i, j$. By Lemma 4.3(iv), we obtain $x(us_i) + x(f_i) = x(us_j) + x(f_j)$, and hence $x(f_i) = x(f_j)$, contradicting (iv). ∎

We break the proof of Theorem 6.1 into a series of lemmas.

Lemma 6.3. *If $T_2/S = F_6$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_5, v_6)$ and $s^* = v_2$. Clearly, $C = v_1v_3v_4v_1$ is the unique cycle contained in $T_2\backslash v_6$, which is a triangle. Since $\tau_w(T_2\backslash v_6) > 0$ by (α), we have $w(a) > 0$ for each arc a on C. Therefore $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 4.8. ∎

Lemma 6.4. *If $T_2/S = F_0$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_1, v_2)$ and $s^* = v_3$ or v_4. We only consider the case when $s^* = v_3$, as the proof in other case goes along the same line. To establish the statement, by Lemma 4.6(ii), it suffices to prove that

(1) the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. By Lemma 4.4(i), we have $w(e) = \lceil z(e) \rceil$ for each arc e in T. By (α) and Lemma 6.2(i) and (vi), there exists precisely one vertex s_k in S such that $z(v_1s_k) > 0$, which implies $y(v_1s_kv_4v_1) > 0$. By Lemma 6.2(i), we may assume that $s_k = s_1$, the sink of $T[S]$. Observe that T is also the 1-sum of two smaller Möbius-free tournaments T_1' and T_2' with the same hub b, where T_2' arises from T_2 by deleting $S\backslash s_1$. Since $v_1s_1v_4v_1$ is the unique cycle contained in $T_2'\backslash v_2$, which is a triangle, (1) follows instantly from Lemma 4.8. ∎

Lemma 6.5. *If $T_2/S = F_3$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_5, v_2)$, $s^* = v_1$, and $v_0 = v_3$. To establish the statement, by Lemma 4.6(ii), it suffices to prove that

(1) the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral.

Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 6.2(i) and (vi), we have

(2) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

(3) There exist precisely two vertices s_i's in S with $\varphi(s_i) \neq \emptyset$.

In view of (2) and the structure of F_3, there are at most two vertices s_i's in S with $\varphi(s_i) \neq \emptyset$. Suppose on the contrary that there exists precisely one vertex $s_i \in S$ with $\varphi(s_i) \neq \emptyset$. By Lemma 6.2(i), we may assume that $s_i = s_1$, the sink of $T[S]$. Let T' be obtained from T by reversing the direction of the arc v_4v_j for each j with $1 < j \leq r$. Define the weight of each new arc to be zero. As $w(v_4v_j) = 0$ for each j with $1 < j \leq r$ by Lemma 4.4(i), the optimal value of $\mathbb{D}(T', \boldsymbol{w})$ equals

$\nu^*_w(T)$. Observe that T' is the 1-sum of two smaller Möbius-free tournaments T'_1 and T'_2 with the same hub b, where T'_2 arises from T_2 by deleting $S\backslash s_1$. Since $T'_2 = F_3$ and $\tau_w(T'_2\backslash v_2) > 0$, statement (1) follows instantly from Lemma 5.3. So we may assume that (3) holds.

By (3) and Lemma 6.2(i), we may further assume that $\varphi(s_1) = \{v_5\}$ and $\varphi(s_2) = \{v_4\}$ for any optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$.

In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(4) $y(\mathcal{C}_2)$ is maximized; and

(5) subject to (4), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically.

Let us make some observations about $\boldsymbol{y}$. By Lemma 6.2(v), we have

(6) $\mathcal{C}_2^y \subseteq \{v_5s_1v_3v_5, v_5s_1v_3v_4v_5, v_4s_2v_3v_4\}$.

In view of $\varphi(s_i)$ for $i = 1, 2$ and Lemma 6.2(iii), we obtain

(7) $w(v_5s_1) \ge z(v_5s_1) > 0$, $w(v_4s_2) \ge z(v_4s_2) > 0$, and $w(s_iv_3) = z(s_iv_3) > 0$ for $i = 1, 2$.

From Lemma 4.7(v) we see that

(8) *if $y(v_5s_1v_3v_4v_5) > 0$, then v_3v_5 is saturated by $\boldsymbol{y}$ in T_2.*

(9) If $w(v_3v_4) > 0$, then $y(v_4s_2v_3v_4)$ is a positive integer.

To justify this, observe that s_2v_3 is contained in some cycle $C \in \mathcal{C}_0^y$, for otherwise, s_2v_3 is saturated by $\boldsymbol{y}$ in T_2 and hence, by (6), we have $y(v_4s_2v_3v_4) = w(s_2v_3)$, which is a positive integer by (7). If C contains v_4s_2, then it also contains v_3v_5. By Lemma 4.7(iv), v_3v_4 is saturated by $\boldsymbol{y}$ in T_2. By (8), we have $y(v_5s_1v_3v_4v_5) = 0$. From (6) we deduce that $y(v_4s_2v_3v_4) = w(v_3v_4)$, which is a positive integer. So we assume that v_4s_2 is outside C. Furthermore, v_4s_2 is outside $\mathcal{C}_0^y$, because every cycle containing v_4s_2 passes through s_2v_3. If v_4s_2 is saturated by $\boldsymbol{y}$ in T_2, then $y(v_4s_2v_3v_4) = w(v_4s_2)$ by (6), as desired. So we assume that v_4s_2 is not saturated by $\boldsymbol{y}$ in T and that C contains v_3v_5. By Lemma 4.7(iii) and (iv), v_3v_4 is saturated by $\boldsymbol{y}$ in T_2. By (8), we have $y(v_5s_1v_3v_4v_5) = 0$. From (6) we see that $y(v_4s_2v_3v_4) = w(v_3v_4)$. Hence (9) holds.

By (9) and Lemma 4.4(iii), we may assume that $w(v_3v_4) = 0$. Let us show that

(10) $y(v_5s_1v_3v_5)$ is a positive integer.

If s_1v_3 is outside $\mathcal{C}_0^y$, then s_1v_3 is saturated by $\boldsymbol{y}$ in T_2. Thus $y(v_5s_1v_3v_5) = w(s_1v_3) > 0$. If s_1v_3 is contained in some cycle in $\mathcal{C}_0^y$, then, by Lemma 4.7(iv), v_5s_1 is saturated by $\boldsymbol{y}$ in T_2. So $y(v_5s_1v_3v_5) = w(v_5s_1) > 0$. Hence (10) holds in either case.

Using (10) and Lemma 4.4(iii), we conclude that the optimal value $\nu^*_w(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral, as described in (1) above. ∎

Lemma 6.6. *If $T_2/S = F_4$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_5, v_6)$, $s^* = v_2$, and $v_0 = v_3$. To establish the statement, by Lemma 4.6(ii), it suffices to prove that

(1) the optimal value $\nu^*_w(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral.

Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 6.2(i) and (vi), we have

(2) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \ne j$.

(3) There exist at least two and at most three vertices s_i's in S with $\varphi(s_i) \ne \emptyset$.

In view of (2) and the structure of F_4, there are at most three vertices s_i's in S with $\varphi(s_i) \ne \emptyset$. Suppose on the contrary that there exists precisely one vertex $s_i \in S$ with $\varphi(s_i) \ne \emptyset$. Then (1) follows immediately from Lemma 5.4; the argument can be found in that of (3) in the proof of Lemma 6.5.

Lemma 6.2(i) allows us to assume that

(4) if $\varphi(s_i) \neq \emptyset$, then $i \in \{1, 2, 3\}$.

Let t be the subscript in $\{1, 2, 3\}$ with $v_5 \in \varphi(s_t)$, if any. By (2), t is well defined. In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized;

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically; and

(7) subject to (5) and (6), $y(v_1v_5s_tv_3v_1) + y(v_1v_5v_3v_4v_1)$ is minimized.

Let us make a few observations about $\boldsymbol{y}$ before proceeding.

(8) If $y(v_1v_5s_iv_3v_4v_1) > 0$ for some $i \in \{1, 2, 3\}$, then each arc in the set $\{v_1s_i, v_3v_1, v_4s_i, v_4v_5, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in T_2. Furthermore, $y(v_1s_jv_3v_1) = y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$ for any $j \in \{1, 2, 3\} \backslash \{i\}$.

To justify this, note that each arc in the given set is a chord of the cycle $v_1v_5s_iv_3v_4v_1$. So the first half follows instantly from Lemma 4.7(v). Once again let $\uplus$ stand for the multiset sum. Then $v_1v_5s_iv_3v_4v_1 \uplus v_1s_jv_3v_1 = v_1v_5s_iv_3v_1 \uplus v_1s_jv_3v_4v_1$, $v_1v_5s_iv_3v_4v_1 \uplus v_1v_5v_3v_1 = v_1v_5s_iv_3v_1 \uplus v_1v_5v_3v_4v_1$, and $v_1v_5s_iv_3v_4v_1 \uplus v_3v_4v_5v_3 = v_1v_5v_3v_4v_1 \uplus v_5s_iv_3v_4v_5$. Since $\boldsymbol{y}$ satisfies (6), we deduce that $y(v_1s_jv_3v_1) = y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$.

(9) If $y(v_1v_5s_iv_3v_1) > 0$ for some $i \in \{1, 2, 3\}$, then both v_1s_i and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2; so are v_4s_i and v_4v_5 if $y(v_1s_jv_3v_4v_1) > 0$. Furthermore, $y(v_3v_4v_5v_3) = 0$.

Since both v_1s_i and v_5v_3 are chords of the cycle $v_1v_5s_iv_3v_1$, the first half follows instantly from Lemma 4.7(v). To establish the second half, observe that $v_1v_5s_iv_3v_1 \uplus v_3v_4v_5v_3 = v_1v_5v_3v_1 \uplus v_5s_iv_3v_4v_5$. Hence $y(v_3v_4v_5v_3) = 0$ by (7). Suppose $y(v_1s_jv_3v_4v_1) > 0$. Since the multiset sum of the cycles $v_1v_5s_iv_3v_1$, $v_1s_jv_3v_4v_1$, and the arc v_4v_5 (resp. v_4s_i) contains arc-disjoint cycles $v_1s_jv_3v_1$ and $v_5s_iv_3v_4v_5$ (resp. $v_4s_iv_3v_4$), from (7) we deduce that both v_4s_i and v_4v_5 are are saturated by $\boldsymbol{y}$ in T_2.

(10) If $y(v_1v_5v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in T_2. Furthermore, $y(v_1s_iv_3v_1) = 0$ for any $i \in \{1, 2, 3\}$.

Since both v_3v_1 and v_4v_5 are chords of the cycle $v_1v_5v_3v_4v_1$, the first half follows instantly from Lemma 4.7(v). To establish the second half, observe that $v_1v_5v_3v_4v_1 \uplus v_1s_iv_3v_1 = v_1v_5v_3v_1 \uplus v_1s_iv_3v_4v_1$. Since $\boldsymbol{y}$ satisfies (7), we have $y(v_1s_iv_3v_1) = 0$.

The following two statements can be seen from Lemma 4.7(v).

(11) If $y(v_1s_iv_3v_4v_1) > 0$, then both v_3v_1 and v_4s_i are saturated by $\boldsymbol{y}$ in T_2, for $i \in \{1, 2, 3\}$.

(12) If $y(v_5s_iv_3v_4v_5) > 0$, then both v_4s_i and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2, for $i \in \{1, 2, 3\}$.

We proceed by considering two cases, depending on whether $\varphi(s_k) = \{v_4\}$ for some $k \in \{1, 2, 3\}$ (see (4)).

Case 1. $\varphi(s_k) = \{v_4\}$ for some $k \in \{1, 2, 3\}$.

By Lemma 6.2(i), we may assume that $k = 1$; that is, $\varphi(s_1) = \{v_4\}$. Let i and j be the subscripts in $\{2, 3\}$, if any (possibly $i = j$), such that $v_5 \in \varphi(s_i)$ and $v_1 \in \varphi(s_j)$. Then

(13) $\mathcal{C}_2^y \subseteq \{v_4s_1v_3v_4, v_1s_jv_3v_1, v_1s_jv_3v_4v_1, v_1v_5s_iv_3v_1, v_5s_iv_3v_4v_5, v_1v_5s_iv_3v_4v_1, v_1v_5v_3v_1, v_3v_4v_5v_3, v_1v_5v_3v_4v_1\}$.

We propose to show that

(14) if $w(v_3v_4) > 0$, then $y(v_4s_1v_3v_4)$ is a positive integer.

For this purpose, note that $z(s_1v_3) = w(s_1v_3) > 0$ by Lemma 6.2(iii). If s_1v_3 is outside $\mathcal{C}_0^y$, then $y(v_4s_1v_3v_4) = w(s_1v_3) > 0$. So we assume that s_1v_3 is contained in some cycle $C \in \mathcal{C}_0^y$. If C

contains v_4s_1, then v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii). Moreover, the multiset sum of C and each cycle in the set $\{v_1s_jv_3v_4v_1, v_5s_iv_3v_4v_5, v_1v_5s_iv_3v_4v_1, v_3v_4v_5v_3, v_1v_5v_3v_4v_1\}$ contains the cycle $v_4s_1v_3v_4$, a cycle in $\{v_1s_jv_3v_1, v_1v_5s_iv_3v_1, v_1v_5v_3v_1\}$, and a cycle $C' \in \mathcal{C}_0$ that are arc-disjoint, where $C' = C[v_5, v_4] \cup \{v_4v_5\}$ or $C[v_5, v_4] \cup \{v_4v_1, v_1v_5\}$. From the optimality of $\boldsymbol{y}$, we thus deduce that $y(v_1s_jv_3v_4v_1)$, $y(v_5s_iv_3v_4v_5)$, $y(v_1v_5s_iv_3v_4v_1)$, $y(v_3v_4v_5v_3)$, and $y(v_1v_5v_3v_4v_1)$ are all zero. Hence $y(v_4s_1v_3v_4) = w(v_3v_4) > 0$. So we assume that C does not contain v_4s_1. Furthermore, v_4s_1 is outside $\mathcal{C}_0^y$, because every cycle using v_4s_1 passes through s_1v_3. Note that v_4s_1 is not saturated by $\boldsymbol{y}$ in T, for otherwise $y(v_4s_1v_3v_4) = w(v_4s_1) > 0$, as desired. By Lemma 4.7(vii), v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 and C contains v_3v_1. It follows from (8), (10) and (11) that $y(v_1v_5s_iv_3v_4v_1)$, $y(v_1v_5v_3v_4v_1)$ and $y(v_1s_jv_3v_4v_1)$ are all zero. As the multiset sum of C, each of $v_5s_iv_3v_4v_5$ and $v_3v_4v_5v_3$, and the unsaturated arc v_4s_1 contains arc-disjoint cycles $v_4s_1v_3v_4$ and one of $v_1v_5s_iv_3v_1$ and $v_1v_5v_3v_1$, both $y(v_5s_iv_3v_4v_5)$ and $y(v_3v_4v_5v_3)$ are zero by Lemma 4.7(vi). So $y(v_4s_1v_3v_4) = w(v_3v_4) > 0$. This proves (14).

By (14) and Lemma 4.4(iii), we may assume that $w(v_3v_4) = 0$. It follows that $w(v_3v_1) \geq z(v_3v_1) > 0$, for otherwise, $\tau_w(T_2 \backslash a_2) = w(v_3v_1) + w(v_3v_4) = 0$, contradicting (α). Since $z(v_4s_1) > 0$ and $w(v_3v_4) = 0$, the arc v_4s_1 is contained in some cycle in $\mathcal{C}_0^y$. From the proof of (14) we see that

(15) $y(v_1s_jv_3v_4v_1)$, $y(v_5s_iv_3v_4v_5)$, $y(v_1v_5s_iv_3v_4v_1)$, $y(v_3v_4v_5v_3)$, and $y(v_1v_5v_3v_4v_1)$ are all zero.

(16) If $w(v_1s_j) \geq z(v_1s_j) > 0$, then $y(v_1s_jv_3v_1)$ is a positive integer.

To justify this, note that $z(s_jv_3) = w(s_jv_3) > 0$ by Lemma 6.2(iii). Assume first that s_jv_3 is outside $\mathcal{C}_0^y$. If $i \neq j$, then $y(v_1s_jv_3v_1) = w(s_jv_3) > 0$. So we assume that $i = j$. Then $y(v_1s_iv_3v_1) + y(v_1v_5s_iv_3v_1) = w(s_iv_3)$. If $y(v_1v_5s_iv_3v_1) > 0$, then v_1s_i is saturated by $\boldsymbol{y}$ in T_2 by (9). Thus $y(v_1s_iv_3v_1) = w(v_1s_i)$. Next assume that s_jv_3 is contained in some cycle $C \in \mathcal{C}_0^y$. Since $w(v_3v_4) = 0$, cycle C contains v_3v_1. It follows that v_1s_j is saturated by $\boldsymbol{y}$ in T_2. So $y(v_1s_jv_3v_1) = w(v_1s_j) > 0$ and hence (16) is established.

By (16) and Lemma 4.4(iii), we may assume that $w(v_1s_j) = 0$. By (3), we have $z(v_5s_i) > 0$ and $\varphi(s_i) = \{v_5\}$. By (13)-(16), we obtain

(17) $\mathcal{C}_2^y \subseteq \{v_1v_5s_iv_3v_1, v_1v_5v_3v_1\}$.

(18) $y(v_1v_5s_iv_3v_1)$ is a positive integer.

To justify this, note that $z(s_iv_3) = w(s_iv_3) > 0$ by Lemma 6.2(iii). If s_iv_3 is outside $\mathcal{C}_0^y$, then $y(v_1v_5s_iv_3v_1) = w(s_iv_3) > 0$ by (17), as desired. So we assume that s_iv_3 is contained in some cycle $C \in \mathcal{C}_0^y$. Applying Lemma 4.7(iii) to the cycle $v_1v_5s_iv_3v_1$, we deduce that (v_5, s_i) is saturated by $\boldsymbol{y}$ in T_2. So $y(v_1v_5s_iv_3v_1) = w(v_5s_i) > 0$ and hence (18) holds.

By (18) and Lemma 4.4(iii), $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution, which implies (1).

Case 2. $\varphi(s_k) \neq \{v_4\}$ for any $k \in \{1, 2, 3\}$.

By (3), the hypothesis of the present case, and Lemma 6.2(i), we may assume that $v_1 \in \varphi(s_1)$ and $v_5 \in \varphi(s_2)$. Then

(19) $\mathcal{C}_2^y \subseteq \{v_1s_1v_3v_1, v_1s_1v_3v_4v_1, v_1v_5s_2v_3v_1, v_5s_2v_3v_4v_5, v_1v_5s_2v_3v_4v_1, v_1v_5v_3v_1, v_3v_4v_5v_3, v_1v_5v_3v_4v_1, v_4s_1v_3v_4, v_4s_2v_3v_4\}$.

By Lemma 6.2(vi), we have

(20) if $v_4 \in \varphi(s_i)$, then $z(v_4s_{3-i}) = 0$ and $y(v_4s_{3-i}v_3v_4) = 0$ for $i = 1, 2$.

Claim 1. $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$.

To justify this, observe that

(21) if K is an FAS of $T_2\backslash a_2$ such that $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 5.3.)

In view of Lemma 6.2(iii), we distinguish among three subcases, depending on whether $s_i v_3$ is contained in a cycle in $\mathcal{C}_0^y$.

Subcase 1.1. Both s_1v_3 and s_2v_3 are outside $\mathcal{C}_0^y$. In this subcase, s_iv_3 is saturated by $\boldsymbol{y}$ in T_2 for $i = 1, 2$. If v_5v_3 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_5v_3, s_1v_3, s_2v_3\}$. Since K is an FAS of $T_2\backslash a_2$, it is an MFAS by (21) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. So we assume that v_5v_3 is not saturated by $\boldsymbol{y}$ in T_2.

(22) Both v_3v_1 and v_3v_4 are outside $\mathcal{C}_0^y$. Furthermore, at least one of them is not saturated by $\boldsymbol{y}$ in T_2.

Indeed, the first half follows directly from Lemma 4.7(iii). To justify the second half, assume the contrary. Then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_3v_4\}$. Thus K is an MFAS of $T_2\backslash a_2$ by (21) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

By (22), (8), (9), and (12), we have

(23) $y(v_1v_5s_2v_3v_1)$, $y(v_5s_2v_3v_4v_5)$, and $y(v_1v_5s_2v_3v_4v_1)$ are all zero.

Since $\mathcal{C}_0^y \neq \emptyset$, some cycle $C \in \mathcal{C}_0^y$ contains v_1v_5 or v_4v_5. Thus there are two possibilities to consider.

• C contains v_1v_5. Now by (22) and Lemma 4.7(iii), v_3v_1 is saturated by $\boldsymbol{y}$ in T_2 and hence v_3v_4 is not saturated by $\boldsymbol{y}$ in T_2. It follows from Lemma 4.7(i) and (iii) that both v_4v_1 and v_4v_5 are saturated by $\boldsymbol{y}$ in T_2. If $z(v_4s_i) = w(v_4s_i)$ for $i = 1, 2$, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_5, v_4s_1, v_4s_2\}$. Thus K is an MFAS of $T_2\backslash a_2$ by (21) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. So we assume that $0 < z(v_4s_i) < w(v_4s_i)$ for $i = 1$ or 2. Then $z(v_4s_{3-i}) = w(v_4s_{3-i}) = 0$ by (2). If $i = 2$, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_5, v_4s_1, s_2v_3\}$, and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. If $i = 1$, then $y(v_1s_1v_3v_4v_1) = 0$ by (11). Since the multiset sum of the cycles $v_1s_1v_3v_1$, C, and the unsaturated arcs $\{v_4s_1, v_5v_3, v_3v_4\}$ contains arc-disjoint cycles $v_4s_1v_3v_4$ and $v_1v_5v_3v_1$, we have $y(v_1s_1v_3v_1) = 0$ by Lemma 4.7(vi). Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_5, s_1v_3, v_4s_2\}$. It follows that $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

• C contains v_4v_5. Now by (22) and Lemma 4.7(iii), v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 and hence v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2. It follows from Lemma 4.7(i) and (iii) that v_1v_5 is saturated by $\boldsymbol{y}$ in T_2. By (10) and (11), we have $y(v_1v_5v_3v_4v_1) = y(v_1s_1v_3v_4v_1) = 0$. If v_1s_1 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_5, v_3v_4, v_1s_1\}$. Thus $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. So we assume that v_1s_1 is not saturated by $\boldsymbol{y}$ in T_2 and hence not in T by (22). Since the multiset sum of the cycles C, $v_4s_1v_3v_4$, and the unsaturated arcs $\{v_3v_1, v_5v_3, v_1s_1\}$ contains arc-disjoint cycles $v_1s_1v_3v_1$ and $v_3v_4v_5v_3$, we have $y(v_4s_1v_3v_4) = 0$ by Lemma 4.7(vi). So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_5, v_3v_4, s_1v_3\}$. It follows that $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

Subcase 1.2. s_1v_3 is contained in some cycle $C \in \mathcal{C}_0^y$; subject to this, we choose C so that it contains as many edges in $T_2\backslash a_2$ as possible.

Assume first that C contains v_1s_1. Then C contains the path $v_1s_1v_3v_4v_5$. By Lemma 4.7(iii), each arc in the set $\{v_3v_1, v_4v_1, v_4s_1, v_5v_3\}$ is saturated by $\boldsymbol{y}$ in T_2. By (2), (8) and (10), we have $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$. Since the multiset sum of C and one of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$ contains arc-disjoint cycles $v_3v_4v_5v_3$, $C' = C[v_5, v_1] \cup \{v_1v_5\}$, and one of $v_1s_1v_3v_1$ and $v_5s_2v_3v_4v_5$, from the optimality of $\boldsymbol{y}$ we deduce that $y(v_1v_5v_3v_1) = y(v_1v_5s_2v_3v_1) = 0$. If s_2v_3 is outside $\mathcal{C}_0^y$, then s_2v_3 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 6.2(iii). So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4s_1, s_2v_3, v_5v_3\}$. Hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. So we assume that s_2v_3 is contained in some cycle in $\mathcal{C}_0^y$. Since v_3v_1 is saturated by $\boldsymbol{y}$ in T_2, every cycle in $\mathcal{C}_0^y$ containing s_2v_3 passes through v_3v_4. By Lemma 4.7(iii), both v_4s_2 and v_5s_2 are saturated by $\boldsymbol{y}$ in T_2. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4s_1, v_4s_2, v_5s_2, v_5v_3\}$. It follows that $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$.

Assume next that v_1s_1 is not on C. Then we may further assume that v_1s_1 is outside $\mathcal{C}_0^y$. We proceed by considering three subsubcases.

• C contains v_3v_1. Now v_1s_1 and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii). Hence $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = y(v_1s_1v_3v_4v_1) = 0$ by (8), (10) and (11). If v_4s_1 is not saturated by $\boldsymbol{y}$ in T_2, then v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii). Moreover, for each $D \in \{v_3v_4v_5v_3, v_5s_2v_3v_4v_5\}$, if v_4s_1 is on C, then the multiset sum of C and D contains arc-disjoint cycles $v_4s_1v_3v_4$, $C' = C[v_5, v_4] \cup \{v_4v_5\}$, and one of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$; if v_4s_1 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of C, D and the arc v_4s_1 contains $v_4s_1v_3v_4$ and one of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$ that are arc-disjoint. It follows from the optimality of $\boldsymbol{y}$ or Lemma 4.7(iv) that $y(v_3v_4v_5v_3) = y(v_5s_2v_3v_4v_5) = 0$. So $y(\mathcal{C}_2) = w(K)$ if s_2v_3 is contained in some cycle in $\mathcal{C}_0^y$ and $y(\mathcal{C}_2) = w(J)$ otherwise, where $K = \{v_1s_1, v_3v_4, v_5v_3, s_2v_3\}$ and $J = \{v_1s_1, v_3v_4, v_5v_3, v_5s_2\}$. Hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$. So we assume that v_4s_1 is saturated by $\boldsymbol{y}$ in T_2. If s_2v_3 is outside $\mathcal{C}_0^y$, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, s_2v_3\}$, which implies that $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$. So we further assume that s_2v_3 is contained in some cycle in $\mathcal{C}_0^y$. By Lemma 4.7(iii), v_5s_2 is saturated by $\boldsymbol{y}$ in T_2. If v_4s_2 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$; otherwise, v_3v_4 is saturated by $\boldsymbol{y}$ in T_2, and $w(v_4s_1) = z(v_4s_1) = 0$. Similar to the case when v_4s_1 is not saturated by $\boldsymbol{y}$ in T_2, we can show that $y(v_3v_4v_5v_3) = y(v_5s_2v_3v_4v_5) = 0$. Thus $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1s_1, v_3v_4, v_5v_3, v_5s_2\}$. Therefore $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$ in either situation.

• C contains both v_3v_4 and v_4v_1. Now v_1s_1, v_4s_1 and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii). If s_2v_3 is outside $\mathcal{C}_0^y$, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, s_2v_3\}$; otherwise, v_5s_2 and v_4s_2 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii). So $y(\mathcal{C}_2) = w(J)$, where $J = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$. Therefore $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$ in either situation.

• C contains both v_3v_4 and v_4v_5. Now v_4s_1 and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii) and $y(v_1v_5v_3v_4v_1) = y(v_1v_5s_2v_3v_4v_1) = 0$ by (8) and (10). If v_1s_1 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$ or $w(J)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, s_2v_3\}$ and $J = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$; otherwise, both v_3v_1 and v_4v_1 are saturated by $\boldsymbol{y}$ in T_2, and every cycle in $\mathcal{C}_0^y$ containing s_2v_3 traverses $v_3v_4v_5$. Since the multiset sum of C, each of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$, and the unsaturated arc v_1s_1 contains $v_1s_1v_3v_1$ and one of $v_3v_4v_5v_3$ and $v_5s_2v_3v_4v_5$ that are arc-disjoint, we have $y(v_1v_5v_3v_1) = y(v_1v_5s_2v_3v_1) = 0$ by Lemma 4.7(iv). So $y(\mathcal{C}_2) = w(K)$ if s_2v_3 is outside $\mathcal{C}_0^y$ and $y(\mathcal{C}_2) = w(J)$ otherwise, where $K = \{v_3v_1, v_4v_1, v_4s_1, v_5v_3, s_2v_3\}$ and $J = \{v_1s_1, v_4s_1, v_5v_3, v_4s_2, v_5s_2\}$. Therefore $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$ in either situation.

Subcase 1.3. s_2v_3 is contained in some cycle $C \in \mathcal{C}_0^y$ and s_1v_3 is saturated by $\boldsymbol{y}$ in T_2. In this subcase, both v_5s_2 and v_5v_3 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii). If v_4s_2 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{s_1v_3, v_5v_3, v_4s_2, v_5s_2\}$; otherwise, $z(v_4s_2) > 0$ and $w(v_4s_1) = z(v_4s_1) = 0$ by Lemma 6.2(vii). In this case C contains v_3v_1, so v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii). By (8) and (10)-(12), we have $y(v_1v_5s_2v_3v_4v_1)$, $y(v_1v_5v_3v_4v_1)$, $y(v_1s_1v_3v_4v_1)$, and $y(v_5s_2v_3v_4v_5)$ are all zero. Since the multiset sum of the cycles C, $v_3v_4v_5v_3$, and the unsaturated arc v_4s_2 contains arc-disjoint cycles $v_4s_2v_3v_4$ and $v_1v_5v_3v_1$, by Lemma 4.7(iv), we have $y(v_3v_4v_5v_3) = 0$. It follows that $y(\mathcal{C}_2) = w(K)$, where $K = \{s_1v_3, v_5v_3, v_3v_4, v_5s_2\}$.

Combining the above three subcases, we see that the equality $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2 \backslash a_2)$ holds. So Claim 1 is established.

Claim 2. $y(C)$ is a positive integer for some $C \in \mathcal{C}_2$ or $\nu_w^*(T)$ is an integer.

To justify this, note that $y(\mathcal{C}_2) = w(K)$ for some MFAS K of $T_2 \backslash a_2$ by Claim 1. Depending on what K is, we distinguish among eight cases.

Subcase 2.1. K is one of $\{v_1v_5, v_3v_4, v_1s_1\}$, $\{v_1s_1, v_3v_4, s_2v_3, v_5v_3\}$, $\{v_1s_1, v_3v_4, v_5s_2, v_5v_3\}$, $\{v_1v_5, v_3v_4, s_1v_3\}$, and $\{s_1v_3, v_3v_4, v_5s_2, v_5v_3\}$.

In this case, by Lemma 4.3(i), we have $y(C) = 0$ for some cycles C listed in (19). By Lemma 4.3(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) = w(v_1s_1)$ or $w(s_1v_3)$, each of them is positive by Lemma 6.2(iii) and the assumption that $v_1 \in \varphi(s_1)$.

Subcase 2.2. $K = \{v_3v_1, v_4v_1, v_4s_1, s_2v_3, v_5v_3\}$.

In this case, by Lemma 4.3(i), we have $y(C) = 0$ for some cycles C listed in (19). By Lemma 4.3(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) = w(v_3v_1)$, $y(v_1s_1v_3v_4v_1) = w(v_4v_1)$, $y(v_4s_1v_3v_4) = w(v_4s_1)$, $y(v_4s_2v_3v_4) + y(v_5s_2v_3v_4v_5) = w(s_2v_3)$, $y(v_3v_4v_5v_3) = w(v_5v_3)$. If $y(v_5s_2v_3v_4v_5) = 0$, then $y(v_4s_2v_3v_4) = w(s_2v_3) > 0$ by Lemma 6.2(iii). If $y(v_5s_2v_3v_4v_5) > 0$, then v_4s_2 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii). So $w(v_4s_2) = y(\mathcal{C}_2(v_4s_2))$. It follows that $y(v_4s_2v_3v_4) = w(v_4s_2)$, and hence $y(v_5s_2v_3v_4v_5)$ is a positive integer.

Subcase 2.3. $K = \{v_3v_1, v_4v_1, v_4s_1, v_4s_2, v_5s_2, v_5v_3\}$.

In this case, by Lemma 4.3(i), we have $y(C) = 0$ for some cycles C listed in (19). By Lemma 4.3(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3) = w(v_3v_1)$, $y(v_1s_1v_3v_4v_1) = w(v_4v_1)$, $y(v_4s_1v_3v_4) = w(v_4s_1)$, $y(v_4s_2v_3v_4) = w(v_4s_2)$, $y(v_5s_2v_3v_4v_5) = w(v_5s_2)$, and $y(v_3v_4v_5) = w(v_5v_3)$. Since $v_5 \in \varphi(s_2)$, we have $w(v_5s_2) > 0$. So $y(v_5s_2v_3v_4v_5)$ is a positive integer.

Subcase 2.4. $K = \{v_3v_1, v_4v_1, v_4v_5, v_4s_1, s_2v_3\}$ or $\{v_3v_1, v_4v_1, v_4v_5, s_1v_3, v_4s_2\}$.

In this case, by Lemma 4.3(i), we have $y(C) = 0$ for some cycles C listed in (19). By Lemma 4.3(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_4s_2v_3v_4) = w(s_2v_3) > 0$ or $y(v_4s_1v_3v_4) = w(s_1v_3) > 0$ by Lemma 6.2(iii).

Subcase 2.5. $K = \{v_1s_1, v_4s_1, s_2v_3, v_5v_3\}$ or $\{v_1s_1, v_4s_1, v_4s_2, v_5s_2, v_5v_3\}$.

We only consider the subcase when $K = \{v_1s_1, v_4s_1, s_2v_3, v_5v_3\}$, as the other subcase can be justified likewise.

By Lemma 4.3(i), we have $y(C) = 0$ for some cycles C listed in (19). By Lemma 4.3(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) + y(v_1s_1v_3v_4v_1) = w(v_1s_1)$, $y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1) = w(v_5v_3)$, $y(v_4s_1v_3v_4) = w(v_4s_1)$, and $y(v_4s_2v_3v_4) + y(v_1v_5s_2v_3v_1) + y(v_5s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_1) = w(s_2v_3)$. We may assume that $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$, for otherwise, by (8) or (10), we have $y(v_1s_1v_3v_1) = 0$ and hence $y(v_1s_1v_3v_4v_1) = w(v_1s_1) > 0$.

If $y(v_1v_5s_2v_3v_1) = 0$, then $y(v_5s_2v_3v_4v_5) + y(v_4s_2v_3v_4) = w(s_2v_3)$. Observe that $y(v_4s_2v_3v_4) > 0$, for otherwise, $y(v_5s_2v_3v_4v_5) = w(s_2v_3) > 0$. By (6), we obtain $y(v_4s_2v_3v_4) = w(s_2v_3)$ or $w(v_4s_2)$, which is a positive integer. So we assume that $y(v_1v_5s_2v_3v_1) > 0$. Then $y(v_3v_4v_5v_3) = 0$ by (9). Note that $y(v_1s_1v_3v_4v_1) > 0$, for otherwise, $y(v_1s_1v_3v_1) = w(v_1s_1) > 0$. Thus, by (9), both v_4s_2 and v_4v_5 are saturated by $\boldsymbol{y}$ in T_2. It follows that $y(v_4s_2v_3v_4) = w(v_4s_2)$ and

$y(v_5s_2v_3v_4v_5) = w(v_4v_5)$. So $y(v_1v_5s_2v_3v_1) = w(s_2v_3) - y(v_4s_2v_3v_4) - y(v_5s_2v_3v_4v_5)$. Since $w(s_2v_3) > 0$, at least one of $y(v_4s_2v_3v_4)$, $y(v_5s_2v_3v_4v_5)$, and $y(v_1v_5s_2v_3v_1)$ is a positive integer.

Subcase 2.6. $K = \{s_1v_3, v_4s_2, v_5s_2, v_5v_3\}$ or $\{s_1v_3, s_2v_3, v_5v_3\}$.

We only consider the subcase when $K = \{s_1v_3, s_2v_3, v_5v_3\}$, as the other subcase can be justified likewise.

By Lemma 4.3(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_4s_1v_3v_4) + y(v_1s_1v_3v_1) + y(v_1s_1v_3v_4v_1) = w(s_1v_3)$, $y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1) = w(v_5v_3)$, and $y(v_4s_2v_3v_4) + y(v_1v_5s_2v_3v_1) + y(v_5s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_1) = w(s_2v_3)$.

We may assume that $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$, for otherwise, by (8) or (10), we have $y(v_1s_1v_3v_1) = 0$ and hence $y(v_4s_1v_3v_4) + y(v_1s_1v_3v_4v_1) = w(v_1s_1) > 0$, which together with (6) implies that $y(v_4s_1v_3v_4) = w(s_1v_3)$ or $w(v_4s_1)$, so $y(v_1s_1v_3v_4v_1) = w(v_1s_1) - y(v_4s_1v_3v_4)$. Since $w(s_1v_3) > 0$, at least one of $y(v_4s_1v_3v_4)$ and $y(v_1s_1v_3v_4v_1)$ is a positive integer.

If $y(v_1v_5s_2v_3v_1) = 0$, then $y(v_5s_2v_3v_4v_5) + y(v_4s_2v_3v_4) = w(s_2v_3)$, which together with (6) implies that $y(v_4s_2v_3v_4) = w(s_2v_3)$ or $w(v_4s_2)$, so $y(v_5s_2v_3v_4v_5) = w(s_2v_3) - y(v_4s_2v_3v_4)$. Since $w(s_2v_3) > 0$, at least one of $y(v_4s_2v_3v_4)$ and $y(v_5s_2v_3v_4v_5)$ is a positive integer. So we assume that $y(v_1v_5s_2v_3v_1) > 0$. Thus, by (9), we have $y(v_1v_5v_3v_1) = w(v_5v_3)$. If $y(v_1s_1v_3v_4v_1) > 0$, then $y(v_4s_2v_3v_4) = w(v_4s_2)$, $y(v_5s_2v_3v_4v_5) = w(v_4v_5)$, and $y(v_1v_5s_2v_3v_1) = w(s_2v_3) - y(v_4s_2v_3v_4) - y(v_5s_2v_3v_4v_5)$. Since $w(s_2v_3) > 0$, at least one of $y(v_4s_2v_3v_4)$, $y(v_5s_2v_3v_4v_5)$, and $y(v_1v_5s_2v_3v_1)$ is a positive integer. So we further assume that $y(v_1s_1v_3v_4v_1) = 0$. Then $y(v_1s_1v_3v_1) + y(v_4s_1v_3v_4) = w(s_1v_3)$. If $y(v_4s_1v_3v_4) = 0$, then $y(v_1s_1v_3v_1) = w(s_1v_3) > 0$. So we assume that $y(v_4s_1v_3v_4) > 0$. By Lemma 6.2(vii), we have $y(v_4s_2v_3v_4) = 0$, so $y(v_1v_5s_2v_3v_1) + y(v_5s_2v_3v_4v_5) = w(s_2v_3)$. Observe that if $y(v_1s_1v_3v_1)$ or $y(v_1v_5s_2v_3v_1)$ is an integer, then accordingly $y(v_4s_1v_3v_4)$ or $y(v_5s_2v_3v_4v_5)$ is an integer. Since $w(s_iv_3) > 0$ for $i = 1, 2$ by Lemma 6.2(iii), at least one of $y(v_1s_1v_3v_1)$, $y(v_4s_1v_3v_4)$, $y(v_1v_5s_2v_3v_1)$, and $y(v_5s_2v_3v_4v_5)$ is a positive integer, as claimed.

It remains to consider the subcase when neither $y(v_1s_1v_3v_1)$ nor $y(v_1v_5s_2v_3v_1)$ is an integer. We propose to show that

(24) $\nu_w^*(T)$ is an integer.

To justify this, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since $0 < y(v_1s_1v_3v_1) < w(v_1s_1)$ and $0 < y(v_4s_1v_3v_4) < w(v_4s_1)$, by Lemma 4.3(i) and (ii), we have $x(v_1s_1) = x(v_4s_1) = 0$ and $x(v_1s_1v_3v_1) = x(v_4s_1v_3v_4) = 1$, which implies $x(v_3v_1) = x(v_3v_4)$. Furthermore, since $y(v_1v_5s_2v_3v_1) > 0$ and $y(v_5s_2v_3v_4v_5) > 0$, we have $x(v_1v_5s_2v_3v_1) = x(v_5s_2v_3v_4v_5) = 1$, which implies $x(v_3v_1) + x(v_1v_5) = x(v_3v_4) + x(v_4v_5)$. Thus $x(v_1v_5) = x(v_4v_5)$. Similarly, for each vertex $u \in V \backslash (V(T_2) \backslash a_2)$, we deduce that $x(uv_1) = x(uv_4)$. Let $T' = (V', A')$ be obtained from T by identifying v_1 and v_4; the resulting vertex is still denoted by v_1. Let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ by setting $w'(v_1v_5) = w(v_1v_5) + w(v_4v_5)$, $w'(v_3v_1) = w(v_3v_1) + w(v_3v_4)$, $w'(v_1s_i) = w(v_1s_i) + w(v_4s_i)$ for $1 \le i \le r$, and $w'(uv_1) = w(uv_1) + w(uv_4)$ for each $u \in V \backslash (V(T_2) \backslash a_2)$. By the LP-duality theorem, $\boldsymbol{x}$ and $\boldsymbol{y}$ naturally correspond to solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$ respectively with the same optimal value $\nu_w^*(T)$. From the hypothesis of Theorem 4.1, we deduce that $\nu_w^*(T)$ is an integer. This proves (24).

Subcase 2.7. $K = \{v_3v_1, v_4v_1, v_4v_5, v_4s_1, v_4s_2\}$.

In this case, by Lemma 4.3(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_4s_iv_3v_4) = w(v_4s_i)$ for $i = 1, 2$, $y(v_1s_1v_3v_1) + y(v_1v_5v_3v_1) +$

$y(v_1v_5s_2v_3v_1) = w(v_3v_1)$, $y(v_1s_1v_3v_4v_1) + y(v_1v_5v_3v_4v_1) + y(v_1v_5s_2v_3v_4v_1) = w(v_4v_1)$, and $y(v_3v_4v_5v_3)+y(v_5s_2v_3v_4v_5) = w(v_4v_5)$. We may assume that $w(v_4s_i) = 0$ for $i = 1, 2$, for otherwise, $y(v_4s_1v_3v_4)$ or $y(v_4s_2v_3v_4)$ is a positive integer. Note that both s_1v_3 and s_2v_3 are outside $\mathcal{C}_0^y$. So s_iv_3 is saturated by $\boldsymbol{y}$ in T_2 for $i = 1, 2$, and hence $y(v_1s_1v_3v_1)+y(v_1s_1v_3v_4v_1) = w(s_1v_3)$ and $y(v_1v_5s_2v_3v_1) + y(v_5s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_1) = w(s_2v_3)$. If $y(v_1v_5s_2v_3v_4v_1) > 0$ or $y(v_1v_5v_3v_4v_1) > 0$, then $y(v_1s_1v_3v_4v_1) = w(s_1v_3) > 0$ by (8) or (10). So we assume that $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$. Then $y(v_1s_1v_3v_4v_1) = w(v_4v_1)$ and $y(v_1s_1v_3v_1) = w(s_1v_3) - y(v_1s_1v_3v_4v_1)$. Since $w(s_1v_3) > 0$, at least one of $y(v_1s_1v_3v_1)$ and $y(v_1s_1v_3v_4v_1)$ is a positive integer.

Subcase 2.8. $K = \{v_3v_1, v_3v_4\}$.

In this case, by Lemma 4.3(iii), we obtain $w(e) = y(\mathcal{C}_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) + y(v_1v_5v_3v_1) + y(v_1v_5s_2v_3v_1) = w(v_3v_1)$, $y(v_4s_1v_3v_4) + y(v_4s_2v_3v_4) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_1) + y(v_1s_1v_3v_4v_1) + y(v_5s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_1) = w(v_3v_4)$. Since both s_1v_3 and s_2v_3 are outside $\mathcal{C}_0^y$, we see that s_iv_3 is saturated by $\boldsymbol{y}$ in T_2 for $i = 1, 2$. Hence $y(v_1s_1v_3v_1)+y(v_4s_1v_3v_4)+y(v_1s_1v_3v_4) = w(s_1v_3)$ and $y(v_4s_2v_3v_4)+y(v_1v_5s_2v_3v_1)+ y(v_5s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_1) = w(s_2v_3)$.

If $y(v_1v_5s_2v_3v_4v_1) > 0$ or $y(v_1v_5v_3v_4v_1) > 0$, then $y(v_4s_1v_3v_4) + y(v_1s_1v_3v_4v_1) = w(s_1v_3)$ by (8) and (10). It follows from (6) that either $y(v_4s_1v_3v_4) = w(s_1v_3) > 0$ or $y(v_4s_1v_3v_4) = w(v_4s_1)$ and $y(v_1s_1v_3v_4v_1) = w(s_1v_3) - y(v_4s_1v_3v_4)$. Since $w(s_1v_3) > 0$, at least one of $y(v_4s_1v_3v_4)$ and $y(v_1s_1v_3v_4)$ is a positive integer. So we assume that $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = 0$. If $y(v_1v_5s_2v_3v_1) = 0$, then either $y(v_4s_2v_3v_4) = w(s_2v_3)$ or $y(v_4s_2v_3v_4) = w(v_4s_2)$ by (12), so $y(v_5s_2v_3v_4v_5) = w(s_2v_3) - w(v_4s_2)$. Since $w(s_2v_3) > 0$, at least one of $y(v_4s_2v_3v_4)$ and $y(v_5s_2v_3v_4v_5)$ is a positive integer.

Suppose $y(v_1v_5s_2v_3v_1) > 0$. Then $y(v_1v_5v_3v_1) = w(v_5v_3)$ by (9). If $y(v_1s_1v_3v_4v_1) > 0$, then $y(v_4s_2v_3v_4) = w(v_4s_2)$, $y(v_5s_2v_3v_4v_5) = w(v_4v_5)$, and $y(v_4s_1v_3v_4) = w(v_4s_1)$ by (9) and (11). It follows that $y(v_1v_5s_2v_3v_1) = w(s_2v_3) - y(v_4s_2v_3v_4) - y(v_5s_2v_3v_4v_5)$. Since $w(s_2v_3) > 0$, at least one of $y(v_4s_2v_3v_4)$, $y(v_5s_2v_3v_4v_5)$, and $y(v_1v_5s_2v_3v_1)$ is a positive integer. So we assume that $y(v_1s_1v_3v_4v_1) = 0$. If $y(v_5s_2v_3v_4v_5) = 0$, then $y(v_4s_1v_3v_4) + y(v_4s_2v_3v_4) = w(v_3v_4)$. By Lemma 6.2(vii), at most one of $w(v_4s_1)$ and $w(v_4s_2)$ is nonzero. Thus either $y(v_4s_1v_3v_4) = 0$ or $y(v_4s_2v_3v_4) = 0$, and hence either $y(v_1s_1v_3v_1) = w(s_1v_3) > 0$ or $y(v_1v_5s_2v_3v_1) = w(s_2v_3) > 0$. So we further assume that $y(v_5s_2v_3v_4v_5) > 0$. If $y(v_1s_1v_3v_1)$ or $y(v_1v_5s_2v_3v_1)$ is an integer, then accordingly $y(v_4s_1v_3v_4)$ or $y(v_5s_2v_3v_4v_5)$ is an integer. Since $w(s_iv_3) > 0$ for $i = 1, 2$, at least one of $y(v_1s_1v_3v_1)$, $y(v_4s_1v_3v_4)$, $y(v_1v_5s_2v_3v_1)$, and $y(v_5s_2v_3v_4v_5)$ is a positive integer, as claimed.

It remains to consider the subcase when neither $y(v_1s_1v_3v_1)$ nor $y(v_1v_5s_2v_3v_1)$ is an integer. Now we can prove that $\nu_w^*(T)$ is an integer. Since the proof is the same as that of (24), we omit the details here.

Combining the above subcases, we see that Claim 2 holds. Hence, by Lemma 4.4(iii), the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral, as described in (1) above. ∎

Lemma 6.7. *If $T_2/S = G_2$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_4, v_5)$, $s^* = v_2$, and $v_0 = v_4$. To establish the statement, by Lemma 4.6(ii), it suffices to prove that

(1) the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral.

Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 6.2 (i) and (vi), we have

(2) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

(3) There exist at least two and at most three vertices s_i's in S with $\varphi(s_i) \neq \emptyset$.

In view of (2) and the structure of G_2, there are at most three vertices s_i's in S with $\varphi(s_i) \neq \emptyset$. Suppose on the contrary that there exists precisely one vertex $s_i \in S$ with $\varphi(s_i) \neq \emptyset$. Then (1) follows immediately from Lemma 5.5; the argument can be found in that of (3) in the proof of Lemma 6.5.

Lemma 6.2(i) allows us to assume that

(4) if $\varphi(s_i) \neq \emptyset$, then $i \in \{1, 2, 3\}$.

In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized;

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically;

(7) subject to (5) and (6), $y(v_1v_6v_3v_4v_1)$ is minimized; and

(8) subject to (5)-(7), $y(v_1v_6v_4v_1)$ is minimized.

Let us make some observations about $\boldsymbol{y}$ before proceeding.

(9) If K is an FAS of $T_2\backslash a_2$ such that $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 5.3.)

The statements below follow instantly from Lemma 4.7(v).

(10) If $y(v_1v_6v_3v_4v_1) > 0$, then both v_3v_1 and v_6v_4 are saturated by $\boldsymbol{y}$ in T_2.

(11) If $y(v_1v_6s_iv_4v_1) > 0$ for some $i \in \{1, 2, 3\}$, then both v_1s_i and v_6v_4 are saturated by $\boldsymbol{y}$ in T_2.

(12) If $y(v_1v_6v_3s_iv_4v_1) > 0$ for some $i \in \{1, 2, 3\}$, then each arc in the set $\{v_3v_1, v_3v_4, v_6v_4, v_1s_i, v_6s_i\}$ is saturated by $\boldsymbol{y}$ in T_2.

Claim 1. $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

To justify this, we may assume that

(13) at most one of v_3v_1 and v_4v_1 is saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1\}$. Since K is an FAS of $T_2\backslash a_2$, it is an MFAS by (9) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

We proceed by considering two cases, depending on whether $v_1 \in \varphi(s_i)$ for some i.

Case 1.1. $v_1 \notin \varphi(s_i)$ for any $i \in \{1, 2, 3\}$.

By (2), (3) and Lemma 6.2(i), we may assume that $\varphi(s_1) = \{v_6\}$ and $\varphi(s_2) = \{v_3\}$. Thus

(14) $\mathcal{C}_2^y \subseteq \{v_1v_6v_3v_1, v_1v_6v_4v_1, v_1v_6v_3v_4v_1, v_1v_6s_1v_4v_1, v_1v_6v_3s_2v_4v_1\}$.

By Lemma 6.2(iii), $z(s_iv_4) = w(z_iv_4) > 0$. If s_iv_4 is outside $\mathcal{C}_0^y$ for $i = 1$ or 2, then s_iv_4 is saturated by $\boldsymbol{y}$ in T_2. In view of (14), we have $y(v_1v_6s_1v_4v_1) = w(s_1v_4) > 0$ or $y(v_1v_6v_3s_2v_4v_1) = w(s_2v_4) > 0$, and hence (1) follows from Lemma 4.4(iii). Similarly, if v_6s_1 or v_3s_2 is saturated by $\boldsymbol{y}$ in T_2, then $y(v_1v_6s_1v_4v_1) = w(v_6s_1) > 0$ or $y(v_1v_6v_3s_2v_4v_1) = w(v_3s_2) > 0$, and hence (1) follows from Lemma 4.4(iii). So we assume that

(15) s_iv_4 is contained in some cycle in $\mathcal{C}_0^y$ for $i = 1$ and 2. Furthermore, neither v_6s_1 nor v_3s_2 is saturated by $\boldsymbol{y}$ in T_2.

By (15) and Lemma 4.7(iii), at least one of v_1v_6 and v_4v_1 is saturated by $\boldsymbol{y}$ in T_2. If v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(v_1v_6)$. By (9), $\{v_1v_6\}$ is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. If v_4v_1 is saturated by $\boldsymbol{y}$ in T_2, then v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2

by (13). So, by Lemma 4.7(vi), v_6v_3 is saturated by $\boldsymbol{y}$ in T_2 and, by (10) and (12), we have $y(v_1v_6v_3s_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3\}$. Since K is an FAS of $T_2\backslash a_2$, it is an MFAS by (9) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$.

Case 1.2. $v_1 \in \varphi(s_i)$ for some $i \in \{1, 2, 3\}$.

By (2), (3) and Lemma 6.2(i), we may assume that $v_1 \in \varphi(s_1)$, $v_6 \in \varphi(s_i)$, and $v_3 \in \varphi(s_j)$, with $\{1\} \neq \{i, j\} \subseteq \{1, 2, 3\}$. Furthermore,

(16) $\mathcal{C}_2^y \subseteq \{v_1v_6v_3v_1, v_1v_6v_4v_1, v_1v_6v_3v_4v_1, v_1s_1v_4v_1, v_1v_6s_iv_4v_1, v_1v_6v_3s_jv_4v_1\}$.

We may further assume that s_1v_4 is contained in some cycle in $\mathcal{C}_0^y$ and v_1s_1 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_1s_1v_4v_1) = w(s_1v_4) > 0$ or $y(v_1s_1v_4v_1) = w(v_1s_1) > 0$. Hence (1) follows instantly from Lemma 4.4(iii). It follows from Lemma 4.7(vii) that v_4v_1 is saturated by $\boldsymbol{y}$ in T_2 and hence, by (13), v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2. By (10) and (12), we obtain $y(v_1v_6v_3s_jv_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. If v_6v_3 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3\}$. Since K is an FAS of $T_2\backslash a_2$, it is an MFAS by (9) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. So we assume that v_6v_3 is not saturated by $\boldsymbol{y}$ in T_2. Thus, by Lemma 4.7(vii), v_1v_6 is saturated by $\boldsymbol{y}$ in T_2. We propose to show that

(17) $y(v_1v_6v_4v_1) = y(v_1v_6s_iv_4v_1) = 0$.

Assume the contrary: $y(v_1v_6v_4v_1) > 0$ or $y(v_1v_6s_iv_4v_1) > 0$. Then v_1s_1 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_1s_1. Then the multiset sum of the cycles C and $v_1v_6v_4v_1$ (resp. $v_1v_6s_iv_4v_1$) contains arc-disjoint cycles $v_1s_1v_4v_1$ and $C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_4\}$ (resp. $C' = C[v_4, v_1] \cup \{v_1v_6, v_6s_i, s_iv_4\}$). Set $\theta = \min\{y(v_1v_6v_4v_1), y(C)\}$ (resp. $\min\{y(v_1v_6s_iv_4v_1), y(C)\}$). Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_4v_1)$ (resp. $y(v_1v_6s_iv_4v_1)$), $y(v_1v_2v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_4v_1) - \theta$ (resp. $y(v_1v_6s_iv_4v_1) - \theta$), $y(v_1v_2v_4v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. It is easy to see that $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$ or $y'(v_1v_6s_iv_4v_1) < y(v_1v_6s_iv_4v_1)$, contradicting (8) or (6). Since v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, every cycle in $\mathcal{C}_0^y$ containing v_3v_1 passes through v_1s_1. Thus v_3v_1 is outside $\mathcal{C}_0^y$, and neither v_1s_1 nor v_3v_1 is saturated by $\boldsymbol{y}$ in T.

Observe that v_6v_3 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ containing v_6v_3. Then the multiset sum of the cycles C, $v_1v_6v_4v_1$ (resp. $v_1v_6s_iv_4v_1$), and the unsaturated arc v_3v_1 contain arc-disjoint cycles $v_1v_6v_3v_1$ and $C' = C[v_4, v_6] \cup \{v_6v_4\}$ (resp. $C' = C[v_4, v_6] \cup \{v_6s_i, s_iv_4\}$). Set $\theta = \min\{y(v_1v_6v_4v_1), y(C), w(v_3v_1) - z(v_3v_1)\}$ (resp. $\theta = \min\{y(v_1v_6s_iv_4v_1), y(C), w(v_3v_1) - z(v_3v_1)\}$). Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_4v_1)$ (resp. $y(v_1v_6s_iv_4v_1)$), $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_4v_1) - \theta$ (resp. $y(v_1v_6s_iv_4v_1) - \theta$), $y(v_1v_6v_3v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. It is easy to see that $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$ or $y'(v_1v_6s_iv_4v_1) < y(v_1v_6s_iv_4v_1)$, contradicting (8) or (6). Hence v_6v_3 is not saturated by $\boldsymbol{y}$ in T.

Let C be a cycle in $\mathcal{C}_0^y$ containing s_1v_4. Then the multiset sum of the cycles C, each of the cycles $v_1v_6v_4v_1$ and $v_1v_6s_iv_4v_1$, and the unsaturated arcs v_6v_3, v_3v_1, and v_1s_1 contains arc-disjoint cycles $v_1s_1v_4v_1$ and $v_1v_6v_3v_1$. So, by Lemma 4.7(vi), we have $y(v_1v_6v_4v_1) = y(v_1v_6s_iv_4v_1) = 0$; this contradiction establishes (17).

Using (17), we obtain $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_1\}$. Since K is an FAS of $T_2\backslash a_2$, it is an MFAS by (9) and hence $y(\mathcal{C}_2) = \tau_{\boldsymbol{w}}(T_2\backslash a_2)$. This proves Claim 1.

The above proof yields the following statement, which will be used later.

(18) If Case 1.1 occurs, then every MFAS comes from $\{\{v_3v_1, v_4v_1\}, \{v_1v_6\}, \{v_4v_1, v_6v_3\}\}$. If Case 1.2 occurs, then every MFAS comes from $\{\{v_3v_1, v_4v_1\}, \{v_1v_6, v_4v_1\}, \{v_4v_1, v_6v_3\}\}$.

Claim 2. $y(C)$ is a positive integer for some $C \in \mathcal{C}_2$ or $\nu_w^*(T)$ is an integer.

To justify this, we first show that

(19) if $v_3 \in \varphi(s_i)$ for $i \in \{1,2,3\}$, then $y(v_1v_6v_3s_iv_4v_1) = 0$.

Assume the contrary: $y(v_1v_6v_3s_iv_4v_1) > 0$. Then $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1v_6v_3v_4v_1) = w(v_3v_4)$, and $y(v_1v_6v_4v_1) = w(v_6v_4)$ by (12). So Lemma 4.4(iii) allows us to assume that $w(v_3v_1) = w(v_3v_4) = w(v_6v_4) = 0$. Let j and k be subscripts in $\{1,2,3\}$, if any, such that $v_6 \in \varphi(s_j)$ and $v_1 \in \varphi(s_k)$. If both $y(v_1s_kv_4v_1)$ and $y(v_1v_6s_jv_4v_1)$ are integral, then, by Claim 1, $y(v_1v_6v_3s_iv_4v_1)$ is a positive integer, so Claim 2 holds. Thus we may assume that $y(v_1s_kv_4v_1)$ or $y(v_1v_6s_jv_4v_1)$ is not integral. Then, by (11) and Lemma 4.4(iii), we have $j, k \neq i$. Furthermore, both v_1s_k and v_6s_j are outside $\mathcal{C}_0^y$, for otherwise, we can construct an optimal solution $\boldsymbol{y}'$ to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3s_jv_4v_1) < y(v_1v_6v_3s_jv_4v_1)$, contradicting (6).

Consider first the case when $y(v_1v_6s_jv_4v_1)$ is not integral. If $j = k$ and $y(v_1s_kv_4v_1) > 0$, then $y(v_1s_kv_4v_1) = w(v_1s_k) > 0$ by (11), so Claim 2 holds. Thus we may assume that $j \neq k$ if $y(v_1s_kv_4v_1) > 0$. Let us show that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1v_6s_jv_4v_1)$ and $y(v_1v_6v_3s_iv_4v_1)$ are positive, $x(v_1v_6s_jv_4v_1) = x(v_1v_6v_3s_iv_4v_1) = 1$ by Lemma 4.3(i). By Lemma 6.2(vi), $x(v_6s_j) = x(v_3s_i) = 0$. It follows that $x(s_jv_4) = x(v_6v_3) + x(s_iv_4)$. If v_6v_3 is outside $\mathcal{C}_0^y$, then $x(v_6v_3) = 0$ by Lemma 4.3(ii), because $z(v_6v_3) = y(v_1v_6v_3s_iv_4v_1) < w(v_6v_3)$. Thus $x(s_iv_4) = x(s_jv_4)$, contradicting Lemma 6.2(iv). So we assume that v_6v_3 is contained in some cycle in $\mathcal{C}_0^y$. Since $w(v_3v_4) = w(v_6v_4) = 0$ and (v_6, s_j) is outside $\mathcal{C}_0^y$, for any $u \in V \backslash (V(T_2) \backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_6, then it passes through $v_6v_3s_iv_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_j, then it passes through s_jv_4. By Lemma 4.3(iv), we obtain $x(uv_6) + x(v_6v_3) + x(v_3s_i) + x(s_iv_4) = x(us_j) + x(s_jv_4)$. Hence $x(uv_6) = x(us_j)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting vertex s_j, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_jv_4)$ for each $e \in \{v_6v_3, v_3s_i, s_iv_4\}$ and replacing $w(uv_6)$ with $w(uv_6) + w(us_j)$ for each $u \in V \backslash (V(T_2) \backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6v_3s_iv_4v_1) = y(v_1v_6s_jv_4v_1) + y(v_1v_6v_3s_iv_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_jv_4 for any $u \in V \backslash (V(T_2) \backslash a_2)$, let C' be the cycle arising from C by replacing the path us_jv_4 with the path $uv_6v_3s_iv_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer.

In the other case when $y(v_1v_6s_jv_4v_1) = 0$ and $y(v_1s_kv_4v_1)$ is not integral, the proof goes along the same line, so we omit the details here.

By Claim 1, $y(\mathcal{C}_2) = w(K)$ for some FAS K of $T_2 \backslash a_2$ as described in (18). Recall that

(20) in Case 1.1, we have $v_1 \notin \varphi(s_i)$ for any $i \in \{1,2,3\}$, $\varphi(s_1) = \{v_6\}$, and $\varphi(s_2) = \{v_3\}$; in Case 1.2, we have $v_1 \in \varphi(s_1)$, $v_6 \in \varphi(s_i)$, and $v_3 \in \varphi(s_j)$, with $\{1\} \neq \{i, j\} \subseteq \{1,2,3\}$.

Depending on what K is, we distinguish among four cases.

Case 2.1. $K = \{v_4v_1, v_6v_3\}$ in Case 1.1 or $K = \{v_1v_6, v_4v_1\}$ in Case 1.2.

Consider first the subcase when $K = \{v_4v_1, v_6v_3\}$ in Case 1.1. Now $y(v_1v_6v_3v_1) = w(v_6v_3)$ and $y(v_1v_6v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_4v_1)$ (see (20)). If $y(v_1v_6s_1v_4v_1) = 0$, then $y(v_1v_6v_4v_1) = w(v_4v_1)$. If $y(v_1v_6s_1v_4v_1) > 0$, then $y(v_1v_6v_4v_1) = w(v_6v_4)$ by (11), and hence $y(v_1v_6s_1v_4v_1) = w(v_4v_1) - w(v_6v_4)$. By the hypothesis of the present section, $w(K) = \tau_w(T_2 \backslash a_2) > 0$. So at least one of $y(v_1v_6v_3v_1)$, $y(v_1v_6v_4v_1)$, and $y(v_1v_6s_1v_4v_1)$ is a positive integer.

Next consider the subcase when $K = \{v_1v_6, v_4v_1\}$ in Case 1.2. Now $y(v_1s_1v_4v_1) = w(v_4v_1)$ and $y(v_1v_6v_3v_1) = w(v_1v_6)$. So at least one of $y(v_1s_1v_4v_1)$ and $y(v_1v_6v_3v_1)$ is a positive integer.

Case 2.2. $K = \{v_1v_6\}$ or $\{v_3v_1, v_4v_1\}$ in Case 1.1.

We only consider the subcase when $K = \{v_1v_6\}$, as the proof in the other subcase goes along the same line. Now $y(v_1v_6v_3v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_3v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_1v_6)$, and v_3v_1 is outside $\mathcal{C}_0^y$.

Observe that $y(v_1v_6v_3v_4v_1) > 0$, for otherwise, if $y(v_1v_6s_1v_4v_1) > 0$, then $y(v_1v_6v_4v_1) = w(v_6v_4)$ by (11), and hence $y(v_1v_6v_3v_1)+y(v_1v_6s_1v_4v_1) = w(v_1v_6)-w(v_6v_4)$; if $y(v_1v_6s_1v_4v_1) = 0$, then $y(v_1v_6v_3v_1)+y(v_1v_6v_4v_1) = w(v_1v_6)$. Let us show that $y(v_1v_6v_3v_1)$ is integral. Assume first that $y(v_1v_6s_1v_4v_1) > 0$. If v_6v_3 is outside $\mathcal{C}_0^y$, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$ and $y(v_1v_6s_1v_4v_1)$ with $y(v_1v_6v_3v_1)+[y(v_1v_6s_1v_4v_1)]$ and $\lfloor y(v_1v_6s_1v_4v_1)\rfloor$, respectively; if v_6v_3 is contained in a cycle $C \in \mathcal{C}_0^y$, set $\theta = \min\{y(C), [y(v_1v_6s_1v_4v_1)]\}$ and $C' = C[v_4, v_6]\cup\{v_6s_1, s_1v_4\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$, $y(v_1v_6s_1v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_1) + \theta$, $y(v_1v_6s_1v_4v_1) - \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_1) > y(v_1v_6v_3v_1)$ while $y'(v_1v_6s_1v_4) < y(v_1v_6s_1v_4v_1)$ in either situation, so $\boldsymbol{y}'$ is a better choice than $\boldsymbol{y}$ (see (6)), a contradiction. Assume next that $y(v_1v_6s_1v_4v_1) = 0$. Imitating the above proof, with $y(v_1v_6v_4v_1)$ in place of $y(v_1v_6s_1v_4v_1)$, we can reach a contradiction to (8).

Since $y(v_1v_6v_3v_4v_1) > 0$, by (10), we have $y(v_1v_6v_3v_1) = w(v_3v_1)$ and $y(v_1v_6v_4v_1) = w(v_6v_4)$; so Lemma 4.4(iii) allows us to assume that $w(v_3v_1) = w(v_6v_4) = 0$. Thus the previous equality concerning $w(v_1v_6)$ becomes $y(v_1v_6s_1v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$. So we may assume that neither $y(v_1v_6s_1v_4v_1)$ nor $y(v_1v_6v_3v_4v_1)$ is integral, for otherwise, at least one of them is a positive integer. Observe that v_6s_1 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle in $\mathcal{C}_0^y$ that contains v_6s_1, let $C' = C[v_4, v_6] \cup \{v_6v_3, v_3v_4\}$, and let $\theta = \min\{y(C), y(v_1v_6v_3v_4v_1)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6s_1v_4v_1)$, $y(v_1v_6v_3v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6s_1v_4v_1)+\theta$, $y(v_1v_6v_3v_4v_1) - \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (7).

We propose to show that $\nu_w^*(T)$ is an integer. For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1v_6s_1v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ are positive, $x(v_1v_6s_1v_4v_1) = x(v_1v_6v_3v_4v_1) = 1$ by Lemma 4.3(i). Since $y(v_1v_6s_1v_4v_1) < w(v_6s_1)$, we have $x(v_6s_1) = 0$ by Lemma 4.3(ii). Thus $x(s_1v_4) = x(v_6v_3) + x(v_3v_4)$. Since $w(v_6v_4) = 0$, for any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_6, then it passes through $v_6v_3v_4$ or $v_6s_1v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_1, then it passes through s_1v_4. By Lemma 4.3(iv), we obtain $x(uv_6) + x(v_6v_3) + x(v_3v_4) = x(us_1) + x(s_1v_4)$ or $x(uv_6) + x(v_6s_1) + x(s_1v_4) = x(us_1) + x(s_1v_4)$. Hence $x(uv_6) = x(us_j)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting vertex s_1, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_1v_4)$ for $e = v_6v_3$ and v_3v_4 and replacing $w(uv_6)$ with $w(uv_6) + w(us_1)$ for any $u \in V\backslash V(T_2)\backslash a_2$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6v_3v_4v_1) = y(v_1v_6s_1v_4v_1) + y(v_1v_6v_3v_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_1v_4 for any $u \in V\backslash(V(T_2)\backslash a_2)$, let C' be the cycle arising from C by replacing the path us_1v_4 with the path $uv_6v_3v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$

is an integer.

Case 2.3. $K = \{v_4v_1, v_6v_3\}$ in Case 1.2.

In this case, $y(v_1v_6v_3v_1) = w(v_6v_3)$ and $y(v_1s_1v_4v_1) + y(v_1v_6v_4v_1) + y(v_1v_6s_iv_4v_1) = w(v_4v_1)$ (see (20)). By Lemma 4.4(iii), we may assume that $w(v_6v_3) = 0$. Let us show that

(21) $y(v_1v_6s_iv_4v_1) = 0$.

Assume the contrary. Then, by (11), we have $y(v_1v_6v_4v_1) = w(v_6v_4)$, and v_1s_i is saturated by $\boldsymbol{y}$ in T_2. Lemma 4.4(iii) allows us to assume that $w(v_6v_4) = 0$ and that $y(v_1v_6s_iv_4v_1)$ is not integral. It follows from (6) and Lemma 4.7(v) that $i \neq 1$ and v_1s_1 is outside $\mathcal{C}_0^y$. We propose to prove that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1s_1v_4v_1)$ and $y(v_1v_6s_iv_4v_1)$ are positive, by Lemma 4.3(i), we have $x(v_1s_1v_4v_1) = x(v_1v_6s_iv_4v_1) = 1$. Since $y(v_1s_1v_4v_1) < w(v_1s_1)$, by Lemma 4.3(ii), we obtain $x(v_1s_1) = 0$, so $x(s_1v_4) = x(v_1v_6) + x(v_6s_i) + x(s_iv_4)$. If v_1v_6 is outside $\mathcal{C}_0^y$, then $x(v_1v_6) = 0$, because $z(v_1v_6) = y(v_1v_6s_iv_4v_1) < w(v_1v_6)$. By Lemma 6.2(vi), $x(v_1s_1) = x(v_6s_i) = 0$. Hence, $x(s_1v_4) = x(s_iv_4)$, contradicting Lemma 6.2(iv). So we assume that v_1v_6 is contained in some cycle in $\mathcal{C}_0^y$. Since $w(v_6v_3) = w(v_6v_4) = 0$, for any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_1, then it passes through $v_1v_6s_iv_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_1, then it passes through s_1v_4. By Lemma 4.3(iv), we obtain $x(uv_1) + x(v_1v_6) + x(v_6s_i) + x(s_iv_4) = x(us_1) + x(s_1v_4)$. Hence $x(uv_1) = x(us_1)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting vertex s_1, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_1v_4)$ for $e \in \{v_1v_6, v_6s_i, s_iv_4\}$ and replacing $w(uv_1)$ with $w(uv_1) + w(us_1)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$, Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A', and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6s_iv_4v_1) = y(v_1s_1v_4v_1) + y(v_1v_6s_iv_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_1v_4, let C' arise from C by replacing the path us_1v_4 with the path $uv_1v_6s_iv_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer. So we may assume that (21) holds.

By (21), the equality concerning $w(v_4v_1)$ becomes $y(v_1s_1v_4v_1) + y(v_1v_6v_4v_1) = w(v_4v_1)$. As $w(v_4v_1) = w(K) = \tau_w(T_2\backslash a_2) > 0$, neither $y(v_1s_1v_4v_1)$ nor $y(v_1v_6v_4v_1)$ is integral. Observe that v_1s_1 is outside $\mathcal{C}_0^y$, for otherwise, let C be a cycle containing v_1s_1 in $\mathcal{C}_0^y$, let $C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_4\}$, and let $\theta = \min\{y(C), y(v_1v_6v_4v_1)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1s_1v_4v_1)$, $y(v_1v_6v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1s_1v_4v_1) + \theta$, $y(v_1v_6v_4v_1) - \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1)$, contradicting (8). Moreover, $i \neq 1$, for otherwise, it can be shown similarly that v_6s_1 is outside $\mathcal{C}_0^y$, which implies $z(v_6s_1) = 0$, contradicting that $v_6 \in \varphi(s_1)$. Let us show that

(22) $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1s_1v_4v_1)$ and $y(v_1v_6v_4v_1)$ are positive, we have $x(v_1s_1v_4v_1) = x(v_1v_6v_4v_1) = 1$ by Lemma 4.3(i). By (16) and Lemma 4.4(iii), we have $y(v_1s_1v_4v_1) < w(v_1s_1)$ and hence $x(v_1s_1) = 0$. So $x(s_1v_4) = x(v_1v_6) + x(v_6v_4)$. Note that if a cycle in $\mathcal{C}_0^y$ contains us_1, then it passes through s_1v_4. For any $u \in V\backslash(V(T_2)\backslash a_2)$, if there exists a cycle $C \in \mathcal{C}_0^y$ containing uv_1 and passing through $v_1v_6v_4$, then by Lemma 4.3(iv), we obtain $x(uv_1) + x(v_1v_6) + x(v_6v_4) = x(us_1) + x(s_1v_4)$, and hence $x(uv_1) = x(us_1)$. Otherwise, since $w(v_6v_3) = 0$, if a cycle in $\mathcal{C}_0^y$ contains uv_1, then it passes through $v_1v_6s_iv_4$. By Lemma 4.3(i) and (iv), we have $x(v_6v_4) \geq x(v_6s_i) + x(s_iv_4)$ and

$x(uv_1)+x(v_1v_6)+x(v_6s_i)+x(s_iv_4) = x(us_1)+x(s_1v_4)$. Since $x(v_1v_6v_4v_1) = 1$ and $x(v_1v_6s_iv_4v_1) \geq 1$, we see that $x(v_6v_4) \leq x(v_6s_i) + x(s_iv_4)$. Hence, $x(uv_1) = x(us_1)$ also holds. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting vertex s_1, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_1v_4)$ for $e = v_1v_6$ and v_6v_4 and replacing $w(uv_1)$ with $w(uv_1) + w(us_1)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6v_4v_1) = y(v_1s_1v_4v_1) + y(v_1v_6v_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_1v_4 for any $u \in V\backslash(V(T_2)\backslash a_2)$, let C' arise from C by replacing the path us_1v_4 with the path $uv_1v_6v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value as $\boldsymbol{x}$ and $\boldsymbol{y}$. From the hypothesis of Theorem 4.1, (22) follows.

Case 2.4. $K = \{v_3v_1, v_4v_1\}$ in Case 1.2.

In this case, $y(v_1v_6v_3v_1) = w(v_3v_1)$ and $y(v_1s_1v_4v_1)+y(v_1v_6v_4v_1)+y(v_1v_6s_iv_4v_1)+y(v_1v_6v_3v_4v_1) = w(v_4v_1)$ (see (20)). By Lemma 4.4(iii), we may assume that $w(v_3v_1) = 0$.

If $y(v_1v_6s_iv_4v_1) = y(v_1v_6v_3v_4v_1) = 0$, then $y(v_1s_1v_4v_1) + y(v_1v_6v_4v_1) = w(v_4v_1)$. Since $w(v_4v_1) = w(K) = \tau_w(T_2\backslash a_2) > 0$, we see that $y(v_1s_1v_4v_1)$ is not integral. Imitating the proof of (22), it can be shown that $\nu_w^*(T)$ is an integer. So we assume that at least one of $y(v_1v_6v_3v_4v_1)$ and $y(v_1v_6s_iv_4v_1)$ is positive. By (10) or (11), v_6v_4 is saturated by $\boldsymbol{y}$ in T_2, and hence $y(v_1v_6v_4v_1) = w(v_6v_4)$. By Lemma 4.4(iii), we may assume that $w(v_6v_4) = 0$. If neither $y(v_1v_6s_iv_4v_1)$ nor $y(v_1v_6v_3v_4v_1)$ is integral then, imitating the proof in Case 2.2, it can be shown that $\nu_w^*(T)$ is an integer. It remains to consider the subcase when precisely one of them is positive. Now it can be shown that $\nu_w^*(T)$ is an integer. Since the proof is the same as that contained in the argument of (21), we omit the routine details here.

Combining the above four cases, we see that Claim 2 holds. Hence, by Lemma 4.4(iii), the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral, as described in (1) above. ∎

Lemma 6.8. *If $T_2/S = G_3$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_4, v_5)$, $s^* = v_2$, and $v_0 = v_4$. To establish the statement, by Lemma 4.4(iii) and Lemma 4.6(ii), it suffices to prove that

(1) $y(C)$ is a positive integer for some $C \in \mathcal{C}_2$ or the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is an integer.

Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 6.2 (i) and (vi), we have

(2) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

(3) There exist at least two and at most three vertices s_i's in S with $\varphi(s_i) \neq \emptyset$. (The statement is exactly the same as (3) in the proof of Lemma 6.7.)

Lemma 6.2(i) allows us to assume that

(4) if $\varphi(s_i) \neq \emptyset$, then $i \in \{1, 2, 3\}$.

Let t be the subscript in $\{1, 2, 3\}$ with $v_1 \in \varphi(s_t)$, if any. By (2), t is well defined. In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized;

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically;

(7) subject to (5) and (6), $y(v_1v_6v_3v_4v_1)$ is minimized; and

(8) subject to (5)-(7), $y(v_1s_tv_4v_1) + y(v_3v_4v_6v_3)$ is minimized.

Let us make some observations about $\boldsymbol{y}$ before proceeding.

(9) If K is an FAS of $T_2\backslash a_2$ such that $y(\mathcal{C}_2) = w(K)$, then K is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 5.3.)

The statements below follow instantly from Lemma 4.7(v) and the choice of $\boldsymbol{y}$.

(10) If $y(v_1v_6v_3v_4v_1) > 0$, then both v_3v_1 and v_4v_6 are saturated by $\boldsymbol{y}$ in T_2. Furthermore, for any $i \in \{1,2,3\}$, we have $y(v_6s_iv_4v_6) = 0$; if $y(v_3s_iv_4v_6v_3) > 0$, then v_1s_i is saturated by $\boldsymbol{y}$ in T_2.

(11) If $y(v_1v_6s_iv_4v_1) > 0$ for some $i \in \{1,2,3\}$, then both v_1s_i and v_4v_6 are saturated by $\boldsymbol{y}$ in T_2. Furthermore, if $y(v_3v_4v_6v_3) > 0$, then v_3v_1 is saturated by $\boldsymbol{y}$ in T_2; for any $1 \le j \ne i \le 3$, if $y(v_3s_jv_4v_6v_3) > 0$, then both v_3v_1 and v_1s_j are saturated by $\boldsymbol{y}$ in T_2.

(12) If $y(v_3s_iv_4v_6v_3) > 0$ for some $i \in \{1,2,3\}$, then both v_3v_4 and v_6s_i are saturated by $\boldsymbol{y}$ in T_2.

(13) If $v_1 \in \varphi(s_i)$ for some $i \in \{1,2,3\}$, then $y(v_1s_iv_4v_6v_3v_1) = 0$.

Assume the contrary: $y(v_1s_iv_4v_6v_3v_1) > 0$. Then v_1v_6, v_3v_4, and v_4v_1 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(v). Let j and k be subscripts in $\{1,2,3\}$, if any, such that $v_3 \in \varphi(s_j)$ and $v_6 \in \varphi(s_k)$ (possibly $j = k$). As before, let $\uplus$ denote the multiset sum. Then $v_1s_iv_4v_6v_3v_1 \uplus v_1v_6v_3v_4v_1 = v_1s_iv_4v_1 \uplus v_1v_6v_3v_1 \uplus v_3v_4v_6v_3$, $v_1s_iv_4v_6v_3v_1 \uplus v_1v_6s_kv_4v_1 = v_1s_iv_4v_1 \uplus v_1v_6v_3v_1 \uplus v_6s_kv_4v_6$, and $v_1s_iv_4v_6v_3v_1 \cup v_1v_6v_3s_jv_4v_1 = v_1s_iv_4v_1 \uplus v_1v_6v_3v_1 \uplus v_3s_jv_4v_6v_3$. Thus, from the optimality of $\boldsymbol{y}$, we deduce that $y(v_1v_6v_3v_4v_1)$, $y(v_1v_6s_kv_4v_1)$, and $y(v_1v_6v_3s_jv_4v_1)$ are all zero. So $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_1s_iv_4v_1) = w(v_4v_1)$, and $y(v_3v_4v_6v_3) = w(v_3v_4)$. Clearly, we may assume that $w(v_1v_6) = w(v_4v_1) = w(v_3v_4) = 0$, otherwise (1) holds. By (3), we have $\{j,k\} \ne \{i\}$. Let us show that one of $y(v_6s_kv_4v_6)$, $y(v_3s_jv_4v_6v_3)$, and $y(v_1s_iv_4v_6v_3v_1)$ is a positive integer or $\nu_w^*(T)$ is an integer. We proceed by considering two cases.

- k exists and $i \ne k$. In this case, observe first that v_6s_k is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_6s_kv_4v_6) = w(v_6s_k) > 0$ and hence (1) holds. Next, v_kv_4 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, if $k \ne j$, then $y(v_6s_kv_4v_6) = w(s_kv_4) > 0$; if $k = j$, then $y(v_6s_kv_4v_6) + y(v_3s_kv_4v_6v_3) = w(s_kv_4) > 0$, and $y(v_6s_kv_4v_6) = w(v_6s_k) > 0$ by Lemma 4.7(v) provided $y(v_3s_kv_4v_6v_3) > 0$. So $y(v_6s_kv_4v_6)$ is a positive integer, and hence (1) also holds. Moreover, both v_6s_k and v_3s_j are outside $\mathcal{C}_0^y$, for otherwise, let C_1 (resp. C_2) be a cycle in $\mathcal{C}_0^y$ containing v_6s_k (resp. v_3s_j). Since $C_1 \uplus v_1s_iv_4v_6v_3v_1 = v_6s_kv_4v_6 \uplus C_1'$ and $C_2 \uplus v_1s_iv_4v_6v_3v_1 = v_3s_jv_4v_6v_3 \uplus C_2'$, where $C_1' = C_1[v_4,v_6] \cup \{v_6v_3, v_3v_1, v_1s_i, s_iv_4\}$ and $C_2' = C_2[v_4,v_3] \uplus \{v_3v_1, v_1s_i, s_iv_4\}$, by Lemma 4.7(viii), we have $y(C_i) = 0$ for $i = 1,2$, a contradiction. It follows that v_6s_k is not saturated by $\boldsymbol{y}$ in T, and s_kv_4 is contained in some cycle in $\mathcal{C}_0^y$. By Lemma 4.7(vii), v_4v_6 is saturated by $\boldsymbol{y}$ in T_2, so $y(v_1s_iv_4v_6v_3v_1) + y(v_6s_kv_4v_6) + y(v_3s_jv_4v_6v_3) = w(v_4v_6)$. If $j = k$ and $y(v_3s_kv_4v_6v_3) > 0$, then v_6s_k is saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(v), a contradiction. So either $j \ne k$ or $j = k$ and $y(v_3s_kv_4v_6v_3) = 0$. Since $w(v_6s_k) > 0$ and v_6s_k is outside $\mathcal{C}_0^y$, we have $y(v_6s_kv_4v_6) > 0$. Assume $y(v_6s_kv_4v_6)$ is not integral. Let us show that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T,\boldsymbol{w})$. Since both $y(v_6s_kv_4v_6)$ and $y(v_1s_iv_4v_6v_3v_1)$ are positive, by Lemma 4.3(i), we have $x(v_6s_kv_4v_6) = x(v_1s_iv_4v_6v_3v_1) = 1$. By Lemma 4.3(ii), we obtain $x(v_6s_k) = 0$. Hence $x(s_kv_4) = x(v_6v_3) + x(v_3v_1) + x(v_1s_i) + x(s_iv_4)$. Since $w(v_3v_4) = 0$ and v_6s_k is outside $\mathcal{C}_0^y$, for any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_6, then it passes through $v_6v_3v_1s_iv_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_k, then it passes through s_kv_4. By Lemma 4.3(iv), we obtain $x(uv_6) + x(v_6v_3) + x(v_3v_1) + x(v_1s_i) + x(s_iv_4) = x(us_k) + x(s_kv_4)$. Hence $x(uv_6) = x(us_k)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting s_k, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(v_4 s_k)$ for $e \in \{v_6 v_3, v_3 v_1, v_1 s_i, s_i v_4\}$ and replacing $w(uv_6)$ with $w(uv_6) + w(us_k)$ for any $u \in V \backslash (V(T_2) \backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A', and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1 s_i v_4 v_6 v_3 v_1) = y(v_1 s_i v_4 v_6 v_3 v_1) + y(v_6 s_k v_4 v_6)$; for each $C \in \mathcal{C}_0^y$ passing through $us_k v_4$, let C' arise from C by replacing the path $us_k v_4$ with the path $uv_6 v_3 v_1 s_i v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer.

• Either k does not exist or $i = k$. In this case, by (3), we see that j exists; that is, $v_3 \in \varphi(s_j)$. Similar to the above case, we can show that either $y(v_3 s_j v_4 v_6 v_3)$ is a positive integer or $\nu_w^*(T)$ is an integer. Since the proof goes along the same line (with $v_3 s_j$ and $y(v_3 s_j v_4 v_6 v_3)$ in place of $v_6 s_k$ and $y(v_6 s_k v_4 v_6)$, respectively), we omit the details here. Hence we may assume that (13) holds.

(14) If $v_3 \in \varphi(s_j)$ for some $j \in \{1, 2, 3\}$, then $y(v_1 v_6 v_3 s_j v_4 v_1) = 0$.

Assume the contrary: $y(v_1 v_6 v_3 s_j v_4 v_1) > 0$. Then $v_3 v_1$, $v_3 v_4$, and $v_4 v_6$ are saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(v). Let i and k be subscripts in $\{1, 2, 3\}$, if any, such that $v_1 \in \varphi(s_i)$ and $v_6 \in \varphi(s_k)$ (possibly $i = k$). Since $v_1 v_6 v_3 s_j v_4 v_1 \uplus v_3 v_4 v_6 v_3 = v_1 v_6 v_3 v_4 v_1 \uplus v_3 s_j v_4 v_6 v_3$, and $v_1 v_6 v_3 s_j v_4 v_1 \cup v_6 s_k v_4 v_6 = v_1 v_6 s_k v_4 v_1 \uplus v_3 s_j v_4 v_6 v_3$, from the optimality of $\boldsymbol{y}$, we deduce that $y(v_3 v_4 v_6 v_3) = y(v_6 s_k v_4 v_6) = 0$. So $y(v_1 v_6 v_3 v_1) = w(v_3 v_1)$, $y(v_1 v_6 v_3 v_4 v_1) = w(v_3 v_4)$, and $y(v_3 s_j v_4 v_6 v_3) = w(v_4 v_6)$. Clearly, we may assume that $w(v_3 v_1) = w(v_3 v_4) = w(v_4 v_6) = 0$, otherwise (1) holds. By (3), we have $\{i, k\} \neq \{j\}$. Let us show that one of $y(v_1 s_i v_4 v_1)$, $y(v_1 v_6 s_k v_4 v_1)$, and $y(v_1 v_6 v_3 s_j v_4 v_1)$ is a positive integer or $\nu_w^*(T)$ is an integer. We proceed by considering two cases.

• i exists and $i \neq j$. In this case, observe first that $v_1 s_i$ is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_1 s_i v_4 v_1) = w(v_1 s_i) > 0$ and hence (1) holds. Next, $s_i v_4$ is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, if $i \neq k$, then $y(v_1 s_i v_4 v_1) = w(s_i v_4) > 0$; if $i = k$, then $y(v_1 s_i v_4 v_1) + y(v_1 v_6 s_i v_4) = w(s_i v_4) > 0$, and $y(v_1 s_i v_4 v_1) = w(v_1 s_i) > 0$ by Lemma 4.7(v) provided $y(v_1 v_6 s_i v_4 v_1) > 0$. So $y(v_1 s_i v_4 v_1)$ is a positive integer, and hence (1) also holds. Moreover, both $v_1 s_i$ and $v_6 s_k$ are outside $\mathcal{C}_0^y$, for otherwise, let C_1 (resp. C_2) be a cycle in $\mathcal{C}_0^y$ containing $v_1 s_i$ (resp. $v_6 s_k$). Since $C_1 \uplus v_1 v_6 v_3 s_j v_4 v_1 = v_1 s_i v_4 v_1 \cup C_1'$ and $C_2 \uplus v_1 v_6 v_3 s_j v_4 v_1 = v_1 v_6 s_k v_4 v_1 \uplus C_2'$, where $C_1' = C_1[v_4, v_1] \cup \{v_1 v_6, v_6 v_3, v_3 s_j, s_j v_4\}$ and $C_2' = C_2[v_4, v_6] \cup \{v_6 v_3, v_3 s_j, s_j v_4\}$, by Lemma 4.7(viii), we have $y(C_i) = 0$ for $i = 1, 2$, a contradiction. It follows that $v_1 s_i$ is not saturated by $\boldsymbol{y}$ in T and $s_i v_4$ is contained in some cycle in $\mathcal{C}_0^y$. By Lemma 4.7(vii), $v_4 v_1$ is saturated by $\boldsymbol{y}$ in T_2, so $y(v_1 s_i v_4 v_1) + y(v_1 v_6 s_k v_4 v_1) + y(v_1 v_6 v_3 s_j v_4 v_1) = w(v_4 v_1)$. If $i = k$ and $y(v_1 v_6 s_k v_4 v_1) > 0$, then $v_1 s_i$ is saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(v), a contradiction. So either $i \neq k$ or $i = k$ and $y(v_1 v_6 s_k v_4 v_1) = 0$. Since $w(v_1 s_i) > 0$ and $v_1 s_i$ is outside $\mathcal{C}_0^y$, we have $y(v_1 s_i v_4 v_1) > 0$. Assume $y(v_1 s_i v_4 v_1)$ is not integral. Let us show that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1 s_i v_4 v_1)$ and $y(v_1 v_6 v_3 s_j v_4 v_1)$ are positive, by Lemma 4.3(i), we have $x(v_1 s_i v_4 v_1) = y(v_1 v_6 v_3 s_j v_4 v_1) = 1$. By Lemma 4.3(ii), we obtain $x(v_1 s_i) = 0$. Hence $x(s_i v_4) = x(v_1 v_6) + x(v_6 v_3) + x(v_3 s_j) + x(s_j v_4)$. Since $w(v_3 v_1) = w(v_3 v_4) = 0$, for any $u \in V \backslash (V(T_2) \backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_1, then it passes through $v_6 v_3 s_j v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_i, then it passes through $s_i v_4$. By Lemma 4.3(iv), we obtain $x(uv_1) + x(v_1 v_6) + x(v_6 v_3) + x(v_3 s_j) + x(s_j v_4) = x(us_i) + x(s_i v_4)$. Hence $x(uv_1) = x(us_i)$. Clearly, we may assume that this equality holds in any other situation.

Let $T' = (V', A')$ be obtained from T by deleting s_i, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(v_4s_i)$ for $e \in \{v_1v_6, v_6v_3, v_3s_j, s_jv_4\}$ and replacing $w(uv_1)$ with $w(uv_1) + w(us_i)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6v_3s_jv_4v_1) = y(v_1v_6v_3s_jv_4v_1) + y(v_1s_iv_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_iv_4, let C' be obtained from C by replacing the path us_iv_4 with the path $uv_1v_6v_3s_jv_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer.

• Either i does not exist or $i = j$. In this case, by (3), we see that k exists; that is, $v_6 \in \varphi(s_k)$. Similar to the above case, we can show that either $y(v_1v_6s_kv_4v_1)$ is a positive integer or $\nu_w^*(T)$ is an integer. Since the proof goes along the same line (with v_6s_k and $y(v_1v_6s_kv_4v_1)$ in place of v_1s_i and $y(v_1s_iv_4v_1)$, respectively), we omit the details here. Hence we may assume that (14) holds.

We proceed by considering two cases, depending on whether $\varphi(s_i) = \{v_1\}$ for some i.

Case 1. $\varphi(s_i) = \{v_1\}$ for some $i \in \{1, 2, 3\}$.

By Lemma 6.2(i), we may assume that $\varphi(s_1) = \{v_1\}$. Let j and k be subscripts in $\{1, 2, 3\}$, if any, such that $v_3 \in \varphi(s_j)$ and $v_6 \in \varphi(s_k)$ (possibly $j = k$). By (13) and (14), we have

(15) $\mathcal{C}_2^y \subseteq \{v_1v_6v_3v_4v_1, v_1v_6s_kv_4v_1, v_3s_jv_4v_6v_3, v_1s_1v_4v_1, v_6s_kv_4v_6, v_1v_6v_3v_1, v_3v_4v_6v_3\}$.

Observe that neither s_1v_4 nor v_1s_1 is saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_1s_1v_4v_1) = w(s_1v_4)$ or $w(v_1s_1)$; both of them are positive, so (1) holds. By Lemma 6.2(iii), $z(s_1v_4) = w(z_1v_4) > 0$. Thus there exists a cycle $C \in \mathcal{C}_0^y$ containing s_1v_4; subject to this, C is chosen to contain v_1s_1 if possible. If v_1s_1 is outside C, then v_1s_1 is not saturated by $\boldsymbol{y}$ in T. By Lemma 4.7(vii), v_4v_1 is saturated by $\boldsymbol{y}$ in T_2 and hence $y(v_1s_1v_4v_1) + y(v_1v_6s_kv_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1)$.

(16) If $w(v_4v_1) > 0$, then either $y(v_1s_1v_4v_1)$ is a positive integer or $\nu_w^*(T)$ is an integer.

To justify this, assume $y(v_1s_1v_4v_1)$ is not a positive integer. Then at least one of $y(v_1v_6s_kv_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ is positive. Observe that v_1s_1 is outside $\mathcal{C}_0^y$, for otherwise, let D be a cycle in $\mathcal{C}_0^y$ containing v_1s_1. If $y(v_1v_6v_3v_4v_1) > 0$ then, using $D \uplus v_1v_6v_3v_4v_1 = v_1s_1v_4v_1 \uplus D'$, where $D' = D[v_4, v_1] \cup \{v_1v_6, v_6v_3, v_3v_4\}$, and applying Lemma 4.7(viii), we deduce that $y(D) = 0$, a contradiction. If $y(v_1v_6s_kv_4v_1) > 0$, then a contradiction can be reached similarly. Since $w(v_1s_1) > 0$, we obtain $y(v_1s_1v_4v_1) > 0$. As $y(v_1s_1v_4v_1)$ is not integral, at least one of $y(v_1v_6s_kv_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ is not integral. Let us show that $\nu_w^*(T)$ is an integer.

We only consider the case when $y(v_1v_6v_3v_4v_1)$ is not integral, as the proof in the other case when $y(v_1v_6v_3v_4v_1) = 0$ and $y(v_1v_6s_kv_4v_1) > 0$ goes along the same line.

Let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1s_1v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ are positive, by Lemma 4.3(i), we have $x(v_1s_1v_4v_1) = x(v_1v_6v_3v_4v_1) = 1$. By Lemma 4.3(ii), we obtain $x(v_1s_1) = 0$, because v_1s_1 is not saturated by $\boldsymbol{y}$. It follows that $x(s_1v_4) = x(v_1v_6) + x(v_6v_3) + x(v_3v_4)$. Observe that there is no cycle D in $\mathcal{C}_0^y$ that contains the path $v_1v_6s_kv_4$, for otherwise, let $\theta = \min\{y(D), y(v_1v_6v_3v_4v_1)\}$, let $D' = D[v_4, v_1] \cup \{v_1v_6, v_6v_3, v_3v_4\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(D)$, $y(D')$, $y(v_1v_6v_3v_4v_1)$, and $y(v_1v_6s_kv_4v_1)$ with $y(D) - \theta$, $y(D') + \theta$, $y(v_1v_6v_3v_4v_1) - \theta$, and $y(v_1v_6s_kv_4v_1) + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (7). For any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_1, then it passes through $v_1v_6v_3v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_1, then it passes through s_1v_4. By Lemma 4.3(iv), we obtain $x(uv_1) +$

$x(v_1v_6) + x(v_6v_3) + x(v_3v_4) = x(us_1) + x(s_1v_4)$. Hence $x(uv_1) = x(us_1)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting s_1, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_1v_4)$ for $e \in \{v_1v_6, v_6v_3, v_3v_4\}$ and replacing $w(uv_1)$ with $w(uv_1) + w(us_1)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A', and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1) + y(v_1s_1v_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_1v_4, let C' be obtained from C by replacing the path us_1v_4 with the path $uv_1v_6v_3v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer. So (16) follows.

By (16) and Lemma 4.4(iii), we may assume that $w(v_4v_1) = 0$ hereafter.

(17) If k exists (so $v_6 \in \varphi(s_k)$) and $w(v_4v_6) > 0$, then either $y(v_6s_kv_4v_6)$ is a positive integer or $\nu_w^*(T)$ is an integer.

To justify this, observe first that v_6s_k is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_6s_kv_4v_6) = w(v_6s_k) > 0$, so (17) holds. Next, s_kv_4 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, if $j \neq k$, then $y(v_6s_kv_4v_6) = w(s_kv_4) > 0$; if $j = k$, then $y(v_6s_kv_4v_6) + y(v_3s_kv_4v_6v_3) = w(s_kv_4)$, and $y(v_6s_kv_4v_6) = w(v_6s_k) > 0$ by Lemma 4.7(v) provided $y(v_3s_kv_4v_6v_3) > 0$, so (17) also holds. By Lemma 6.2(iii), s_kv_4 is saturated by $\boldsymbol{y}$ in T, so s_kv_4 is contained in some cycle $C \in \mathcal{C}_0^y$; subject to this, C is chosen to contain v_6s_k if possible. Clearly, if v_6s_k is not on C, then v_6s_k is not saturated by $\boldsymbol{y}$ in T. By Lemma 4.7(vii), v_4v_6 is saturated by $\boldsymbol{y}$ in T_2, and hence $y(v_6s_kv_4v_6) + y(v_3v_4v_6v_3) + y(v_3s_jv_4v_6v_3) = w(v_4v_6)$.

Assume $y(v_6s_kv_4v_6)$ is not a positive integer. Then at least one of $y(v_3v_4v_6v_3)$ and $y(v_3s_jv_4v_6v_3)$ is positive, say the former. Note that v_6s_k is outside $\mathcal{C}_0^y$, for otherwise, let D be a cycle in $\mathcal{C}_0^y$ containing v_6s_k. Set $D' = D[v_4, v_6] \cup \{v_6v_3, v_3v_4\}$ and $\theta = \min\{y(v_3v_4v_6v_3), y(C)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_6v_3)$, $y(v_6s_kv_4v_6)$, $y(C)$, and $y(C')$ with $y(v_3v_4v_6v_3) - \theta$, $y(v_6s_kv_4v_6) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (8). Since $w(v_6s_k) > 0$, we have $y(v_6s_kv_4v_6) > 0$. As $y(v_6s_kv_4v_6)$ is not integral, $y(v_3v_4v_6v_3)$ or $y(v_3s_jv_4v_6v_3)$ is not integral. If $y(v_3s_jv_4v_6v_3) > 0$, then v_3v_4 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(v), so $y(v_3v_4v_6v_3) = w(v_3v_4)$. Hence we may assume that exactly one of $y(v_3v_4v_6v_3)$ and $y(v_3s_jv_4v_6v_3)$ is positive. Let us show that $\nu_w^*(T)$ is an integer.

We only consider the case when $y(v_3v_4v_6v_3)$ is not integral, because the proof in the other case when $y(v_3v_4v_6v_3) = 0$ and $y(v_3s_jv_4v_6v_3) > 0$ goes along the same line.

Let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_6s_kv_4v_6)$ and $y(v_3v_4v_6v_3)$ are positive, we have $x(v_6s_kv_4v_6) = x(v_3v_4v_6v_3) = 1$ by Lemma 4.3(i). Since v_6s_k is not saturated by $\boldsymbol{y}$ in T, we obtain $x(v_6s_k) = 0$ by Lemma 4.3(ii). It follows that $x(s_kv_4) = x(v_6v_3) + x(v_3v_4)$. For any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_6, then it passes through $v_6v_3v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_k, then it passes through s_kv_4. By Lemma 4.3(iv), we obtain $x(uv_6) + x(v_6v_3) + x(v_3v_4) = x(us_k) + x(s_kv_4)$. Hence $x(uv_6) = x(us_k)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting s_k, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_kv_4)$ for $e = v_6v_3$ and v_3v_4 and replacing $w(uv_6)$ with $w(uv_6) + w(us_k)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_3v_4v_6v_3) = y(v_3v_4v_6v_3) + y(v_6s_kv_4v_6)$; for each $C \in \mathcal{C}_0^y$ passing through us_iv_4, let C' be the cycle arising from C by replacing the path us_kv_4 with the path $uv_6v_3v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer. So (17) holds.

By (17) and Lemma 4.4(iii), we may assume that if $w(v_4v_6) > 0$, then k does no exist, and hence j exists (so $v_3 \in \varphi(s_j)$) by (3).

(18) If $w(v_4v_6) > 0$, then at least one of $y(v_1v_6v_3v_1)$, $y(v_3v_4v_6v_3)$, and $y(v_3s_jv_4v_6v_3)$ is a positive integer.

To justify this, note that neither s_jv_4 nor v_3s_j is saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_3s_jv_4v_6v_3) = w(s_jv_4)$ or $w(v_3s_j)$; both of them are positive, so (18) holds. By Lemma 6.2(iii), s_jv_4 is saturated by $\boldsymbol{y}$ in T, so s_jv_4 is contained in a cycle $C \in \mathcal{C}_0^y$; subject to this, C is chosen to contain v_3s_j if possible. Clearly, if v_3s_j is not on C, then v_3s_j is not saturated by $\boldsymbol{y}$ in T. By Lemma 4.7(iii), at least one of v_4v_6 and v_6v_3 is saturated by $\boldsymbol{y}$ in T_2. Furthermore, by Lemma 4.7(iv), if v_6v_3 is contained in some cycle in $\mathcal{C}_0^y$, then v_4v_6 is saturated by $\boldsymbol{y}$ in T_2. If v_4v_6 is saturated by $\boldsymbol{y}$ in T_2, then $y(v_3v_4v_6v_3) + y(v_3s_jv_4v_6v_3) = w(v_4v_6)$, and $y(v_3v_4v_6v_3) = w(v_3v_4)$ by Lemma 4.7(v) provided $y(v_3s_jv_4v_6v_3) > 0$. So at least one of $y(v_3v_4v_6v_3)$ and $y(v_3s_jv_4v_6v_3)$ is a positive integer, and hence (18) holds. Thus we may assume that v_4v_6 is not saturated by $\boldsymbol{y}$ in T_2, which implies that v_6v_3 saturated by $\boldsymbol{y}$ in T_2. It follows that $y(v_1v_6v_3v_1)+y(v_3v_4v_6v_3)+y(v_3s_jv_4v_6v_3) = w(v_6v_3)$. If $w(v_6v_3) = 0$, then $K = \{v_4v_1, v_6v_3, v_6s_j\}$ is an FAS of T with total weight zero, so $\tau_w(T_2 \backslash a_2) = 0$, contradicting the hypothesis (α) of this section. Therefore $w(v_6v_3) > 0$. If $y(v_3s_jv_4v_6v_3) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (15) and Lemma 4.7(v). So we may further assume that exactly one of $y(v_3v_4v_6v_3)$ and $y(v_3s_jv_4v_6v_3)$ is positive, and thus $y(v_1v_6v_3v_1) > 0$.

Let us show that $y(v_1v_6v_3v_1)$ is an integer. Suppose not. Then $y(v_3v_4v_6v_3)$ or $y(v_3s_jv_4v_6v_3)$ is not integral, say the former (the proof in the other case goes along the same line). Since v_6v_3 is saturated by $\boldsymbol{y}$ in T_2 and $w(v_6s_j) = 0$, the arc v_1v_6 is outside $\mathcal{C}_0^y$. If v_3v_1 is also outside $\mathcal{C}_0^y$, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_6v_3)$ and $y(v_1v_6v_3v_1)$ with $y(v_3v_4v_6v_3)-\theta$ and $y(v_1v_6v_3v_1)+\theta$, respectively, where $\theta = \min\{w(v_1v_6) - z(v_1v_6), w(v_3v_1) - z(v_3v_1), y(v_3v_4v_6v_3)\}$; if v_3v_1 is contained in some cycle $C \in \mathcal{C}_0^y$, let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3v_4v_6v_3)$, $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_3v_4v_6v_3) - \sigma$, $y(v_1v_6v_3v_1) + \sigma$, $y(C) - \sigma$, $y(C') + \sigma$, respectively, where $C' = C[v_4, v_3] \cup \{v_3v_4\}$ and $\sigma = \min\{w(v_1v_6) - z(v_1v_6), y(C), y(v_3v_4v_6v_3)\}$. It is easy to see that in either situation $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (8). This proves (18).

By (16)-(18), we may assume that $w(v_4v_1) = w(v_4v_6) = 0$. Since each of $\{v_4v_1, v_4v_6, v_1v_6\}$, $\{v_4v_1, v_4v_6, v_6v_3\}$, and $\{v_4v_1, v_4v_6, v_3v_1\}$ is a minimal FAS of $T_2 \backslash a_2$,

$$\epsilon = \min\{w(v_1v_6), w(v_6v_3), w(v_3v_1)\} > 0$$

by the hypothesis (α) of this section. By Lemma 4.7(vii), we obtain $y(v_1v_6v_3v_1) = \epsilon > 0$. Thus (1) is established in the present case.

Case 2. $\varphi(s_i) \neq \{v_1\}$ for any $i \in \{1, 2, 3\}$.

By the hypothesis of the present case, we may assume that $v_6 \in \varphi(s_1)$, $v_3 \in \varphi(s_2)$, and $v_1 \in \varphi(s_i)$ for $i = 1$ or 2. By (13) and (14), we have

(19) $\mathcal{C}_2^y \subseteq \{v_1v_6v_3v_1, v_3v_4v_6v_3, v_1v_6v_3v_4v_1, v_6s_1v_4v_6, v_1v_6s_1v_4v_1, v_3s_2v_4v_6v_3, v_1s_1v_4v_1, v_1s_2v_4v_1\}$

and $y(v_1s_iv_4v_1) = 0$ for $i = 1$ or 2.

Claim 1. $y(\mathcal{C}_2) = \tau_w(T_2 \backslash a_2)$.

To justify this, note that $z(s_iv_4) = w(s_iv_4) > 0$ for $i = 1$ and 2 by Lemma 6.2(iii). Depending on the saturation of s_1v_4 and s_2v_4, we distinguish among three subcases.

Subcase 1.1. s_1v_4 is contained in some cycle $C \in \mathcal{C}_0^y$. In this subcase, v_4v_6 is saturated by $\boldsymbol{y}$ in T_2, for otherwise, v_4v_6 is not saturated by $\boldsymbol{y}$ in T, because it is outside $\mathcal{C}_0^y$. By Lemma 4.7(iii), v_6s_1 is saturated by $\boldsymbol{y}$ in T_2. By (11), we have $y(v_1v_6s_1v_4v_1) = 0$, which together with (19) implies $y(v_6s_1v_4v_6) = w(v_6s_1) > 0$, so (1) holds. Clearly, v_4v_1 is outside $\mathcal{C}_0^y$. We proceed by considering two subsubcases.

Assume first that v_4v_1 is not saturated by $\boldsymbol{y}$ in T_2 (and hence in T). Then, by Lemma 4.7(iii), v_1s_1 and at least one of v_1s_2 and s_2v_4 are saturated by $\boldsymbol{y}$ in T_2. Furthermore, v_1s_2 is outside $\mathcal{C}_0^y$. If v_1s_2 is not saturated by $\boldsymbol{y}$ in T, then $y(v_3s_2v_4v_6v_3) = 0$, for otherwise, let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1s_2v_4v_1)$ and $y(v_3s_2v_4v_6v_3)$ with $y(v_1s_2v_4v_1) + \theta$ and $y(v_3s_2v_4v_6v_3) - \theta$, where $\theta = \min\{w(v_4v_1) - z(v_4v_1), w(v_1s_2) - z(v_1s_2), y(v_3s_2v_4v_6v_3)\} > 0$. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, contradicting (6). It follows from (19) that $y(v_1s_2v_4v_1) = w(s_2v_4) > 0$, so (1) holds. Thus we may assume that v_1s_2 is saturated by $\boldsymbol{y}$ in T_2. If v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_6, v_1s_1, v_1s_2\}$. By (9), K is an MFAS of $T_2 \backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2 \backslash a_2)$. By Lemma 4.7(iii), v_1v_6 is outside $\mathcal{C}_0^y$, for otherwise, v_4v_1 would be saturated by $\boldsymbol{y}$ in T_2, a contradiction. So we may assume that v_1v_6 is not saturated by $\boldsymbol{y}$ in T. By Lemma 4.7(iii), v_6s_1 is saturated by $\boldsymbol{y}$ in T_2. If v_6v_3 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_6v_3, v_6s_1, v_1s_1, v_1s_2\}$. So we assume that v_6v_3 is not saturated by $\boldsymbol{y}$ in T_2. By Lemma 4.7(iii), v_6v_3 is outside $\mathcal{C}_0^y$. Furthermore, v_3v_1, v_3s_2, and v_3v_4 are all saturated by $\boldsymbol{y}$ in T_2. So $y(\mathcal{C}_2) = w(J)$, where $J = \{v_3v_1, v_3v_4, v_6s_1, v_1s_1, v_1s_2, v_3s_2\}$. By (9), J is an MFAS of $T_2 \backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2 \backslash a_2)$.

Next assume that v_4v_1 is saturated by $\boldsymbol{y}$ in T_2. We may assume that v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_6\}$. By (9), K is an MFAS of $T_2 \backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2 \backslash a_2)$. Thus, by (10), we have $y(v_1v_6v_3v_4v_1) = 0$. If $y(v_1v_6s_1v_4v_1) = 0$ and v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_1, v_4v_6\}$. So we may assume that $y(v_1v_6s_1v_4v_1) > 0$ or v_1v_6 is not saturated by $\boldsymbol{y}$ in T_2. Consider the situation when $y(v_1v_6s_1v_4v_1) > 0$. Now, by (11), v_1s_1 is saturated by $\boldsymbol{y}$ in T_2, and $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$. Moreover, at least one of v_1s_2 and s_2v_4 is saturated by $\boldsymbol{y}$ in T_2 (otherwise, $y(v_1s_2v_4v_1)$ can be made larger). If v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_6, v_1s_1, v_1s_2\}$ or $\{v_1v_6, v_4v_6, v_1s_1, s_2v_4\}$; if v_1v_6 is not saturated by $\boldsymbol{y}$ in T_2, then v_6v_3 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iiv). So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_6v_3\}$. By (9), K is an MFAS of $T_2 \backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2 \backslash a_2)$. So we may assume that $y(v_1v_6s_1v_4v_1) = 0$ and v_1v_6 is not saturated by $\boldsymbol{y}$ in T_2. By Lemma 4.7(vii), v_6v_3 is saturated by $\boldsymbol{y}$ in T_2. If v_6s_1 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6s_1, v_6v_3\}$. So we further assume that v_6s_1 is not saturated by $\boldsymbol{y}$ in T_2. We propose to show that

(20) $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$.

We only prove that $y(v_3s_2v_4v_6v_3) = 0$, as the proof of the other equality $y(v_3v_4v_6v_3) = 0$ goes along the same line. Assume the contrary: $y(v_3s_2v_4v_6v_3) > 0$. Depending on the saturation of v_1v_6 and v_3v_1, we consider several possibilities.

• Both v_1v_6 and v_3v_1 are not saturated by $\boldsymbol{y}$ in T. Define $\theta = \min\{w(v_1v_6) - z(v_1v_6), w(v_3v_1) - z(v_3v_1), y(v_3s_2v_4v_6v_3)\}$. Then $\theta > 0$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_3s_2v_4v_6v_3)$ and

$y(v_1v_6v_3v_1)$ with $y(v_3s_2v_4v_6v_3)-\theta$ and $y(v_1v_6v_3v_1)+\theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$ with $y'(v_3s_2v_4v_6v_3) < y(v_3s_2v_4v_6v_3)$, contradicting (6).

• v_3v_1 is not saturated by $\boldsymbol{y}$ in T and v_1v_6 is contained in some cycle $C \in \mathcal{C}_0^y$. Since v_6v_3 is saturated by $\boldsymbol{y}$ in T_2, cycle C passes through $v_6s_1v_4$. Thus the multiset sum of the cycles C, $v_3s_2v_4v_6v_3$ and the unsaturated arc v_3v_1 contains arc-disjoint cycles $v_6s_1v_4v_6$ and $v_1v_6v_3v_1$. From Lemma 4.7(vi) we deduce that $y(v_3s_2v_4v_6v_3) = 0$, a contradiction.

• v_1v_6 is is not saturated by $\boldsymbol{y}$ in T and v_3v_1 is contained in some cycle $D \in \mathcal{C}_0^y$. It is clear that D passes through $v_1s_iv_4$ for $i=1$ or 2. Furthermore, the multiset sum of D, $v_3s_2v_4v_6v_3$, and the unsaturated arc v_1v_6 contains arc-disjoint cycles $v_1v_6v_3v_1$ and $D' = D[v_4,v_3]\cup\{v_3s_2, s_2v_4\}$. Define $\theta = \min\{y(D), y(v_3s_2v_4v_6v_3), w(v_1v_6)-z(v_1v_6)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(D)$, $y(D')$, $y(v_3s_2v_4v_6v_3)$, and $y(v_1v_6v_3v_1)$ with $y(D)-\theta$, $y(D')+\theta$, $y(v_3s_2v_4v_6v_3)-\theta$, and $y(v_1v_6v_3v_1)+\theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$ with $y'(v_3s_2v_4v_6v_3) < y(v_3s_2v_4v_6v_3)$, contradicting (6).

• v_1v_6 and v_3v_1 are contained in some cycles C and D in $\mathcal{C}_0^y$, respectively. If v_3v_1 is on C, then the multiset sum of C and $v_3s_2v_4v_6v_3$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_6s_1v_4v_6$, and $C' = C[v_4,v_3]\cup\{v_3s_2,s_2v_4\}$; if v_3v_1 is outside C, then the multiset sum of C, D, and $v_3s_2v_4v_6v_3$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_6s_1v_4v_6$, $C' = C[v_4,v_1]\cup\{v_1s_i, s_iv_4\}$ for $i=1$ or 2, and $D' = D[v_4,v_3]\cup\{v_3s_2,s_2v_4\}$. In either situation from the optimality of $\boldsymbol{y}$ we deduce that $y(v_3s_2v_4v_6v_3)=0$.

Combining the above observations, we see that (20) holds. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_6v_3\}$. By (9), K is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$.

Subcase 1.2. s_1v_4 is saturated by $\boldsymbol{y}$ in T_2 and s_2v_4 is contained in some cycle $C \in \mathcal{C}_0^y$; subject to this, C is chosen to contain v_3s_2 if possible. In this subcase, observe first that both v_1s_1 and v_6s_1 are outside $\mathcal{C}_0^y$. Next, v_3s_2 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(v_3s_2v_4v_6v_3) = w(v_3s_2) > 0$, so (1) holds. If both v_6v_3 and v_1s_2 are saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{s_1v_4, v_1s_2, v_6v_3\}$. By (9), K is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$. We proceed by considering two subsubcases.

(a) v_6v_3 is not saturated by $\boldsymbol{y}$ in T_2. Now v_4v_6 is saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii).

Assume first that v_4v_1 is not saturated by $\boldsymbol{y}$ in T. Then both v_1v_6 and v_1s_2 are saturated by $\boldsymbol{y}$ in T_2 by Lemma 4.7(iii). If v_1s_1 is also saturated by $\boldsymbol{y}$ in T_2, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_1v_6, v_4v_6, v_1s_1, v_1s_2\}$; otherwise, v_1s_1 is not saturated by $\boldsymbol{y}$ in T. By (11), we have $y(v_1v_6s_1v_4v_1) = 0$. Let us show that

(21) $y(v_6s_1v_4v_6) = 0$.

Indeed, if v_6v_3 is not saturated by $\boldsymbol{y}$ in T, then the multiset sum of the cycles C, $v_6s_1v_4v_6$, and the unsaturated arcs v_4v_1, v_1s_1, and v_6v_3 (or v_3s_2 if it is outside C) contains arc-disjoint cycles $v_1s_1v_4v_1$ and $v_3s_2v_4v_6v_3$. Thus, by Lemma 4.7(vi), we have $y(v_6s_1v_4v_6) = 0$. If v_6v_3 is contained in some cycle $C \in \mathcal{C}_0^y$, then C contains v_3v_4 or v_3s_2. Thus the multiset sum of cycles C, $v_6s_1v_4v_6$, and the unsaturated arcs v_4v_1 and v_1s_1 contains arc-disjoint cycles $v_1s_1v_4v_1$ and one of $v_3v_4v_6v_3$ and $v_3s_2v_4v_6v_3$. Thus, by Lemma 4.7(vi), we have $y(v_6s_1v_4v_6) = 0$. This proves (21).

It follows from (19) and (21) that $y(v_1s_1v_4v_1) = w(s_1v_4) > 0$, so (1) holds. Thus we may assume that v_4v_1 is saturated by $\boldsymbol{y}$ in T (and hence in T_2). Then we may further assume that v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2, for otherwise, $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_3v_1\}$. Thus $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$. By Lemma 4.7(vii), v_1v_6 is saturated by $\boldsymbol{y}$ in T_2 and hence, by (10), we have $y(v_1v_6v_3v_4v_1) = 0$. Let us show that

(22) $y(v_1v_6s_1v_4v_1) = 0$.

To justify this, we consider four possibilities, depending on the saturation of v_6v_3 and v_3v_1.

• Both v_6v_3 and v_3v_1 are saturated by $\boldsymbol{y}$ in T. Now define $\theta = \min\{w(v_6v_3)-z(v_6v_3), w(v_3v_1)-z(v_3v_1), y(v_1v_6s_1v_4v_1)\}$. Then $\theta > 0$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$ and $y(v_1v_6s_1v_4v_1)$ with $y(v_1v_6v_3v_1)+\theta$ and $y(v_1v_6s_1v_4v_1)-\theta$, respectively. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6s_1v_4v_1) < y(v_1v_6s_1v_4v_1)$, contradicting (6).

• v_3v_1 is not saturated by $\boldsymbol{y}$ in T and v_6v_3 is contained in some cycle $C \in \mathcal{C}_0^y$. Now the multiset sum of the cycles C, $v_1v_6s_1v_4v_1$ and the unsaturated arc v_3v_1 contains arc-disjoint cycles $v_1v_6v_3v_1$ and $C' = C[v_4, v_6] \cup \{v_6s_1, s_1v_4\}$. Define $\theta = \min\{w(v_3v_1) - z(v_3v_1), y(C), y(v_1v_6s_1v_4v_1)\}$. Then $\theta > 0$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6s_1v_4v_1)$, $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6s_1v_4v_1) - \theta$, $y(v_1v_6v_3v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y'}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6s_1v_4v_1) < y(v_1v_6s_1v_4v_1)$, contradicting (6).

• v_6v_3 is not saturated by $\boldsymbol{y}$ in T and v_3v_1 is contained in some cycle $D \in \mathcal{C}_0^y$. Now D passes through $v_1s_2v_4$. Since the multiset sum of the cycles D, $v_1v_6s_1v_4v_1$, and the unsaturated arc v_6v_3 contains arc-disjoint cycles $v_1v_6v_3v_1$ and $v_1s_2v_4v_1$, by Lemma 4.7(vi), we have $y(v_1v_6s_1v_4v_1) = 0$, a contradiction.

• v_6v_3 and v_3v_1 are contained in some cycles C and D in $\mathcal{C}_0^y$, respectively. Now if v_3v_1 is on C, then the multiset sum of the cycles C and $v_1v_6s_1v_4v_1$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_1s_2v_4v_1$, and $C' = C[v_4, v_6] \cup \{v_6s_1, s_1v_4\}$; otherwise, the multiset sum of the cycles C, D, and $v_1v_6s_1v_4v_1$ contains arc-disjoint cycles $v_1v_6v_3v_1$, $v_1s_2v_4v_1$, and $C' = C[v_4, v_6] \cup \{v_6s_1, s_1v_4\}$, and $D' = D[v_4, v_3] \cup C[v_3, v_4]$. In each situation from the optimality of $\boldsymbol{y}$ we deduce that $y(v_1v_6s_1v_4v_1) = 0$.

Combining the above observations, we see that (22) holds. Thus $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_1v_6\}$. By (9), K is an MFAS of $T_2\backslash a_2$ and hence $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$.

(b) v_6v_3 is saturated by $\boldsymbol{y}$ in T_2. Now v_1s_2 is not saturated by $\boldsymbol{y}$ in T_2. By Lemma 4.7(vii), v_4v_1 is saturated by $\boldsymbol{y}$ in T_2. Since $z(v_1s_2) > 0$, by Lemma 6.2(vii), we have $z(v_1s_1) = 0$. Furthermore, we may assume that $y(v_1v_6v_3v_4v_1) = 0$, for otherwise, both v_3v_1 and v_4v_6 saturated by $\boldsymbol{y}$ in T_2 by (10). Hence $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_4v_6, v_3v_1\}$. If $y(v_1v_6s_1v_4v_1) = 0$, then $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_6v_3, s_1v_4\}$; if $y(v_1v_6s_1v_4v_1) > 0$ then, by (11), v_4v_6 is saturated by $\boldsymbol{y}$ in T_2, and either v_3v_1 is saturated by $\boldsymbol{y}$ in T_2 or $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$. Thus $y(\mathcal{C}_2) = w(J)$, where $J = \{v_4v_1, v_4v_6, v_3v_1\}$ or $\{v_4v_1, v_4v_6, v_6v_3\}$. Therefore $y(\mathcal{C}_2) = \tau_w(T_2\backslash a_2)$.

Subcase 1.3. s_iv_4 is saturated by $\boldsymbol{y}$ in T_2 for $i = 1$ and 2. In this subcase, since $\mathcal{C}_0^y \neq \emptyset$, v_3v_4 is contained in some cycle in $\mathcal{C}_0^y$. By (12), we have $y(v_3s_2v_4v_6v_3) = 0$. Thus $y(v_1s_2v_4v_1) = w(s_2v_4) > 0$ and (1) holds. This completes the proof of Claim 1.

Claim 2. $y(C)$ is a positive integer for some $C \in \mathcal{C}_2^y$ or $\nu_w^*(T)$ is an integer.

To justify this, note that $y(\mathcal{C}_2) = w(K)$ for some MFAS K of $T_2\backslash a_2$ by Claim 1. From the proof of Claim 1, we see that K has ten possibilities. So we proceed by considering them accordingly.

Subcase 2.1. K is one of $\{v_1v_6, v_4v_6, v_1s_1, s_2v_4\}$, $\{v_4v_1, v_6v_3, v_6s_1\}$, and $\{v_4v_1, v_6v_3, s_1v_4\}$.

In this subcase, by (15) and (19), we have $y(v_1s_2v_4v_1) = w(s_2v_4) > 0$ if $K = \{v_1v_6, v_4v_6, v_1s_1, s_2v_4\}$, $y(v_6s_1v_4v_6) = w(v_6s_1) > 0$ if $K = \{v_4v_1, v_6v_3, v_6s_1\}$, and $y(v_6s_1v_4v_6) = w(s_1v_4) > 0$ if $K = \{v_4v_1, v_6v_3, s_1v_4\}$, as desired.

Subcase 2.2. $K = \{v_3v_1, v_3v_4, v_6s_1, v_1s_1, v_1s_2, v_3s_2\}$.

In this subcase, by (15) and (19), we have $y(v_6s_1v_4v_6) + y(v_1v_6s_1v_4v_1) = w(v_6s_1) > 0$ and $y(v_3v_4v_6v_3)+y(v_1v_6v_3v_4v_1) = w(v_3v_4)$. So we may assume that $y(v_1v_6s_1v_4v_1) > 0$, for otherwise, $y(v_6s_1v_4v_6) = w(v_6s_1) > 0$. It follows from Lemma 4.7(v) that v_4v_6 is saturated by $\boldsymbol{y}$ in T_2. If $y(v_1v_6v_3v_4v_1) > 0$, then $y(v_6s_1v_4v_6) = 0$ by (10), and hence $y(v_1v_6s_1v_4v_1) = w(v_6s_1) > 0$; if $y(v_1v_6v_3v_4v_1) = 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ and so $y(v_6s_1v_4v_6) = w(v_4v_6) - y(v_3v_4v_6v_3)$. Since $w(v_6s_1) > 0$, at least one of $y(v_6s_1v_4v_6)$ and $y(v_1v_6s_1v_4v_1)$ is a positive integer.

Subcase 2.3. $K = \{v_6v_3, v_6s_1, v_1s_1, v_1s_2\}$ or $\{v_6v_3, s_1v_4, v_1s_2\}$.

In this subcase, we only consider the situation when $K = \{v_6v_3, s_1v_4, v_1s_2\}$, as the proof in the other situation goes along the same line.

Given the arcs in K, we have $y(v_1s_2v_4v_1) = w(v_1s_2)$, $y(v_1s_1v_4v_1)+y(v_6s_1v_4v_6)+y(v_1v_6s_1v_4v_1) = w(s_1v_4) > 0$, and $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) + y(v_3s_2v_4v_6v_3) = w(v_6v_3)$. If $y(v_1v_6v_3v_4v_1) > 0$, then $y(v_6s_1v_4v_6) = 0$ by (10). Thus $y(v_1s_1v_4v_1) + y(v_1v_6s_1v_4) = w(s_1v_4)$. If $y(v_1v_6s_1v_4v_1) > 0$, then one more equality $y(v_1s_1v_4v_1) = w(v_1s_1)$ holds by (11). Since $w(s_1v_4) > 0$, at least one of $y(v_1s_1v_4v_1)$ and $y(v_1v_6s_1v_4v_1)$ is a positive integer. So we assume that $y(v_1v_6v_3v_4v_1) = 0$ in the following discussion.

Assume first that $y(v_1v_6s_1v_4v_1) > 0$. Then $y(v_1s_1v_4v_1) = w(v_1s_1)$ and $y(v_6s_1v_4v_6) + y(v_3v_4v_6v_3) + y(v_3s_2v_4v_6v_3) = w(v_4v_6)$ by (11). If $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$, then $y(v_6s_1v_4v_6) = w(v_4v_6)$, and hence $y(v_1v_6s_1v_4v_1) = w(s_1v_4) - y(v_1s_1v_4v_1) - y(v_6s_1v_4v_6)$. Since $w(s_1v_4) > 0$, at least one of $y(v_1s_1v_4v_1)$, $y(v_6s_1v_4v_6)$, and $y(v_1v_6s_1v_4v_1)$ is a positive integer. So we assume that $y(v_3v_4v_6v_3)$ or $y(v_3s_2v_4v_6v_3)$ is positive. By (11), we have $y(v_1v_6v_3v_1) = w(v_3v_1)$; by (12), one more equality $y(v_3v_4v_6v_3) = w(v_3v_4)$ holds if $y(v_3s_2v_4v_6v_3) > 0$. Thus $y(v_6s_1v_4v_6)$, $y(v_1v_6s_1v_4v_1)$, $y(v_3v_4v_6v_3)$, and $y(v_3s_2v_4v_6v_3)$ are all integers.

Assume next that $y(v_1v_6s_1v_4v_1) = 0$. Then $y(v_1s_1v_4v_1) + y(v_6s_1v_4v_6) = w(s_1v_4)$. If $y(v_3s_2v_4v_6v_3) > 0$, then $y(v_6v_3v_4v_6) = w(v_3v_4)$ by (12), so $y(v_1v_6v_3v_1) + y(v_3s_2v_4v_6v_3) = w(v_6v_3) - w(v_3v_4)$; if $y(v_3s_2v_4v_6v_3) = 0$, then $y(v_1v_6v_3v_1) + y(v_6v_3v_4v_6) = w(v_6v_3)$. Since both v_1v_6 and v_3v_1 are outside $\mathcal{C}_0^y$, from the choice of $\boldsymbol{y}$, we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_1v_6)\}$. This implies that in either situation $y(v_3s_2v_4v_6v_3)$ and $y(v_6v_3v_4v_6)$ are integers. On the other hand, since both v_4v_6 and v_6s_1 are outside $\mathcal{C}_0^y$, by (8), we obtain $y(v_6s_1v_4v_6) = \min\{w(v_6s_1), w(v_4v_6)-y(v_6v_3v_4v_6)-y(v_3s_2v_4v_6v_3)\}$, which is also an integer. Since $w(s_1v_4) > 0$, at least one of $y(v_1s_1v_4v_1)$ and $y(v_6s_1v_4v_6)$ is a positive integer.

Subcase 2.4. $K = \{v_1v_6, v_4v_6, v_4v_1\}$.

In this subcase, we have $y(v_1v_6v_3v_1) = w(v_1v_6)$, $y(v_1s_1v_4v_1) + y(v_1s_2v_4v_1) = w(v_4v_1)$, and $y(v_3v_4v_6v_3) + y(v_6s_1v_4v_6) + y(v_3s_2v_4v_6v_3) = w(v_4v_6)$. By Lemma 4.4(iii) and Lemma 6.2(vi), we may assume that $w(v_1v_6) = w(v_4v_1) = 0$ and thus $w(v_4v_6) = w(K) > 0$. If $y(v_3s_2v_4v_6v_3) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (12), and thus we may assume that $w(v_3v_4) = 0$. Hence $y(v_3v_4v_6v_3) + y(v_6s_1v_4v_6) = w(v_4v_6)$ or $y(v_6s_1v_4v_6) + y(v_3s_2v_4v_6v_3) = w(v_4v_6)$. If $y(v_6s_1v_4v_6)$ is an integer, then one of $y(v_3v_4v_6v_3)$, $y(v_6s_1v_4v_6)$, and $y(v_6s_2v_4v_6v_3)$ is a positive integer. So we assume that $y(v_6s_1v_4v_6)$ is not integral. Then we can prove that $\nu_w^*(T)$ is an integer; for a proof, see the argument of the same statement contained in the proof of (17) (with $y(v_6s_1v_4v_6)$ in place of $y(v_6s_iv_4v_6)$).

Subcase 2.5. $K = \{v_1v_6, v_4v_6, v_1s_1, v_1s_2\}$.

In this subcase, we have $y(v_1s_1v_4v_1) = w(v_1s_1)$, $y(v_1s_2v_4v_1) = w(v_1s_2)$, $y(v_1v_6v_3v_1) + y(v_1v_6v_3v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_1v_6)$, and $y(v_3v_4v_6v_3) + y(v_6s_1v_4v_6) + y(v_3s_2v_4v_6v_3) =$

$w(v_4v_6)$. By Lemma 4.4(iii), we may assume that $w(v_1s_1) = w(v_1s_2) = 0$.

Assume first that $y(v_1v_6v_3v_4v_1) > 0$. Then $y(v_6s_1v_4v_6) = 0$ and $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (10). So $y(v_3v_4v_6v_3) + y(v_3s_2v_4v_6v_3) = w(v_4v_6)$. By (12), one more equality $y(v_3v_4v_6v_3) = w(v_3v_4)$ holds if $y(v_3s_2v_4v_6v_3) > 0$. So both $y(v_3v_4v_6v_3)$ and $y(v_3s_2v_4v_6v_3)$ are integers. By Lemma 4.4(iii), we may assume that $w(v_3v_1)$ and $w(v_4v_6)$ are both zero. Thus $y(v_1v_6v_3v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_1v_6) > 0$. By Lemma 4.4(iii), we may assume that neither $y(v_1v_6v_3v_4v_1)$ nor $y(v_1v_6s_1v_4v_1)$ is integral. Observe that v_6s_1 is outside $\mathcal{C}_0^y$, for otherwise, let $C \in \mathcal{C}_0^y$ be a cycle containing v_6s_1. Then C contains s_1v_4. Let $C' = C[v_4, v_6] \cup \{v_6v_3, v_3v_4\}$ and $\theta = \min\{y(C), y(v_1v_6v_3v_4v_1)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_4v_1)$, $y(v_1v_6s_1v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_4v_1) - \theta$, $y(v_1v_6s_1v_4v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (7). Let us show that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1v_6s_1v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ are positive, we have $x(v_1v_6s_1v_4v_1) = x(v_1v_6v_3v_4v_1) = 1$ by Lemma 4.3(i). So $x(v_6s_1)+x(s_1v_4) = x(v_6v_3)+x(v_3v_4)$. Since $y(v_1v_6s_1v_4v_1) < w(v_6s_1)$, by Lemma 4.3(ii), we have $x(v_6s_1) = 0$, which implies $x(s_1v_4) = x(v_6v_3) + x(v_3v_4)$. For any $u \in V \backslash (V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_6, then it passes through $v_6v_3v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_1, then it passes through s_1v_4. By Lemma 4.3(iv), we obtain $x(uv_6)+x(v_6v_3)+x(v_3v_4) = x(us_1)+x(s_1v_4)$. Hence $x(uv_6) = x(us_1)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting vertex s_1, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by setting $w'(uv_6) = w(uv_6) + w(us_1)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through us_1v_4 with $u \in V\backslash(V(T_2)\backslash a_2)$, let C' arise from C by replacing the path us_1v_4 with $uv_6v_3v_4$, and set $y'(C') = y(C) + y(C')$ and $y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1) + y(v_1v_6s_1v_4v_1)$. It is easy to see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer.

Assume next that $y(v_1v_6v_3v_4v_1) = 0$. Then both $y(v_1v_6v_3v_1)$ and $y(v_1v_6s_1v_4v_1)$ are integers, for otherwise, neither of them is integral, because their sum is $w(v_1v_6)$. If $y(v_3v_4v_6v_3)$ or $y(v_3s_2v_4v_6v_3)$ is positive, then $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (11), a contradiction. So $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$. Since v_1v_6 is saturated by $\boldsymbol{y}$ in T_2, the arc v_3v_1 is outside $\mathcal{C}_0^y$. If v_3v_1 is is saturated by $\boldsymbol{y}$ in T_2, then $y(v_1v_6v_3v_1) = w(v_3v_1)$; this contradiction implies that v_3v_1 is not saturated by $\boldsymbol{y}$ in T_2 (and hence in T). If v_6v_3 is outside $\mathcal{C}_0^y$, then from the choice of $\boldsymbol{y}$ we see that $y(v_1v_6v_3v_1) = \min\{w(v_6v_3), w(v_3v_1)\}$, a contradiction again. So we assume that v_6v_3 is contained in some cycle $C \in \mathcal{C}_0^y$. Define $\theta = \min\{w(v_3v_1) - z(v_3v_1), y(C), y(v_1v_6s_1v_4v_1)\}$. Let $C' = C[v_4, v_6] \cup \{v_6s_1, s_1v_4\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_6v_3v_1)$, $y(v_1v_6s_1v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_1) + \theta$, $y(v_1v_6s_1v_4v_1) - \theta$, $y(C) - \theta$, $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6s_1v_4v_1) < y(v_1v_6s_1v_4v_1)$, contradicting (6). By Lemma 4.4(iii), we may assume $w(v_1v_6) = 0$. Thus $z(v_4v_1) = w(v_4v_1) = 0$; the remainder of the proof is exactly the same as that in the preceding subcase.

Subcase 2.6. $K = \{v_4v_1, v_4v_6, v_6v_3\}$.

In this subcase, we have $y(v_1v_6v_3v_1) = w(v_6v_3)$, $y(v_6s_1v_4v_6) = w(v_4v_6)$, and $y(v_1s_1v_4v_1) + y(v_1s_2v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_4v_1)$. Since $w(K) = \tau_w(T_2\backslash a_2) > 0$, we have $w(v_4v_1) > 0$. By Lemma 6.2(vi), $y(v_1s_1v_4v_1)$ or $y(v_1s_2v_4v_1)$ is zero. By Lemma 4.4(iii), we may assume that $w(v_6v_3) = w(v_4v_6) = 0$ and $y(v_1v_6s_1v_4v_1) > 0$. So $y(v_1s_1v_4v_1) = w(v_1s_1)$ by (11). By

Lemma 4.4(iii), we may further assume that $w(v_1s_1) = 0$. Thus $y(v_1s_2v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_4v_1)$, and hence neither $y(v_1s_2v_4v_1)$ nor $y(v_1v_6s_1v_4v_1)$ is integral. Observe that v_1s_2 is outside $\mathcal{C}_0^y$, for otherwise, let $C \in \mathcal{C}_0^y$ be a cycle containing v_1s_2. Then C contains s_2v_4. Let $C' = C[v_4, v_1] \cup \{v_1v_6, v_6s_1, s_1v_4\}$ and $\theta = \min\{y(C), y(v_1v_6s_1v_4v_1)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1s_2v_4v_1)$, $y(v_1v_6s_1v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1s_2v_4v_1)+\theta$, $y(v_1v_6s_1v_4v_1)-\theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(v_1v_6s_1v_4v_1) < y(v_1v_6s_1v_4v_1)$, contradicting (6). Furthermore, since $w(v_1s_1) = 0$, the arc v_3v_1 is also outside $\mathcal{C}_0^y$. Thus $w(v_3v_1) = z(v_3v_1) = 0$. Let us show that $\nu_w^*(T)$ is an integer.

For this purpose, let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. Since both $y(v_1s_2v_4v_1)$ and $y(v_1v_6s_1v_4v_1)$ are positive, we have $x(v_1s_2v_4v_1) = x(v_1v_6s_1v_4v_1) = 1$ by Lemma 4.3(i). Since $y(v_1s_2v_4v_1) < w(v_1s_2)$, we have $x(v_1s_2) = 0$ by Lemma 4.3(ii). It follows that $x(s_2v_4) = x(v_1v_6)+x(v_6s_1)+x(s_1v_4)$. Since $w(v_1s_1) = 0$ and v_1s_2 is outside $\mathcal{C}_0^y$, for any $u \in V\backslash(V(T_2)\backslash a_2)$, if a cycle in $\mathcal{C}_0^y$ contains uv_1, then it passes through $v_1v_6s_1v_4$. Moreover, if a cycle in $\mathcal{C}_0^y$ contains us_2, then it passes through s_2v_4. By Lemma 4.3(iv), we obtain $x(uv_1) + x(v_1v_6) + x(v_6, s_1) + x(s_1v_4) = x(us_2) + x(s_2v_4)$. Hence $x(uv_1) = x(us_2)$. Clearly, we may assume that this equality holds in any other situation. Let $T' = (V', A')$ be obtained from T by deleting s_2, and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to A' by replacing $w(e)$ with $w(e) + w(s_2v_4)$ for $e \in \{v_1v_6, v_6s_1, s_1v_4\}$, replacing $w(uv_1)$ with $w(uv_1) + w(us_2)$ for any $u \in V\backslash(V(T_2)\backslash a_2)$, and replacing $w(v_3v_1)$ with $w(v_3v_1) + w(v_3s_2)$. Let $\boldsymbol{x}'$ be obtained from $\boldsymbol{x}$ by setting $x(v_3v_1) = x(v_3s_2)$. Since $w(v_3v_1) = 0$ and $w'(v_3v_1) = w(v_3s_2)$, we have $(\boldsymbol{w}')^T\boldsymbol{x}' = \boldsymbol{w}^T\boldsymbol{x}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: set $y'(v_1v_6s_1v_4v_1) = y(v_1v_6s_1v_4v_1)+y(v_1s_2v_4v_1)$; for each $C \in \mathcal{C}_0^y$ passing through us_2v_4, let C' arise from C by replacing the path us_2v_4 with the path $uv_1v_6s_1v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer.

Subcase 2.7. $K = \{v_4v_1, v_4v_6, v_3v_1\}$.

In this subcase, we have $y(v_1v_6v_3v_1) = w(v_3v_1)$, $y(v_1s_1v_4v_1) + y(v_1s_2v_4v_1) + y(v_1v_6s_1v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_4v_1)$, and $y(v_6s_1v_4v_6) + y(v_3v_4v_6v_3) + y(v_3s_2v_4v_6v_3) = w(v_4v_6)$. By Lemma 4.4(iii), we may assume that $w(v_3v_1) = 0$.

Assume first that $y(v_1v_6v_3v_4v_1) > 0$. Then $y(v_6s_1v_4v_6) = 0$ by (10). If $y(v_3s_2v_4v_6v_3) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (12); otherwise, $y(v_3v_4v_6v_3) = w(v_4v_6)$. So both $y(v_3v_4v_6v_3)$ and $y(v_3s_2v_4v_6v_3)$ are integers in either situation. Thus we may assume that $w(v_4v_6) = 0$. The remainder of the proof is exactly the same as that of (16).

Assume next that $y(v_1v_6v_3v_4v_1) = 0$. Consider first the subsubcase when $w(v_4v_1) = 0$. Then $w(v_4v_6) = w(K) > 0$. If $y(v_3s_2v_4v_6v_3) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (12), so $y(v_6s_1v_4v_6) + y(v_3s_2v_4v_6v_3) = w(v_4v_6) - w(v_3v_4)$; if $y(v_3s_2v_4v_6v_3) = 0$, then $y(v_6s_1v_4v_6) + y(v_3v_4v_6v_3) = w(v_4v_6)$. It can be shown that $\nu_w^*(T)$ is an integer; for a proof, see the argument of the same statement contained in the proof of (17).

Consider next the subsubcase when $w(v_4v_1) > 0$. Observe that $y(v_1v_6s_1v_4v_1) > 0$ and $y(v_3s_2v_4v_6v_3) = 0$, for otherwise, since $w(v_1s_1)w(v_1s_2) = 0$ by Lemma 6.2(vi), at most one of $y(v_1s_1v_4v_1)$ and $y(v_1s_2v_4v_1)$ is positive. Hence, if $y(v_1v_6s_1v_4v_1) = 0$, then either $y(v_1s_1v_4v_1) = w(v_4v_1)$ or $y(v_1s_2v_4v_1) = w(v_4v_1)$; if $y(v_1v_6s_1v_4v_1) > 0$ and $y(v_3s_2v_4v_6v_3) > 0$, then, by (11), we have $y(v_1s_1v_4v_1) = w(v_1s_1)$, $y(v_1s_2v_4v_1) = w(v_1s_2)$. So $y(v_1v_6s_1v_4v_1) = w(v_4v_1) - w(v_1s_1) - w(v_1s_2)$. By Lemma 4.4(iii), we see that $\nu_w^*(T)$ is an integer. The preceding observation together with (11) implies that $y(v_1s_1v_4v_1) = w(v_1s_1)$, $y(v_1s_2v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_4v_1) - w(v_1s_1)$, and $y(v_6s_1v_4v_6) + y(v_3v_4v_6v_3) = w(v_4v_6)$. Lemma 4.4(iii) allows us to assume that $w(v_1s_1) = 0$ and that neither $y(v_1s_2v_4v_1)$ nor $y(v_1v_6s_1v_4v_1)$ is integral.

It can then be shown that v_1s_2 is outside $\mathcal{C}_0^y$ and $\nu_w^*(T)$ is an integer; for a proof, see the argument of the same statement contained in the preceding case.

Combining the above seven subcases, we see that Claim 2 holds. Hence, by Lemma 4.4(iii), the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral, as described in (1) above. ∎

To establish the corresponding lemmas for the cases when $T_2/S \in \{G_4, G_5, G_6\}$, we need some further preparations.

Lemma 6.9. *If $T_2/S \in \{G_5, G_6\}$, then we may assume that* $\min\{w(v_1v_3), w(v_3v_4), w(v_4v_1)\} = 0$.

Proof. Let $\theta = \min\{w(v_1v_3), w(v_3v_4), w(v_4v_1)\}$ and $C_0 = v_1v_3v_4v_1$. Assume the contrary: $\theta > 0$. Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(1) $y(\mathcal{C}_2)$ is maximized; and

(2) subject to (1), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically.

Let $\mathcal{C}_2' = \mathcal{C}_2 \backslash \{C_0\}$. Note that every cycle in $\mathcal{C}_2'$ passes through b. By Lemma 4.7(vii), at least one of v_1v_3, v_3v_4, and v_4v_1 is saturated by $\boldsymbol{y}$ in T_2, say v_1v_3 (by symmetry). Thus $w(v_1v_3) = \theta$. We propose to show that

(3) there is no cycle $C \in \mathcal{C}_2'$ with $y(C) > 0$ passing through v_1v_3.

Assume the contrary: v_1v_3 is contained in some cycle $C_1 \in \mathcal{C}_2'$ with $y(C_1) > 0$. Clearly, $|C_1| \geq 4$. If neither v_3v_4 nor v_4v_1 is saturated by $\boldsymbol{y}$ in T, then $\theta_1 = \min\{w(v_3v_4) - z(v_3v_4), w(v_4v_1) - z(v_4v_1)\} > 0$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C_1)$ and $y(C_0)$ with $y(C_1) - \theta_1$ and $y(C_0) + \theta_1$, respectively. Then $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(C_1) < y(C_1)$, contradicting (2). Thus at least one of v_3v_4 and v_4v_1 is saturated by $\boldsymbol{y}$ in T. We proceed by considering two cases.

• Both v_3v_4 and v_4v_1 are saturated by $\boldsymbol{y}$ in T. In this case, let $C_2 \in \mathcal{C}_0^y \cup \mathcal{C}_2'$ be a cycle containing v_3v_4 with $y(C_2) > 0$; subject to this, C_2 is chosen to contain v_4v_1, if possible. If v_4v_1 is on C_2, then the multiset sum of C_1 and C_2 contains three arc-disjoint cycles C_0, $C_1' = \{bv_1\} \cup C_2[v_1, b]$, and $C_2' = C_2[b, v_3] \cup C_1[v_3, b]$. Define $\epsilon = \min\{y(C_1), y(C_2)\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C_0)$ with $y(C_0) + \epsilon$, and replacing $y(C_i)$ and $y(C_i')$ with $y(C_i) - \epsilon$ and $y(C_i') + \epsilon$, respectively, for $i = 1, 2$. Then $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $(\boldsymbol{y'})^T\mathbf{1} = \boldsymbol{y}^T\mathbf{1} + \epsilon$, a contradiction. If v_4v_1 is outside C_2, then there exists a cycle $C_3 \in \mathcal{C}_0^y \cup \mathcal{C}_2'$ containing v_4v_1 with $y(C_3) > 0$. Observe that the multiset sum of C_1, C_2, and C_3 contains four arc-disjoint cycles C_0, $C_1' = \{bv_1\} \cup C_3[v_1, b]$, $C_2' = C_2[b, v_3] \cup C_1[v_3, b]$, and $C_3' = C_3[b, v_4] \cup C_2[v_4, b]$. Define $\epsilon = \min_{1 \leq i \leq 3} y(C_i)$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C_0)$ with $y(C_0) + \epsilon$, and replacing $y(C_i)$ and $y(C_i')$ with $y(C_i) - \epsilon$ and $y(C_i') + \epsilon$, respectively, for $1 \leq i \leq 3$. Then $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $(\boldsymbol{y'})^T\mathbf{1} = \boldsymbol{y}^T\mathbf{1} + \epsilon$, a contradiction again.

• Exactly one of v_3v_4 and v_4v_1 is saturated by $\boldsymbol{y}$ in T. In this case, by symmetry, we may assume that v_3v_4 is saturated while v_4v_1 is not. Let $C_2 \in \mathcal{C}_0^y \cup \mathcal{C}_2'$ be a cycle containing v_3v_4 with $y(C_2) > 0$. Then the multiset sum of C_1, C_2, and the unsaturated arc v_4v_1 contains two arc-disjoint cycles C_0 and $C_2' = C_2[b, v_3] \cup C_1[v_3, b]$. Clearly, $C_2' \in \mathcal{C}_2'$ if $C_2 \in \mathcal{C}_2'$. Define $\epsilon = \min\{y(C_1), y(C_2), w(v_4v_1) - z(v_4v_1)\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C_0)$ with $y(C_0)+$

ϵ, replacing $y(C_1)$ with $y(C_1) - \epsilon$, and replacing $y(C_2)$ and $y(C_2')$ with $y(C_2) - \epsilon$ and $y(C_2') + \epsilon$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ with $y'(C_1) < y(C_1)$, contradicting (2).

Combining the above two cases, we see that (3) holds. So $y(C_0) = \theta > 0$, and hence $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 4.4(iii). This proves the lemma. ∎

Let $Q = V(T_2)\backslash(S \cup \{b_2, a_2\})$. Then $Q = \{v_2, v_3\}$ if $T_2/S = G_4$, $Q = \{v_1, v_3, v_4\}$ if $T_2/S = G_5$, and $Q = \{v_1, v_2, v_3, v_4\}$ if $T_2/S = G_6$. Moreover, $v_1v_3v_4v_1$ is the unique cycle in $T[Q]$ when $T_2/S = G_5$ or G_6. Let $T' = T$ if $T_2/S = G_4$, and let T' be obtained from T be reversing precisely one arc e on $v_1v_3v_4v_1$ with $w(e) = 0$ (see Lemma 6.9) so that $T[Q]$ is acyclic if $T_2/S = G_5$ and G_6. From Lemma 2.3 we see that T' is also Möbius-free. Note that every integral optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ naturally corresponds to an integral optimal solution to $\mathbb{D}(T', \boldsymbol{w})$ with the same value, and vice versa. So we shall not make effort to distinguish between $\mathbb{D}(T, \boldsymbol{w})$ and $\mathbb{D}(T', \boldsymbol{w})$. Let us label the vertices in Q as $q_1, q_2, \ldots, q_t$ such that q_jq_i is an arc in T' for $1 \le i < j \le t$, where $t = |Q|$.

Lemma 6.10. *Suppose $T_2/S \in \{G_4, G_5, G_6\}$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively. Then we may assume that the following statements hold:*

(i) For each $q_i \in Q$, there exists exactly one $s_k \in S$ such that $z(q_is_k) > 0$;

(ii) $z(q_jq_i) = w(q_jq_i) = 0$ for $1 \le i < j \le t$, where $t = |Q|$;

(iii) If $z(q_is_k)z(q_js_k) > 0$ for some $1 \le i < j \le t$ and $s_k \in S$, then $x(q_is_k) \ne x(q_js_k)$.

Proof. As remarked above the lemma, we may simply treat T, $\mathbb{P}(T, \boldsymbol{w})$, and $\mathbb{D}(T, \boldsymbol{w})$ as T' and $\mathbb{P}(T', \boldsymbol{w})$, and $\mathbb{D}(T', \boldsymbol{w})$, respectively, in our proof.

(i) By Lemma 6.2(vi), for each vertex $q_i \in Q$, there exists at most one $s_k \in S$ with $z(q_is_k) > 0$. Assume on the contrary that $z(q_is_k) = 0$ for all $s_k \in S$. Then no cycle in $\mathcal{C}^y$ passes through q_i. Let $G = T\backslash q_i$ and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to the arcs of G. By the hypothesis of Theorem 4.1, $\mathbb{D}(G, \boldsymbol{w}')$ has an integral optimal solution, and so does $\mathbb{D}(T', \boldsymbol{w})$. Hence we assume that (i) holds.

(ii) Assume the contrary: $z(q_jq_i) > 0$; subject to this, $j + i$ is minimized. If there exists exactly one $s_k \in S$ such that $z(q_is_k)z(q_js_k) > 0$, then the proof is the same as that of Lemma 6.2(i) (with s_k, q_i, and q_j in place of v_0, s_i, and s_j, respectively), so we omit the details here. In view of Lemma 6.2(i), we may assume that $z(q_is_1)z(q_js_2) > 0$. We proceed by considering two cases.

Case 1. $x(q_jq_i) = 0$. In this case, we may assume that $x(uq_j) = x(uq_i)$ for any $u \in V\backslash(S \cup Q)$. Indeed, if $z(uq_j)z(uq_i) > 0$, then Lemma 4.3(iv) implies $x(uq_j) = x(uq_i)$; if $z(uq_j)z(uq_i) = 0$, then $w(us_i')w(us_j') = 0$ by Lemma 4.4(i). Thus we may modify $x(uq_j)$ and $x(uq_i)$ so that they become equal. Let $T' = (V', A')$ be obtained from T by identifying q_j with q_i; we still use q_i to denote the resulting vertex. Let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(uq_i)$ with $w(uq_j) + w(uq_i)$ for any $u \in V\backslash(S \cup Q)$. Let $\boldsymbol{x}'$ and $\boldsymbol{y}'$ be the projections of $\boldsymbol{x}$ and $\boldsymbol{y}$ onto T', respectively. From the LP-duality theorem, it is easy to see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w}')$ and $\mathbb{D}(T, \boldsymbol{w}')$, respectively, with the same value as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer. It follows from Lemma 4.6(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

Case 2. $x(q_jq_i) > 0$. In this case, $z(q_jq_i) = w(q_jq_i) > 0$ by Lemma 4.3(iii). Let C_1 and C_2 be two cycles in $\mathcal{C}^y$ that passes through q_jq_i and q_js_2, respectively. Clearly, both C_1 and C_2 pass through b. By Lemma 4.3(iv), we have $x(q_jq_i)+x(q_is_1)+x(s_1b)=x(q_js_2)+x(s_2b)$. Let $\boldsymbol{w}'$ be obtained from $\boldsymbol{w}$ by replacing $w(e_1)$ with $w(e_1)+w(q_jq_i)$ for $e_1=q_js_2$ and s_2b and replacing $w(e_2)$ with $w(e_2)-w(q_jq_i)$ for $e_2=q_jq_i$, q_is_1, and s_1b. Let $\boldsymbol{x}'=\boldsymbol{x}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle passing through q_jq_i, let C' be the cycle arising from C by replacing the path $q_jq_is_1b$ with q_js_2b. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T,\boldsymbol{w}')$ and $\mathbb{D}(T,\boldsymbol{w}')$, respectively, with the same value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Since $w'(A)<w(A)$, by the hypothesis of Theorem 4.1, $\nu_w^*(T)$ is an integer. It follows from Lemma 4.6(ii) that $\mathbb{D}(T,\boldsymbol{w})$ has an integral optimal solution.

Combining the above two cases, we may assume that $z(q_jq_i)=0$.

(iii) Since the proof is the same as that of Lemma 6.2(iv) (with s_k, q_i, and q_j in place of v_0, s_i, and s_j, respectively), we omit the routine details here. ∎

Lemma 6.11. *If $T_2/S=G_4$, then $\mathbb{D}(T,\boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2,a_2)=(v_1,v_5)$, $s^*=v_4$, and $Q=\{v_2,v_3\}$. Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T,\boldsymbol{w})$, set $\varphi(s_i)=\{u: z(us_i)>0 \text{ for } u\in V(T_2)\backslash a_2\}$ for each $s_i\in S$. By Lemma 6.2(i) and (vi), we have

(1) $\varphi(s_i)\cap\varphi(s_j)=\emptyset$ whenever $i\neq j$.

From (1) and Lemma 6.10(i), we see that

(2) there exists at least one and at most two vertices s_i's in S with $\varphi(s_i)\neq\emptyset$.

Lemma 6.2(i) allows us to assume that

(3) if $\varphi(s_i)\neq\emptyset$, then $i\in\{1,2\}$.

By Lemma 6.10(ii), we obtain

(4) $w(v_2v_3)=z(v_2v_3)=0$.

In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized; and

(6) subject to (5), $(y(\mathcal{D}_q),y(\mathcal{D}_{q-1}),\ldots,y(\mathcal{D}_3))$ is minimized lexicographically.

Claim. $y(C)$ is integral for some $C\in\mathcal{C}_2^y$.

To justify this, we distinguish between two cases.

Case 1. $\varphi(s_i)=\{v_2\}$ for $i=1$ or 2.

In this case, by Lemma 6.2(i) and Lemma 6.10(i), we may assume that $\varphi(s_1)=\{v_2\}$ and $\varphi(s_2)=\{v_3\}$. By (4), we obtain

(7) $\mathcal{C}_2^y\subseteq\{v_1v_2s_1v_1, v_1v_3s_2v_1\}$.

From Lemma 4.7(vii), we deduce that $y(v_1v_2s_1v_1)=\min\{w(v_1v_2),w(v_2s_1),w(s_1v_1)\}$ and $y(v_1v_3s_2v_1)=\min\{w(v_1v_3),w(v_3s_2),w(s_2v_1)\}$. If both $y(v_1v_2s_1v_1)$ and $y(v_1v_3s_2v_1)$ are zero, then $\tau_w(T_2\backslash a_2)=\min\{w(v_1v_2),w(v_2s_1),w(s_1v_1)\}+\min\{w(v_1v_3),w(v_3s_2),w(s_2v_1)\}=0$, contradicting (α). Therefore, $y(v_1v_2s_1v_1)$ or $y(v_1v_3s_2v_1)$ is a positive integer.

Case 2. $\varphi(s_i)\neq\{v_2\}$.

In this case, Lemma 6.10(i), (2) and (3) allow us to assume that $\varphi(s_1)=\{v_2,v_3\}$. By (4), we have

(8) $\mathcal{C}_2^y\subseteq\{v_1v_2s_1v_1, v_1v_3s_1v_1\}$.

By Lemma 6.2(iii), we also obtain $z(s_1v_1)=w(s_1v_1)>0$. Assume first that s_1v_1 is outside $\mathcal{C}_0^y$. Then both v_2s_1 and v_3s_1 are outside $\mathcal{C}_0^y$, and s_1v_1 is saturated by $\boldsymbol{y}$ in T_2. So $y(v_1v_2s_1v_1)+y(v_1v_3s_1v_1)=w(s_1v_1)>0$. Observe that both $y(v_1v_2s_1v_1)$ and $y(v_1v_3s_1v_1)$ are integral, for otherwise, $0 < y(v_1v_is_1v_1) < w(v_is_1)$ for $i = 2, 3$, by Lemma 4.3(i) and (ii), we have $x(v_2s_1) = x(v_3s_1) = 0$, contradicting Lemma 6.9(iii). Hence $y(v_1v_2s_1v_1)$ or $y(v_1v_3s_1v_1)$ is a positive integer.

Assume next that s_1v_1 is contained in some cycle $C \in \mathcal{C}_0^y$. From Lemma 4.7(vii), we see that $y(v_1v_is_1v_1) = \min\{w(v_1v_i), w(v_is_1)\}$ for $i = 2, 3$. If $y(v_1v_is_1v_1) = 0$ for $i = 2, 3$, then $\tau_w(T_2\backslash a_2) = \sum_{i=1}^{2} \min\{w(v_1v_i), w(v_is_1)\} = 0$, contradicting (α). Therefore $y(v_1v_2s_1v_1)$ or $y(v_1v_3s_1v_1)$ is a positive integer. So the above Claim is established.

From the above Claim and Lemma 4.4(iii), we conclude that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. ∎

Lemma 6.12. *If $T_2/S = G_5$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_2, v_6)$, $s^* = v_5$, and $Q = \{v_1, v_3, v_4\}$. Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 6.2(i) and (vi), we have

(1) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

From (1) and Lemma 6.10(i), we see that

(2) there exists at least one and at most three vertices s_i's in S with $\varphi(s_i) \neq \emptyset$.

Lemma 6.2(i) allows us to assume that

(3) if $\varphi(s_i) \neq \emptyset$, then $i \in \{1, 2, 3\}$.

By Lemma 6.10(ii), we obtain

(4) $w(e) = z(e) = 0$ for $e \in \{v_1v_3, v_3v_4, v_4v_1\}$.

In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized; and

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically.

Claim. $y(C)$ is integral for some $C \in \mathcal{C}_2^y$.

To justify this, we consider three possible cases (see the structure of G_5), depending on the size of $\varphi(s_i)$ for $1 \leq i \leq 3$.

Case 1. $|\varphi(s_i)| = 1$ for each $1 \leq i \leq 3$.

In this case, by Lemma 6.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1\}$, $\varphi(s_2) = \{v_3\}$, and $\varphi(s_3) = \{v_4\}$. By (4), we obtain

(7) $\mathcal{C}_2^y \subseteq \{v_2v_1s_1v_2, v_2v_3s_2v_2, v_2v_4s_3v_2\}$.

From Lemma 4.7(vii), we deduce that $y(v_2v_1s_1v_2) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\}$, $y(v_2v_3s_2v_2) = \min\{w(v_2v_3), w(v_3s_2), w(s_2v_2)\}$, and $y(v_2v_4s_3v_2) = \min\{w(v_2v_4), w(v_4s_3), w(s_3v_2)\}$. If $y(v_2v_1s_1v_2)$, $y(v_2v_3s_2v_2)$, and $y(v_2v_4s_3v_2)$ are all zero, then $\tau_w(T_2\backslash a_2) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\} + \min\{w(v_2v_3), w(v_3s_2), w(s_2v_2)\} + \min\{w(v_2v_4), w(v_4s_3), w(s_3v_2)\} = 0$, contradicting (α). Therefore, at least one of $y(v_2v_1s_1v_2)$, $y(v_2v_3s_2v_2)$, and $y(v_2v_4s_3v_2)$ is a positive integer.

Case 2. $|\varphi(s_i)| = 1$ for exactly one $i \in \{1, 2, 3\}$.

In this case, by Lemma 6.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1\}$, $\varphi(s_2) = \{v_3, v_4\}$. By (4), we have

(8) $\mathcal{C}_2^y \subseteq \{v_2v_1s_1v_2, v_2v_3s_2v_2, v_2v_4s_2v_2\}$.

From Lemma 4.7(vii), we see that $y(v_2v_1s_1v_2) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\}$. If $y(v_2v_1s_1v_2) > 0$, we are done. So we assume that $y(v_2v_1s_1v_2) = 0$. Since $w(v_1s_1)w(s_1v_2) > 0$, we obtain $w(v_2v_1) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\} = 0$. By Lemma 6.2(iii), we have $z(s_2v_2) = w(s_2v_2) > 0$.

Assume first that s_2v_2 is outside $\mathcal{C}_0^y$. Then both v_3s_2 and v_4s_2 are outside $\mathcal{C}_0^y$, and s_2v_2 is saturated by $\boldsymbol{y}$ in T_2. Hence $y(v_2v_3s_2v_2) + y(v_2v_4s_2v_2) = w(s_2v_2) > 0$. Observe that both $y(v_2v_3s_2v_2)$ and $y(v_2v_4s_2v_2)$ are integral, for otherwise, since $0 < y(v_2v_is_2v_2) < w(v_is_2)$ for $i = 3, 4$, by Lemma 4.3(i) and (ii), we have $x(v_3s_2) = x(v_4s_2) = 0$, contradicting Lemma 6.9(iii). Hence both $y(v_2v_3s_2v_2)$ and $y(v_2v_4s_2v_2)$ are positive integers.

Assume next that s_2v_2 is contained in some cycle $C \in \mathcal{C}_0^y$. From Lemma 4.7(vii), we see that $y(v_2v_is_2v_2) = \min\{w(v_2v_i), w(v_is_2)\}$ for $i = 3, 4$. If $y(v_2v_is_2v_2) = 0$ for $i = 3, 4$, then $\tau_w(T_2\backslash a_2) = w(v_2v_1) + \sum_{i=3}^{4} \min\{w(v_2v_i), w(v_is_2)\} = 0$, contradicting (α). Therefore $y(v_2v_3s_2v_2)$ or $y(v_2v_4s_2v_2)$ is a positive integer.

Case 3. $|\varphi(s_i)| \neq 1$ for any $i \in \{1, 2, 3\}$.

In this case, by Lemma 6.10(i), (2), and (3), we may assume that $\varphi(s_1) = \{v_1, v_3, v_4\}$ (see the structure of G_5). By (4), we obtain

(9) $\mathcal{C}_2^y \subseteq \{v_2v_1s_1v_2, v_2v_3s_1v_2, v_2v_4s_1v_2\}$.

By Lemma 6.2(iii), we have $z(s_1v_2) = w(s_1v_2) > 0$.

Assume first that s_1v_2 is outside $\mathcal{C}_0^y$. Then v_is_1 is outside $\mathcal{C}_0^y$ for each $i \in \{1, 3, 4\}$, and s_1v_2 is saturated by $\boldsymbol{y}$ in T_2. So $\sum_{i\in\{1,3,4\}} y(v_2v_is_1v_2) = w(s_1v_2) > 0$. Observe that $y(v_2v_is_1v_2)$ is integral for each $i \in \{1, 3, 4\}$, for otherwise, symmetry allows us to assume that $y(v_2v_1s_1v_2)$ is not integral. Then $y(v_2v_3s_1v_2)$ or $y(v_2v_4s_1v_2)$ is not integral, say $y(v_2v_3s_1v_2)$. Since $0 < y(v_2v_is_1v_2) < w(v_is_1)$ for $i = 1, 3$, by Lemma 4.3(i) and (ii), we have $x(v_1s_1) = x(v_3s_1) = 0$, contradicting Lemma 6.9(iii). It follows that $y(v_2v_is_1v_2)$ is a positive integer for each $i \in \{1, 3, 4\}$.

Assume next that s_1v_2 is contained in some cycle $C \in \mathcal{C}_0^y$. From Lemma 4.7(vii), we deduce that $y(v_2v_is_1v_2) = \min\{w(v_2v_i), w(v_is_1)\}$ for $i \in \{1, 3, 4\}$. If $y(v_2v_is_1v_2) = 0$ for each $i \in \{1, 3, 4\}$, then $\tau_w(T_2\backslash a_2) = \sum_{i\in\{1,3,4\}} \min\{w(v_2v_i), w(v_is_1)\} = 0$, contradicting (α). Hence $y(v_2v_is_1v_2)$ is a positive integer for some $i \in \{1, 3, 4\}$. This proves the Claim.

From the Claim and Lemma 4.4(iii), we conclude that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. ∎

Lemma 6.13. *If $T_2/S = G_6$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_6, v_7)$, $s^* = v_5$, and $Q = \{v_1, v_2, v_3, v_4\}$. Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2)\backslash a_2\}$ for each $s_i \in S$. By Lemma 6.2(i) and (vi), we have

(1) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

From (1) and Lemma 6.10(i), we see that

(2) there exists at least one and at most four vertices s_i's in S with $\varphi(s_i) \neq \emptyset$.

Lemma 6.2(i) allows us to assume that

(3) if $\varphi(s_i) \neq \emptyset$, then $1 \leq i \leq 4$.

By Lemma 6.10(ii), we obtain

(4) $w(e) = z(e) = 0$ for $e \in \{v_1v_3, v_3v_4, v_4v_1, v_1v_2, v_3v_2, v_4v_2\}$.

In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(5) $y(\mathcal{C}_2)$ is maximized; and

(6) subject to (5), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically.

Claim. $y(C)$ is integral for some $C \in \mathcal{C}_2^y$.

To justify this, we consider five possible cases (see the structure of G_6), depending on the size of $\varphi(s_i)$ for $1 \leq i \leq 4$.

Case 1. $|\varphi(s_i)| = 1$ for each $1 \le i \le 4$.

In this case, by Lemma 6.10(i), (2) and (3), we may assume that $\varphi(s_i) = \{v_i\}$ for each $1 \le i \le 4$. By (4), we obtain

(7) $\mathcal{C}_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_2v_6, v_6v_3s_3v_6, v_6v_4s_4v_6\}$.

From Lemma 4.7(vii), we deduce that $y(v_6v_is_iv_6) = \min\{w(v_6v_i), w(v_is_i), w(s_iv_6)\}$ for each $1 \le i \le 4$. If $y(v_6v_is_iv_6) = 0$ for $1 \le i \le 4$, then $\tau_w(T_2 \backslash a_2) = \sum_{i=1}^4 \min\{w(v_6v_i), w(v_is_i), w(s_iv_6)\} = 0$, contradicting ($\alpha$). Hence $y(v_6v_is_iv_6)$ is a positive integer for some $i \in \{1,2,3,4\}$.

Case 2. $|\varphi(s_i)| = 1$ for exactly one $i \in \{1,2,3,4\}$.

In this case, by Lemma 6.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1\}$, $\varphi(s_2) = \{v_2, v_3, v_4\}$. By (4), we have

(8) $\mathcal{C}_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_2v_6, v_6v_3s_2v_6, v_6v_4s_2v_6\}$.

From Lemma 4.7(vii), we see that $y(v_6v_1s_1v_6) = \min\{w(v_6v_1), w(v_1s_1), w(s_1v_6)\}$. If $y(v_6v_1s_1v_6) > 0$, we are done. So we assume that $y(v_6v_1s_1v_6) > 0$. Since $w(v_1s_1)w(s_1v_6) > 0$, we obtain $w(v_6v_1) = \min\{w(v_6v_1), w(v_1s_1), w(s_1v_6)\} = 0$. By Lemma 6.2(iii), we have $z(s_2v_6) = w(s_2v_6) > 0$.

Assume first that s_2v_6 is outside $\mathcal{C}_0^y$. Then v_is_2 is outside $\mathcal{C}_0^y$ for $i \in \{2,3,4\}$, and s_2v_6 is saturated by $\boldsymbol{y}$ in T_2. So $\sum_{i=2}^4 y(v_6v_is_2v_6) = w(s_2v_6) > 0$. Observe that $y(v_6v_is_2v_6)$ is integral for each $i \in \{2,3,4\}$, for otherwise, symmetry allows us to assume that $y(v_6v_2s_2v_6)$ is not integral. Then one of $y(v_6v_3s_2v_6)$ and $y(v_6v_4s_2v_6)$ is not integral, say $y(v_6v_3s_2v_6)$. Since $0 < y(v_6v_is_2v_6) < w(v_is_2)$ for $i = 2,3$, by Lemma 4.3(i) and (ii), we have $x(v_2s_2) = x(v_3s_2) = 0$, contradicting Lemma 6.9(iii). It follows that $y(v_6v_is_2v_6)$ is a positive integer for each $i \in \{2,3,4\}$.

Assume next that s_2v_6 is contained in some cycle $C \in \mathcal{C}_0^y$. By Lemma 4.7(vii), we obtain $y(v_6v_is_2v_6) = \min\{w(v_6v_i), w(v_is_2)\}$ for $i \in \{2,3,4\}$. If $y(v_6v_is_2v_6) = 0$ for $i \in \{2,3,4\}$, then $\tau_w(T_2 \backslash a_2) = w(v_6v_1) + \sum_{i=2}^4 \min\{w(v_6v_i), w(v_is_2)\} = 0$, contradicting ($\alpha$). Hence $y(v_6v_is_2v_6)$ is a positive integer for some $i \in \{2,3,4\}$.

Case 3. $|\varphi(s_i)| = 1$ for exactly two i's in $\{1,2,3,4\}$.

In this case, by Lemma 6.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_i\}$ for $i = 1,2$ and $\varphi(s_3) = \{v_3, v_4\}$. By (4), we obtain

(9) $\mathcal{C}_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_2v_6, v_6v_3s_3v_6, v_6v_4s_3v_6\}$.

From Lemma 4.7(vii), we see that $y(v_6v_is_iv_6) = \min\{w(v_6v_i), w(v_is_i), w(s_iv_6)\}$ for $i = 1,2$. If $y(v_6v_is_iv_6) > 0$, we are done. So we assume that $y(v_6v_is_iv_6) = 0$. Since $w(v_is_i)w(s_iv_6) > 0$, we obtain $w(v_6v_i) = \min\{w(v_6v_i), w(v_is_i), w(s_iv_6)\} = 0$ for $i = 1,2$. By Lemma 6.2(iii), we have $z(s_3v_6) = w(s_3v_6) > 0$.

Assume first that s_3v_6 is outside $\mathcal{C}_0^y$. Then v_is_3 is outside $\mathcal{C}_0^y$ for $i = 3,4$, and s_3v_6 is saturated by $\boldsymbol{y}$ in T_2. So $y(v_6v_3s_3v_6) + y(v_6v_4s_3v_6) = w(s_3v_6) > 0$. Observe that both $y(v_6v_3s_3v_6)$ and $y(v_6v_4s_3v_6)$ are integral, for otherwise, since $0 < y(v_6v_is_3v_6) < w(v_is_3)$ for $i = 3,4$, by Lemma 4.3(i) and (ii), we have $x(v_3s_3) = x(v_4s_3) = 0$, contradicting Lemma 6.9(iii). It follows that $y(v_6v_is_3v_6)$ is a positive integer for $i = 3,4$.

Assume next that s_3v_6 is contained in some cycle $C \in \mathcal{C}_0^y$. By Lemma 4.7(vii), we obtain $y(v_6v_is_3v_6) = \min\{w(v_6v_i), w(v_is_2)\}$ for $i = 3,4$. If $y(v_6v_is_3v_6) = 0$ for $i = 3,4$, then $\tau_w(T_2 \backslash a_2) = \sum_{i=1}^2 w(v_6v_i) + \sum_{i=3}^4 \min\{w(v_6v_i), w(v_is_3)\} = 0$, contradicting ($\alpha$). Hence $y(v_6v_is_3v_6)$ is a positive integer for $i = 3$ or 4.

Case 4. $1 < |\varphi(s_i)| < 4$ if $\varphi(s_i) \neq \emptyset$, for $i \in \{1,2,3,4\}$.

In this case, by Lemma 6.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1, v_2\}$ and $\varphi(s_2) = \{v_3, v_4\}$. By (4), we obtain

(10) $\mathcal{C}_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_1v_6, v_6v_3s_2v_6, v_6v_4s_2v_6\}$.

By Lemma 6.2(iii), we have $z(s_iv_6) = w(s_iv_6) > 0$ for $i = 1, 2$.

Assume first that s_1v_6 is outside $\mathcal{C}_0^y$. Then both v_1s_1 and v_2s_1 are outside $\mathcal{C}_0^y$, and s_1v_6 is saturated by $\boldsymbol{y}$ in T_2. So $y(v_6v_1s_1v_6) + y(v_6v_2s_1v_6) = w(s_1v_6) > 0$. Observe that both $y(v_6v_1s_1v_6)$ and $y(v_6v_2s_1v_6)$ are integral, for otherwise, since $0 < y(v_6v_is_1v_6) < w(v_is_1)$ for $i = 1, 2$, by Lemma 4.3(i) and (ii), we have $x(v_1s_1) = x(v_2s_1) = 0$, contradicting Lemma 6.9(iii). It follows that $y(v_6v_is_1v_6)$ is a positive integer for $i = 1, 2$. Similarly, we can show that if s_2v_6 is outside $\mathcal{C}_0^y$, then $y(v_6v_is_2v_6)$ is a positive integer for $i = 3, 4$.

Assume next that s_iv_6 is contained in some cycle in $\mathcal{C}_0^y$ for $i = 1, 2$. By Lemma 4.7(vii), we have $y(v_6v_is_1v_6) = \min\{w(v_6v_i), w(v_is_1)\}$ for $i = 1, 2$, and $y(v_6v_is_2v_6) = \min\{w(v_6v_i), w(v_is_2)\}$ for $i = 3, 4$. If $y(v_6v_1s_1v_6)$, $y(v_6v_2s_1v_6)$, $y(v_6v_3s_2v_6)$ and $y(v_6v_4s_2v_6)$ are all zero, then $\tau_w(T_2\backslash a_2) = \sum_{i=1}^{2} \min\{w(v_6v_i), w(v_is_1)\} + \sum_{i=3}^{4} \min\{w(v_6v_i), w(v_is_2)\} = 0$, contradicting (α). So at least one of $y(v_6v_1s_1v_6)$, $y(v_6v_2s_1v_6)$, $y(v_6v_3s_2v_6)$, and $y(v_6v_4s_2v_6)$ is a positive integer.

Case 5. $|\varphi(s_i)| > 2$ if $\varphi(s_i) \neq \emptyset$, for $i \in \{1, 2, 3, 4\}$.

In this case, by Lemma 6.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1, v_2, v_3, v_4\}$. By (4), we obtain

(11) $\mathcal{C}_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_1v_6, v_6v_3s_1v_6, v_6v_4s_1v_6\}$.

By Lemma 6.2(iii), we have $z(s_1v_6) = w(s_1v_6) > 0$.

Assume first that s_1v_6 is outside $\mathcal{C}_0^y$. Then $\sum_{i=1}^{4} y(v_6v_is_1v_6) = w(s_1v_6)$. If $y(v_6v_is_1v_6)$ is a positive integer for some $i \in \{1, 2, 3, 4\}$, we are done. So we assume the contrary. Thus at least two of $y(v_6v_1s_1v_6)$, $y(v_6v_2s_1v_6)$, $y(v_6v_3s_1v_6)$, and $y(v_6v_4s_1v_6)$ are not integral, say $y(v_6v_1s_1v_6)$ and $y(v_6v_2s_1v_6)$. Since $0 < y(v_6v_is_1v_6) < w(v_is_1)$ for $i = 1, 2$, by Lemma 4.3 (i) and (ii), we have $x(v_1s_1) = x(v_2s_1) = 0$, contradicting Lemma 6.9(iii).

Assume next that s_1v_6 is contained in some cycle of $\mathcal{C}_0^y$. By Lemma 4.7(vii), we have $y(v_6v_is_1v_6) = \min\{w(v_6v_i), w(v_is_1)\}$ for $1 \leq i \leq 4$. If $y(v_6v_is_1v_6)$ is zero for $1 \leq i \leq 4$, then $\tau_w(T_2\backslash a_2) = \sum_{i=1}^{4} \min\{w(v_6v_i), w(v_is_1)\} = 0$, contradicting (α). So $y(v_6v_is_1v_6)$ is a positive integer for some $i \in \{1, 2, 3, 4\}$. This proves the Claim.

From the above Claim and Lemma 4.4(iii), we conclude that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. ∎

With the aid of the above lemmas, we can now derive the desired total-dual integrality.

Proof of Theorem 6.1. By the hypothesis of this section, T is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 with properties (α) and (β). Since $T_2/S \in \mathcal{T}_3$, the statement follows instantly from Lemmas 6.3-6.8 and Lemmas 6.11-6.13. ∎

7 Proof: Last Step

In the preceding two sections we have carried out a series of reduction operations, and finished the main body of the proof of Theorem 4.1. To complete the proof, we still need to consider two more cases.

Lemma 7.1. *Let $G = (V, A)$ be a digraph with a nonnegative integral weight $c(e)$ on each arc e, and let v be a vertex of G. If each positive cycle in G contains v, then $\mathbb{D}(G, \boldsymbol{c})$ has an integral optimal solution.*

Proof. Construct a flow network $N = (V', A')$ with vertex set $V' = (V \backslash v) \cup \{s, t\}$ as follows:

- for each arc $ab \in A$ with $a \neq v \neq b$, there is an arc $ab \in A'$ with capacity $c(ab)$;
- for each arc $va \in A$, there is an arc $sa \in A'$ with capacity $c(va)$; and
- for each arc $av \in A$, there is an arc $at \in A'$ with capacity $c(av)$.

Then there is a one-to-one correspondence between cycles containing v in G and s-t paths in N. So, by the max-flow min-cut theorem, $\mathbb{D}(G, \boldsymbol{c})$ has an integral optimal solution. ∎

Lemma 7.2. *Tournament G_1 is cycle Mengerian.*

For a computer-assisted proof of this lemma, see Appendix [11].

Proof of Theorem 4.1. Clearly, we may assume that T is strong, $T \neq C_3$, and $\tau_w(T) > 0$. Since F_1 can be obtained from G_1 by deleting vertex v_6 (see the labeling in Figure 4), from Lemma 7.2 we deduce that F_1 is also cycle Mengerian. So we may further assume that $F_1 \neq T \neq G_1$.

By Theorems 3.1 and 3.2 and Lemma 3.4, $\{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$ is the list of all *i2s* Möbius-free tournaments. Hence

(1) if T is *i2s*, then $T \in \{F_0, F_2, F_3, F_4, G_2, G_3\} = \mathcal{T}_2 \backslash \{F_6\}$.

We claim that T can be expressed as a 1-sum of two strong Möbius-free tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2), such that one of the following three cases occurs:

(2) $\tau_w(T_2 \backslash a_2) > 0$ and $T_2 \in \mathcal{T}_2$;

(3) $\tau_w(T_2 \backslash a_2) > 0$ and there exists a vertex subset S of $T_2 \backslash \{a_2, b_2\}$ with $|S| \geq 2$, such that $T[S]$ is acyclic, $T_2/S \in \mathcal{T}_3$, and the vertex s^* arising from contracting S is a near-sink in T/S; and

(4) every positive cycle in T crosses the hub b of the 1-sum.

Indeed, if T is not *i2s*, then the statement follows from Lemma 4.2. It remains to consider the case when T is *i2s*. By (1), we have $T \in \mathcal{T}_2 \backslash \{F_6\}$. Since each tournament in $\mathcal{T}_2 \backslash \{F_6\}$ has a special arc, we may view T as a 1-sum of T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2), where T_1 is a triangle and $T_2 = T$. If $\tau_w(T_2 \backslash a_2) > 0$, then (2) holds. If $\tau_w(T_2 \backslash a_2) = 0$, then every positive cycle in T contains the hub of the 1-sum. So (4) occurs.

Applying Theorem 5.1, Theorem 6.1, and Lemma 7.1 to (2), (3), and (4), respectively, we conclude that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution in any case. ∎

Proof of Theorem 1.1. Implication (*iii*) $\Rightarrow$ (*ii*) holds, because total-dual integrality implies primal integrality (see Edmonds-Giles theorem [18] stated in Section 1). Implication (*ii*) $\Rightarrow$ (*i*) is established in Lemma 2.1. Implication (*i*) $\Rightarrow$ (*iii*) follows instantly from Theorem 4.1. ∎

8 Concluding Remarks

In this paper we have characterized all tournaments with the min-max relation on packing and covering cycles. Our characterization yields a polynomial-time algorithm for the minimum-weight feedback arc set problem on cycle Mengerian tournaments. But this algorithm is based on the ellipsoid method for linear programming, and therefore very much unlike the typical combinatorial optimization procedures. It would be interesting to know whether it can be replaced by a strongly polynomial-time algorithm of a transparent combinatorial nature. In combinatorial optimization, there are some other min-max results that are obtained using the "structure-driven" approach. Despite availability of structural descriptions, combinatorial polynomial-time algorithms for the corresponding optimization problems have yet to be found, for instance, those on matroids with the max-flow min-cut property; see Seymour [31] for a characterization and Truemper [35] for efficient algorithms once again based on the ellipsoid method. Certainly, these types of problems deserve more research efforts.

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