Ranking Tournaments with No Errors I: Structural Description

Xujin Chen[a*] Guoli Ding[b†] Wenan Zang[c‡] Qiulan Zhao[d§]

[a] Institute of Applied Mathematics, Chinese Academy of Sciences
Beijing 100190, China

[b] Mathematics Department, Louisiana State University
Baton Rouge, LA 70803, USA

[c] Department of Mathematics, The University of Hong Kong
Hong Kong, China

[d] Department of Mathematics, Nanjing University
Nanjing 210093, China

Abstract

In this series of two papers we examine the classical problem of ranking a set of players on the basis of a set of pairwise comparisons arising from a sports tournament, with the objective of minimizing the total number of upsets, where an *upset* occurs if a higher ranked player was actually defeated by a lower ranked player. This problem can be rephrased as the so-called minimum feedback arc set problem on tournaments, which arises in a rich variety of applications and has been a subject of extensive research. In this series we study this NP-hard problem using structure-driven and linear programming approaches. Let $T = (V, A)$ be a tournament with a nonnegative integral weight $w(e)$ on each arc e. A subset F of arcs is called a *feedback arc set* if $T\backslash F$ contains no cycles (directed). A collection $\mathcal{C}$ of cycles (with repetition allowed) is called a *cycle packing* if each arc e is used at most $w(e)$ times by members of $\mathcal{C}$. We call T *cycle Mengerian* (CM) if, for every nonnegative integral function $\boldsymbol{w}$ defined on A, the minimum total weight of a feedback arc set is equal to the maximum size of a cycle packing. The purpose of these two papers is to show that a tournament is CM iff it contains none of four Möbius ladders as a subgraph; such a tournament is referred to as Möbius-free. In this first paper we present a structural description of all Möbius-free tournaments.

MSC 2000 subject classification. Primary: 90C10, 90C27, 90C57.
OR/MS subject classification. Primary: Programming/graphs.
Key words. Tournament, feedback arc set, cycle packing, minimax relation, characterization.

*Supported in part by NSF of China under Grant 11531014.
†Supported in part by NSF grant DMS-1500699.
‡Supported in part by the Research Grants Council of Hong Kong.
§Corresponding author. E-mail: qiulanzhao@163.com.

1 Introduction

Consider a sports tournament in which each of n players is required to play precisely one game with each other player, and assume that each game ends in a win or a loss. After completion of the tournament, it is desirable to find a ranking of all n players that minimizes the number of upsets, where an *upset* occurs if a higher ranked player was actually defeated by a lower ranked player. This problem can be rephrased as the so-called minimum feedback arc set problem on tournaments, and will be investigated in the more general weighted setting in this series of two papers.

Let $G = (V, A)$ be a digraph with a nonnegative integral weight $w(e)$ on each arc e. A subset F of arcs is called a *feedback arc set* (FAS) of G if $G \backslash F$ contains no cycles (directed). The *minimum-weight FAS problem* (or simply *FAS problem*) is to find an FAS in G with minimum total weight. Digraph G is called a *tournament* if there is precisely one arc between any two vertices in G. The FAS problem on tournaments, abbreviated FAST, dates back to as early as the 1780s when Borda [7] and Condorcet [11] each proposed voting systems for elections with more than two candidates. Since the FAST arises in a rich variety of applications in sports, databases, and statistics, where it is necessary to effectively combine rankings from different sources, FAS's in tournaments have been studied extensively from the combinatorial [17, 18, 31, 35], statistical [30], and algorithmic [1, 2, 12, 26, 33, 34] points of view, and thus have produced a vast body of literature. In [1], Ailon, Charikar, and Newman proved that the FAST is NP-hard under randomized reductions even in the unweighted case. In [3], Alon showed that this unweighted version is in fact NP-hard; in [10], Charbit, Thomassé, and Yeo established this result independently. In [26], Mathieu and Schudy devised a polynomial time approximation scheme (PTAS) for the FAST. Given these results, it is natural to ask the following question: When can the FAST be solved exactly in polynomial time? Inspired by the title of Mathieu and Schudy's paper [26], this is equivalent to asking: Which tournaments can be ranked with no errors? The purpose of this series of two papers is to resolve this problem using structure-driven and linear programming approaches.

We introduce some terminology before proceeding. Let $G = (V, A)$ be a weighted digraph as described above. A collection $\mathcal{C}$ of cycles (with repetition allowed) in G is called a *cycle packing* of G if each arc e is used at most $w(e)$ times by members of $\mathcal{C}$. The *cycle packing problem* consists in finding a cycle packing with maximum size, which can be viewed as the dual version of the FAS problem. Let $\nu_w(G)$ be the maximum size of a cycle packing, and let $\tau_w(G)$ be the minimum total weight of an FAS. Clearly, $\nu_w(G) \leq \tau_w(G)$; this inequality, however, need not hold with equality in general (as we shall see in a moment). We call G *cycle Mengerian* (CM) if $\nu_w(G) = \tau_w(G)$ for every nonnegative integral function $\boldsymbol{w}$ defined on A. It is worthwhile pointing out that a characterization of CM digraphs can yield not only a beautiful minimax theorem but also a polynomial-time algorithm for the FAS problem on such digraphs, by a general theorem of Grötschel, Lovász, and Schrijver [20]. So the study of CM digraphs has both great theoretical interest and practical value. Initiated in the early 1960s [13, 35], it has inspired many minimax theorems in combinatorial optimization, such as Lucchesi and Younger [25], Seymour [28, 29], Geelen and Guenin [19], Guenin [21, 22], Guenin and Thomas [23], Cai, Deng, and Zang [8, 9], and Ding, Xu, and Zang [15, 16]. Interestingly, such minimax theorems have also found applications in the design of approximation algorithm; see, for instance, Mnich, Williams, and Végh [27]. Despite tremendous research efforts, only some special classes of CM digraphs [4, 5, 21, 23, 25] have been identified to date, and a complete characterization seems extremely hard to obtain.

Let D_5 be the digraph obtained from K_5 (the complete graph with five vertices) by replacing each edge ij with a pair of opposite arcs (i, j) and (j, i). Applegate, Cook, and McCormick [4] and Barahona, Fonlupt, and Mahjoub [5] independently proved that D_5 is CM, thereby confirming a conjecture posed in both Barahona and Mahjoub [6] and Jünger [24]. This theorem is equivalent to saying that every tournament with five vertices is CM.

In this series of two papers we shall give a complete characterization of all CM tournaments. We call a tournament *Möbius-free* if it contains none of $K_{3,3}$, $K'_{3,3}$, M_5, and M_5^* depicted in Figure 1 as a subgraph. (Actually, M_5^* arises from M_5 by reversing the direction of each arc.) This class of tournaments is so named because the forbidden structures are all Möbius ladders.

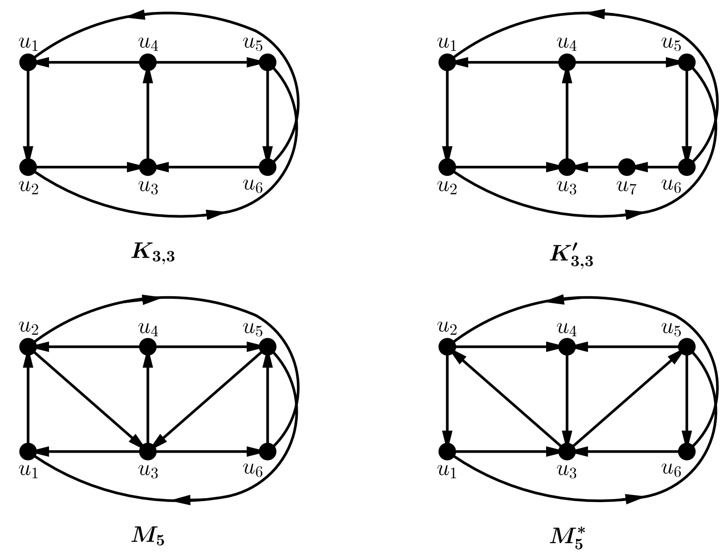

Figure 1. Forbidden Structures

Theorem 1.1. *A tournament is CM iff it is Möbius-free.*

Observe that every CM tournament is Möbius-free: Let $T = (V, A)$ be a tournament containing a member $D = (U, B)$ of $\{K_{3,3}, K'_{3,3}, M_5, M_5^*\}$. Define $w(e) = 1$ if $e \in B$ and $w(e) = 0$ if $e \in A \backslash B$. It is a routine matter to check that

- $\nu_w(T) = 1$ while $\tau_w(T) = 2$ if D is $K_{3,3}$ or $K'_{3,3}$, and
- $\nu_w(T) = 2$ while $\tau_w(T) = 3$ if D is M_5 or M_5^*.

So T does not satisfy the desired minimax relation. Our theorem asserts that actually these four Möbius ladders are the only obstructions to CM tournaments. Since the whole proof takes about 100 pages, we split this work into two papers. In this first paper we give a structural description of all Möbius-free tournaments.

Let us define a few more terms before presenting our structural theorems. Let $G = (V, A)$ be a digraph. For each $v \in V$, we use $d^+_G(v)$ and $d^-_G(v)$ to denote the out-degree and in-degree of v, respectively. We call v a *near-sink* of G if its out-degree is one, and call v a *near-source* if its in-degree is one. For simplicity, an arc $e = (u, v)$ of G is also denoted by uv. Arc e is called *special* if either u is a near-sink or v is a near-source of G. A *dicut* of G is a partition (X, Y) of $V(G)$ such that all arcs between X and Y are directed to Y. A dicut (X, Y) is *trivial* if $|X| = 1$ or $|Y| = 1$. Recall that G is called *weakly connected* if its underlying undirected graph is connected, and is called *strongly connected* or *strong* if each vertex is reachable from each other vertex. Clearly, a weakly connected digraph G is strong iff G has no dicut. Furthermore, a weakly connected digraph G is called *internally strong* if every dicut of G is trivial, and is called *internally 2-strong* (*i2s*) if G is strong and $G\backslash v$ is internally strong for every vertex v. By definition, a strong digraph is internally strong.

Let $T_1 = (V_1, A_1)$ and $T_2 = (V_2, A_2)$ be two tournaments. We say that T_1 is *smaller* than T_2 if $|V_1| < |V_2|$. Suppose that both T_1 and T_2 are strong, with $|V_i| \geq 3$ for $i = 1, 2$, and suppose further that (a_1, b_1) is a special arc of T_1 with $d^+_{T_1}(a_1) = 1$ and (b_2, a_2) is a special arc of T_2 with $d^-_{T_2}(a_2) = 1$. The *1-sum* of T_1 and T_2 over (a_1, b_1) and (b_2, a_2) is the tournament arising from the disjoint union of $T_1\backslash a_1$ and $T_2\backslash a_2$ by identifying b_1 with b_2 (the resulting vertex is denoted by b) and adding all arcs from $T_1\backslash\{a_1, b_1\}$ to $T_2\backslash\{a_2, b_2\}$. We call b the *hub* of the 1-sum. See Figure 2 for an illustration. Note that if $|V_i| = 3$ for $i = 1$ or 2, then T_i is a triangle (a directed cycle of length three), and thus $T = T_{3-i}$.

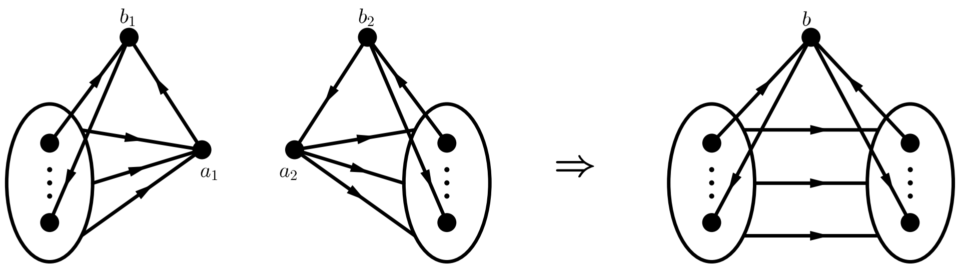

Figure 2. 1-sum of T_1 and T_2.

Let C_3 (resp. F_0) denote the strong tournament with three (resp. four) vertices (see Figure 3), let F_1, F_2, F_3, F_4, F_5 be the five tournaments depicted in Figure 4, and let G_1, G_2, G_3 be the three tournaments shown in Figure 5. In these two papers, we reserve the symbols

$$\mathcal{T}_0 = \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$$

and

$$\mathcal{T}_1 = \{C_3, F_0, F_2, F_3, F_4, G_2, G_3\} = \mathcal{T}_0\backslash\{F_1, G_1\}.$$

Now we are ready to present the main results of this paper. (Obviously, to verify that a tournament T is CM, we may restrict our attention to the case when T is strong.)

Theorem 1.2. *Let $T = (V, A)$ be an i2s tournament with $|V| \geq 3$. Then T is Möbius-free iff $T \in \mathcal{T}_0$.*

Theorem 1.3. *Let $T = (V, A)$ be a strong Möbius-free tournament with $|V| \geq 3$. Then either $T \in \{F_1, G_1\}$ or T can be obtained by repeatedly taking* 1*-sums starting from the tournaments in $\mathcal{T}_1$.*

Throughout this paper we shall repeatedly use the following notations and terminology.

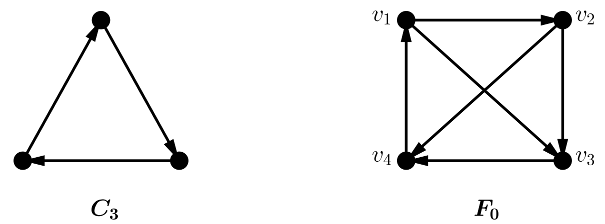

Figure 3. Strong tournaments with three or four vertices.

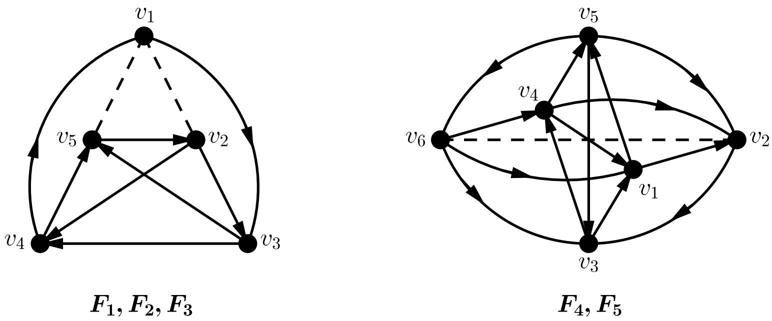

Figure 4. $v_1v_2, v_5v_1 \in F_1$; $v_2v_1, v_1v_5 \in F_2$; $v_2v_1, v_5v_1 \in F_3$; $v_6v_2 \in F_4$; $v_2v_6 \in F_5$.

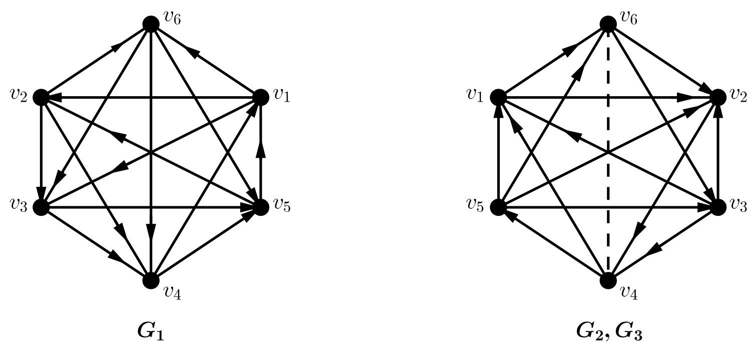

Figure 5. $v_6v_4 \in G_2$ and $v_4v_6 \in G_3$.

For a digraph G, we use $V(G)$ and $A(G)$ to denote its vertex set and arc set, respectively, if they are not specified. For each $U \subseteq V(G)$, we use $G[U]$ to denote the subgraph of G induced by U, and use $\delta^+(U)$ (resp. $\delta^-(U)$) to denote the set of all arcs from U to $V(G)\backslash U$ (resp. from $V(G)\backslash U$ to U); we write $\delta^+(U) = \delta^+(u)$ and $\delta^-(U) = \delta^-(u)$ if $U = \{u\}$. Moreover, we use G/U to denote the digraph obtained from G by first deleting arcs between any two vertices in U, then identifying all vertices in U, and finally deleting the parallel arcs except one from each vertex to each other vertex; we say that G/U is obtained from G by *contracting* U. Note that G/U may contain pairs of opposite arcs but contains no parallel arcs. For each arc $e = (u, v)$

of G, the digraph obtained from G by *contracting* e, denoted by G/e, is exactly $G/\{u, v\}$. A *strong component* of G is a maximal strong subgraph, where the adjective maximal is meant with respect to set-inclusion rather than size. Note that each vertex of G belongs to exactly one strong component. Thus the strong components of G can be ordered as $A_1, A_2, \ldots, A_p$, such that the arcs between A_i and A_j are all directed from A_i to A_j for any $1 \le i < j \le p$; we refer to $(A_1, A_2, \ldots, A_p)$ as a *strong partition* of G. The *reverse* of G, denoted by G^*, is obtained from G be reversing the direction of each arc.

By a cycle or a path in a digraph we always mean a directed one. Let P be a directed path from a to b and let c and d be two vertices on P such that a, b, c, d (not necessarily distinct) occur on P in order as we traverse P in its direction from a. Then $P[c, d]$ denotes the subpath of P from c to d, and $P(c, d) = P[c, d]\backslash\{c, d\}$. Let C be a directed cycle. For each vertex a on C, we use a^- (resp. a^+) to denote the vertex precedes (resp. succeeds) a as we traverse C in its direction. For each pair of vertices a and b on C, we use $C[a, b]$ to denote the segment of C from a to b.

The remainder of this paper is organized as follows. In Section 2, we exhibit some important properties enjoyed by the 1-sum operation. In Section 3, we prove a chain theorem, which says that every *i2s* tournament can be constructed from some small tournaments by repeatedly adding vertices so that all the intermediate tournaments are also *i2s*. In Section 4, we give a structural description of all strong Möbius-free tournaments based on this chain theorem.

2 Preliminaries

In this section, we show that if a strong tournament is not *i2s*, then it can be expressed as the 1-sum of two smaller strong tournaments (so the connectivity can be lifted by using this operation). We also prove that being Möbius-free is preserved under 1-sum operation and under contracting special arcs.

Lemma 2.1. *Let $T = (V, A)$ be a strong tournament. If T is not i2s, then T is the 1-sum of two smaller strong tournaments.*

Proof. Since T is not *i2s*, it contains a vertex b such that $T\backslash b$ has a nontrivial dicut (X, Y). As T is strong, there exist $a_1 \in Y$ and $a_2 \in X$ such that $\{(a_1, b), (b, a_2)\} \subseteq A$. Set $T_1 = T\backslash(Y\backslash a_1)$, $T_2 = T\backslash(X\backslash a_2)$, and rename b as b_i in T_i for $i = 1, 2$. Clearly, a_1 has out-degree one in T_1 and a_2 has in-degree one in T_2. From the definition we see that T is the 1-sum of T_1 and T_2 over (a_1, b_1) and (b_2, a_2). Furthermore, T_i is strong and has fewer vertices than T for $i = 1, 2$. ∎

Let us show that being Möbius-free is maintained under the 1-sum operation.

Lemma 2.2. *Let $T = (V, A)$ be the 1-sum of two tournaments T_1 and T_2. Then T is Möbius-free iff both T_1 and T_2 are Möbius-free.*

Proof. Since both T_1 and T_2 are sub-tournaments of T, the "only if" part holds trivially. To establish the "if" part, assume the contrary: T contains a member D of $\{K_{3,3}, K'_{3,3}, M_5, M_5^*\}$; subject to this, the number of vertices in D is minimum. Let b be the hub of the 1-sum. Then b is contained in D. Observe that

(1) if $D = K'_{3,3}$, then $(u_3, u_6) \in A$ (see the labeling in Figure 1), for otherwise T would contain $K_{3,3}$, contradicting the minimality assumption on D.

Set $D' = D \cup \{(u_3, u_6)\}$ if $D = K'_{3,3}$ and set $D' = D$ otherwise. It it a routine matter to check that D' is $i2s$ (while $K'_{3,3}$ is not). Since T is the 1-sum of T_1 and T_2 and since T contains D' by (1), either $T_1 \backslash b$ or $T_2 \backslash b$ contains precisely one vertex from $D' \backslash b$. Therefore, either T_1 or T_2 contains a subgraph isomorphic to D' and hence is not Möbius-free. ∎

In the remainder of this section, we show that being Möbius-free is also preserved under the operation of contracting a special arc. (Recall that the resulting digraph may contain pairs of opposite arcs.) This lemma will not be used in subsequent sections but will be employed in our second paper.

Lemma 2.3. *Let $T = (V, A)$ be a Möbius-free tournament with a special arc $a = (x, y)$. Then T/a is also Möbius-free.*

Proof. Replacing T by its reverse T^* if necessary, we may assume that x is a near-sink of T. Thus y is the only out-neighbor of x. Let z be the vertex obtained by identifying x and y in T/a and let $\mathcal{F} = \{K_{3,3}, K'_{3,3}, M_5, M_5^*\}$. Assume the contrary: T/a contains a subdigraph $D \in \mathcal{F}$. Then z is in D. We use D' to denote the digraph obtained from $D \backslash z$ by adding two vertices x and y and adding all arcs in $\{(x, y)\} \cup \{(y, u) : (z, u) \in A(D)\} \cup \{(u, x) : u \in V(D) \backslash z\}$. Clearly, D' is a subgraph of T. We propose to prove that

(1) T contains a member of $\mathcal{F}$.

We have a computer-assisted verification of (1). Nevertheless, the proof given below is computer-free.

Let us label the vertices of D as in Figure 1. Depending on the structure of D, we distinguish among four cases.

Case 1. $D = K_{3,3}$. In this case, symmetry allows us to assume that $z = u_4$ or u_5.

• $z = u_4$. Then u_1 and u_5 are the only out-neighbors of y in D'. Thus the union of the three cycles $u_1u_2u_5u_6u_1$, xyu_1u_2x, and xyu_5u_6x forms a $K_{3,3}$ in T.

• $z = u_5$. Then u_6 is the only out-neighbor of y in D'. If $(u_4, y) \in A$, then the union of the three cycles $u_1u_2u_3u_4u_1$, $u_4yu_6u_3u_4$, and $u_1u_2xyu_6u_1$ forms a $K'_{3,3}$ in T. Similarly, if $(u_2, y) \in A$, then the union of the three cycles $u_1u_2u_3u_4u_1$, $u_1u_2yu_6u_1$, and $u_4xyu_6u_3u_4$ also forms a $K'_{3,3}$ in T. So we assume that $\{(y, u_4), (y, u_2)\} \subseteq A$. Thus the union of the three cycles $u_4u_1xyu_4$, $u_1u_2u_3u_4u_1$, and $u_2u_3xyu_2$ forms a $K_{3,3}$ in T.

Case 2. $D = K'_{3,3}$. In this case, we may assume that $(u_3, u_6) \in A$, for otherwise the present case reduces to Case 1.

• $z = u_2$. Then u_5 and u_3 are the only out-neighbors of y in D'. It follows that the union of the three cycles $u_3u_4u_5u_6u_7u_3$, xyu_3u_4x, and xyu_5u_6x forms a $K'_{3,3}$ in T.

• $z = u_3$. Then u_4 is the only out-neighbor of y in D'. If $(u_6, y) \in A$, then the union of the three cycles $u_1u_2u_5u_6u_1$, $yu_4u_5u_6y$, and $xyu_4u_1u_2x$ forms a $K'_{3,3}$ in T; if $(u_2, y) \in A$, then the union of the three cycles $u_1u_2u_5u_6u_1$, $yu_4u_1u_2y$, and $xyu_4u_5u_6x$ also forms a $K'_{3,3}$ in T. So we assume that $\{(y, u_6), (y, u_2)\} \subseteq A$. It follows that a $K_{3,3}$ is formed in T by the three cycles xyu_6u_1x, xyu_2u_5x, and $u_1u_2u_5u_6u_1$.

• $z = u_4$. Then u_1 and u_5 are the only out-neighbors of y in D'. Thus the union of the three cycles $u_1u_2u_5u_6u_1$, xyu_5u_6x, and xyu_1u_2x forms a $K_{3,3}$ in T.

• $z = u_6$. Then u_1 and u_7 are the only out-neighbors of y in D'. It follows that the union of the three cycles $u_1u_2u_3u_4u_1$, $xyu_7u_3u_4x$, and xyu_1u_2x forms a $K'_{3,3}$ in T.

• $z = u_1$. Then u_2 is the only out-neighbor of y in D'. If $\{(u_4, y), (u_6, y)\} \subseteq A$, then the union of the three cycles $yu_2u_3u_4y$, $yu_2u_5u_6y$, and $u_3u_4u_5u_6u_7u_3$ forms a $K'_{3,3}$ in T. So we assume that at least one of (y, u_4) and (y, u_6) is in A.

Consider the first subcase when $(y, u_4) \in A$. If $(u_6, u_2) \in A$, then the union of the three cycles xyu_2u_3x, xyu_4u_5x, and $u_2u_3u_4u_5u_6u_2$ forms a $K'_{3,3}$ in T; if $(y, u_7) \in A$, then the union of the three cycles xyu_7u_3x, xyu_4u_5x, and $u_3u_4u_5u_6u_7u_3$ forms a $K'_{3,3}$ in T. So we assume that $\{(u_2, u_6), (u_7, y)\} \subseteq A$. If $(u_4, u_6) \in A$, then a $K'_{3,3}$ is formed by the three cycles $yu_2u_6u_7y$, $u_3u_4u_6u_7u_3$, and $xyu_2u_3u_4x$; if $(u_3, u_5) \in A$, then a $K'_{3,3}$ is formed by the three cycles $yu_2u_6u_7y$, $u_3u_5u_6u_7u_3$, and $xyu_2u_3u_5x$. So we further assume that $\{(u_6, u_4), (u_5, u_3)\} \subseteq A$. It follows that the union of the three cycles $u_3u_6u_4u_5u_3$, xyu_4u_5x, and $xyu_2u_3u_6x$ forms a $K'_{3,3}$.

Consider the second subcase when $(y, u_6) \in A$. If $(u_7, u_2) \in A$, then the union of the three cycles xyu_2u_3x, $u_2u_3u_6u_7u_2$, and xyu_6u_7x forms a $K_{3,3}$; if $(y, u_3) \in A$, then a $K'_{3,3}$ is formed by the three cycles xyu_6u_7x, xyu_3u_4x, and $u_3u_4u_5u_6u_7u_3$; if $(u_4, u_6) \in A$, then a $K'_{3,3}$ is formed by the three cycles xyu_6u_7x, $u_3u_4u_6u_7$, and $xyu_2u_3u_4x$. So we assume that $\{(u_2, u_7), (u_3, y), (u_6, u_4)\} \subseteq A$. If $(u_5, u_3) \in A$, then a $K'_{3,3}$ is formed by the three cycles $u_3u_6u_4u_5u_3$, xyu_6u_4x, and $xyu_2u_5u_3x$; if $(u_4, u_2) \in A$, then a $K_{3,3}$ is formed by the three cycles xyu_2u_3x, xyu_6u_4x, and $u_2u_3u_6u_4u_2$. So we further assume that $\{(u_3, u_5), (u_2, u_4)\} \subseteq A$. Now if $(y, u_5) \in A$, then the union of the three cycles $u_3u_5u_6u_7u_3$, xyu_5u_6x, and $xyu_2u_7u_3x$ forms a $K'_{3,3}$; if $(u_5, y) \in A$, then the union of the three cycles $yu_2u_7u_3y$, $yu_2u_4u_5y$, and $u_3u_4u_5u_6u_7u_3$ also forms a $K'_{3,3}$.

• $z = u_5$. Then u_6 is the only out-neighbor of y in D'. If $(y, u_2) \in A$, then the union of the three cycles $u_1u_2u_3u_6u_1$, xyu_2u_3x, and xyu_6u_1x forms a $K_{3,3}$ in T. So we assume that $(u_2, y) \in A$. If $(u_4, y) \in A$, then the union of the three cycles $yu_6u_1u_2y$, $u_1u_2u_3u_4u_1$, and $yu_6u_7u_3u_4y$ forms a $K'_{3,3}$ in T. So we also assume that $(y, u_4) \in A$. If $(u_1, u_7) \in A$, then a $K'_{3,3}$ is formed by the three cycles $u_1u_7u_3u_4u_1$, xyu_4u_1x, and $xyu_6u_7u_3x$. So we further assume that $(u_7, u_1) \in A$. If $(y, u_7) \in A$, then a $K'_{3,3}$ is formed by the three cycles $u_1u_2u_3u_4u_1$, $u_1u_2yu_7u_1$, and $xyu_7u_3u_4x$. Similarly, if $(y, u_3) \in A$, then a $K'_{3,3}$ is formed by the three cycles $u_1u_2u_3u_4u_1$, xyu_3u_4x, and $xyu_6u_1u_2x$; if $(y, u_1) \in A$, then a $K'_{3,3}$ is formed by the three cycles xyu_1u_2x, xyu_6u_7x, and $u_1u_2u_3u_6u_7u_1$. Thus it remains to consider the subcase when $\{(u_7, y), (u_3, y), (u_1, y)\} \subseteq A$. If $(u_2, u_7) \in A$, then a $K'_{3,3}$ is formed by the three cycles $u_1u_2yu_6u_1$, $yu_6u_7u_3y$, and $u_1u_2u_7u_3u_4u_1$. So we assume that $(u_7, u_2) \in A$. If $(u_6, u_4) \in A$, then a $K'_{3,3}$ is formed by the three cycles $yu_6u_4u_1y$, $u_1u_2u_3u_4u_1$, and $yu_6u_7u_2u_3y$. So we also assume that $(u_4, u_6) \in A$. If $(u_4, u_7) \in A$, then a $K'_{3,3}$ is formed by the three cycles $yu_4u_6u_1y$, $u_1u_2u_3u_6u_1$, and $yu_4u_7u_2u_3y$. So we further assume that $(u_7, u_4) \in A$.

From the above observations, we conclude that u_7 has a unique in-neighbor u_6 in the subtournament T' of T induced by $V(D')$. If $\{(u_6, u_2), (u_2, u_4)\} \subseteq A$, then an M_5^* is formed by the five cycles yu_4u_1y, $u_1u_2u_4u_1$, $u_2u_4u_6u_2$, $u_4u_6u_7u_4$, and $yu_6u_7u_1y$. If $(u_2, u_6) \in A$, then a $K_{3,3}$ is formed by $u_1u_2u_3u_4u_1$, $u_3u_4u_6u_7u_3$, and $u_1u_2u_6u_7u_1$; if $(u_4, u_2) \in A$, then the union of the three cycles $u_1u_2u_3u_6u_1$, $yu_4u_2u_3y$, and $yu_4u_6u_1y$ also forms a $K_{3,3}$ in T.

• $z = u_7$. Then u_3 is the only out-neighbor of y in D'. If $(u_6, y) \in A$, then the union of the three cycles $u_1u_2u_5u_6u_1$, $u_1u_2u_3u_4u_1$, and $yu_3u_4u_5u_6y$ forms a $K'_{3,3}$ in T; if $(y, u_1) \in A$, then a $K_{3,3}$ is formed in T by the three cycles $u_1u_2u_3u_4u_1$, xyu_1u_2x, and xyu_3u_4x. So we assume that $\{(y, u_6), (u_1, y)\} \subseteq A$. If $(u_1, u_3) \in A$, then a $K'_{3,3}$ is formed by xyu_6u_1x, xyu_3u_4x, and $u_1u_3u_4u_5u_6u_1$; if $(u_4, u_6) \in A$, then the union of the three cycles xyu_6u_1x, xyu_3u_4x, and $u_1u_2u_3u_4u_6u_1$ forms a $K'_{3,3}$ in T; if $(y, u_2) \in A$, then the union of the three cycles $u_1u_2u_5u_6u_1$, xyu_6u_1x, and xyu_2u_5x forms a $K_{3,3}$ in T. So we further assume that $\{(u_3, u_1), (u_6, u_4), (u_2, y)\} \subseteq A$. Depending on whether $(u_5, y) \in A$, we distinguish between two subcases.

Consider the first subcase when $(u_5, y) \in A$. If $(u_4, u_2) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles $u_1u_2u_5u_6u_1$, $yu_3u_6u_1y$, and $yu_3u_4u_2u_5y$. So we assume that $(u_2, u_4) \in A$. If $(u_6, u_2) \in A$, then a $K_{3,3}$ is formed by the three cycles $u_2u_4u_5u_6u_2$, $yu_3u_6u_2y$, and $yu_3u_4u_5y$. So we further assume that $(u_2, u_6) \in A$. If $(u_4, y) \in A$, then a $K_{3,3}$ is formed in T by the three cycles $u_1u_2u_6u_4u_1$, $yu_3u_6u_4y$, and $yu_3u_1u_2y$ in T; if $(y, u_4) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_6u_1x, xyu_4u_5x, and $u_1u_2u_4u_5u_6u_1$.

Consider the second subcase when $(y, u_5) \in A$. If $(y, u_4) \in A$, then a $K'_{3,3}$ is formed by the three cycles xyu_5u_6x, xyu_4u_1x, and $u_1u_2u_5u_6u_4u_1$. So we assume that $(u_4, y) \in A$. If $(u_2, u_6) \in A$, then a $K_{3,3}$ is formed by the three cycles $u_1u_2u_6u_4u_1$, $yu_3u_6u_4y$, and $yu_3u_1u_2y$. So we also assume that $(u_6, u_2) \in A$. If $(u_2, u_4) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles $u_2u_4u_5u_6u_2$, xyu_6u_2x, and $xyu_3u_4u_5x$; if $(u_5, u_3) \in A$, then a $K_{3,3}$ is formed in T by the three cycles xyu_6u_2x, xyu_5u_3x, and $u_2u_5u_3u_6u_2$. Thus we further assume that $\{(u_4, u_2), (u_3, u_5)\} \subseteq A$. It follows that a $K'_{3,3}$ is formed in T by the three cycles $u_2u_5u_6u_4u_2$, $yu_3u_4u_2y$, and $yu_3u_5u_6u_1y$.

Case 3. $D = M_5$. In this case, u_1 and u_6 are symmetric, so are u_2 and u_5.

• $z = u_4$. Then vertices u_2 and u_5 are the only out-neighbors of y in D'. If $(u_3, y) \in A$, then an M_5 is formed in T by the five cycles $u_3u_6u_5u_3$, yu_5u_3y, yu_2u_3y, $u_1u_2u_3u_1$, and $u_1u_2u_6u_5u_1$. If $(y, u_3) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_5u_1x, xyu_3u_6x, and $u_1u_2u_3u_6u_5u_1$.

• $z = u_5$. Then u_1 and u_3 are the only out-neighbors of y in D'. If $(u_4, u_1) \in A$, then a $K_{3,3}$ is formed by the three cycles $u_1u_2u_3u_4u_1$, xyu_1u_2x, and xyu_3u_4x. So we assume that $(u_1, u_4) \in A$. If $(y, u_4) \in A$, then a $K_{3,3}$ is formed by the three cycles $u_1u_4u_2u_3u_1$, xyu_4u_2x, and xyu_3u_1x. Thus we further assume that $(u_4, y) \in A$. It follows that an M_5 is formed in T by the five cycles $u_1u_2u_3u_1$, $u_2u_3u_4u_2$, yu_3u_4y, xyu_3x, and xyu_1u_2x.

• $z = u_6$. Then u_5 is the only out-neighbor of y in D'. If $\{(u_2, y), (u_3, y)\} \subseteq A$, then an M_5 is formed in T by the five cycles $u_3u_4u_5u_3$, yu_5u_3y, $u_2u_3u_4u_2$, $u_1u_2u_3u_1$, and $yu_5u_1u_2y$. Otherwise, if $(y, u_2) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_5u_1x, xyu_2u_3x, and $u_1u_2u_3u_4u_5u_1$; if $(y, u_3) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_5u_1x, xyu_3u_4x, and $u_1u_2u_3u_4u_5u_1$.

• $z = u_3$. Then u_1, u_4, and u_6 are the only out-neighbors of y in D'. If $\{(u_5, y), (u_2, y)\} \subseteq A$, then an M_5 is formed in T by the five cycles yu_6u_5y, yu_4u_5y, yu_4u_2y, yu_1u_2y, and $u_1u_2u_6u_5u_1$. Suppose at least one of (y, u_5) and (y, u_2) is in T. If both (y, u_5) and (y, u_2) are in T, then a $K_{3,3}$ is formed by the three cycles $u_1u_2u_6u_5u_1$, xyu_5u_1x, and xyu_2u_6x. So we assume that either $\{(y, u_5)(u_2, y)\} \subseteq A$ or $\{(y, u_2), (u_5, y)\} \subseteq A$.

Consider the first subcase when $\{(y, u_5), (u_2, y)\} \subseteq A$. If $(u_6, u_4) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_5u_1x, xyu_6u_4x, and $u_1u_2u_6u_4u_5u_1$. So we may assume that $(u_4, u_6) \in A$. If $(u_1, u_4) \in A$, then a $K_{3,3}$ is formed in T by the three cycles $u_1u_4u_6u_5u_1$, xyu_5u_1x, and xyu_4u_6x. If $(u_4, u_1) \in A$, then a $K'_{3,3}$ is formed in T by the the three cycles

$u_1u_2u_6u_5u_1$, $yu_4u_1u_2y$, and $xyu_4u_6u_5x$.

Consider the second subcase when $\{(y,u_2),(u_5,y)\} \subseteq A$. If $(u_6,u_4) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_4u_5x, xyu_2u_6x, and $u_1u_2u_6u_4u_5u_1$. If $(u_1,u_4) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_2u_6x, xyu_1u_4x, and $u_1u_4u_2u_6u_5u_1$. So we assume that $\{(u_4,u_6),(u_4,u_1)\} \subseteq A$. Then a $K'_{3,3}$ is formed in T by the three cycles $u_1u_2u_6u_5u_1$, $yu_4u_6u_5y$, and $xyu_4u_1u_2x$.

Case 4. $D = M_5^*$. In this case, u_1 and u_6 are symmetric, so are u_2 and u_5.

• $z = u_3$. Then u_5 and u_2 are the only out-neighbors of y in D'. Thus a $K_{3,3}$ is formed in T by the three cycle $u_1u_5u_6u_2u_1$, xyu_2u_1x, and xyu_5u_6x.

• $z = u_4$. Then u_3 is the only out-neighbor of y in D'. If both $\{(u_2,y),(u_5,y)\} \subseteq A$, then an M_5^* is formed by the five cycles $u_1u_3u_2u_1$, yu_3u_2y, yu_3u_5y, $u_3u_5u_6u_3$, and $u_1u_5u_6u_2u_1$. So we assume that at most one of (u_2,y) and (u_5,y) is in T. If $(y,u_2) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles $u_1u_3u_5u_6u_2u_1$, xyu_2u_1x, and xyu_3u_5x; if $(y,u_5) \in T$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_3u_2x, xyu_5u_6x, and $u_1u_5u_6u_3u_2u_1$.

• $z = u_6$. Then u_2 and u_3 are the only out-neighbors of y in D'. If $(u_5,y) \in A$, then an M_5^* is formed in T by the five cycles $u_1u_3u_2u_1$, $u_2u_4u_3u_2$, $u_3u_5u_4u_3$, yu_3u_5y, and $yu_2u_1u_5y$. If $(y,u_5) \in A$, then a $K'_{3,3}$ is formed in T by the three cycles xyu_5u_4x, xyu_2u_1x, and $u_1u_5u_4u_3u_2u_1$.

• $z = u_5$. Then u_4 and u_6 are the only out-neighbors of y in D'. Observe that if both (u_3,y) and (u_1,y) are arcs in T, then an M_5^* is formed in T by the five cycles $u_1u_3u_2u_1$, $u_2u_4u_3u_2$, yu_4u_3y, yu_6u_3y, and $yu_6u_2u_1y$. So we assume that at least one of (y,u_3) and (y,u_1) is in T.

Suppose $(u_4,u_1) \in A$. If $(u_1,u_6) \in A$, then a $K_{3,3}$ is formed in T by the three cycles $u_1u_6u_2u_4u_1$, xyu_4u_1x, and xyu_6u_2x; if $(y,u_3) \in A$, then a $K_{3,3}$ is formed in T by the three cycles $u_1u_3u_2u_4u_1$, xyu_4u_1x, and xyu_3u_2x. So we assume that $\{(u_6,u_1),(u_3,y)\} \subseteq A$. Then a $K'_{3,3}$ is formed in T by the three cycles $u_1u_3u_2u_4u_1$, $yu_6u_1u_3y$, and $xyu_6u_2u_4x$.

Suppose $(u_1,u_4) \in A$. If $(y,u_2) \in A$, then a $K_{3,3}$ is formed by the three cycles $u_1u_4u_3u_2u_1$, xyu_2u_1x, and xyu_4u_3x. So we assume that $(u_2,y) \in A$. Consider the subcase when $(u_1,y) \in A$. Now $(y,u_3) \in A$. If $(u_4,u_6) \in A$, then the union of the three cycles $u_2u_4u_6u_3u_2$, xyu_4u_6x, and xyu_3u_2x forms a $K_{3,3}$ in T; if $(u_6,u_4) \in A$, then the union of the three cycles $u_1u_4u_3u_2u_1$, $yu_6u_2u_1y$, and $xyu_6u_4u_3x$ forms a $K'_{3,3}$ in T. Next, consider the subcase when $(y,u_1) \in A$. If $(y,u_3) \in A$, then the union of the three cycles $u_1u_4u_3u_2u_1$, xyu_1u_4x, and xyu_3u_2x forms a $K_{3,3}$ in T; if $(u_4,u_6) \in A$, then the union of the three cycles $u_2u_4u_6u_3u_2$, $yu_1u_3u_2y$, and $xyu_1u_4u_6x$ forms a $K'_{3,3}$ in T. Suppose $\{(u_3,y),(u_6,u_4)\} \subseteq A$. Then a $K'_{3,3}$ is formed in T by the three cycles $u_1u_4u_3u_2u_1$, $yu_6u_4u_3y$, and $xyu_6u_2u_1x$.

Combining the above four cases, we establish (1). Therefore T is not Möbius-free, a contradiction. ∎

3 A Chain Theorem

In this section we show that every *i2s* tournament $T = (V,A)$ with $|V| \geq 5$ can be constructed from $\{F_1,F_2,F_3,F_4,F_5\}$ (see Figure 4) by repeatedly adding vertices such that all the intermediate tournaments are also *i2s*.

Theorem 3.1. *Let $T = (V,A)$ be an i2s tournament with $|V| \geq 3$. Then the following statements hold:*

(i) *If* $|V| = 3$*, then* $T = C_3$*; if* $|V| = 4$*, then* $T = F_0$*;*

(ii) *If* $|V| = 5$*, then* $T \in \{F_1, F_2, F_3\}$*;*

(iii) *If* $|V| = 6$*, then either* T *has a vertex* z *with* $T \backslash z \in \{F_1, F_2, F_3\}$ *or* $T \in \{F_4, F_5\}$*;*

(iv) *If* $|V| \geq 7$*, then* T *has a vertex* z *such that* $T \backslash z$ *remains to be i2s.*

We break the proofs of this theorem into a series of lemmas.

Lemma 3.2. *Let* $T = (V, A)$ *be a strong tournament. If* $|V| = 3$*, then* T *is* C_3*; if* $|V| = 4$*, then* T *is* F_0*. (So* T *is strong iff it is i2s when* $|V| = 3$ *or* 4*.)*

Proof. Since every strong tournament has a Hamilton cycle, it is clear that $T = C_3$ if $|V| = 3$ and $T = F_0$ if $|V| = 4$. Note that both C_3 and F_0 are *i2s*, so T is strong iff it is *i2s* when $|V| = 3$ or 4. ∎

Lemma 3.3. *Let* $T = (V, A)$ *be an i2s tournament. If* $|V| = 5$*, then* $T \in \{F_1, F_2, F_3\}$*.*

Proof. If $T \backslash u$ is strong for each $u \in V$, then both the in-degree and out-degree of each vertex equal two, and hence T is isomorphic to F_1.

So we assume that $T \backslash u$ has a trivial dicut (X, Y) for some $u \in V$. Since each F_i is isomorphic to its reverse for $i = 1, 2, 3$, replacing T by its reverse if necessary, we may assume that $|X| = 1$ and $|Y| = 3$. Let $X = \{x\}$ and $Y = \{y_1, y_2, y_3\}$. Since $T \backslash u$ is internally strong, Y induces a C_3. Since T is strong, $(u, x) \in A$, and u has at most two out-neighbors in Y. If u has exactly two out-neighbors in Y, say y_1 and y_2 (by symmetry), then $(\{u, x\}, \{y_1, y_2\})$ would be a nontrivial dicut of $T \backslash y_3$, a contradiction. So u has at most one out-neighbor in Y. If u has no out-neighbors in Y, then all arcs between Y and u are directed to u, so T is isomorphic to F_2. If u has only one out-neighbor in Y, then T is isomorphic to F_3.

Combining the above observations, we conclude that $T \in \{F_1, F_2, F_3\}$. ∎

The following lemma strengthens a classical theorem, asserting that every strong tournament contains a Hamilton cycle.

Lemma 3.4. *Let* $T = (V, A)$ *be a strong tournament and let* x *and* y *be two distinct vertices of* T*. Then* T *has a third vertex* z *such that* $T \backslash z$ *is still strong, unless* T *has a Hamilton path between* x *and* y *such that the remaining arcs are all backward.*

Proof. Since T is strong, it has a Hamilton cycle C. Let us first consider the case when

(1) T has a strong subgraph S containing both x and y with $|V(S)| < |V|$.

For notational simplicity, we assume that, subject to (1), S is chosen so that $|V(S)|$ is as large as possible. Then the vertices of S are consecutive on C. Let $P = C \backslash V(S)$. If P has only one vertex, then we are done. So we assume that P has two or more vertices. Let s and t be the initial and terminal vertices of P, respectively. Using the maximality assumption on S, we see that $\{(v, s), (t, v)\} \subseteq A$ for any vertex v in S. We claim that P contains no vertex other than s and t, for otherwise, let z be an internal vertex of P and let v be a vertex in S. Then either $S \cup C[s^-, z] \cup \{(z, v)\}$ or $S \cup C[z, t^+] \cup \{(v, z)\}$ would be a strong subgraph of T properly containing S; this contradiction to (1) justifies the claim. Since $\{(v, s), (t, v)\} \subseteq A$ for all vertices v in S, we deduce that $T \backslash z$ is strong for any vertex z in $S \backslash \{x, y\}$.

Next, let us consider the case when (1) does not occur. Renaming x and y if necessary, we may assume that $(x, y) \in A$. From the hypothesis of the present case, we deduce that (x, y) is an arc on C, $\{(x, y^+), (x^-, y)\} \subseteq A$, and $\{(x, v), (v, y)\} \subseteq A$ for any $v \in V \backslash \{x, y, x^-, y^+\}$. Thus $C \backslash (x, y)$ is a Hamilton path from y to x such that the remaining arcs are all backward. ∎

Corollary 3.5. *Let $T = (V, A)$ be a strong tournament with $|V| \geq 4$ and let x be a vertex in T. Then there exists a vertex $z \neq x$ such that $T \backslash z$ is strong.*

Proof. Let y be a vertex of T with $y \neq x$. By Lemma 2.3,
• either T has a vertex $z \neq x, y$ such that $T \backslash z$ is strong
• or T has a Hamilton path between x and y such that the remaining arcs are all backward.
In the former case z is a desired vertex, and in the latter case y is as desired. ∎

A digraph is called *trivial* if it contains only one vertex. The following lemma on strong partitions of tournaments (see Section 1) is straightforward, so we omit its proof here.

Lemma 3.6. *Let $T = (V, A)$ be an internally strong tournament and let $(A_1, A_2, \ldots, A_p)$ be the strong partition of T. If $|V| \geq 3$, then one of the following statements holds:*

(i) $p = 1$; A_1 is nontrivial;

(ii) $p = 2$; exactly one of A_1 and A_2 is nontrivial;

(iii) $p = 3$; both A_1 and A_3 are trivial.

The lemma below follows instantly from the preceding one.

Lemma 3.7. *Let $T = (V, A)$ be an i2s tournament, let x be a vertex in T, and let $(A_1, A_2, \ldots, A_p)$ be the strong partition of $T \backslash x$. Then $1 \leq p \leq 3$. (The value of p is called the* type *of x in T).*

For convenience, we shall not distinguish each A_i from its vertex set $V(A_i)$ in subsequent proofs, if there is no risk of confusion. Thus $|A_i| = |V(A_i)|$.

The following two lemmas guarantee the existence of a vertex z in an *i*2*s* tournament T with at least six vertices such that $T \backslash z$ remains to be *i*2*s*.

Lemma 3.8. *Let $T = (V, A)$ be an i2s tournament with $|V| \geq 6$. If T contains a vertex x of type 3 (see Lemma 3.7), then it contains a vertex z such that $T \backslash z$ remains to be i2s.*

Proof. Let (A_1, A_2, A_3) be the strong partition of $T \backslash x$. Since x is of type 3, $|A_1| = |A_3| = 1$ by Lemma 3.6. So $|A_2| \geq 3$. Let z_i be the only vertex in A_i for $i = 1, 3$. Since T is *i*2*s*, both (x, z_1) and (z_3, x) are arcs in T. Furthermore, x has at least one in-neighbor x_1 and at least one out-neighbor x_2 in A_2. If there exists $z \in A_2 \backslash \{x_1, x_2\}$ such that $A_2 \backslash z$ is strong, then $T \backslash z$ is *i*2*s*. Otherwise, by Lemma 3.4, A_2 has a Hamilton path between x_1 and x_2 such that the remaining arcs of A_2 are all backward. Let $z = x_2$ if x_1 is the only in-neighbor of x in A_2 and let $z = x_1$ otherwise. Then $A_2 \backslash z$ is strong and has at least one in-neighbor and at least one out-neighbor of x. Therefore $T \backslash z$ is *i*2*s*. ∎

Lemma 3.9. *Let $T = (V, A)$ be an i2s tournament with $|V| \geq 6$ and $T \notin \{F_4, F_5\}$ (see Figure 4). Then T contains a vertex z such that $T \backslash z$ remains to be i2s.*

Proof. We proceed by contradiction. By a *triple* $(T; x, y)$ we mean an *i2s* tournament $T = (V, A)$ with $|V| \geq 6$ and $T \notin \{F_4, F_5\}$ such that $T\backslash z$ is not *i2s* for any vertex z, together with two distinguished vertices x and y in T. Choose a triple $(T; x, y)$ such that

(1) $T\backslash x$ is strong while $T\backslash\{x, y\}$ is not internally strong;

(2) subject to (1), letting $(A_1, A_2, \ldots, A_p)$ be the strong partition of $T\backslash\{x, y\}$, A_1 contains an out-neighbor x' of x; and

(3) subject to (1) and (2), the tuple $(|A_1|, |A_2|, ..., |A_p|)$ is minimized lexicographically.

Let us show that such a triple is available. By Corollary 3.5, there exists a triple $(T; x, y)$ satisfying (1). To verify the existence of a triple $(T; x, y)$ satisfying both (1) and (2), note that if x has no out-neighbor in A_1, then it must have an in-neighbor in A_p, for otherwise, y would be of type 3, and hence $T\backslash z$ would be *i2s* for some vertex z by Lemma 3.8, a contradiction. Since each of F_4 and F_5 is isomorphic to its reverse, replacing T by T^* if necessary, we see that a triple $(T; x, y)$ with properties (1) and (2) (and hence the desired one) exists.

Let us make some simple observations about the triple $(T; x, y)$. Since $|V| \geq 6$, by (1) we have

(4) $p \geq 2$, and y has an out-neighbor y' in A_1 and an in-neighbor y'' in A_p.

(5) If $p = 2$, then x has an in-neighbor in A_p.

Otherwise, since $|V| \geq 6$ and $T\backslash y$ is internally strong, $|A_2| = 1$ and $|A_1| \geq 3$, which implies that $T\backslash\{x, y\}$ is internally strong, this contradiction justifies (5).

Once again, since $T\backslash y$ is internally strong, the statement below follows instantly from Lemma 3.6.

(6) If $p \geq 3$ and x has no in-neighbor in A_p, then $|A_p| = 1$ and x has an in-neighbor in A_{p-1}.

Since A_i is strong, either $|A_i| = 1$ or $|A_i| \geq 3$ for $1 \leq i \leq p$. Let $A_i = \{a_i\}$ for each i with $|A_i| = 1$ hereafter. We divide the remainder of the proof into a series of claims.

Claim 1. $|A_1| = 1$.

Assume the contrary: $|A_1| \geq 3$. Replacing x' (resp. y') by a second out-neighbor of x (resp. y) in A_1 if necessary, we may assume that $x' \neq y'$, for otherwise, $x' = y'$ is the unique out-neighbor of both x and y in A_1. Since $T\backslash x'$ is internally strong and $A_1\backslash x'$ has no incoming arcs, $|A_1\backslash x'| \leq 1$ and thus $|A_1| \leq 2$, contradicting the assumption on $|A_1|$. By Lemma 3.4, one of (7), (8), and (9) holds:

(7) $A_1\backslash\{x', y'\}$ has a vertex z such that $A_1\backslash z$ is strong.

(8) $|A_1| = 3$. Renaming the vertices in A_1 as x', y', z if necessary, we assume that both (x, x') and (y, y') are arcs in T, and that if three vertices in A_1 are all out-neighbors of x, then (y', x') is an arc in T; otherwise, if three vertices in A_1 are all out-neighbors of y, then (x', y') is an arc in T.

(9) $|A_1| \geq 4$ and A_1 has a Hamilton path P between x' and y' such that the remaining arcs in A_1 are all backward. Furthermore, we may assume that both (v, x) and (v, y) are arcs in T for any $v \in A_1\backslash\{x', y'\}$, for otherwise, (7) holds true by replacing x' or y' (which is z) with v.

Let z be as specified in (7) or (8), whichever holds, and let z be the terminal vertex of $P\backslash\{x', y'\}$ if (9) holds. Clearly, $T\backslash z$ is strong. We propose to prove that $T\backslash z$ is *i2s*, which amounts to saying that

(10) $T\backslash\{w, z\}$ is internally strong for each $w \in V\backslash z$.

From (5), (6), and the definition of z, we see that (10) holds trivially for any $w \in \cup_{i=2}^{p-1} A_i \cup \{x, y\}$. It remains to consider the following two cases.

Case 1.1. $w \in A_p$.

Depending on whether $w = y''$ (see (4)), we distinguish between two subcases.

• $w \neq y''$. In this subcase, $|A_p| \geq 2$. Thus x has at least one in-neighbor in A_p by (5) and (6). Let $(B_1, B_2, \ldots, B_q)$ be the strong partition of $A_p \backslash w$, let r be the largest subscript such that B_r contains an in-neighbor of x or y, and let $B = \cup_{i=r+1}^{q} B_i$. Since B has no outgoing arcs in $T \backslash w$ (which is internally strong), $|B| \leq 1$. Let us show that $T \backslash \{w, z\}$ is internally strong, for otherwise, x is a source and x' is a near-source of $T \backslash \{w, z\}$; in particular, $(x', y') \in A$. From the descriptions of (7)-(9), we deduce that $|A_1| = 3$ and $(z, x) \in A$. Consider the triple $(T; z, w)$. Let $(A'_1, A'_2, \ldots, A'_t)$ be the strong partition of $T \backslash \{z, w\}$. Then $A'_1 = \{x\}$. Since $T \backslash z$ is strong while $T \backslash \{z, w\}$ is not internally strong, and $|A'_1| < |A_1|$, the existence of the triple $(T; z, w)$ contradicts the minimality assumption on $(|A_1|, |A_2|, ..., |A_p|)$ in the choice of $(T; x, y)$ (see (1)-(3)).

• $w = y''$. In this subcase, we may assume that y'' is the only in-neighbor of y in A_p, for otherwise, replacing y'' by a second in-neighbor of y in A_p, we reduce the present subcase to the preceding one. If x has an in-neighbor in $A_p \backslash w$, then $T \backslash y$ is strong. Interchanging the roles of x and y, we reduce the present subcase to the preceding one as well. Thus we further assume that $A_p \backslash w$ contains no in-neighbors of x. Since $T \backslash w$ is internally strong, $A_p = \{w\}$. If w is an in-neighbor of x, then the existence of the triple $(T^*; x, y)$ contradicts the minimality assumption on $(|A_1|, |A_2|, ..., |A_p|)$ in the choice of $(T; x, y)$ (see (1)-(3)). So w is an out-neighbor of x. By (5) and (6), A_{p-1} contains an in-neighbor of x. Let us show that $T \backslash \{w, z\}$ is internally strong, for otherwise, y is a source and y' is a near-source of $T \backslash \{w, z\}$; in particular, both (y', x') and (y', x) are arcs in T. From the descriptions of (7)-(9), we deduce that $|A_1| = 3$ and $(z, y) \in A$. Thus the existence of the triple $(T; z, w)$ contradicts the minimality assumption on $(|A_1|, |A_2|, ..., |A_p|)$ in the choice of $(T; x, y)$ (see (1)-(3)).

Case 1.2. $w \in A_1 \backslash z$.

Depending on whether (7), (8), or (9) holds, we distinguish between two subcases.

• (7) holds. In this subcase, let $(B_1, B_2, ..., B_q)$ be the strong partition of $A_1 \backslash \{w, z\}$, let r be the smallest subscript such that B_r contains an out-neighbor of x or y, and let $B = \cup_{i=1}^{r-1} B_i$. Then $(T \backslash \{w, z\}) \backslash B$ is strong. If $|B| \leq 1$, then $T \backslash \{w, z\}$ is internally strong. So we assume that $|B| \geq 2$. Since $T \backslash w$ is internally strong and since B has no incoming arcs in $T \backslash \{w, z\}$, $T \backslash w$ contains at least one arc from z to B. Thus the triple $(T; z, w)$ is a better choice than $(T; x, y)$ (see (1)-(3)) because $|B| < |A_1|$, a contradiction.

• (8) or (9) holds. In this subcase, if $w = x'$, then $T \backslash \{w, x, z\}$ is strong, so $T \backslash \{w, z\}$ is internally strong. If $w = y'$ and x has an in-neighbor contained in A_p, then $T \backslash \{w, y, z\}$ is strong, so $T \backslash \{w, z\}$ is also internally strong; if $w = y'$ and x has no in-neighbor contained in A_p, then x has an in-neighbor x'' contained in A_{p-1} by (5) and (6), and y has an out-neighbor contained in $\{x\} \cup (A_1 \backslash y') \cup (\cup_{i=2}^{p-1} A_i)$ (as $T \backslash y'$ is internally strong), and hence $T \backslash \{w, z\}$ is strong. Suppose $w \notin \{x', y'\}$. In view of (5) and (6), it is clear that $T \backslash \{w, z\}$ is strong.

Combining the above two cases, we establish (10) for all $w \in A_p \cup (A_1 \backslash z)$ and hence for all $w \in V \backslash z$. So $T \backslash z$ is *i2s*; this contradiction justifies Claim 1.

Claim 2. $|A_2| = 1$.

Assume the contrary: $|A_2| \geq 3$. Since $T \backslash a_1$ is internally strong, A_2 contains a vertex a_2

which is an out-neighbor of x or y. If $|A_2| \geq 4$, let z be a vertex in $A_2 \backslash a_2$ such that $A_2 \backslash z$ is strong (see Corollary 3.5); if $|A_2| = 3$, let z be the vertex in A_2 with $(z, a_2) \in A$. Since T is $i2s$ and since x has an in-neighbor in $A_{p-1} \cup A_p$ by (5) and (6), $T \backslash z$ is strong. We propose to show that $T \backslash z$ is $i2s$, which amounts to saying that

(11) $T \backslash \{w, z\}$ is internally strong for each $w \in V \backslash z$.

From (5), (6), and the definition of z, we see that (11) holds trivially for any $w \in \{x, y\} \cup (A_2 \backslash z) \cup (\cup_{i=3}^{p-1} A_i)$. It remains to consider the following two cases.

Case 2.1. $w = a_1$.

In this case, if a_2 is an out-neighbor of y, then $T \backslash \{a_1, x, z\}$ is strong and hence $T \backslash \{a_1, z\}$ is internally strong. So we assume that a_2 is an out-neighbor of x. If x has an in-neighbor in A_p, then $T \backslash \{a_1, y, z\}$ is strong and hence $T \backslash \{a_1, z\}$ is internally strong. So we further assume that x has no in-neighbor in A_p. Then $|A_p| = 1$ and x has an in-neighbor in A_{p-1} by (5) and (6). We claim that y has an out-neighbor in $\{x\} \cup (A_2 \backslash z) \cup (\cup_{i=3}^{p-1} A_i)$, for otherwise, let $B = \{y, y''\}$ and $\bar{B} = V \backslash \{a_1, y, y'', z\}$. Then $(\bar{B}, B)$ is a nontrivial dicut in $T \backslash \{a_1, z\}$, so $T \backslash \{a_1, z\}$ is not internally strong. Therefore the existence of the triple $(T^*; z, a_1)$ contradicts the minimality assumption on $(|A_1|, |A_2|, ..., |A_p|)$ in the choice of $(T; x, y)$ (see (1)-(3)). It follows instantly from the claim that $T \backslash \{a_1, z\}$ is strong.

Case 2.2. $w \in A_p$.

Depending on whether $w = y''$, we distinguish between two subcases.

• $w \neq y''$. In this subcase, $|A_p| \geq 2$. So x has an in-neighbor in A_p by (5) and (6). Let $(B_1, B_2, ..., B_q)$ be the strong partition of $A_p \backslash w$, let r be the largest subscript such that B_r contains an in-neighbor of x or y, and let $B = \cup_{i=r+1}^{q} B_i$. Since B has no outgoing arcs in $T \backslash w$ (which is internally strong), $|B| \leq 1$. Clearly, $(T \backslash \{w, z\}) \backslash B$ is strong, so $T \backslash \{w, z\}$ is internally strong.

• $w = y''$. In this subcase, we may assume that y'' is the only in-neighbor of y in A_p, for otherwise, replacing y'' by a second in-neighbor of y in A_p, we reduce the present subcase to the preceding one. If x has an in-neighbor in $A_p \backslash w$, then $T \backslash y$ is strong. Interchanging the roles of x and y, we reduce the present subcase to the preceding one as well. So we assume that $A_p \backslash w$ contains no in-neighbors of x. Since $T \backslash w$ is internally strong, $|A_p \backslash w| \leq 1$, so $|A_p| \leq 2$. Since A_p is strong, we have $A_p = \{w\}$. If w is an out-neighbor of x, then x has an in-neighbor in A_{p-1} by (5) and (6). Thus $T \backslash \{w, y, z\}$ is strong and hence $T \backslash \{w, z\}$ is internally strong. So we further assume that w is an in-neighbor of x. If A_{p-1} contains an in-neighbor of x or y, then $T \backslash \{w, z\}$ is also internally strong; if A_{p-1} contains no in-neighbor of x or y, then $(\cup_{i=1}^{p-2} A_i \cup \{x, y\}, A_{p-1})$ is a dicut in $T \backslash w$. Since $T \backslash w$ is internally strong, $|A_{p-1}| = 1$. Thus the existence of the triple $(T^*; x, y)$ contradicts the minimality assumption on $(|A_1|, |A_2|, ..., |A_p|)$ in the choice of $(T; x, y)$ (see (1)-(3)).

Combining the above two cases, we establish (11) for all $w \in \{a_1\} \cup A_p$ and hence for all $w \in V \backslash z$. So $T \backslash z$ is $i2s$; this contradiction justifies Claim 2.

Claim 3. At least one of (x, a_2) and (y, a_2) is an arc in T.

Assume the contrary: both (a_2, x) and (a_2, y) are arcs in T. By (5) and (6), x has an in-neighbor in $A_{p-1} \cup A_p$, so $T \backslash a_2$ is strong. We propose to show that $T \backslash a_2$ is $i2s$, which amounts to saying that

(12) $T \backslash \{w, a_2\}$ is internally strong for each $w \in V \backslash a_2$.

Clearly, (12) holds for $w \in \cup_{i=3}^{p-1} A_i \cup \{x, y\}$. It remains to consider the following two cases.

Case 3.1. $w = a_1$.

Since $T \backslash a_1$ is internally strong, A_3 contains an out-neighbor of x or y. If A_3 contains an out-neighbor of y, then $T \backslash \{a_1, x, a_2\}$ is strong, and hence $T \backslash \{a_1, a_2\}$ is internally strong. So we assume that A_3 contains an out-neighbor of x. If A_p contains an in-neighbor of x, then $T \backslash \{a_1, y, a_2\}$ is strong, so $T \backslash \{a_1, a_2\}$ is internally strong. If A_p contains no in-neighbor of x, then $|A_p| = 1$ and x has an in-neighbor in A_{p-1} by (5) and (6). Thus $\{x\} \cup A_3 \cup \ldots \cup A_{p-1}$ induces a strong sub-tournament. Since $T \backslash a_1$ is internally strong, y has an out-neighbor in $\{x\} \cup A_4 \cup \ldots \cup A_{p-1}$. It follows that $T \backslash \{a_1, a_2\}$ is strong.

Case 3.2. $w \in A_p$.

Depending on whether $w = y''$, we distinguish between two subcases.

• $w \neq y''$. In this subcase, the argument is exactly the same as the one employed in Case 2.2 when $w \neq y''$.

• $w = y''$. In this subcase, we may assume that $A_p = \{w\}$ and w is an in-neighbor of x (see the proof in Case 2.2 when $w = y''$). If A_{p-1} contains an in-neighbor of x or y, then $T \backslash \{w, a_2\}$ is internally strong; otherwise, $(\cup_{i=1}^{p-2} A_i \cup \{x, y\}, A_{p-1})$ is a dicut in $T \backslash w$, so $|A_{p-1}| = 1$. If $p = 4$, then T is isomorphic to F_4 (see its labeling in Figure 3) under the mapping

$$(a_1, a_2, a_3, a_4, \{x, y\}) \to (v_5, v_6, v_2, v_3, \{v_1, v_4\}),$$

contradicting the hypothesis. So $p \geq 5$. Thus A_{p-2} contains an in-neighbor of x or y, for otherwise $(\cup_{i=1}^{p-3} A_i \cup \{x, y\}, A_{p-1} \cup A_{p-2})$ would a nontrivial dicut in $T \backslash w$, contradicting the fact that $T \backslash w$ is internally strong. It follows that $T \backslash \{w, a_2\}$ is internally strong, in which a_{p-1} is a sink and possible y is a source.

Combining the above two cases, we establish (12) for all $w \in \{a_1\} \cup A_p$ and hence for all $w \in V \backslash a_2$. So $T \backslash a_2$ is $i2s$; this contradiction justifies Claim 3.

Claim 4. Let k be the largest subscript such that A_k contains an in-neighbor of x. Then $k = 3$.

Assume the contrary: $k \neq 3$. Since $|V| \geq 6$ and $|A_1| = |A_2| = 1$ by Claims 1 and 2, we have $p \geq 3$. If $p = 3$, then $|A_p| \geq 2$, so x has an in-neighbor in A_p by (5) and (6) and hence $k = 3$, this contradiction implies that $p \geq 4$. We propose to show that

(13) $T \backslash z$ is $i2s$ for some vertex z of T.

Depending on the size of A_p and value of p, we distinguish among three cases.

Case 4.1. $|A_p| \geq 3$.

In this case, x has an in-neighbor x'' in A_p by (5) and (6). Let z be an arbitrary vertex in A_3. Clearly, $T \backslash z$ is strong. We aim to show that (13) holds for this z. By Claim 3, at least one of (x, a_2) and (y, a_2) is in T. Thus $T \backslash \{w, z\}$ is internally strong for $w \in \cup_{i=3}^{p-1} A_i \cup \{x, y, a_1, a_2\}$. To establish this statement for $w \in A_p$, we consider two subcases.

• $w \neq y''$. In this subcase, the argument is exactly the same as that employed in Case 2.2 when $w \neq y''$.

• $w = y''$. In this subcase, we may assume that w is the only in-neighbor of y in A_p. Observe that x has an in-neighbor in $A_p \backslash w$, for otherwise, since $T \backslash w$ is internally strong, $|A_p \backslash w| \leq 1$, so $|A_p| \leq 2$, a contradiction. Interchanging the roles of x and y, we reduce the present subcase to the preceding one.

Case 4.2. $|A_p| = 1$ and $p \geq 5$.

In this case, x has an in-neighbor in $A_{p-1} \cup A_p$ by (5) and (6). To prove (13), we proceed by considering two subcases.

• A_3 contains an out-neighbor of y. In this subcase, let us show that $T\backslash a_2$ is $i2s$. Clearly, $T\backslash a_2$ is strong, and $T\backslash\{a_2, w\}$ is internally strong for any $w \in \cup_{i=3}^{p-1} A_i \cup \{a_1, x, y\}$. If A_{p-1} contains an in-neighbor of x or y, then $T\backslash\{a_2, a_p\}$ is internally strong. So we assume that A_{p-1} contains no in-neighbor of x or y. Note that $(\cup_{i=1}^{p-2} A_i \cup \{x, y\}, A_{p-1})$ is a dicut in $T\backslash a_p$. Since $T\backslash a_p$ is internally strong, $|A_{p-1}| = 1$. Since $p \geq 5$, A_{p-2} contains an in-neighbor of x or y, for otherwise $(\cup_{i=1}^{p-3} A_i \cup \{x, y\}, A_{p-1} \cup A_{p-2})$ would be a nontrivial dicut in $T\backslash a_p$, a contradiction. It follows that $T\backslash\{a_2, a_p\}$ is internally strong, in which a_{p-1} is a sink and possible one of x and y is a source.

• All vertices in A_3 are in-neighbors of y. In this subcase, let z be an arbitrary vertex in A_3; let us show that $T\backslash z$ is $i2s$. Clearly, $T\backslash z$ is strong. Observe that if a_p is an out-neighbor of x, then $\cup_{i=4}^{p-1} A_i \cup \{a_2, x\}$ contains an out-neighbor of y, for otherwise $(\cup_{i=3}^{p-1} A_i \cup \{a_2, x\}, \{y, a_p\})$ would be an nontrivial dicut in $T\backslash a_1$, a contradiction. It follows that $T\backslash\{w, z\}$ is internally strong for any $w \in \cup_{i=3}^{p-1} A_i \cup \{x, y, a_2\}$ no matter whether (x, a_p) is an arc in T. Let us make two more observations.

(14) $T\backslash\{a_1, z\}$ is internally strong. To justify this, note that if (y, a_2) is an arc in T, then $T\backslash\{a_1, x, z\}$ is strong, so $T\backslash\{a_1, z\}$ is internally strong. Thus we may assume that (a_2, y) is an arc in T. By Claim 3, (x, a_2) is also in T. If a_p is an in-neighbor of x, then $T\backslash\{a_1, y, z\}$ is strong and hence $T\backslash\{a_1, z\}$ is internally strong; if a_p is an out-neighbor of x, then x contains an in-neighbor in A_{p-1} by (6), and $\cup_{i=4}^{p-1} A_i \cup \{a_2, x\}$ contains an out-neighbor of y as observed in the preceding paragraph. Thus (14) follows.

(15) $T\backslash\{a_p, z\}$ is internally strong. To justify this, note that if A_{p-1} contains an in-neighbor of x or y, then $T\backslash\{a_p, z\}$ is internally strong. If A_{p-1} contains no in-neighbor of x or y, then $(\cup_{i=1}^{p-2} A_i \cup \{x, y\}, A_{p-1})$ is a dicut in $T\backslash a_p$, which implies that $|A_{p-1}| = 1$. Since $p \geq 5$, A_{p-2} contains an in-neighbor of x or y, for otherwise $(\cup_{i=1}^{p-3} A_i \cup \{x, y\}, A_{p-1} \cup A_{p-2})$ is a nontrivial dicut in $T\backslash a_p$, a contradiction. Thus (15) holds.

Case 4.3. $|A_p| = 1$ and $p = 4$.

In this case, (a_4, x) is an arc in T by (5), (6), and the assumption $k \neq 3$. Depending on the size of A_3, we consider two subcases.

• $|A_3| \geq 3$. In this subcase, A_3 contains a vertex a_3 which is an in-neighbor of x or y, because $T\backslash a_4$ is internally strong. If $|A_3| = 3$, let z be the vertex such that $(a_3, z) \in A_3$; if $|A_3| \geq 4$, Corollary 3.5 guarantees the existence of a vertex $z \in A_3\backslash a_3$ such that $A_3\backslash z$ is strong. Let us show that $T\backslash z$ is $i2s$. Clearly, $T\backslash z$ is strong, and $T\backslash\{w, z\}$ is internally strong for any $w \neq a_1$. If (y, a_2) is an arc in T, then $T\backslash\{a_1, x, z\}$ is strong and hence $T\backslash\{a_1, z\}$ is internally strong. If (a_2, y) is an arc in T, then so is (x, a_2) by Claim 3. Since $T\backslash\{a_1, y, z\}$ is strong, $T\backslash\{a_1, z\}$ is internally strong.

• $|A_3| = 1$. In this subcase, if exactly one of (y, a_3) and (x, a_3) is an arc in T, then $T\backslash a_2$ is $i2s$. So we assume that either both (a_3, y) and (a_3, x) are in T or both (y, a_3) and (x, a_3) are in T. If exactly one of (x, a_2) and (y, a_2) is in T, then $T\backslash a_3$ is $i2s$. So we further assume that both (x, a_2) and (y, a_2) are in T by Claim 3. Thus both (a_3, y) and (a_3, x) are in T, for otherwise, $(\{x, y\}, \{a_1, a_2, a_3\})$ would be a dicut in $T\backslash a_4$, a contradiction. Now we can see that

T is isomorphic to F_5 (see its labeling in Figure 3) under the mapping

$$(a_1, a_2, a_3, a_4, \{x, y\}) \rightarrow (v_5, v_2, v_6, v_3, \{v_1, v_4\}),$$

contradicting the hypothesis of the present lemma.

Combining the above three cases, we have proved (13); this contradiction justifies Claim 4.

Claim 5. $p = 4$.

Assume the contrary: $p \neq 4$. Since $|V| \geq 6$ and $|A_i| = 1$ for $i = 1, 2$, we have $p \geq 3$. By Claim 4, (5), and (6), we also have $p \leq 4$. So $p = 3 = k$. Let x'' be an in-neighbor of x in A_3. Replacing x'' (resp. y'') by a second in-neighbor of x (resp. y) in A_3 if necessary, we may assume that $x'' \neq y''$, for otherwise, x'' is the only in-neighbor of x and y in A_3. Since $T \backslash x''$ is internally strong, $|A_3 \backslash x''| \leq 1$, so $|A_3| \leq 2$ and hence $|A_3| = 1$, contradicting the hypothesis that $|V| \geq 6$. If all vertices in A_3 are in-neighbors of both x and y, then $T \backslash z$ is *i2s* for any $z \in A_3$ by Claim 3.

So we assume that A_3 contains an out-neighbor of x or y. We propose to show that $T \backslash a_2$ is *i2s*. Clearly, $T \backslash a_2$, $T \backslash \{x, a_2\}$, and $T \backslash \{y, a_2\}$ are all strong. By the hypothesis of the present case, $A_3 \cup \{x\}$ or $A_3 \cup \{y\}$ induces a strong sub-tournament of T, so $T \backslash \{a_1, a_2\}$ is internally strong. Let w be an arbitrary vertex in A_3. Since $x'' \neq y''$, symmetry allows us to assume that $w \neq x''$. If $A_3 \backslash w$ is strong, then $T \backslash \{w, a_2\}$ is internally strong; otherwise, let $(B_1, B_2, ..., B_q)$ be the strong partition of $A_3 \backslash w$. Then $q \geq 2$. Let r be the largest subscript such that B_r contains an in-neighbor of x or y and let $B = \cup_{i=r+1}^{q} B_i$. Since $(\cup_{i=1}^{r} B_i \cup \{a_1, a_2, x, y\}, B)$ is a dicut in $T \backslash w$, we have $|B| \leq 1$. If $T \backslash (B \cup \{a_2, w\}) = \cup_{i=1}^{r} B_i \cup \{a_1, x, y\}$ is strong, then $T \backslash \{w, a_2\}$ is internally strong; otherwise, w is the only in-neighbor of y in $A_3 \cup \{x\}$. Since $\cup_{i=1}^{r} B_i \cup \{a_1, x\}$ is strong, $T \backslash \{w, a_2\}$ is also internally strong.

Combining the above observations, we see that $T \backslash z$ is *i2s* for some vertex z of T; this contradiction justifies Claim 5.

From (6) and Claims 4 and 5, we deduce that $|A_4| = 1$ and (x, a_4) is an arc in T. Depending on the size of A_3, we distinguish between two cases.

• $|A_3| \geq 3$. In this case, let x'' be an in-neighbor of x in A_3 (see Claim 4). If $|A_3| = 3$, let z be the vertex in A_3 such that (x'', z) is an arc; otherwise, let z be a vertex in $A_3 \backslash x''$ such that $A_3 \backslash z$ is strong (see Corollary 3.5). Clearly, $T \backslash z$ is *i2s*. Let us show it is actually *i2s*; that is, $T \backslash \{w, z\}$ is internally strong for any $w \in V \backslash z$. This statement holds trivially when $w \neq a_1$. So we assume that $w = a_1$. If (y, a_2) is an arc in T, then $T \backslash \{a_1, z\}$ is strong; otherwise, by Claim 3, (x, a_2) is an arc in T. So $(A_3 \backslash z) \cup \{a_2, x\}$ induces a strong sub-tournament of T. Since $T \backslash a_1$ is internally, $(A_3 \backslash z) \cup \{a_2, x\}$ contains an out-neighbor of y. Thus $T \backslash \{a_1, z\}$ is strong.

• $|A_3| = 1$. In this subcase, (a_3, x) is an arc in T by Claim 4. If (y, a_3) or (y, x) is an arc in T, then $T \backslash a_2$ is *i2s*. So we assume that both (a_3, y) and (x, y) are arcs in T. Since $T \backslash a_1$ is internally strong, (y, a_2) is an arc in T. Note that (a_2, x) is an arc of T, for otherwise T would be isomorphic to F_4 (see its labeling in Figure 3) under the mapping

$$(a_1, a_2, a_3, a_4, x, y) \rightarrow (v_4, v_1, v_5, v_2, v_6, v_3),$$

contradicting the hypothesis of the present lemma. It follows that $T \backslash a_3$ is *i2s*.

Combining the above two cases, we conclude that T contains a vertex z such that $T \backslash z$ remains to be *i2s*; this contradiction proves the lemma. ∎

With the above preparations, we can establish the main result of this section now.

Proof of Theorem 3.1. The desired statements follow instantly from Lemmas 3.2, 3.3 and 3.9. ∎

4 Structural Description

In this section we show that every $i2s$ Möbius-free tournament comes from the list $\mathcal{T}_0$ of nine sporadic tournaments, and every strong Möbius-free tournament different from F_1 and G_1 can be obtained by repeatedly taking 1-sums starting from the seven tournaments in $\mathcal{T}_1$.

Proof of Theorem 1.2. Our proof is based on the chain theorem (Theorem 3.1), so it consists in handling tournaments with at most seven vertices; in principle it can be carried out by computer, and we indeed have such a proof. Nevertheless, the proof given below is computer-free.

By Lemmas 3.2 and 3.3, we may assume that $|V| \geq 6$. For convenience, we say that an $i2s$ Möbius-free tournament T' is an *extension* of T if $T' \backslash v$ is isomorphic to T for some vertex v of T'.

Claim 1. G_1 is the only extension of F_1.

To justify this, let T be an extension of F_1, let v_6 be a vertex of T such that $T \backslash v_6$ is isomorphic to F_1, and label the vertices of $T \backslash v_6$ as in Figure 3 for F_1. We propose to show that T is isomorphic to G_1. Since the in-degree and out-degree of each vertex in F_1 are two, F_1 enjoys a high degree of symmetry in which all vertices behave likewise.

Since T is strong, symmetry allows us to assume that v_1 is an in-neighbor of v_6. Then at most one of (v_6, v_2) and (v_6, v_5) is in T, for otherwise, the union of the five cycles $v_1v_6v_5v_1$, $v_1v_3v_5v_1$, $v_2v_3v_5v_2$, $v_2v_4v_5v_2$, and $v_1v_6v_2v_4v_1$ would form an M_5^* in T, a contradiction. Thus we may proceed by considering the following three cases.

• Both (v_2, v_6) and (v_5, v_6) are in T. In this case, since T is strong, at most one of (v_3, v_6) and (v_4, v_6) is contained in T. If both (v_6, v_3) and (v_4, v_6) are in T, then the five cycles $v_1v_2v_4v_1$, $v_5v_2v_4v_5$, $v_1v_3v_4v_1$, $v_6v_3v_4v_6$, and $v_2v_6v_3v_5v_2$ would form an M_5. Similarly, if both (v_3, v_6) and (v_6, v_4) are in T, then the five cycles $v_1v_3v_5v_1$, $v_2v_3v_5v_2$, $v_2v_4v_5v_2$, $v_6v_4v_5v_6$, and $v_1v_3v_6v_4v_1$ would form an M_5 in T as well. So both (v_6, v_3) and (v_6, v_4) are in T. Thus T is isomorphic to G_1, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_2, v_6, v_4, v_5, v_1, v_3)$ in G_1 as labeled in Figure 4.

• Both (v_6, v_2) and (v_5, v_6) are in T. In this case, (v_6, v_3) is in T, for otherwise, the five cycles $v_1v_3v_4v_1$, $v_1v_3v_5v_1$, $v_2v_3v_5v_2$, $v_2v_3v_6v_2$, and $v_1v_6v_2v_4v_1$ would form an M_5, a contradiction. If (v_6, v_4) is in T, then T is isomorphic to G_1, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_2, v_3, v_4, v_5, v_1, v_6)$ in G_1 as labeled in Figure 3. If (v_4, v_6) is in T, then T is also isomorphic to G_1, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_6, v_4, v_5, v_1, v_2, v_3)$ in G_1 as labeled in Figure 4.

• Both (v_2, v_6) and (v_6, v_5) are in T. In this case, (v_6, v_4) is in T, for otherwise, the five cycles $v_1v_2v_4v_1$, $v_1v_3v_4v_1$, $v_1v_3v_5v_1$, $v_1v_6v_5v_1$, and $v_2v_4v_6v_5v_2$ would form an M_5, a contradiction. If (v_6, v_3) is in T, then T is isomorphic to G_1, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_1, v_2, v_3, v_4, v_5, v_6)$ in G_1 as labeled in Figure 3. If (v_3, v_6) is in T, then T is also isomorphic to G_1, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_1, v_2, v_6, v_4, v_5, v_3)$ in G_1 as labeled in Figure 4.

Combining the above observations, we see that G_1 is the only extension of F_1.

Claim 2. F_2 has no extension.

Assume the contrary: T is an extension of F_2 such that $T\backslash v_6$ is isomorphic to F_2 for some vertex v_6 of T. Let us label the vertices of $T\backslash v_6$ as in Figure 3 for F_2. Since T is $i2s$, v_6 has an in-neighbor in $\{v_1, v_3, v_4\}$, for otherwise, $(\{v_2, v_6\}, \{v_1, v_3, v_4\})$ would be a nontrivial dicut in $T\backslash v_5$, a contradiction. By symmetry, we may assume that (v_1, v_6) is an arc in T. Next, v_3 or v_4 is an out-neighbor of v_6, for otherwise $(\{v_1, v_3, v_4\}, \{v_5, v_6\})$ would be a nontrivial dicut in $T\backslash v_2$. Depending on the direction of the arc between v_6 and v_3, we consider two cases.

• (v_6, v_3) is in T. In this case, if (v_6, v_5) is an arc in T, then the union of the three cycles $v_1v_6v_3v_4v_1$, $v_2v_3v_4v_5v_2$, and $v_1v_6v_5v_2v_1$ is a $K_{3,3}$. So (v_5, v_6) is an arc in T. If (v_4, v_6) is an arc in T, then the union of the five cycles $v_2v_3v_5v_2$, $v_6v_3v_5v_6$, $v_6v_3v_4v_6$, $v_1v_3v_4v_1$, and $v_1v_5v_2v_4v_1$ would form an M_5^*; if (v_6, v_4) is an arc in T, then the union of the five cycles $v_2v_4v_5v_2$, $v_6v_4v_5v_6$, $v_6v_4v_1v_6$, $v_3v_4v_1v_3$, and $v_1v_3v_5v_2v_1$ would form an M_5^* as well. Thus we reach a contradiction in either subcase.

• (v_3, v_6) is in T. In this case, (v_6, v_4) is in T. If (v_6, v_5) is in T, then the union of the three cycles $v_1v_3v_6v_4v_1$, $v_2v_3v_6v_5v_2$, and $v_1v_5v_2v_4v_1$ would form a $K_{3,3}$. Thus (v_5, v_6) is in T. But then the union of the five cycles $v_2v_4v_5v_2$, $v_6v_4v_5v_6$, $v_6v_4v_1v_6$, $v_3v_4v_1v_3$, and $v_1v_3v_5v_2v_1$ would form an M_5^*, a contradiction.

Combining the above observations, we see that F_2 has no extension.

Claim 3. G_2 and G_3 are the only extensions of F_3.

To justify this, let T be an extension of F_3 such that $T\backslash v_6$ is isomorphic to F_3 for some vertex v_6 of T. Let us label the vertices of $T\backslash v_6$ as in Figure 3 for F_3. Since T is $i2s$, v_6 has at least one in-neighbor in $\{v_1, v_3, v_4\}$, for otherwise $(\{v_2, v_6\}, \{v_1, v_3, v_4\})$ would be a nontrivial dicut in $T\backslash v_5$, a contradiction.

• (v_6, v_1) is in T. In this case, at most one of (v_6, v_3) and (v_6, v_4) is in T. Let us first consider the subcase when (v_4, v_6) is in T. Now at most one of (v_2, v_6) and (v_5, v_6) is in T, for otherwise, $(\{v_2, v_4, v_5\}, \{v_1, v_6\})$ would be a nontrivial dicut in $T\backslash v_3$. Next, (v_5, v_6) is in T, for otherwise, the three cycles $v_1v_3v_4v_6v_1$, $v_2v_4v_6v_5v_2$, and $v_1v_3v_5v_2v_1$ would form a $K_{3,3}$. It follows that (v_6, v_2) is also in T. If (v_6, v_3) is in T, then the five cycles $v_1v_3v_4v_1$, $v_6v_3v_4v_6$, $v_2v_4v_6v_2$, $v_2v_4v_5v_2$, and $v_1v_3v_5v_2v_1$ would form an M_5; if (v_3, v_6) is in T, then the three cycles $v_1v_3v_6v_2v_1$, $v_1v_3v_4v_5v_1$, and $v_6v_2v_4v_5v_6$ would form a $K_{3,3}$, a contradiction. It remains to consider the subcase when (v_6, v_4) is in T. Thus (v_3, v_6) is also in T. If (v_5, v_6) is in T, then the five cycles $v_1v_3v_5v_1$, $v_2v_3v_5v_2$, $v_2v_4v_5v_2$, $v_6v_4v_5v_6$, and $v_1v_3v_6v_4v_1$ would form an M_5, this contradiction implies that (v_6, v_5) is an arc of T. If (v_2, v_6) is in T, then the three cycles $v_1v_3v_6v_4v_1$, $v_2v_6v_4v_5v_2$, and $v_1v_3v_5v_2v_1$ would form a $K_{3,3}$. So (v_6, v_2) is in T and thus T is isomorphic to G_3, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_2, v_3, v_4, v_1, v_6, v_5)$ in G_3 as labeled in Figure 4.

• (v_1, v_6) is in T. Let us first consider the subcase when (v_6, v_3) is in T. Now (v_5, v_6) is in T, for otherwise, the three cycles $v_1v_6v_3v_4v_1$, $v_1v_6v_5v_2v_1$, and $v_2v_3v_4v_5v_2$ would form a $K_{3,3}$. It follows that (v_4, v_6) is in T, for otherwise the five cycles $v_2v_4v_5v_2$, $v_6v_4v_5v_6$, $v_1v_6v_4v_1$, $v_1v_3v_4v_1$, and $v_1v_3v_5v_2v_1$ would form an M_5^*. Thus (v_4, v_6) is in T, which in turn implies that (v_6, v_2) is in T, for otherwise $(\{v_2, v_4, v_5\}, \{v_1, v_6\})$ would be a nontrivial dicut in $T\backslash v_3$. But then the five cycles $v_2v_3v_5v_2$, $v_2v_4v_5v_2$, $v_2v_4v_6v_2$, $v_2v_1v_6v_2$, and $v_1v_6v_3v_5v_1$ would form an M_5, a contradiction. It remains to consider the subcase when (v_3, v_6) is in T. If (v_6, v_2) is in T, then the five cycles $v_1v_3v_4v_1$, $v_1v_3v_5v_1$, $v_2v_3v_5v_2$, $v_2v_3v_6v_2$, and $v_1v_6v_2v_4v_1$ would form an M_5. Thus (v_2, v_6) is in T. If (v_6, v_4) is in T, then the three cycles $v_1v_3v_6v_4v_1$, $v_1v_3v_5v_2v_1$, and $v_2v_6v_4v_5v_2$ would form a $K_{3,3}$. Thus (v_4, v_6) is in T. Since T is strong, (v_6, v_5) must be in T. Therefore, T is isomorphic to G_2, where $(v_1, v_2, v_3, v_4, v_5, v_6)$ in T corresponds to $(v_1, v_5, v_6, v_3, v_4, v_2)$ in G_2 as labeled in Figure 4.

Combining the above observations, we see that G_2 and G_3 are the only extensions of F_3.

Claim 4. F_4 is Möbius-free while F_5 is not.

It is routine to check that F_4 contains none of the digraphs displayed in Figure 1, so F_4 is Möbius-free. Let us label F_5 as in Figure 3. Then the union of the three cycles $v_1v_5v_3v_4v_1$, $v_2v_6v_3v_4v_2$, and $v_1v_5v_2v_6v_1$ forms a $K_{3,3}$. Thus F_5 is not Möbius-free.

Claim 5. G_1 has no extension.

Assume the contrary: T is an extension of G_1 such that $T\backslash v_7$ is isomorphic to G_1 for some vertex v_7 of T. Let us label the vertices of $T\backslash v_7$ as in Figure 4 for G_1. Depending on the direction of the arc between v_7 and v_1, we distinguish between two cases.

• (v_1, v_7) is in T. In this case, (v_5, v_7) is in T, for otherwise, the union of the three cycles $v_1v_7v_5v_2v_4v_1$, $v_2v_6v_3v_5v_2$, and $v_1v_6v_3v_4v_1$ would form a $K'_{3,3}$. If (v_7, v_6) is in T, then the union of the three cycles $v_1v_7v_6v_4v_1$, $v_7v_6v_3v_5v_7$, and $v_1v_3v_5v_2v_4v_1$ would form a $K'_{3,3}$. Thus (v_6, v_7) is in T, which in turn implies that (v_2, v_7) is in T, for otherwise, the union of the three cycles $v_3v_5v_7v_2v_3$, $v_1v_6v_7v_2v_4v_1$, and $v_1v_6v_3v_5v_1$ would form a $K'_{3,3}$. If (v_7, v_4) is in T, then the union of the three cycles $v_2v_6v_3v_5v_2$, $v_1v_3v_5v_7v_4v_1$, and $v_1v_2v_6v_4v_1$ would form a $K'_{3,3}$. Thus (v_4, v_7) is in T. Since T is strong, (v_7, v_3) is in T. It follows that the union of the five cycles $v_1v_2v_4v_1$, $v_5v_2v_4v_5$, $v_1v_3v_4v_1$, $v_7v_3v_4v_7$, and $v_2v_7v_3v_5v_2$ would form an M_5, a contradiction. Therefore G_1 has no extension.

• (v_7, v_1) is in T. Note that G_1 is isomorphic to its reverse under the mapping

$$(v_1, v_2, v_3, v_4, v_5, v_6) \to (v_5, v_4, v_6, v_2, v_1, v_3).$$

So if T is an extension of G_1, then T^* is also an extension of G_1. If (v_7, v_5) appears in T, then (v_1, v_7) is in T^* and hence the present case reduces to the preceding one. So we may assume that (v_5, v_7) is in T, which implies that (v_7, v_3) is in T, for otherwise the union of the three cycles $v_2v_3v_4v_5v_2$, $v_1v_2v_3v_7v_1$, and $v_1v_6v_4v_5v_7v_1$ would form a $K'_{3,3}$. Thus (v_7, v_2) is in T, for otherwise, the union of the five cycles $v_1v_2v_7v_1$, $v_1v_2v_4v_1$, $v_1v_3v_4v_1$, $v_1v_3v_5v_1$, and $v_2v_7v_3v_5v_2$ would form an M_5^*. But then the union of the three cycles $v_2v_6v_5v_7v_2$, $v_3v_4v_5v_7v_3$, and $v_1v_2v_6v_3v_4v_1$ would form a $K'_{3,3}$, a contradiction.

Combining the above observations, we see that G_1 has no extension.

Claim 6. Neither G_2 nor G_3 has an extension.

To justify this, observe that G_3 is isomorphic to G_2^* under the mapping

$$(v_1, v_2, v_3, v_4, v_5, v_6) \to (v_3, v_5, v_1, v_4, v_2, v_6).$$

So if T is an extension of G_2, then T^* is an extension of G_3. Hence it suffices to show that G_2 has no extension. Assume the contrary: T is an extension of G_2 such that $T\backslash v_7$ is isomorphic to

G_2 for some vertex v_7 of T. Let us label the vertices of $T\backslash v_7$ as in Figure 4 for G_2. Depending the direction of the arc between v_7 and v_1, we distinguish between two cases.

• (v_7, v_1) is in T. Let us first consider the subcase when (v_3, v_7) is in T. Now (v_4, v_7) is in T, for otherwise, the union of the three cycles $v_1v_6v_4v_5v_1$, $v_1v_6v_3v_7v_1$, and $v_3v_7v_4v_5v_3$ would form a $K_{3,3}$. Next, (v_7, v_5) is in T, for otherwise, the union of the three cycles $v_1v_6v_3v_7v_1$, $v_3v_4v_5v_6v_3$, and $v_1v_2v_4v_5v_7v_1$ would form a $K'_{3,3}$. If (v_7, v_6) is in T, then the union of the five cycles $v_1v_6v_3v_1$, $v_7v_6v_3v_7$, $v_3v_7v_5v_3$, $v_3v_4v_5v_3$, and $v_1v_6v_4v_5v_1$ would form an M_5; if (v_6, v_7) is in T, the the union of the three cycles $v_1v_6v_7v_5v_1$, $v_1v_6v_3v_4v_1$, and $v_7v_5v_3v_4v_7$ would form a $K_{3,3}$, a contradiction. It remains to consider the subcase when (v_7, v_3) is in T. Now (v_7, v_2) is in T, for otherwise, the union of the three cycles $v_1v_6v_3v_4v_1$, $v_1v_6v_2v_7v_1$, and $v_2v_7v_3v_4v_5v_2$ would form a $K'_{3,3}$. Since T is $i2s$, (v_6, v_7) must be in T, for otherwise $(\{v_5, v_7\}, \{v_1, v_6, v_3, v_2\})$ would be a nontrivial dicut in $T\backslash v_4$. Thus (v_7, v_4) is in T, for otherwise the union of the three cycles $v_1v_6v_7v_3v_1$, $v_2v_4v_7v_3v_2$, and $v_1v_6v_2v_4v_5v_1$ would form a $K'_{3,3}$. But then the union of the three cycles $v_1v_6v_7v_3v_1$, $v_4v_5v_6v_7v_4$, and $v_1v_2v_4v_5v_3v_1$ also forms a $K'_{3,3}$, a contradiction.

• (v_1, v_7) is in T. Let us first consider the subcase when (v_7, v_6) is in T. If (v_7, v_4) is in T, then the union of the three cycles $v_1v_7v_6v_3v_1$, $v_1v_7v_4v_5v_1$, and $v_3v_4v_5v_6v_3$ would form a $K_{3,3}$; if (v_4, v_7) is in T, then the union of the three cycles $v_1v_7v_6v_3v_1$, $v_2v_4v_7v_6v_2$, and $v_1v_2v_4v_5v_3v_1$ would form a $K'_{3,3}$, a contradiction. It remains to consider the subcase when (v_6, v_7) is in T. Now (v_5, v_7) is in T, for otherwise the union of the five cycles $v_1v_6v_3v_1$, $v_1v_6v_4v_1$, $v_4v_5v_6v_4$, $v_7v_5v_6v_7$, and $v_1v_7v_5v_3v_1$ would form an M_5. Since T is $i2s$, (v_7, v_3) is in T, for otherwise $(\{v_1, v_3, v_5, v_6\}, \{v_2, v_7\})$ would be a nontrivial dicut in $T\backslash v_4$. But then the union of the three cycles $v_1v_6v_7v_3v_1$, $v_1v_6v_4v_5v_1$, and $v_3v_4v_5v_7v_3$ forms a $K_{3,3}$, a contradiction.

Combining the above observations, we see that G_2 has no extension.

Claim 7. F_4 has no extension.

Assume the contrary: T is an extension of F_4 such that $T\backslash v_7$ is isomorphic to F_4 for some vertex v_7 of T. Let us label the vertices of $T\backslash v_7$ as in Figure 3 for F_4. Depending on the direction of the arc between v_2 and v_7, we distinguish between two cases.

• (v_2, v_7) is in T. In this case, (v_5, v_7) appears in T, for otherwise, the union of the three cycles $v_1v_2v_3v_4v_1$, $v_3v_4v_5v_6v_3$, and $v_1v_2v_7v_5v_6v_1$ would form a $K'_{3,3}$. Next, (v_6, v_7) is in T, for otherwise, if (v_3, v_7) is in T, then the union of the three cycles $v_1v_5v_2v_3v_1$, $v_1v_5v_6v_4v_1$, and $v_2v_3v_7v_6v_4v_2$ would form a $K'_{3,3}$; if (v_7, v_3) is in T, then the union of the three cycles $v_1v_5v_3v_4$, $v_2v_7v_3v_4v_2$, and $v_1v_5v_2v_7v_6v_1$ would also form a $K'_{3,3}$, a contradiction. Since T is $i2s$, at least one of (v_7, v_1) and (v_7, v_4) is in T, for otherwise $(\{v_6, v_5, v_1, v_4\}, \{v_2, v_7\})$ would be a nontrivial dicut in $T\backslash v_3$. Assume that (v_7, v_1) is in T. If (v_7, v_3) is in T, then the union of the three cycles $v_1v_5v_3v_4v_1$, $v_2v_7v_3v_4v_2$, and $v_1v_5v_2v_7v_1$ would form a $K_{3,3}$; if (v_3, v_7) is in T, then the union of the three cycles $v_1v_5v_6v_7v_1$, $v_1v_2v_3v_7v_1$, and $v_2v_3v_4v_5v_6v_2$ would form a $K'_{3,3}$, a contradiction. Thus (v_1, v_7) is in T and hence so is (v_7, v_4). Consequently, the union of the three cycles $v_1v_5v_6v_3v_1$, $v_7v_4v_5v_6v_7$, and $v_1v_7v_4v_2v_3v_1$ forms a $K'_{3,3}$, a contradiction.

• (v_7, v_2) is in T. Observe that F_4 is isomorphic to its reverse under the mapping

$$(v_1, v_2, v_3, v_4, v_5, v_6) \to (v_4, v_6, v_5, v_1, v_3, v_2).$$

If T is an extension of F_4, then T^* is also an extension of F_4. If (v_7, v_6) occurs in T, then (v_2, v_7) occurs in T^*, and hence the present case reduces to the preceding case. So we may assume

(v_6, v_7) is in T.

Let us first consider the subcase when (v_3, v_7) is in T. Then (v_5, v_7) is in T, for otherwise, the union of the five cycles $v_1v_5v_6v_4v_1$, $v_1v_2v_3v_4v_1$, and $v_2v_3v_7v_5v_6v_2$ would form a $K'_{3,3}$, a contradiction. If (v_7, v_4) is in T, then the union of the three cycles $v_1v_5v_7v_4v_1$, $v_2v_3v_7v_4v_2$, and $v_1v_5v_2v_3v_1$ would form a $K_{3,3}$. So (v_4, v_7) is in T. Since T is $i2s$, (v_7, v_1) is in T, for otherwise $(\{v_6, v_5, v_1, v_4\}, \{v_2, v_7\})$ would be a nontrivial dicut in $T\backslash v_3$. Thus the union of the three cycles $v_1v_2v_3v_7v_1$, $v_1v_5v_6v_7v_1$, and $v_2v_3v_4v_5v_6v_2$ would form a $K'_{3,3}$, a contradiction.

It remains to consider the second subcase when (v_7, v_3) is in T. Assume that (v_1, v_7) is in T. Then (v_4, v_7) is in T, for otherwise, the union of the three cycles $v_1v_5v_6v_3v_1$, $v_7v_4v_5v_6v_7$, and $v_1v_7v_4v_2v_3v_1$ would form a $K'_{3,3}$, a contradiction. Since T is $i2s$, (v_7, v_5) is in T, for otherwise $(\{v_6, v_5, v_1, v_4\}, \{v_2, v_7\})$ would be a nontrivial dicut in $T\backslash v_3$. But then the union of the three cycles $v_2v_3v_4v_7v_2$, $v_4v_7v_5v_6v_4$, and $v_1v_5v_6v_2v_3v_1$ would form a $K'_{3,3}$, a contradiction. So (v_7, v_1) must appear in T. Since T is $i2s$, (v_4, v_7) is in T, for otherwise $(\{v_6, v_7\}, \{v_1, v_2, v_3, v_4\})$ would be a nontrivial dicut in $T\backslash v_5$. But then the union of the three cycles $v_1v_5v_2v_3v_1$, $v_7v_2v_3v_4v_7$, and $v_1v_5v_6v_4v_7v_1$ would form a $K'_{3,3}$, a contradiction again. So Claim 7 is justified.

From Claims 1-4, we conclude that G_1, G_2, G_3, and F_4 are the only $i2s$ Möbius-free tournaments on six vertices. By Claims 5-7 and Theorem 3.1(iv), there is no $i2s$ Möbius-free tournament on seven or more vertices. This completes the proof of Theorem 1.2. ∎

Proof of Theorem 1.3. We apply induction on $|V|$. By Lemma 3.2, $T = C_3$ if $|V| = 3$ and $T = F_0$ if $|V| = 4$, so $T \in \mathcal{T}_1$ if $|V| \leq 4$. Let us proceed to the induction step.

If T is $i2s$, then $T \in \mathcal{T}_1$ by Theorem 1.2. So we assume that T is not $i2s$. Thus T can be expressed as the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 by Lemmas 2.1 and 2.2. Note that $T_i \notin \{F_1, G_1\}$ because neither F_1 nor G_1 contains a special arc for $i = 1, 2$. By induction hypothesis, both T_1 and T_2 can be constructed by repeatedly taking 1-sums starting from tournaments in $\mathcal{T}_1$, and hence so can T. ∎

References

[1] N. Ailon, M. Charikar, and A. Newman, Aggregating inconsistent information: ranking and clustering, *J. ACM* **55** (2008), no. 5, Art. 23, 27 pp.

[2] I. Ali, W. Cook, and M. Kress, On the minimum violations ranking of a tournaments, *Management Sci.* **32** (1986), 660-672.

[3] N. Alon, Ranking tournaments, *SIAM J. Discrete Math.* **20** (2006), 137-142.

[4] D. Applegate, W. Cook, and S. McCormick, Integral infeasibility and testing total dual integrality, *Oper. Res. Lett.* **10** (1991), 37-41.

[5] F. Barahona, J. Fonlupt, and A. Mahjoub, Compositions of graphs and polyhedra IV: Acyclic spanning subgraphs, *SIAM J. Discrete Math.* **7** (1994), 390-402.

[6] F. Barahona and A. Mahjoub, Composition in the acyclic subdigraph polytope, Report No. 85371-OR, Institut für Ökonometrie und Operations Research, Universität Bonn, 1985.

[7] J. Borda, Mémoire sur les élections an scrutin, *Histoire de l'Académie Royale des Sciences*, 1781.

[8] M. Cai, X. Deng, and W. Zang, A TDI system and its Application to approximation algorithms, in: *Proc.* 39^{th} *IEEE Symposium on Foundations of Computer Science* (FOCS), Palo Alto, CA, 1998, pp. 227-233.

[9] M. Cai, X. Deng, and W. Zang, An approximation algorithm for feedback vertex sets in tournaments, *SIAM J. Comput.* **30** (2001), 1993-2007.

[10] P. Charbit, P. Thomassé, and A. Yeo, The minimum feedback arc set problem is NP-hard for tournaments, *Combin. Probab. Comput.* **16** (2007), 1-4.

[11] M. Condorcet, Éssai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix, 1785.

[12] D. Coppersmith, L. Fleischer, and A. Rudra, Ordering by weighted number of wins gives a good ranking for weighted tournaments, in: *Proc. 17th Annual ACM-SIAM Symposium on Discrete Algorithms* (SODA), 2006, pp. 776-782.

[13] S. Deshu and M. Reed, *Linear Graphs and Electrical Networks*, Addison-Wesley, Reading, Mass., 1961.

[14] G. Ding, L. Feng, and W. Zang, The complexity of recognizing linear systems with certain integrality properties, *Math. Program. Ser. A* **114** (2008), 321-334.

[15] G. Ding, Z. Xu, and W. Zang, Packing cycles in graphs, II, *J. Combin. Theory Ser. B* **87** (2003), 244-253.

[16] G. Ding and W. Zang, Packing cycles in graphs, *J. Combin. Theory Ser. B* **86** (2002), 381-407.

[17] P. Erdős and J. Moon, On sets of consistent arcs in tournaments, *Canad. Math. Bull.* **8** (1965), 269-271.

[18] D. Fulkerson, Upsets in round robin tournaments, *Canad. J. Math.* **17** (1965), 957-969.

[19] J. Geelen and B. Guenin, Packing odd circuits in Eulerian graphs, *J. Combin. Theory Ser. B* **86** (2002), 280-295.

[20] M. Grötschel, L. Lovász, and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* **1** (1981), 169-197.

[21] B. Guenin, Circuit Mengerian directed graphs, in: *Integer Programming and Combinatorial Optimization* (Utrecht, 2001), Lecture Notes in Comput. Sci. 2081, pp. 185-195.

[22] B. Guenin, A short proof of Seymour's characterization of the matroids with the max-flow min-cut property, *J. Combin. Theory Ser. B* **86** (2002), 273-279.

[23] B. Guenin and R. Thomas, Packing directed circuits exactly, *Combinatorica* **31** (2011), 397-421.

[24] M. Jünger, *Polyhedral Combinatorics and the Acyclic Subdigraph Problem*, Heldermann Verlag, Berlin, 1985.

[25] C. Lucchesi and D. Younger, A minimax theorem for directed graphs, *J. London Math. Soc.* **17** (1978), 369-374.

[26] C. Mathieu and W. Schudy, How to rank with few errors: A PTAS for weighted feedback arc set on tournaments, in: *Proc. 39th Annual ACM Symposium on Theory of Comput. (STOC'07)*, pp. 95-103, ACM, New York, 2007.

[27] M. Mnich, V. Williams, and L. Végh, A 7/3-approximation for feedback vertex sets in tournaments, in: *Proc. 24th Annual European Symposium on Algorithms (ESA 2016)*, pp. 67:1-67:14, Dagstuhl Publishing, 2016.

[28] P. Seymour, The matroids with the max-flow min-cut property, *J. Combin. Theory Ser. B* **23** (1977), 189-222.

[29] P. Seymour, Packing circuits in eulerian digraphs, *Combinatorica* **16** (1996), 223-231.

[30] P. Slater, Inconsistencies in a schedule of paired comparisons, *Biometrika* **48** (1961), 303-312.

[31] J. Spencer, Optimal ranking of tournaments, *Networks* **1** (1971), 135-138.

[32] A. Truemper, Max-flow min-cut motroids: polynomial testing and polynomial algorithms for maximum flow and shortest routes, *Math. Oper. Res.* **12** (1987), 72-96.

[33] A. van Zuylen, R. Hegde, K. Jain, and D. Williamson, Deterministic pivoting algorithms for constrained ranking and clustering problems, in: *Proc. 18th Annual ACM-SIAM Symposium on Discrete Algorithms* (SODA), 2007, pp. 405-414.

[34] A. van Zuylen and D. Williamson, Deterministic pivoting algorithms for constrained ranking and clustering problems, *Math. Oper. Res.* **34** (2009), 594-620.

[35] D. Younger, Minimum feedback arc sets for a directed graphs, *IEEE Trans. Circuit Theory* **10** (1963), 238-245.