Ranking Tournaments with No Errors II: Minimax Relation

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Abstract

A tournament $T = (V, A)$ is called *cycle Mengerian* (CM) if it satisfies the minimax relation on packing and covering cycles, for every nonnegative integral weight function defined on A. The purpose of this series of two papers is to show that a tournament is CM iff it contains none of four Möbius ladders as a subgraph; such a tournament is referred to as Möbius-free. In the first paper we have given a structural description of all Möbius-free tournaments, and have proved that every CM tournament is Möbius-free. In this second paper we establish the converse using linear programming approach, which relies heavily on our structural theorems.

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1 Introduction

This is a follow-up of the paper by the same authors [5]. Let us first present the main results of our previous work.

Let $G = (V, A)$ be a digraph with a nonnegative integral weight $w(e)$ on each arc e. A subset F of arcs is called a *feedback arc set* (FAS) of G if $G \backslash F$ contains no cycles (directed). The *FAS problem* is to find an FAS in G with minimum total weight. A collection $\mathcal{C}$ of cycles (with repetition allowed) in G is called a *cycle packing* of G if each arc e is used at most $w(e)$ times by members of $\mathcal{C}$. The *cycle packing problem* consists in finding a cycle packing with maximum size. These two problems form a primal-dual pair. Let $\tau_w(G)$ be the minimum total weight of an FAS, and let $\nu_w(G)$ be the maximum size of a cycle packing. Clearly, $\nu_w(G) \le \tau_w(G)$. We call G *cycle Mengerian* (CM) if $\nu_w(G) = \tau_w(G)$ for every nonnegative integral function $\boldsymbol{w}$ defined on A. As stated in [5], the study of CM digraphs has both great theoretical interest and practical value. Despite tremendous research efforts, only some special classes of CM digraphs [1, 2, 14, 16, 17] have been identified to date, and a complete characterization seems extremely hard to obtain. The interested reader is referred to [3, 4, 9, 10, 12, 14, 15, 16, 20, 21] for some related minimax theorems.

The purpose of this series of two papers is to give a complete characterization of all CM tournaments. We call a tournament *Möbius-free* if it contains none of $K_{3,3}$, $K'_{3,3}$, M_5, and M_5^* depicted in Figure 1 as a subgraph. (Actually, M_5^* arises from M_5 by reversing the direction of each arc.) This class of tournaments is so named because the forbidden structures are all Möbius ladders.

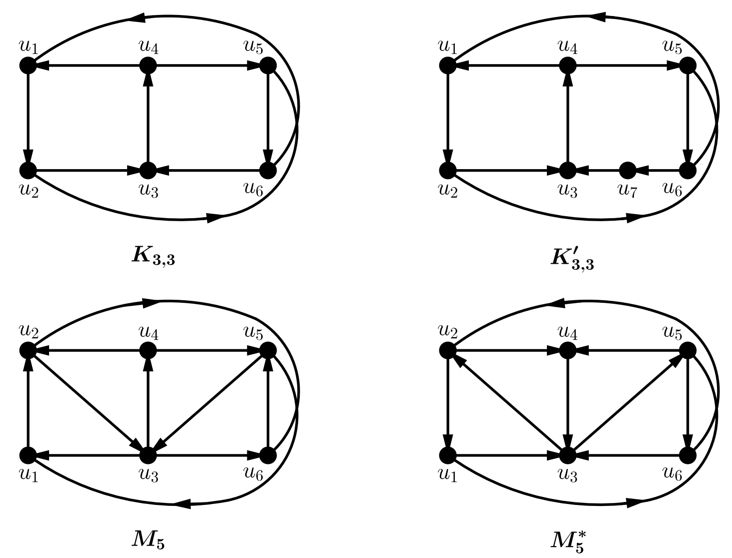


Figure 1. Forbidden Structures

Theorem 1.1. *A tournament is CM iff it is Möbius-free.*

In [5], we have demonstrated that every CM tournament is Möbius-free. The proof of the converse relies heavily on a structural description of Möbius-free tournaments.

Let us recall some terminology and notations introduced in [5]. Let $G = (V, A)$ be a digraph. For each $v \in V$, we use $d^+_G(v)$ and $d^-_G(v)$ to denote the out-degree and in-degree of v, respectively. We call v a *near-sink* of G if its out-degree is one, and call v a *near-source* if its in-degree is one. For simplicity, an arc $e = (u, v)$ of G is also denoted by uv. Arc e is called *special* if either u is a near-sink or v is a near-source of G. For each $U \subseteq V$, we use $G[U]$ to denote the subgraph of G induced by U, and use G/U to denote the digraph obtained from G by *contracting* U. We say that U is a *homogeneous set* of G if $|U| \geq 2$ and the arcs between U and any vertex v outside U are either all directed to U or all directed to v. A *dicut* of G is a partition (X, Y) of $V(G)$ such that all arcs between X and Y are directed to Y. A dicut (X, Y) is *trivial* if $|X| = 1$ or $|Y| = 1$. Recall that G is called *weakly connected* if its underlying undirected graph is connected, and is called *strongly connected* or *strong* if each vertex is reachable from each other vertex. Clearly, a weakly connected digraph G is strong iff G has no dicut. Furthermore, a weakly connected digraph G is called *internally strong* if every dicut of G is trivial, and is called *internally 2-strong* (*i2s*) if G is strong and $G \backslash v$ is internally strong for every vertex v.

Let $T_i = (V_i, A_i)$ be a strong tournament, with $|V_i| \geq 3$ for $i = 1, 2$. Suppose that (a_1, b_1) is a special arc of T_1 with $d^+_{T_1}(a_1) = 1$ and (b_2, a_2) is a special arc of T_2 with $d^-_{T_2}(a_2) = 1$. The *1-sum* of T_1 and T_2 over (a_1, b_1) and (b_2, a_2) is the tournament arising from the disjoint union of $T_1 \backslash a_1$ and $T_2 \backslash a_2$ by identifying b_1 with b_2 (the resulting vertex is denoted by b) and adding all arcs from $T_1 \backslash \{a_1, b_1\}$ to $T_2 \backslash \{a_2, b_2\}$. We call b the *hub* of the 1-sum. See Figure 2 for an illustration. Note that if $|V_i| = 3$ for $i = 1$ or 2, then T_i is a triangle (a directed cycle of length three), and thus $T = T_{3-i}$.

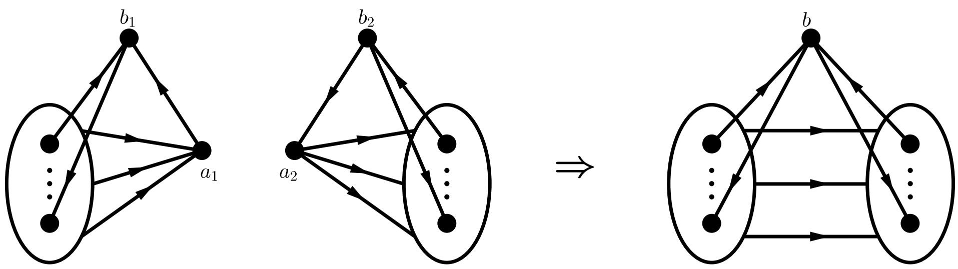


Figure 2. 1-sum of T_1 and T_2.

Let C_3 (resp. F_0) denote the strong tournament with three (resp. four) vertices (see Figure 3), let F_1, F_2, F_3, F_4, F_5 be the five tournaments depicted in Figure 4, and let G_1, G_2, G_3 be the three tournaments shown in Figure 5. In these two papers, we reserve the symbols

$$\mathcal{T}_0 = \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$$

and

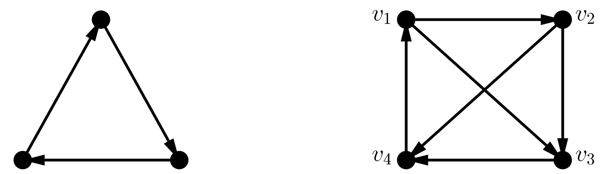
$$\mathcal{T}_1 = \{C_3, F_0, F_2, F_3, F_4, G_2, G_3\} = \mathcal{T}_0 \backslash \{F_1, G_1\}.$$

The following are the structural theorems proved in [5].

Theorem 1.2. *Let $T = (V, A)$ be an i2s tournament with $|V| \geq 3$. Then T is Möbius-free iff $T \in \mathcal{T}_0$.*

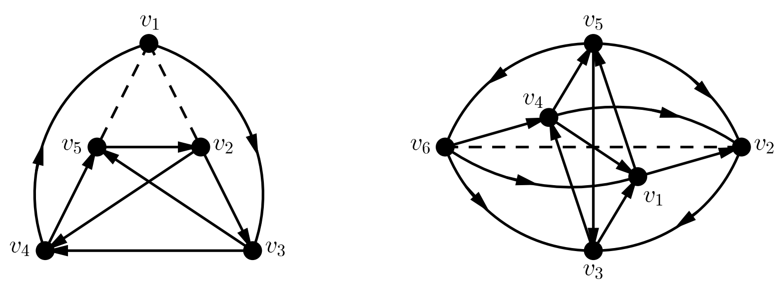
Theorem 1.3. *Let $T = (V, A)$ be a strong Möbius-free tournament with $|V| \geq 3$. Then either $T \in \{F_1, G_1\}$ or T can be obtained by repeatedly taking 1-sums starting from the tournaments in $\mathcal{T}_1$.*

While the proof methods of these two theorems are purely combinatorial, to show that every Möbius-free tournament is CM, we shall appeal to various optimization techniques.



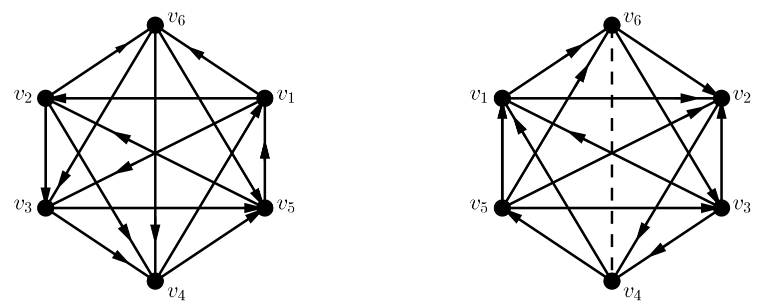
$\boldsymbol{C_3}$ $\boldsymbol{F_0}$

Figure 3. Strong tournaments with three or four vertices.



$\boldsymbol{F_1, F_2, F_3}$ $\boldsymbol{F_4, F_5}$

Figure 4. $v_1v_2, v_5v_1 \in F_1$; $v_2v_1, v_1v_5 \in F_2$; $v_2v_1, v_5v_1 \in F_3$; $v_6v_2 \in F_4$; $v_2v_6 \in F_5$.



$\boldsymbol{G_1}$ $\boldsymbol{G_2, G_3}$

Figure 5. $v_6v_4 \in G_2$ and $v_4v_6 \in G_3$.

Let $C\boldsymbol{x} \geq \boldsymbol{d}$, $\boldsymbol{x} \geq \boldsymbol{0}$ be a rational linear system and let P denote the polyhedron $\{\boldsymbol{x} : C\boldsymbol{x} \geq \boldsymbol{d}, \boldsymbol{x} \geq \boldsymbol{0}\}$. We call P *integral* if it is the convex hull of all integral vectors contained in P. As shown by Edmonds and Giles [11], P is integral iff the minimum in the LP-duality equation

$$\min\{\boldsymbol{w}^T\boldsymbol{x} : C\boldsymbol{x} \geq \boldsymbol{d},\ \boldsymbol{x} \geq \boldsymbol{0}\} = \max\{\boldsymbol{y}^T\boldsymbol{d} : \boldsymbol{y}^T C \leq \boldsymbol{w}^T,\ \boldsymbol{y} \geq \boldsymbol{0}\}$$

has an integral optimal solution, for every integral vector $\boldsymbol{w}$ for which the optimum is finite. If, instead, the maximum in the equation enjoys this property, then the system $C\boldsymbol{x} \geq \boldsymbol{d}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is called *totally dual integral* (TDI). It is well known that many combinatorial optimization problems can be naturally formulated as integer programs of the form $\min\{\boldsymbol{w}^T\boldsymbol{x} : \boldsymbol{x} \in P,\ \text{integral}\}$; if P is integral, then such a problem reduces to its LP-relaxation. Edmonds and Giles [11] proved that total dual integrality implies primal integrality: if $C\boldsymbol{x} \geq \boldsymbol{d}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is TDI and $\boldsymbol{d}$ is integer-valued, then P is integral. Thus the model of TDI systems serves as a general framework for establishing many combinatorial min-max theorems. Over the past six decades, these two integrality properties have been the subjects of extensive research and the major concern of polyhedral combinatorics (see Schrijver [18, 19] for comprehensive accounts).

Let us return to the FAS problem. Let M be the cycle-arc incidence matrix of the input digraph G, and let $\pi(G)$ denote the linear system $M\boldsymbol{x} \geq \boldsymbol{1}$, $\boldsymbol{x} \geq \boldsymbol{0}$. We call G *cycle ideal* (CI) if $\pi(G)$ defines an integral polyhedron. From the above Edmonds-Giles theorem, we see that every CM digraph is CI. Furthermore, G is cycle Mengerian (CM) iff $\pi(G)$ is a TDI system, which gives an equivalent definition of CM digraphs. To facilitate better understanding, we give an intuitive interpretation of these concepts. Let $\mathbb{P}(G, \boldsymbol{w})$ stand for the linear program

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{w}^T\boldsymbol{x} \\ \text{Subject to} & M\boldsymbol{x} \geq \boldsymbol{1} \\ & \quad \boldsymbol{x} \geq \boldsymbol{0}, \end{array}$$

and let $\mathbb{D}(G, \boldsymbol{w})$ denote its dual

$$\begin{array}{ll} \text{Maximize} & \boldsymbol{y}^T\boldsymbol{1} \\ \text{Subject to} & \boldsymbol{y}^T M \leq \boldsymbol{w}^T \\ & \boldsymbol{y} \geq \boldsymbol{0}, \end{array}$$

where $\boldsymbol{w} = (w(e) : e \in A)$. Then $\mathbb{P}(G, \boldsymbol{w})$ (resp. $\mathbb{D}(G, \boldsymbol{w})$) is exactly the LP-relaxation of the FAS problem (resp. cycle packing problem), and thus is called the *fractional FAS problem* (resp. *fractional cycle packing problem*). Let $\tau_w^*(G)$ be the optimal value of $\mathbb{P}(G, \boldsymbol{w})$, and let $\nu_w^*(G)$ be the optimal value of $\mathbb{D}(G, \boldsymbol{w})$. Clearly,

$$\nu_w(G) \leq \nu_w^*(G) = \tau_w^*(G) \leq \tau_w(G);$$

these two inequalities, however, need not hold with equalities in general (as we shall see in Section 2). As is well known, G is CI iff $\mathbb{P}(G, \boldsymbol{w})$ has an integral optimal solution for any nonnegative integral $\boldsymbol{w}$ iff $\tau_w^*(G) = \tau_w(G)$ for any nonnegative integral $\boldsymbol{w}$. Since the separation problem of $\mathbb{P}(G, \boldsymbol{w})$ is the minimum-weight cycle problem, which admits a polynomial-time algorithm, it follows from a theorem of Grötschel, Lovász, and Schrijver [13] that $\mathbb{P}(G, \boldsymbol{w})$ is always solvable in polynomial time. Therefore, the FAS problem can be solved in polynomial time for any nonnegative integral $\boldsymbol{w}$, provided its input digraph G is CI.

We shall actually establish the following strengthening of Theorem 1.1 in this paper.

Theorem 1.4. *For a tournament $T = (V, A)$, the following statements are equivalent:*

(i) T is Möbius-free;

(ii) T is cycle ideal; and

(iii) T is cycle Mengerian.

Throughout we shall repeatedly use the following notations and terminology. As usual, $\mathbb{R}_+$ and $\mathbb{Z}_+$ stand for the sets of nonnegative real numbers and nonnegative integers, respectively. For any two sets Ω and K, where Ω is always a set of numbers and K is always finite, we use Ω^K to denote the set of vectors $\boldsymbol{x} = (x(k) : k \in K)$ whose coordinates are members of Ω. If f is a function defined on a finite set S and $R \subseteq S$, then $f(R)$ denotes $\sum_{s\in R} f(s)$. An *instance* $(T, \boldsymbol{w})$ consists of a Möbius-free tournament $T = (V, A)$ together with a weight function $\boldsymbol{w} \in \mathbb{Z}_+^A$. We say that another instance $(T', \boldsymbol{w}')$ is *smaller* than $(T, \boldsymbol{w})$ if $|V'| < |V|$ or if $|V'| = |V|$ but $w(A') < w(A)$, where $T' = (V', A')$. As introduced in our first paper, we also say that a tournament $T_1 = (V_1, A_1)$ is *smaller* than another tournament $T_2 = (V_2, A_2)$ if $|V_1| < |V_2|$.

Recall the fractional problems introduced above. In view of the equivalent definition, to show that every Möbius-free tournament is CM, we shall turn to proving that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution for every instance $(T, \boldsymbol{w})$. To this end, we apply the double induction on V and $w(A)$, where $T = (V, A)$. Since the desired statement holds trivially when $|V| = 1$, we proceed to the induction step, and propose to establish the following statement.

Theorem 1.5. *Let $(T, \boldsymbol{w})$ be an instance, such that $\mathbb{D}(T', \boldsymbol{w}')$ has an integral optimal solution for any smaller instance $(T', \boldsymbol{w}')$ than $(T, \boldsymbol{w})$. Then $\mathbb{D}(T, \boldsymbol{w})$ also has an integral optimal solution.*

It is clear that C_3 is CM. We shall present a computer-assisted proof that G_1 is CM. Thus F_1 is also CM, as it is an induced subgraph of G_1. For $T \notin \{C_3, F_1, G_1\}$, the proof strategy of Theorem 1.5 is described below.

Obviously, we may assume that T is strong and $\tau_w(T) > 0$. We shall prove that T can be expressed as a 1-sum of two strong Möbius-free tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2), such that one of the following three cases occurs:

• $\tau_w(T_2\backslash a_2) > 0$ and $T_2 \in \mathcal{T}_2$, where $\mathcal{T}_2 = (\mathcal{T}_1\backslash\{C_3\}) \cup \{F_6\}$ for some tournament F_6 to be introduced in Section 2;

• $\tau_w(T_2\backslash a_2) > 0$ and there exists a vertex subset S of $T_2\backslash\{a_2, b_2\}$ with $|S| \geq 2$, such that $T[S]$ is acyclic, $T_2/S \in \mathcal{T}_3$, and the vertex s^* arising from contracting S is a near-sink in T/S, where $\mathcal{T}_3 = (\mathcal{T}_2\backslash F_2) \cup \{G_4, G_5, G_6\}$ for some tournaments G_4, G_5, G_6 to be introduced in Section 2; and

• every positive cycle in T contains arcs in both T_1 and T_2, where a cycle C in T is called *positive* if $w(e) > 0$ for each arc e on C.

In the first two cases, we shall prove that $\mathbb{D}(T, \boldsymbol{w})$ has an optimal solution $\boldsymbol{y}$ such that $y(C)$ is a positive integer for some cycle C contained in $T_2\backslash a_2$. Define $w'(e) = w(e)$ if $e \notin C$ and $w'(e) = w(e) - y(C)$ for each $e \in C$. By the induction hypothesis, $\mathbb{D}(T', \boldsymbol{w}')$ has an integral optimal solution $\boldsymbol{y}'$. We can thus obtain an integral optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ by combining $\boldsymbol{y}'$ with $y(C)$ (the details can be found in Lemma 3.2(iii)). As we shall see, this strategy is carried out by performing various reductions.

In the third case, the desired statement can be established directly by using the max-flow min-cut theorem.

The remainder of this paper is organized as follows. In Section 2, we show that every cycle ideal tournament is Möbius-free. We also prove that one of the three cases described in the above proof strategy of Theorem 1.5 occurs (see Lemma 2.5). In Section 3, we make technical preparations for the proof of Theorem 1.5, and derive properties enjoyed by optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$. In Section 4, we prove Theorem 1.5, in the case of (i) exhibited in Lemma 2.5, by preforming a series of basic reduction operations. In Section 5, we prove Theorem 1.5, in the case of (ii) exhibited in Lemma 2.5, by preforming a series of composite reduction operations. In Section 6, we accomplish the last step of our proof of Theorem 1.5 and hence of Theorem 1.4. In Section 7, we conclude this paper with some remarks.

2 Preliminaries

In this section, we first verify that each Möbius ladder displayed in Figure 1 is a forbidden structure of cycle ideal (CI) tournaments. We then show that one of the three cases described in the proof strategy of Theorem 1.5 occurs. Finally, we prove that being Möbius-free is preserved under 1-sum operation and under contracting two vertices in some circumstances.

Lemma 2.1. *Every cycle ideal tournament is Möbius-free.*

Proof. Assume the contrary: Some CI tournament $T = (V, A)$ contains a member D of $\{K_{3,3}, K'_{3,3}, M_5, M_5^*\}$. Let B be the arc set of D and let $\mathcal{C}$ be the family of all cycles in T. Define $w(e) = 1$ if $e \in B$ and $w(e) = 0$ if $e \in A \backslash B$. We propose to show that, for this weight function $\boldsymbol{w}$, the optimal value of $\mathbb{P}(T, \boldsymbol{w})$, denoted by $\tau_w^*(T)$, is not integral. Depending on the structure of D, we consider four cases.

Case 1. $D = K_{3,3}$.

Define $\boldsymbol{x} \in \mathbb{R}_+^A$ and $\boldsymbol{y} \in \mathbb{R}_+^{\mathcal{C}}$ as follows:

- $x(e) = 1$ if $e \in A \backslash B$, $x(e) = 1/2$ if $e \in \{u_1u_2, u_3u_4, u_5u_6\}$, and $x(e) = 0$ otherwise; and
- $y(C) = 1/2$ if $C \in \{u_1u_2u_3u_4u_1, u_3u_4u_5u_6u_3, u_1u_2u_5u_6u_1\}$ and $y(C) = 0$ otherwise.

It is easy to see that $\boldsymbol{x}$ and $\boldsymbol{y}$ are feasible solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively. Since both of their objective values are $3/2$, by the LP-duality theorem, $\boldsymbol{x}$ and $\boldsymbol{y}$ are actually optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively. Thus $\tau_w^*(T) = 3/2$.

Case 2. $D = K'_{3,3}$.

Define $\boldsymbol{x} \in \mathbb{R}_+^A$ and $\boldsymbol{y} \in \mathbb{R}_+^{\mathcal{C}}$ as follows:

- $x(e) = 1$ if $e \in A \backslash B$, $x(e) = 1/2$ if $e \in \{u_1u_2, u_3u_4, u_5u_6\}$, and $x(e) = 0$ otherwise; and
- $y(C) = 1/2$ if $C \in \{u_1u_2u_3u_4u_1, u_3u_4u_5u_6u_7u_3, u_1u_2u_5u_6u_1\}$ and $y(C) = 0$ otherwise.

Similar to Case 1, we can show that $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively, and $\tau_w^*(T) = 3/2$.

Case 3. $D = M_5$.

Define $\boldsymbol{x} \in \mathbb{R}_+^A$ and $\boldsymbol{y} \in \mathbb{R}_+^{\mathcal{C}}$ as follows:

- $x(e) = 1$ if $e \in A \backslash B$, $x(e) = 1/2$ if $e \in \{u_1u_2, u_2u_3, u_3u_4, u_5u_3, u_6u_5\}$, and $x(e) = 0$ otherwise; and
- $y(C) = 1/2$ if $C \in \{u_1u_2u_3u_1, u_2u_3u_4u_2, u_3u_4u_5u_3, u_3u_6u_5u_3, u_1u_2u_6u_5u_1\}$ and $y(C) = 0$

otherwise.

Similar to Case 1, we can show that $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively, and $\tau_w^*(T) = 5/2$.

Case 4. $D = M_5^*$.

Consider the reverse T^*. In view of the $1-1$ correspondence between cycles in T and those in T^* and using the statement established in Case 3, we obtain $\tau_w^*(T) = 5/2$ in this case as well.

Combining the above cases, we conclude that $\tau_w^*(T)$ is not integral. So $\mathbb{P}(T, \boldsymbol{w})$ has no integral optimal solution and hence T is not CI, a contradiction. ∎

The following two lemmas have played an important role in our structural description of Möbius-free tournaments (see Lemmas 2.1 and 2.2 in [5]).

Lemma 2.2. *Let $T = (V, A)$ be a strong tournament. If T is not i2s, then T is the 1-sum of two smaller strong tournaments.*

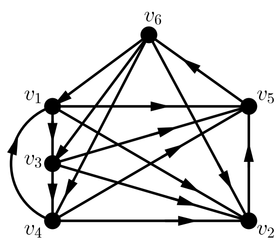
Lemma 2.3. *Let $T = (V, A)$ be the 1-sum of two tournaments T_1 and T_2. Then T is Möbius-free iff both T_1 and T_2 are Möbius-free.*

Let T be a strong Möbius-free tournament. If T is *i2s*, then it comes from a finite set $\mathcal{T}_0$ by Theorem 1.2. In the opposite case, although the statement of Theorem 1.3 is not so strong as this, T can be expressed as the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 by Lemmas 2.2 and 2.3; we can completely determine T_2 if we impose minimality constraint on $|V(T_2)|$, as our next lemma shows.

Let F_6 be the tournament depicted in Figure 6. Observe that it is not *i2s* because $F_6 \backslash v_6$ has a nontrivial dicut. We reserve the symbol

$$\mathcal{T}_2 = \{F_0, F_2, F_3, F_4, F_6, G_2, G_3\}.$$

Notice that $\mathcal{T}_2 = (\mathcal{T}_1 \backslash \{C_3\}) \cup \{F_6\}$.



$\boldsymbol{F_6}$

Figure 6. A minimal tournament involved in 1-sum

Lemma 2.4. *Let $T = (V, A)$ be a strong Möbius-free tournament. Suppose T is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 such that $|V(T_2)|$ is as small as possible. Then $T_2 \in \mathcal{T}_2$.*

Proof. Since T is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2, we have $|V(T_i)| \geq 4$ for $i = 1, 2$. If T_2 is $i2s$, then $T_2 \in \mathcal{T}_1 \backslash \{C_3\}$ by Theorem 1.3, and hence $T_2 \in \mathcal{T}_2$. It remains to consider the case when T_2 is not $i2s$.

Recall the definition of the 1-sum operation in Section 2. There exist a special arc (a_1, b_1) in T_1 and a special arc (b_2, a_2) in T_2, with $d^+_{T_1}(a_1) = d^-_{T_2}(a_2) = 1$, such that T is obtained from the disjoint union of $T_1 \backslash a_1$ and $T_2 \backslash a_2$ by identifying b_1 with b_2 (the resulting vertex is denoted by b) and adding all arcs from $T_1 \backslash \{a_1, b_1\}$ to $T_2 \backslash \{a_2, b_2\}$. We propose to show that

(1) $T_2 \backslash v$ is internally strong for any $v \in V(T_2) \backslash a_2$.

Assume the contrary: $T_2 \backslash v$ has a nontrivial dicut (X, Y) for some $v \in V(T_2) \backslash a_2$. Since a_2 is a near-source in T_2, we have $a_2 \in X$ and $Y \subseteq V(T_2) \backslash \{a_2, b_2\}$. Let x be a vertex in X and y be a vertex in Y such that both (v, x) and (y, v) are arcs in T_2. Set $T'_1 = T \backslash (Y \backslash y)$ and $T'_2 = T_2 \backslash (X \backslash x)$. Then T is the 1-sum of T'_1 and T'_2 over (v, x) and (y, v), with $3 < |V(T'_2)| < |V(T_2)|$, contradicting the minimality hypothesis on T_2. So (1) is justified.

Since T_2 is not $i2s$, $T \backslash a_2$ has a nontrivial dicut (X, Y) by (1). Since T_2 is strong, $b_2 \in Y$. Observe that

(2) $|Y| = 2$, for otherwise, $(X \cup \{a_2\}, Y \backslash b_2)$ would be a nontrivial dicut in $T_2 \backslash b_2$, contradicting (1).

Let c_2 be the vertex in $Y \backslash b_2$. Since T_2 contains no sink, (c_2, b_2) is an arc in T_2. Let S be the sub-tournament induced by X. Then

(3) S is strong, for otherwise, let $(A_1, A_2, ..., A_p)$ be the strong partition of S. Then $p \geq 2$. Thus $(A_1 \cup \{a_2\}, \cup_{i=2}^p A_i \cup \{c_2\})$ would be a nontrivial dicut in $T \backslash b_2$, contradicting (1).

(4) $|X| = 3$.

Suppose not. Then $|X| \geq 4$. Since S is strong by (3), it has a 4-cycle $d_1 d_2 d_3 d_4 d_1$. Note that both (a_2, d_i) and (d_i, b_2) are arcs in T_2 for $1 \leq i \leq 4$. Thus the cycle $d_1 d_2 d_3 d_4 d_1$ together with the five arcs (b_2, a_2), (a_2, d_2), (a_2, d_4), (d_1, b_2), and (d_3, b_2) would form a $K_{3,3}$ in T_2, a contradiction.

Combining (1)-(4), we see that T_2 is isomorphic to F_6. ∎

Based on this technical lemma, we can prove that one of the three cases described in our proof strategy (succeeding Theorem 1.5) occurs. In the next lemma, s^* is the vertex in T/S arising from contracting S, the tournaments G_4, G_5, G_6 are shown in Figure 7, and

$$\mathcal{T}_3 = \{F_0, F_3, F_4, F_6, G_2, G_3, G_4, G_5, G_6\} = (\mathcal{T}_2 \backslash F_2) \cup \{G_4, G_5, G_6\}.$$

Recall that a cycle C in T is called *positive* if $w(e) > 0$ for each arc e on C. We say that C crosses b (the hub of the 1-sum) if it contains an arc between $T_1 \backslash \{b, a_1\}$ and $T_2 \backslash \{b, a_2\}$.

Lemma 2.5. *Let $T = (V, A)$ be a strong Möbius-free tournament with a nonnegative integral weight $w(e)$ on each arc e. Suppose $\tau_w(T) > 0$ and T is not i2s. Then T is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2), such that one of the following three cases occurs:*

(i) $\tau_w(T_2 \backslash a_2) > 0$ and $T_2 \in \mathcal{T}_2$;

(ii) $\tau_w(T_2 \backslash a_2) > 0$ and there exists a vertex subset S of $T_2 \backslash \{a_2, b_2\}$ with $|S| \geq 2$, such that $T[S]$ is acyclic, $T_2/S \in \mathcal{T}_3$, and s^* is a near-sink in T/S. Furthermore,

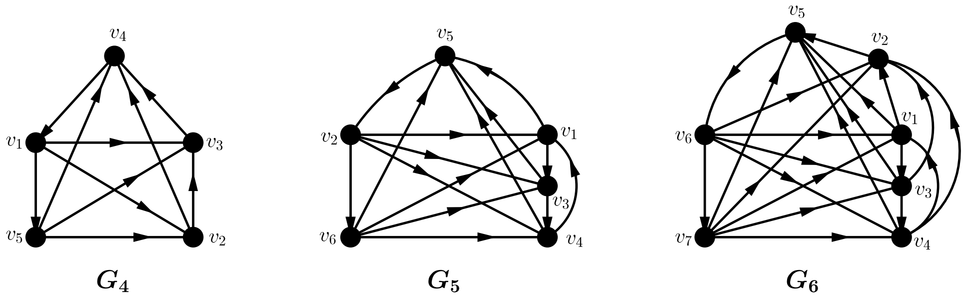


Figure 7. Three more tournaments involved in 1-sum

- $(b_2, a_2) = (v_1, v_5)$ *and* $s^* = v_4$ *if* $T_2/S = G_4$*;*
- $(b_2, a_2) = (v_2, v_6)$ *and* $s^* = v_5$ *if* $T_2/S = G_5$*;*
- $(b_2, a_2) = (v_6, v_7)$ *and* $s^* = v_5$ *if* $T_2/S = G_6$*; and*

(iii) every positive cycle in T *crosses* b*.*

Proof. To establish the statement, we shall construct a sequence of 1-sums of T until one of the three desired cases occurs.

By Lemmas 2.2 and 2.3, T can be expressed as the 1-sum of two smaller strong Möbius-free tournaments T_{11} and T_{12}; subject to this, $|V(T_{12})|$ is as small as possible. Let (a_{11}, b_{11}) in T_{11} and (b_{12}, a_{12}) in T_{12} be the two special arcs involved in the definition of the 1-sum, and let b_1 denote the hub of the 1-sum. By Lemma 2.4, we have $T_{12} \in \mathcal{T}_2$. If $\tau_w(T_{12}\backslash a_{12}) > 0$, then (i) occurs, with $T_1 = T_{11}$ and $T_2 = T_{12}$. So we may assume that $\tau_w(T_{12}\backslash a_{12}) = 0$. Furthermore,

(1) $T_{12}\backslash a_{12}$ is an acyclic tournament in which b_1 is the sink.

To justify this, let K be an MFAS in $T_{12}\backslash a_{12}$. Then $w(K) = 0$ and $T_{12}\backslash K$ is acyclic. Let J be the set of all arcs leaving b_1 in $T_{12}\backslash a_{12}$. Note that no arc in J is contained in any positive cycle in T that crosses b_1. Let T'_{12} be obtained from T_{12} by reversing the directions of all arcs in J and some arcs in K so that $T'_{12}\backslash a_{12}$ is acyclic, and define the weight of each reversed arc in T'_{12} to be zero. Let $T' = (V, A')$ denote the resulting tournament and let $\boldsymbol{w}'$ denote the resulting weight function defined on A'. Then T' remains strong. Since no arc in $K \cup J$ is contained in any positive cycle in T, it is clear that every optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ corresponds to a feasible solution to $\mathbb{D}(T', \boldsymbol{w}')$ with the same objective value, and vice versa. So we may assume that T is T' and that $\boldsymbol{w}$ is $\boldsymbol{w}'$. Thus (1) holds.

At a general step i, suppose T is the 1-sum of two smaller strong Möbius-free tournaments T_{i1} and T_{i2} over two special arcs (a_{i1}, b_{i1}) and (b_{i2}, a_{i2}), such that $T_{i2}\backslash a_{i2}$ is an acyclic tournament in which b_i (the hub of the 1-sum) is the sink. Let S_i be the vertex set of $T_{i2}\backslash\{a_{i2}, b_i\}$, and let T_i be the tournament obtained from T by contracting S_i into a single vertex s_i^*. Clearly, T_i is isomorphic to T_{i1}, in which s_i^* corresponds to a_{i1} and is a near-sink. If $\tau_w(T_i\backslash s_i^*) = 0$, then every positive cycle in T crosses b_i. So (iii) occurs, with $T_1 = T_{i1}$, $T_2 = T_{i2}$, and $b = b_i$. Thus we may assume that $\tau_w(T_i\backslash s_i^*) > 0$. We construct a new 1-sum of T as follows.

Assume first that T_i is *i2s*. In this case, T_i and hence T_{i1} is a member of $\mathcal{T}_2$ by Lemma 2.4. Furthermore, $T_{i1} \neq F_6$. Let T', T'_{i1}, and T'_{i2} be the reverses of T, T_{i1}, and T_{i2}, respectively. Then T' is the 1-sum of two smaller strong Möbius-free tournaments T'_{i2} and T'_{i1}, with $T'_{i1} \in \mathcal{T}_2\backslash F_6$.

Since there is a one-to-one correspondence between cycles in T and those in T', $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution iff so does $\mathbb{D}(T', \boldsymbol{w}')$. Thus we may assume that T is T' and hence (i) occurs.

Assume next that T_i is not $i2s$. By Lemmas 2.2 and 2.3, T_i can be expressed as the 1-sum of two smaller strong Möbius-free tournaments T'_{i1} and T'_{i2}; subject to this, $|V(T'_{i2})|$ is as small as possible. By Lemma 2.4, we have $T'_{i2} \in \mathcal{T}_2$. Let (a'_{i1}, b'_{i1}) in T'_{i1} and (b'_{i2}, a'_{i2}) in T'_{i2} be the two special arcs involved in the definition of the 1-sum, and let b'_i denote the hub of this 1-sum. We proceed by considering two subcases.

• $b'_i \neq s^*_i$. In this subcase, s^*_i is contained in $T'_{i2} \backslash \{a'_{i2}, b'_i\}$, because it is a near-sink in T_i. Hence b_i is contained in $T'_{i2} \backslash a'_{i2}$. Observe that T is the 1-sum of two smaller strong Möbius-free tournaments $T_{(i+1)1}$ and $T_{(i+1)2}$, such that the hub b_{i+1} of this 1-sum is exactly b'_i and that $T_{(i+1)1} = T'_{i1}$. Let $(a_{(i+1)1}, b_{(i+1)1})$ in $T_{(i+1)1}$ and $(b_{(i+1)2}, a_{(i+1)2})$ in $T_{(i+1)2}$ be the two special arcs involved in the definition of this 1-sum. Then $T_{i2} \backslash a_{i2}$ is a proper subtournament of $T_{(i+1)2} \backslash a_{(i+1)2}$. If $\tau_w(T_{(i+1)2} \backslash a_{(i+1)2}) > 0$, then (ii) occurs, with $T_1 = T_{(i+1)1}$, $T_2 = T_{(i+1)2}$, $S = S_i$, and $s^* = s^*_i$. Furthermore, $T_2/S \neq F_2$, because no vertex in $\{v_1, v_3, v_4\}$ (see the labeling in Figure 3) is a near-sink in $F_2 \backslash v_2$ and hence corresponds to s^*. So we assume that $\tau_w(T_{(i+1)2} \backslash a_{(i+1)2}) = 0$. Furthermore,

(2) $T_{(i+1)2} \backslash a_{(i+2)2}$ is an acyclic tournament in which b_{i+1} is the sink.

Since the proof goes along the same line as that of (1), the details are omitted here. In view of (2), we can repeat the construction process by replacing i with $i + 1$.

• $b'_i = s^*_i$. In this subcase, b_i is contained in $T'_{i1} \backslash \{a'_{i1}, b'_i\}$, because it is the only out-neighbor of s^*_i in T_i. Since $T'_{i2} \in \mathcal{T}_2$ (see the labeling in Figures 3-6) and since (b'_{i2}, a'_{i2}) is a special arc of T'_{i2}, and $b'_{i2} = b'_i = s^*_i$ is a sink of $T'_{i2} \backslash a'_{i2}$, it is routine to check that one of (3)-(5) occurs:

(3) $T'_{i2} = F_0$, $(b'_{i2}, a'_{i2}) = (v_4, v_1)$, and $s^*_i = v_4$;

(4) $T'_{i2} = F_2$, $(b'_{i2}, a'_{i2}) = (v_5, v_2)$, and $s^*_i = v_5$; and

(5) $T'_{i2} = F_6$, $(b'_{i2}, a'_{i2}) = (v_5, v_6)$, and $s^*_i = v_5$.

Observe that T is the 1-sum of two smaller strong Möbius-free tournaments $T_{(i+1)1}$ and $T_{(i+1)2}$ along two special arcs $(a_{(i+1)1}, b_{(i+1)1})$ in $T_{(i+1)1}$ and $(b_{(i+1)2}, a_{(i+1)2})$, such that the hub b_{i+1} of this 1-sum is exactly b_i and that $T_{(i+1)1} \backslash a_{(i+1)1} = T'_{i1} \backslash \{s^*_i, a'_{i1}\}$. Clearly, $T_{i2} \backslash a_{i2}$ is a proper subtournament of $T_{(i+1)2} \backslash a_{(i+1)2}$. It is a simple matter to check that $T_{(i+1)2}/S_i$ is isomorphic to G_{t+1} when (t) holds for $t = 3, 4, 5$. If $\tau_w(T_{(i+1)2} \backslash a_{(i+1)2}) > 0$, then (ii) occurs, with $T_1 = T_{(i+1)1}$, $T_2 = T_{(i+1)2}$, $S = S_i$, and $s^* = s^*_i$. So we assume that $\tau_w(T_{(i+1)2} \backslash a_{(i+1)2}) = 0$. Furthermore, $T_{(i+1)2} \backslash a_{(i+2)2}$ is an acyclic tournament in which b_{i+1} is the sink. Since the proof is exactly the same as that of (2), we omit the details here. Thus we can repeat the construction process by replacing i with $i + 1$.

Since $T_{i2} \backslash a_{i2}$ is a proper subtournament of $T_{(i+1)2} \backslash a_{(i+1)2}$ for each step i, the construction process terminates in a finite number of steps. Therefore one of (i)-(iii) holds. ∎

In our proof we occasionally need to contract two vertices. The following two lemmas assert that being Möbius-free is preserved under this operation in some circumstances, and the first is taken from our previous paper (see Lemma 2.3 in [5]).

Lemma 2.6. *Let $T = (V, A)$ be a Möbius-free tournament with a special arc $a = (x, y)$. Then T/a is also Möbius-free.*

Lemma 2.7. *Let $T = (V, A)$ be the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 over the special arcs (a_1, b_1) and (b_2, a_2) such that $T_2 \in \mathcal{T}_2$, and let T' be the digraph obtained from T by contracting two vertices x and y in $T_2 \backslash \{a_2, b_2\}$. Then T' is also Möbius-free.*

Proof. Let T_2' be the digraph obtained from T_2 by contracting x and y. Notice that T_2' may contain opposite arcs. Since $T_2 \in \mathcal{T}$, we have $|V(T_2')| \leq 5$, so T_2' is Möbius-free. Let T_2'' be an arbitrary spanning tournament contained in T_2', and let T'' be the 1-sum of T_1 and T_2''. Then T'' is a spanning tournament contained in T'. By Lemma 2.3, T'' is Möbius-free. It follows that T' is also Möbius-free, because none of $K_{3,3}$, $K_{3,3}'$, M_5, and M_5^* contains a pair of opposite arcs. ∎

3 Reductions: Getting Started

Throughout this section, we assume that $(T, \boldsymbol{w})$ is an instance as described in Theorem 1.5, and that $T = (V, A)$ is the 1-sum of two strong Möbius-free tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2) (see Figure 2).

Let $\mathcal{C}$ be the set of all cycles in T, let $\mathcal{C}_i$ be the set of all cycles in $T_i \backslash a_i$ for $i = 1, 2$, and let $\mathcal{C}_0 = \mathcal{C} \backslash (\mathcal{C}_1 \cup \mathcal{C}_2)$. Note that each cycle in $\mathcal{C}_0$ crosses b, the hub of the 1-sum. For each arc e of T, let $\mathcal{C}(e) = \{C \in \mathcal{C} : e \in C\}$ and $\mathcal{C}_i(e) = \{C \in \mathcal{C}_i : e \in C\}$ for $i = 0, 1, 2$.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$, and let $\nu_w^*(T)$ denote the optimal value of $\mathbb{D}(T, \boldsymbol{w})$. Then $\nu_w^*(T) = \boldsymbol{y}^T \mathbf{1}$. Set $\mathcal{C}^y = \{C \in \mathcal{C} : y(C) > 0\}$ and $\mathcal{C}_i^y = \{C \in \mathcal{C}_i : y(C) > 0\}$ for $i = 0, 1, 2$. For each arc e of T, set $z(e) = y(\mathcal{C}(e))$. We say that e is *saturated* by $\boldsymbol{y}$ if $w(e) = z(e)$ and *unsaturated* otherwise, and say that e is *saturated* by $\boldsymbol{y}$ in T_i if $w(e) = y(\mathcal{C}_i(e))$ for $i = 1, 2$. For each $\mathcal{D} \subseteq \mathcal{C}^y$, we say that arc e is *outside* $\mathcal{D}$ if e is not contained in any cycle in $\mathcal{D}$.

Let us exhibit some properties enjoyed by optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, and make further technical preparations for the proof of Theorem 1.5.

Lemma 3.1. *Let $T = (V, A)$ be a tournament with a nonnegative integral weight $w(e)$ on each arc, and let $\boldsymbol{x}$ (resp. $\boldsymbol{y}$) be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$ (resp. $\mathbb{D}(T, \boldsymbol{w})$). Then the following statements hold:*

(i) $x(C) = 1$ *for any cycle* C *of* T *with* $y(C) > 0$*;*

(ii) $x(e) = 0$ *for all* $e \in A$ *with* $z(e) < w(e)$*;*

(iii) $w(e) = z(e)$ *for all* $e \in A$ *with* $x(e) > 0$*; and*

(iv) Let C_1 and C_2 be two cycles of T with $y(C_i) > 0$ for $i = 1, 2$. Suppose a and b are two common vertices of C_1 and C_2 such that $C_i(a, b)$ is vertex-disjoint from $C_{3-i}(b, a)$ for $i = 1, 2$. Then $\sum_{e \in C_1[a,b]} x(e) = \sum_{e \in C_2[a,b]} x(e)$.

Proof. Statements (i)-(iii) follow directly from the complementary slackness conditions. To justify (iv), let $\theta = \min\{y(C_1), y(C_2)\}$, let $C_i' = C_{3-i}[a, b] \cup C_i[b, a]$ for $i = 1, 2$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C_i)$ with $y(C_i) - \theta$ and replacing $y(C_i')$ with $y(C_i') + \theta$ for $i = 1, 2$. Clearly, $\boldsymbol{y}'$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Using (i), we obtain $x(C_i) = x(C_i') = 1$ for $i = 1, 2$, which implies $\sum_{e \in C_1[a,b]} x(e) = \sum_{e \in C_2[a,b]} x(e)$. ∎

Lemma 3.2. *Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution if one of the following conditions is satisfied:*

(i) $w(e) > \lceil z(e) \rceil$ *for some* $e \in A$*;*

(ii) $\mathcal{C}_0^y = \emptyset$*; and*

(iii) $y(C)$ *is integral for some* $C \in \mathcal{C}^y$*.*

Proof. (i) Define $\boldsymbol{w'} \in \mathbb{Z}_+^A$ by $w'(e) = \lceil z(e) \rceil$ and $w'(a) = w(a)$ for all $a \in A \backslash e$. Then $w(A) > w'(A)$. By the hypothesis of Theorem 1.5, $\mathbb{D}(T, \boldsymbol{w'})$ has an integral optimal solution $\boldsymbol{y'}$. Since $\boldsymbol{y}$ is also a feasible solution to $\mathbb{D}(T, \boldsymbol{w'})$, we have $(\boldsymbol{y'})^T \mathbf{1} \geq \boldsymbol{y}^T \mathbf{1}$. So $\boldsymbol{y'}$ is an integral optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ as well.

(ii) Since $\mathcal{C}_0^y = \emptyset$, each cycle in $\mathcal{C}^y$ is contained in $T_i \backslash a_i$ for $i = 1$ or 2. Let $\boldsymbol{w}_i$ be the restriction of $\boldsymbol{w}$ to $T_i \backslash a_i$. Then the hypothesis of Theorem 1.5 guarantees the existence of an integral optimal solution $\boldsymbol{y}_i$ to $\mathbb{D}(T_i \backslash a_i, \boldsymbol{w}_i)$. Clearly, the union of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ yields an integral optimal solution to $\mathbb{D}(T, \boldsymbol{w})$.

(iii) Define $\boldsymbol{w'} \in \mathbb{Z}_+^A$ by $w'(e) = w(e) - y(C)$ for each arc e on C and $w'(a) = w(a)$ for all other arcs a. Then $w(A) > w'(A)$. By the hypothesis of Theorem 1.5, $\mathbb{D}(T, \boldsymbol{w'})$ has an integral optimal solution $\boldsymbol{y'}$. Clearly, $\boldsymbol{y}$ yields a feasible solution to $\mathbb{D}(T, \boldsymbol{w'})$ with value $\boldsymbol{y}^T \mathbf{1} - y(C)$. So $(\boldsymbol{y'})^T \mathbf{1} \geq \boldsymbol{y}^T \mathbf{1} - y(C)$. Let $\boldsymbol{y}^* \in \mathbb{Z}_+^{\mathcal{C}}$ be defined by $y^*(C) = y(C) + y'(C)$ and $y^*(D) = y'(D)$ for all $D \in \mathcal{C} \backslash C$. Then $\boldsymbol{y}^*$ is an integral feasible solution to $\mathbb{D}(T, \boldsymbol{w})$ with value at least $(\boldsymbol{y'})^T \mathbf{1} + y(C) \geq \boldsymbol{y}^T \mathbf{1}$. Hence $\boldsymbol{y}^*$ is an integral optimal solution to to $\mathbb{D}(T, \boldsymbol{w})$. ∎

Lemma 3.3. *Let* $G = (U, E)$ *be a Möbius-free digraph obtained from a tournament by adding some arcs, and let* $c(e)$ *be a nonnegative integral weight associated with each arc* $e \in E$*. If* $|U| < |V|$ *or if* $|U| = |V|$ *but* $c(E) < w(A)$*, where* V *and* $w(A)$ *are as defined in Theorem 1.5, then* $\mathbb{D}(G, \mathbf{c})$ *has an integral optimal solution.*

Proof. The proof technique employed below is due to Barahona, Fonlupt, and Mahjoub [2].

Let us repeatedly apply the following operations on G whenever possible: For each pair of opposite arcs e and f, replace $c(g)$ by $c(g) - \theta$ for $g = e, f$, where $\theta = \min\{c(e), c(f)\}$, and delete exactly one arc $g \in \{e, f\}$ with $c(g) = 0$ from G. Let $G' = (V', A')$ be the resulting digraph and let $\boldsymbol{c'}$ be the resulting weight function. Clearly, G' is a tournament. Hence, by the hypothesis of Theorem 1.5, G' is CM. Let F' be a minimum FAS of G' and let $\boldsymbol{y'}$ be a maximum cycle packing in G'. Then $c'(F') = (\boldsymbol{y'})^T \mathbf{1}$.

Define $y(C) = y'(C)$ for all cycles C in G'. For each 2-cycle C formed by arcs e and f in G, define $y(C) = \theta$, where $\theta = \min\{c(e), c(f)\}$, and place g and all arcs in F' into F, where g is the arc in $\{e, f\} \backslash A'$. Repeat the process until all 2-cycles in G are exhausted. Clearly, F is an FAS of G, $\boldsymbol{y}$ is a cycle packing of G, and $c(F) = \boldsymbol{y}^T \mathbf{1}$. By the LP-duality theorem, $\boldsymbol{y}$ is an integral optimal solution to $\mathbb{D}(G, \mathbf{c})$. ∎

Lemma 3.4. *Suppose* $a = (s, t)$ *is a special arc of* $T = (V, A)$*, where* s *is a near-sink. Then* $\mathbb{D}(T, \boldsymbol{w})$ *has an integral optimal solution if one of the following conditions is satisfied:*

(i) $w(e) = z(e)$ *for all arcs* $e \in \delta^-(s)$*;*

(ii) $\nu_w^*(T)$ *is an integer;*

(iii) $x(a) = 0$ *for some optimal solution* $\boldsymbol{x}$ *of* $\mathbb{P}(T, \boldsymbol{w})$*;*

(iv) a *is unsaturated by* $\boldsymbol{y}$*; that is,* $z(a) < w(a)$*.*

Proof. (i) By Lemma 3.2(i), we may assume that $w(a) = \lceil z(a) \rceil$. Since $w(e) = z(e)$ for all $e \in \delta^-(s)$ and $z(a) = \sum_{e \in \delta^-(s)} z(e)$, we obtain $w(a) = \sum_{e \in \delta^-(s)} w(e)$. Let $T' = (V', A')$ be the digraph obtained from T by contracting the arc a; we still use t to denote the resulting vertex. By Lemma 2.6, T' is also Möbius-free. Define $\boldsymbol{w'} \in \mathbb{Z}_+^{A'}$ as follows: $w'(e) = w(e)$ if e is not directed to t, $w'(e) = w(f) + w(e)$ if $f = (r, s)$ and $e = (r, t)$ are both in A, and $w'(e) = w(f)$ if $f = (r, s)$ is in A while $e = (r, t)$ is not. It is easy to see that every integral feasible solution of $\mathbb{D}(T, \boldsymbol{w})$ yields an integral feasible solution to $\mathbb{D}(T', \boldsymbol{w'})$ with the same objective value, and vice versa. As $\mathbb{D}(T', \boldsymbol{w'})$ has an integral optimal solution by Lemma 3.3, so does $\mathbb{D}(T, \boldsymbol{w})$.

(ii) By (i), we may assume that $w(e) \neq z(e)$ for some arc $e = (r, s)$ in A. By Lemma 3.2(i), we may assume that $w(e) = \lceil z(e) \rceil$. So $\lceil z(e) \rceil \neq z(e)$. Set $\theta = z(e) - \lfloor z(e) \rfloor$. Then $0 < \theta < 1$. Let $\boldsymbol{w'}$ be obtained from $\boldsymbol{w}$ by replacing $w(e)$ with $w(e) - 1$. Then any optimal solution $\boldsymbol{y}$ of $\mathbb{D}(T, \boldsymbol{w})$ yields a feasible solution of $\mathbb{D}(T, \boldsymbol{w'})$ with value at least $\nu_w^*(T) - \theta$. By the hypothesis of Theorem 1.5, $\mathbb{D}(T, \boldsymbol{w'})$ has an integral optimal solution $\boldsymbol{y'}$ with value at least $\nu_w^*(T) - \theta$ and hence at least $\nu_w^*(T)$. So $\boldsymbol{y'}$ is also an integral optimal solution to $\mathbb{D}(T, \boldsymbol{w})$.

(iii) For each $r \in V \backslash \{s, t\}$ with $e = (r, t) \in A$, we claim that $x(e) = x(f)$, where $f = (r, s)$. If $w(e) = 0$ or $w(f) = 0$, clearly we may assume that $x(e) = x(f)$ (modifying one of them if necessary, the resulting solution remains optimal). Next, consider the case when $w(e) > 0$ and $w(f) > 0$. Let C_1 and C_2 be two cycles passing through e and f, respectively, with $y(C_i) > 0$ for $i = 1, 2$. By Lemma 3.1(iv), $x(e) = x(a) + x(f) = x(f)$. So the claim is justified.

Let $T' = (V', A')$ be the digraph obtained from T by contracting the arc a. By Lemma 2.6, T' is also Möbius-free. Define $\boldsymbol{w'} \in \mathbb{Z}_+^{A'}$ as follows: $w'(e) = w(e)$ if e is not directed to t, $w'(e) = w(f) + w(e)$ if $f = (r, s)$ and $e = (r, t)$ are both in A, and $w'(e) = w(f)$ if $f = (r, s)$ is in A while $e = (r, t)$ is not. Let $\boldsymbol{x'} \in \mathbb{R}_+^{A'}$ be the projection of $\boldsymbol{x}$, and let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through (r, s) in T with $y(C) > 0$, let C' be the cycle in T' arising from C by replacing the path rst with (r, t) and set $y'(C') = y(C) + y(C')$. By the LP-duality theorem, $\boldsymbol{x'}$ and $\boldsymbol{y'}$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w'})$ and $\mathbb{D}(T', \boldsymbol{w'})$, respectively, with the same objective value as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\mathbb{D}(T', \boldsymbol{w'})$ has an integral optimal solution. So $\nu_w^*(T)$ is an integer. Thus (iii) follows from (ii).

(iv) Since $z(a) < w(a)$, we have $x(a) = 0$ by Lemma 3.1(ii). Therefore (iv) can be deduced from (iii). ∎

Recall that $\mathcal{C}_2$ is the set of all cycles in $T_2 \backslash a_2$. In the following lemma, $\mathcal{D}_k$ is the set of all cycles of length k in $T_2 \backslash a_2$, and q is the length of a longest cycle in $T_2 \backslash a_2$. Thus $\mathcal{C}_2 = \cup_{k=3}^{q} \mathcal{D}_k$. Let $H_i = (V_i, E_i)$ be a digraph for $i = 1, 2, \ldots, k$. A digraph $H = (V, E)$ is called a *multiset sum* of these k digraphs if $V = \cup_{i=1}^{k} V_i$ and E is the multiset sum of all these E_i's; that is, if an arc (u, v) is contained in t of these H_i's, then there are precisely t parallel arcs from u to v in H.

Lemma 3.5. *Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that $y(\mathcal{C}_2)$ is maximized and, subject to this, $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically. Then the following statements hold:*

(i) Every $C \in \mathcal{C}$ contains an arc e that is saturated by $\boldsymbol{y}$;

(ii) Every $C \in \mathcal{C}_2$ contains an arc that is outside $\mathcal{C}_0^y$;

(iii) If $C_1 \in \mathcal{C}_0^y$ and $C_2 \in \mathcal{C}_2$ share arcs, then some arc on C_2 but outside C_1 is saturated by $\boldsymbol{y}$;

(iv) If exactly one arc on $C \in \mathcal{C}_2$ is outside $\mathcal{C}_0^y$, then it is saturated by $\boldsymbol{y}$ in T_2;

(v) *Every chord of $C \in \mathcal{C}_2^y$ is saturated by $\boldsymbol{y}$ in T_2;*

(vi) *If the multiset sum of $C_1 \in \mathcal{C}_0$, $C_2 \in \mathcal{C}_2$, and unsaturated arcs in $T_2\backslash a_2$ contains two arc-disjoint cycles in $T_2\backslash a_2$, then $y(C_1)$ or $y(C_2)$ is 0;*

(vii) *Every triangle $C \in \mathcal{C}_2$ contains an arc that is saturated by $\boldsymbol{y}$ in T_2;*

(viii) *If the multiset sum of $C_1 \in \mathcal{C}_0$ and $C_2 \in \mathcal{C}_2$ contains two arc-disjoint cycles $C_1' \in \mathcal{C}_0$ and $C_2' \in \mathcal{C}_2$, with $|C_2'| < |C_2|$, then $y(C_1)$ or $y(C_2)$ is 0.*

Proof. (i) Assume the contrary: $w(e) > z(e)$ for each arc e on C. Set $\theta = \min\{w(e) - z(e) : e \in C\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C)$ with $y(C) + \theta$. Then $\boldsymbol{y'}$ is a feasible solution to $\mathbb{D}(T, \boldsymbol{w})$, with $(\boldsymbol{y'})^T\mathbf{1} = \boldsymbol{y}^T\mathbf{1} + \theta > \boldsymbol{y}^T\mathbf{1}$, contradicting the optimality on $\boldsymbol{y}$.

(ii) Assume the contrary: each arc e_i on C is contained in some $C_i \in \mathcal{C}_0^y$. Observe that b, the hub of the 1-sum, is not on C, for otherwise, let e_j be the arc on C that leaves b. From the definition of the 1-sum, we see that e_j is contained in no cycle in $\mathcal{C}_0$, contradicting the definition of C_j. Let $k = |C|$ and let H be the multiset sum of $C_1, C_2, ..., C_k$. Then H is an even digraph and $d_H^+(b) = d_H^-(b) = k$. Let H' be obtained from H by deleting all arcs on C. Then H' remains even and $d_{H'}^+(b) = d_{H'}^-(b) = k$ because b is outside C. So H' contains k arc-disjoint cycles $C_1', C_2', ..., C_k'$ passing through b and hence in $\mathcal{C}_0$. Set $\theta = \min_{1 \le i \le k}\ y(C_i)$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C)$ with $y(C) + \theta$, replacing $y(C_i)$ with $y(C_i) - \theta$, and replacing $y(C_i')$ with $y(C_i') + \theta$ for $1 \le i \le k$. Clearly, $\boldsymbol{y'}$ is a feasible solution to $\mathbb{D}(T, \boldsymbol{w})$ with $(\boldsymbol{y'})^T\mathbf{1} = \boldsymbol{y}^T\mathbf{1} + \theta > \boldsymbol{y}^T\mathbf{1}$, a contradiction.

(iii) Assume the contrary: $w(e) > z(e)$ for each arc e in B, the set of all arcs on C_2 but outside C_1. Set $\theta = \min\{y(C_1),\ w(e) - z(e) : e \in B\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C_1)$ with $y(C_1) - \theta$ and replacing $y(C_2)$ with $y(C_2) + \theta$. Then $\boldsymbol{y'}$ is also optimal, with $y'(\mathcal{C}_2) = y(\mathcal{C}_2) + \theta$, so the existence of $\boldsymbol{y'}$ contradicts the maximality assumption on $y(\mathcal{C}_2)$ in the choice of $\boldsymbol{y}$.

(iv) Assume the contrary: the only arc $e_0 = (u, v)$ on C outside $\mathcal{C}_0^y$ is not unsaturated by $\boldsymbol{y}$ in T_2. Then $w(e_0) > z(e_0)$. Let C_i a cycle in $\mathcal{C}_0^y$ that passes through each e_i on $C\backslash e_0$. Let $k = |C| - 1$ and let H be the multiset sum of $C_0, C_1, C_2, ..., C_k$, where C_0 is the 2-cycle formed by (u, v) and (v, u). Then H is an even digraph, and $d_H^+(b) = d_H^-(b) = k$ if $b \ne u$ and $d_H^+(b) = d_H^-(b) = k + 1$ otherwise, where b is the hub of the 1-sum. Let H' be obtained from H by deleting all arcs on C. Then H' remains even and contains k arc-disjoint cycles $C_1', C_2', ..., C_k'$ passing through b (and hence in $\mathcal{C}_0$). Clearly, at most one of $C_1', C_2', ..., C_k'$, say C_k' if any, contains the arc (v, u). Then $C_1', C_2', ..., C_{k-1}'$ are all in $\mathcal{C}_0$. Set $\theta = \min\{w(e_0) - z(e_0),\ y(C_i) : 1 \le i \le k\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C)$ with $y(C) + \theta$, replacing $y(C_i)$ with $y(C_i) - \theta$ for $1 \le i \le k$, and replacing $y(C_j')$ with $y(C_j') + \theta$ for $1 \le j \le k - 1$. Then $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y(C) < y'(C)$, the existence of $\boldsymbol{y'}$ contradicts the maximality assumption on $y(\mathcal{C}_2)$ in the choice of $\boldsymbol{y}$.

(v) Assume the contrary: some chord $e = (u, v)$ of C is not saturated by $\boldsymbol{y}$ in T_2. Let $C' = C[v, u] \cup \{(u, v)\}$. Note that $C' \in \mathcal{C}_2$ and $|C'| < |C|$.

We first consider the case when e is outside $\mathcal{C}_0^y$. Then $w(e) - z(e) > 0$. Set $\theta = \min\{y(C), w(e) - z(e)\}$. Let $\boldsymbol{y'}$ be obtained from $\boldsymbol{y}$ by replacing $y(C)$ with $y(C) - \theta$ and replacing $y(C')$ with $y(C') + \theta$. Then $\boldsymbol{y'}$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(C) < y(C)$, the existence of $\boldsymbol{y'}$ contradicts the minimality assumption on $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ in the choice of $\boldsymbol{y}$.

We next consider the case when e is contained in some cycle D in $\mathcal{C}_0^y$. Then the multiset sum of C and D contains a cycle D' in $\mathcal{C}_0$ that is disjoint from C'. Set $\sigma = \min\{y(C), y(D)\}$. Let $\boldsymbol{y'}$

be obtained from $\boldsymbol{y}$ by replacing $y(C)$, $y(D)$, $y(C')$, and $y(D')$ with $y(C)-\sigma$, $y(D)-\sigma$, $y(C')+\sigma$, and $y(D')+\sigma$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$. Since $y'(C) < y(C)$, the existence of $\boldsymbol{y}'$ contradicts the minimality assumption on $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ in the choice of $\boldsymbol{y}$.

(vi) Assume the contrary: $y(C_1)y(C_2) > 0$. Let B be the set of unsaturated arcs in $T_2 \backslash a_2$, and let C_1' and C_2' be two arc-disjoint cycles in $\mathcal{C}_2$ that are contained in the multiset sum of C_1, C_2, and B. Set $\theta = \min\{y(C_1), y(C_2), w(e) - z(e) : e \in B\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C_1)$, $y(C_2)$, $y(C_1')$, and $y(C_2')$ with $y(C_1) - \theta$, $y(C_2) - \theta$, $y(C_1') + \theta$, and $y(C_2') + \theta$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$. Since $y'(\mathcal{C}_2) = y(\mathcal{C}_2) + \theta$, the existence of $\boldsymbol{y}'$ contradicts the maximality assumption on $\boldsymbol{y}$.

(vii) Let $C = ijki$ be a triangle in $T_2 \backslash u_2$. By (ii), at least one arc on C is outside $\mathcal{C}_0^y$, say (i,j). If all arcs on C are outside $\mathcal{C}_0^y$, then by (i) one of the three arcs is saturated by $\boldsymbol{y}$ in T and hence in T_2. If (i,j) is the only arc on C that is outside $\mathcal{C}_0^y$, then (i,j) is saturated by $\boldsymbol{y}$ in T_2 by (iv). If exactly one arc on C, say (j,k), is contained in some cycle in $\mathcal{C}_0^y$, then by (iii) one of (i,j) and (k,i) is saturated by $\boldsymbol{y}$ in T and hence in T_2.

(viii) Assume the contrary: $y(C_1)y(C_2) > 0$. Set $\theta = \min\{y(C_1), y(C_2)\}$. Let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(C_1)$, $y(C_2)$, $y(C_1')$, and $y(C_2')$ with $y(C_1) - \theta$, $y(C_2) - \theta$, $y(C_1') + \theta$, and $y(C_2')+\theta$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$. Since $|C_2'| < |C_2|$ and $y'(C_2) < y(C_2)$, the existence of $\boldsymbol{y}'$ contradicts the minimality assumption on $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ in the choice of $\boldsymbol{y}$. ∎

Lemma 3.6. *Suppose $T_2 \backslash a_2$ contains a unique cycle C, which is a triangle. If $w(a) > 0$ for each arc a on C, then $\mathbb{D}(T,\boldsymbol{w})$ has an integral optimal solution.*

Proof. Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T,\boldsymbol{w})$ such that $y(C)$ is maximized. By Lemma 3.5(vii), some arc e on C is saturated by $\boldsymbol{y}$ in T_2. Since C is the unique cycle in $T_2 \backslash a_2$, we have $y(C) = w(e)$. Thus $\mathbb{D}(T,\boldsymbol{w})$ has an integral optimal solution by Lemma 3.2(iii). ∎

4 Basic Reductions

This section is devoted to the analysis of case (i) exhibited in Lemma 2.5. Throughout this section, we assume that $(T,\boldsymbol{w})$ is an instance as described in Theorem 1.5, and that $T = (V, A)$ is the 1-sum of two strong Möbius-free tournaments T_1 and T_2 over the two special arcs (a_1, b_1) and (b_2, a_2), with $\tau_w(T_2 \backslash a_2) > 0$ and $T_2 \in \mathcal{T}_2$. (Possibly T_1 is a triangle and thus $T = T_2$.) Let us label T_2 as in Figures 3-6. Since (b_2, a_2) is a special arc and a_2 is a near-source of T_2,

- $(b_2, a_2) = (v_1, v_2)$ or (v_4, v_1) if $T_2 = F_0$;
- $(b_2, a_2) = (v_5, v_2)$ if $T_2 = F_2$ or F_3;
- $(b_2, a_2) = (v_5, v_6)$ if $T_2 = F_4$;
- $(b_2, a_2) = (v_5, v_6)$ if $T_2 = F_6$; and
- $(b_2, a_2) = (v_4, v_5)$ if $T_2 = G_2$ or G_3.

Note that $T_2 \backslash a_2$ is a transitive triangle when $T_2 = F_0$ and $(b_2, a_2) = (v_4, v_1)$; in this case, unfortunately, no reduction on $T_2 \backslash a_2$ is available, and the information on $T_2 \backslash a_2$ alone does not lead to a proof of the desired statement; that is, $\mathbb{D}(T,\boldsymbol{w})$ has an integral optimal solution. In fact, the same problem occurs when $\tau_w(T_2 \backslash a_2) = 0$, no matter what T_2 is. That partly explains why the assumption of this section is so made and Lemma 2.5 is so stated.

Theorem 4.1. *For the above instance $(T, \boldsymbol{w})$, problem $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

We shall carry out a proof by performing reductions on $T_2\backslash a_2$. We employ the same notations as introduced before. In particular, $\nu_w^*(T)$ stands for the common optimal value of $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, and $\tau_w(T)$ stands for the minimum total weight of an FAS in T. An FAS K of T is called *minimal* if no proper subset of K is an FAS of T. A minimum-weight FAS is denoted by MFAS. We use $\mathcal{F}_2$ to denote the family of all minimal FAS's in $T_2\backslash a_2$. Recall that $\mathcal{C}_2$ stands for the set of all cycles in $T_2\backslash a_2$, and $\mathcal{D}_k$ is the set of all cycles of length k in $T_2\backslash a_2$. For every real number r, set $[r] = r - \lfloor r \rfloor$.

We break the proof of Theorem 4.1 into a series of lemmas.

Lemma 4.2. *If $T_2 \in \{F_0, F_2, F_6\}$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. By the hypothesis of Theorem 4.1, $\tau_w(T_2\backslash a_2) > 0$. So if $T_2 = F_0$, then $(b_2, a_2) = (v_1, v_2)$ and hence $T_2\backslash a_2$ is a triangle. It is then routine to check that, for each $T_2 \in \{F_0, F_2, F_6\}$, there is a unique cycle contained in $T_2\backslash a_2$, which is a triangle. Therefore $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 3.6. ∎

Lemma 4.3. *If $T_2 = F_3$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. It is routine to check that

• $\mathcal{C}_2 = \{v_1v_3v_4v_1, v_1v_3v_5v_1, v_1v_3v_4v_5v_1\}$ and

• $\mathcal{F}_2 = \{\{v_1v_3\}, \{v_3v_4, v_3v_5\}, \{v_3v_4, v_5v_1\}, \{v_4v_1, v_5v_1\}, \{v_3v_5, v_4v_1, v_4v_5\}\}$.

We also have a computer verification of these results. So $|\mathcal{C}_2| = 3$ and $|\mathcal{F}_2| = 5$. Recall that $(b_2, a_2) = (v_5, v_2)$.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(1) $y(\mathcal{C}_2)$ is maximized;

(2) subject to (1), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \ldots, y(\mathcal{D}_3))$ is minimized lexicographically; and

(3) subject to (1) and (2), $y(v_1v_3v_5v_1)$ is minimized.

Observe that

(4) if $K \in \mathcal{F}_2$ satisfies $y(\mathcal{C}_2) = w(K)$, then K is an MFAS.

Indeed, since $y(\mathcal{C}_2) = \nu_w^*(F_3\backslash v_2)$, we have $w(K) = \nu_w^*(F_3\backslash v_2) \le \tau_w(F_3\backslash v_2) \le w(K)$. So $w(K) = \tau_w(F_3\backslash v_2)$.

Claim 1. $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$.

To justify this, observe that v_1v_3 is a special arc of T and v_1 is a near-sink. By Lemma 3.4(iv), we may assume that v_1v_3 is saturated by $\boldsymbol{y}$ in T. If v_1v_3 is outside $\mathcal{C}_0^y$, then v_1v_3 is saturated by $\boldsymbol{y}$ in F_3. Thus $y(\mathcal{C}_2) = w(v_1v_3)$. By (4), $\{v_1v_3\}$ is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$. So we assume that v_1v_3 is contained in some cycle $C \in \mathcal{C}_0^y$; subject to this, C is chosen to have the maximum number of arcs in $F_3\backslash v_2$. Depending on whether C passes through v_4v_1, we consider two cases.

• C contains v_4v_1. In this case, C contains the path $v_4v_1v_3v_5$. Applying Lemma 3.5(ii) to the triangles $v_1v_3v_4v_1$ and $v_1v_3v_5v_1$ respectively, we see that both v_3v_4 and v_5v_1 are outside $\mathcal{C}_0^y$. By Lemma 3.5(iv), both v_3v_4 and v_5v_1 are saturated by $\boldsymbol{y}$ in F_3. Moreover, $y(v_1v_3v_4v_5v_1) = 0$, for otherwise, by Lemma 3.5(v), v_3v_5 is saturated by $\boldsymbol{y}$ in F_3, contradicting the fact that $v_3v_5 \in C$. So $y(v_1v_3v_4v_1) = w(v_3v_4)$, $y(v_1v_3v_5v_1) = w(v_5v_1)$, and $y(\mathcal{C}_2) = w(K)$, where $K = \{v_3v_4, v_5v_1\}$. By (4), K is an MFAS and hence $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$.

• C does not contain v_4v_1. In this case, we may assume that v_4v_1 is outside $\mathcal{C}_0^y$, for otherwise, let D be a cycle in $\mathcal{C}_0^y$ passing through v_4v_1. Then D contains the path $v_1v_3v_5$. Replacing C by D, we see that the previous case occurs. Since C contains v_1v_3, it also contains v_3v_4 or v_3v_5. If C contains v_3v_4, then it contains the path $v_1v_3v_4v_5$. Using Lemma 3.5(ii) and (iv) and the cycles $v_1v_3v_4v_1$ and $v_1v_3v_4v_5v_1$, we see that both v_4v_1 and v_5v_1 are saturated by $\boldsymbol{y}$ in F_3. So $y(\mathcal{C}_2) = w(K)$, where $K = \{v_4v_1, v_5v_1\}$. Using (4), we obtain $y(\mathcal{C}_2) = \tau_w(F_3\backslash v_2)$. If C contains v_3v_5, then v_5v_1 is saturated by $\boldsymbol{y}$ in F_3 by Lemma 3.5(ii) and (iv). Thus we may assume that v_4v_1 is not saturated by $\boldsymbol{y}$ in F_3, otherwise we are done (as shown above). It follows from Lemma 3.5(v) that $y(v_1v_3v_4v_5v_1) = 0$, and from Lemma 3.5(ii) and (iv) (using the triangle $v_1v_3v_4v_1$) that v_3v_4 is outside $\mathcal{C}_0^y$. So, by Lemma 3.5(iii), v_3v_4 is saturated by $\boldsymbol{y}$ in F_3. Since $y(\mathcal{C}_2) = w(J)$, where $J = \{v_3v_4, v_5v_1\}$, Claim 1 is justified by (4).

Claim 2. $y(C)$ is an integer for each $C \in \mathcal{C}_2$.

To justify this, observe that $y(v_1v_3v_4v_5v_1) = 0$, for otherwise, by Lemma 3.5(v), both v_4v_1 and v_3v_5 are saturated by $\boldsymbol{y}$ in F_3. So $y(v_1v_3v_4v_1) = w(v_4v_1)$ and $y(v_1v_3v_5v_1) = w(v_3v_5)$; both of them are integers. By Claim 1, $y(v_1v_3v_4v_5v_1)$ is also integral, as desired.

From the proof of Claim 1, we see that one of the following three cases occurs:

• $y(v_1v_3v_4v_1) + y(v_1v_3v_5v_1) = w(v_1v_3)$;
• $y(v_1v_3v_4v_1) = w(v_3v_4)$ and $y(v_1v_3v_5v_1) = w(v_5v_1)$; and
• $y(v_1v_3v_4v_1) = w(v_4v_1)$ and $y(v_1v_3v_5v_1) = w(v_5v_1)$.

Thus the desired statement holds trivially in the second and third cases. It remains to consider the first case.

Suppose on the contrary that neither $y(v_1v_3v_4v_1)$ nor $y(v_1v_3v_5v_1)$ is an integer. Then $[y(v_1v_3v_4v_1)] + [y(v_1v_3v_5v_1)] = 1$. By the hypothesis of the present case, v_1v_3 is saturated by $\boldsymbol{y}$ in F_3, so v_4v_1 is outside $\mathcal{C}_0^y$. Thus

$$w(v_4v_1) \geq \lceil y(v_1v_3v_4v_1) \rceil = \lfloor y(v_1v_3v_4v_1) \rfloor + 1 = y(v_1v_3v_4v_1) + [y(v_1v_3v_5v_1)].$$

We propose to show that

(5) v_3v_4 is saturated by $\boldsymbol{y}$ in F_3.

Suppose not. If v_3v_4 is unsaturated in T, set $\theta = \min\{w(v_3v_4) - z(v_3v_4), [y(v_1v_3v_5v_1)]\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_3v_4v_1)$ and $y(v_1v_3v_5v_1)$ with $y(v_1v_3v_4v_1) + \theta$ and $y(v_1v_3v_5v_1) - \theta$, respectively; if v_3v_4 is saturated in T and contained in some $C \in \mathcal{C}_0^y$, set $\theta = \min\{y(C), [y(v_1v_3v_5v_1)]\}$ and $C' = C[v_5, v_3] \cup \{v_3v_5\}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ by replacing $y(v_1v_3v_4v_1)$, $y(v_1v_3v_5v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_3v_4v_1)+\theta$, $y(v_1v_3v_5v_1)-\theta$, $y(C)-\theta$, and $y(C') + \theta$, respectively. Then $\boldsymbol{y}'$ is an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Since $y'(v_1v_3v_5v_1) < y(v_1v_3v_5v_1)$, the existence of $\boldsymbol{y}'$ contradicts the assumption (3) on $\boldsymbol{y}$. So (5) is established.

By (5), we have $y(v_1v_3v_4v_1) = w(v_3v_4)$ and $y(v_1v_3v_5v_1) = w(v_1v_3) - w(v_3v_4)$; both of them are integers. This contradiction proves Claim 2.

Since $\tau_w(F_3\backslash v_2) > 0$, by Claims 1 and 2, $y(C)$ is a positive integer for some $C \in \mathcal{C}_2$. Thus, by Lemma 3.2(iii), $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. ∎

Lemma 4.4. *If $T_2 \in \{F_4, G_2, G_3\}$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Given the length of the whole paper, we omit the proof of this lemma here, and refer the reader to the online appendix [6] (see Lemmas 4.5-4.7). Nevertheless, the proof ideas are all included in the verification of Lemma 4.3. (We hope that this paper could be handled in the same way as Ding and Iverson [7] and Ding, Tan, and Zang [8]; each of these published versions contains only parts of our lengthy proof, and the rest is given in an online appendix at the website of the respective journal.)

Now we are ready to establish the main result of this section.

Proof of Theorem 4.1. By the hypothesis of this section, T is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2, with $T_2 \in \mathcal{T}_2$. Since $\mathcal{T}_2 = \{F_0, F_2, F_3, F_4, F_6, G_2, G_3\}$, the desired statement follows instantly from Lemmas 4.2-4.4. ∎

5 Composite Reductions

This section is devoted to the analysis of case (*ii*) exhibited in Lemma 2.5. Throughout this section, we assume that $(T, \boldsymbol{w})$ is an instance as described in Theorem 1.5, and that $T = (V, A)$ is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2), such that

(α) $\tau_w(T_2 \backslash a_2) > 0$;

(β) there exists a vertex subset S of $T_2 \backslash \{a_2, b_2\}$ with $|S| \geq 2$ and with the following properties:
- $T[S]$ is acyclic and $T_2/S \in \mathcal{T}_3$; and
- the vertex s^* arising from contracting S is a near-sink in T/S.

From (β) we see that S is actually a homogeneous set of T. The purpose of this section is to establish the following statement.

Theorem 5.1. *For the above instance $(T, \boldsymbol{w})$, problem $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Let us label T_2/S as in Figures 3-7. Since (b_2, a_2) is a special arc, a_2 is a near-source of T_2, and s^* is a near-sink in T/S, we have
- $(b_2, a_2) = (v_1, v_2)$ and $s^* = v_3$ or v_4 if $T_2/S = F_0$;
- $(b_2, a_2) = (v_5, v_2)$ and $s^* = v_1$ if $T_2/S = F_3$;
- $(b_2, a_2) = (v_5, v_6)$ and $s^* = v_2$ if $T_2/S = F_4$;
- $(b_2, a_2) = (v_5, v_6)$ and $s^* = v_2$ if $T_2/S = F_6$;
- $(b_2, a_2) = (v_4, v_5)$ and $s^* = v_2$ if $T_2/S = G_2$ or G_3;
- $(b_2, a_2) = (v_1, v_5)$ and $s^* = v_4$ if $T_2/S = G_4$;
- $(b_2, a_2) = (v_2, v_6)$ and $s^* = v_5$ if $T_2/S = G_5$; and
- $(b_2, a_2) = (v_6, v_7)$ and $s^* = v_5$ if $T_2/S = G_6$,

where the last three follow from Lemma 2.5(ii). Observe that if $T_2/S = F_0$, then $(b_2, a_2) \neq (v_4, v_1)$, for otherwise, $T_2 \backslash v_1$ is acyclic, contradicting (α).

Since $T[S]$ is acyclic, we can label the vertices in S as $s_1, s_2, \ldots, s_r$ such that $s_j s_i$ is an arc in T for any $1 \leq i < j \leq r$, where $r = |S|$. For convenience, we use v_0 to denote the only out-neighbor of S in $T_2 \backslash a_2$ (for example, $v_0 = v_3$ if $T_2/S = F_3$), use f_i to denote the arc $s_i v_0$, and use R to denote the vertex subset $V \backslash (S \cup \{v_0\})$.

In this section, we employ the same notations as introduced in Sections 3 and 4. In particular, given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, we use $\mathcal{C}^y$ to denote $\{C \in \mathcal{C} : y(C) > 0\}$ and use $\mathcal{C}_i^y$ to denote $\{C \in \mathcal{C}_i : y(C) > 0\}$ for $i = 0, 1, 2$. For each arc e of T, we use $z(e)$ to denote $y(\mathcal{C}(e))$.

Let G be a digraph with a weight on each arc and let U be a vertex subset of G. By *reorienting $G[U]$ acyclically* we mean the operation of reorienting some arcs of $G[U]$ so that the resulting subgraph induced by U is acyclic, where each new arc is associated with the same weight as its reverse in G.

Lemma 5.2. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively. Then we may assume that the following statements hold:*

(i) $z(s_js_i) = w(s_js_i) = 0$ for any $1 \le i < j \le r$ (so if we reorient $T[S]$ acyclically, then the resulting digraph is isomorphic to T, and the optimal value of the resulting $\mathbb{D}(T, \boldsymbol{w})$ remains the same);

(ii) $x(f_i)z(f_i) > 0$ for any $1 \le i \le r$;

(iii) $z(f_i) = w(f_i) > 0$ for any $1 \le i \le r$;

(iv) $x(f_i) \ne x(f_j)$ for any $1 \le i < j \le r$;

(v) Every cycle $C \in \mathcal{C}^y$ contains at most one vertex from S; and

(vi) $z(us_i)z(us_j) = 0$ for any $u \in R$ and $1 \le i < j \le r$.

Proof. (i) Assume the contrary: $z(s_js_i) > 0$ and, subject to this, $j + i$ is minimized. Then there exists a cycle D passing through $s_js_iv_0$ with $y(D) > 0$.

Consider first the case when $x(s_js_i) = 0$. If $z(f_j) > 0$, then $x(f_j) = x(s_js_i) + x(f_i) = x(f_i)$ by Lemma 3.1(iv). If $z(f_j) = 0$, then $w(f_j) = 0$ by Lemma 3.2(i). Since $x(C) \ge 1$ for any $C \in \mathcal{C}$, we have $x(f_j) \ge x(s_js_i) + x(f_i)$; replacing $x(f_j)$ by $x(s_js_i) + x(f_i)$ if necessary, the resulting $\boldsymbol{x}$ is also an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. So we may assume that $x(f_j) = x(s_js_i) + x(f_i) = x(f_i)$. Similarly, we may assume that $x(us_j) = x(us_i)$ for any $u \in R$. Let $T' = (V', A')$ be obtained from T by deleting s_j. Note that T' also arises from T by identifying s_i with s_j and then deleting some arcs incident with s_j. Let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(f_i)$ with $w(f_i) + w(f_j)$ and replacing $w(us_i)$ with $w(us_i) + w(us_j)$ for every $u \in R$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A', and let $\boldsymbol{y}'$ be the projection of $\boldsymbol{y}$ into the set of all cycles in T'. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w})$ and $\mathbb{D}(T, \boldsymbol{w})$, respectively, having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer. It follows from Lemma 3.4(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

Next consider the case when $x(s_js_i) > 0$. By Lemma 3.1(iii), $w(s_js_i) = z(s_js_i)$. Let $\boldsymbol{w}'$ be obtained from $\boldsymbol{w}$ by replacing $w(f_j)$ with $w(f_j) + w(s_js_i)$ and replacing $w(e)$ with $w(e) - w(s_js_i)$ for $e = s_js_i$ and f_i, let $\boldsymbol{x}' = \boldsymbol{x}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through s_js_i with $y(C) > 0$, let C' be the cycle obtained from C by replacing the path $s_js_iv_0$ with f_j, and set $y'(C) = 0$ and $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T, \boldsymbol{w}')$ and $\mathbb{D}(T, \boldsymbol{w}')$, respectively, having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Since $w'(A) < w(A)$, by the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer. It follows from Lemma 3.4(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

Combining the above two cases, we may assume that $z(s_js_i) = 0$ and hence $w(z_jz_i) = 0$ by Lemma 3.2(i) for any $1 \le i < j \le r$. From (β) we see that S is a homogeneous set of T, so if we reorient $T[S]$ acyclically, then the resulting digraph is isomorphic to T. Given the weights $w(z_jz_i)$ for all $1 \le i < j \le r$, it is clear that the optimal value of the resulting $\mathbb{D}(T, \boldsymbol{w})$ remains the same.

(ii) Assume the contrary: $x(f_i)z(f_i) = 0$ for some i. Consider first the case $z(f_i) = 0$. Let $T' = (V', A')$ be obtained from T by deleting s_i, and let $\boldsymbol{w}'$ be the restriction of $\boldsymbol{w}$ to A'. Then $\mathbb{D}(T', \boldsymbol{w}')$ has an integral optimal solution by the hypothesis of Theorem 1.5. From (i) and the value of $z(f_i)$, we deduce that s_i is contained in no cycle C with $y(C) > 0$, so $\mathbb{D}(T', \boldsymbol{w}')$ has the same optimal value $\nu_w^*(T)$ as $\mathbb{D}(T', \boldsymbol{w}')$. It follows from Lemma 3.4(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution. Thus we may assume that $z(f_j) > 0$ for any $1 \leq j \leq r$.

Next consider the case when $x(f_i) = 0$. Observe that for any $u \in R$ with $uv_0 \in A$, if $z(uv_0)z(us_i) > 0$, then $x(uv_0) = x(us_i) + x(f_i) = x(us_i)$ by Lemma 3.1(iv), so $x(uv_0) = x(us_i)$; if $z(uv_0)z(us_i) = 0$, modifying $x(uv)$ for $v \in \{v_0, s_i\}$ with $z(uv) = 0$ (thus $w(uv) = 0$) so that the equality $x(uv_0) = x(us_i) + x(f_i) = x(us_i)$ holds, the resulting $\boldsymbol{x}$ is also an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. Hence we may assume that $x(uv_0) = x(us_i)$.

Set $U = \{u \in R : z(us_i) > 0 \text{ and } uv_0 \notin A\}$. Let $T' = (V', A')$ be obtained from $T \backslash s_i$ by adding an arc uv_0 for each $u \in U$, and define $w(uv_0) = w(us_i)$ and $x(uv_0) = x(us_i)$ for each $u \in U$. Let $\boldsymbol{w}'$ be obtained from $\boldsymbol{w}$ by replacing $w(uv_0)$ with $w(uv_0) + w(us_i)$ for each $u \in R$ with $uv_0 \in A$, let $\boldsymbol{x}' = \boldsymbol{x}$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle C passing through us_i with $y(C) > 0$, let C' be the cycle arising from C by replacing the path us_iv_0 with uv_0, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. In view of (i), we may assume that $i = 1$. So $f_i = f_1$ is a special arc of T. By Lemma 2.6, $T' = T/f_1$ is a Möbius-free digraph and thus, by Lemma 3.3, $\nu_w^*(T)$ is integral. It follows from Lemma 3.4(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

(iii) The statement follows directly from (ii), Lemma 3.2(i), and Lemma 3.1(iii).

(iv) Assume on the contrary that $x(f_i) = x(f_j)$ for some $1 \leq i < j \leq r$. Observe that for any $u \in R$, if $z(us_i)z(us_j) > 0$, then $x(us_i) + x(f_i) = x(us_j) + x(f_j)$ by Lemma 3.1(iv), so $x(us_i) = x(us_j)$; if $z(us_i)z(us_j) = 0$, letting (k, l) be a permutation of (i, j) with $z(us_k) = 0$, and replacing x_k by x_l if necessary, the resulting $\boldsymbol{x}$ is also an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$. So we may assume that $x(us_i) = x(us_j)$. Let $T' = (V', A')$ be obtained from T by deleting s_i, and let $\boldsymbol{w}'$ be obtained from the restriction of $\boldsymbol{w}$ to A' by replacing $w(us_j)$ with $w(us_j) + w(us_i)$ for any $u \in R$ and replacing $w(f_j)$ with $w(f_j) + w(f_i)$. Let $\boldsymbol{x}'$ be the restriction of $\boldsymbol{x}$ to A' and let $\boldsymbol{y}'$ be obtained from the restriction of $\boldsymbol{y}$ to cycles in T' as follows: for each cycle C passing through us_i with $y(C) > 0$, let C' be obtained from C by replacing the path us_iv_0 with the path us_jv_0, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer. Thus it follows from Lemma 3.4(ii) that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.

(v) Suppose on the contrary that C contains two distinct vertices s_i and s_j in S. Let s_k^+ be the vertex succeeding s_k as we traverse C in its direction, for $k = i, j$. Since $y(C) > 0$, from (i) we deduce that s_i^+ and s_j^+ are two distinct vertices outside S. Thus the vertex s^* arising from contracting S would not be a near-sink in T/S, contradicting (β).

(vi) Assume the contrary: $z(us_i)z(us_j) > 0$ for some $u \in R$ and $1 \leq i < j \leq r$. Consider first the case when $z(us_k) \geq 1$ for $k = i$ or j. In view of (i), we may assume that $z(us_i) \geq 1$. Let T' be obtained from T by adding an arc uv_0 if it is not present in T and define $w(uv_0) = 0$, and let $\boldsymbol{w}'$ be obtained from $\boldsymbol{w}$ by replacing $w(a)$ with $w(a) - \lfloor z(e) \rfloor$ for $a \in \{e, f_i\}$ and replacing $w(uv_0)$ with $w(uv_0) + \lfloor z(e) \rfloor$. Let $\boldsymbol{x}$ be an optimal solution to $\mathbb{P}(T, \boldsymbol{w})$, and let $\boldsymbol{x}'$ be obtained from $\boldsymbol{x}$ by setting $x(uv_0) = x(e) + x(f_i)$. Let $\mathcal{D}$ be the set of all cycles C passing through e with $y(C) > 0$, let $\pi(C)$ be a constant between 0 and $y(C)$ such that $\pi(\mathcal{D}) = \lfloor z(e) \rfloor$, and let $\boldsymbol{y}'$ be obtained from $\boldsymbol{y}$ as follows: for each cycle $C \in \mathcal{D}$, let C' be obtained from C by replacing the path us_iv_0 with uv_0, set $y'(C) = y(C) - \pi(C)$ and $y'(C') = y(C') + \pi(C)$. From the LP-duality theorem, we see that $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are optimal solutions to $\mathbb{P}(T', \boldsymbol{w}')$ and $\mathbb{D}(T', \boldsymbol{w}')$, respectively, having the same objective value $\nu_w^*(T)$ as $\boldsymbol{x}$ and $\boldsymbol{y}$. Let T'' be the tournament obtained from T be adding a new vertex s_0, an arc s_0v_0, and an arc uv_0 for each $u \in V \backslash \{v_0\}$. By Lemma 2.3, T'' is Möbius-free because it is the 1-sum of two smaller Möbius-free tournaments with hub v_0. By Lemma 2.6, the digraph G obtained from T'' by contracting s_0v_0 is also Möbius-free; so is T' because it is a subgraph of G. As $w(A') < w(A)$, from Lemma 3.3 we deduce that $\nu_w^*(T)$ is integral. Therefore, $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 3.4(ii).

So we may assume that $z(us_k) < 1$ for $k = i, j$. Thus $w(us_k) = \lceil z(us_k) \rceil = 1 > z(us_k)$ for $k = i, j$. It follows instantly from Lemma 3.1(ii) that $x(us_k) = 0$ for $k = i, j$. By Lemma 3.1(iv), we obtain $x(us_i) + x(f_i) = x(us_j) + x(f_j)$, and hence $x(f_i) = x(f_j)$, contradicting (iv). ∎

We break the proof of Theorem 5.1 into a series of lemmas.

Lemma 5.3. *If $T_2/S = F_6$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_5, v_6)$ and $s^* = v_2$. Clearly, $C = v_1v_3v_4v_1$ is the unique cycle contained in $T_2 \backslash v_6$, which is a triangle. Since $\tau_w(T_2 \backslash v_6) > 0$ by (α), we have $w(a) > 0$ for each arc a on C. Therefore $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution by Lemma 3.6. ∎

Lemma 5.4. *If $T_2/S = F_0$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_1, v_2)$ and $s^* = v_3$ or v_4. We only consider the case when $s^* = v_3$, as the proof in other case goes along the same line. To establish the statement, by Lemma 3.4(ii), it suffices to prove that

(1) the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral.

Let $\boldsymbol{y}$ be an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$. By Lemma 3.2(i), we have $w(e) = \lceil z(e) \rceil$ for each arc e in T. By (α) and Lemma 5.2(i) and (vi), there exists precisely one vertex s_k in S such that $z(v_1s_k) > 0$, which implies $y(v_1s_kv_4v_1) > 0$. By Lemma 5.2(i), we may assume that $s_k = s_1$, the sink of $T[S]$. Observe that T is also the 1-sum of two smaller Möbius-free tournaments T_1' and T_2' with the same hub b, where T_2' arises from T_2 by deleting $S \backslash s_1$. Since $v_1s_1v_4v_1$ is the unique cycle contained in $T_2' \backslash v_2$, which is a triangle, (1) follows instantly from Lemma 3.6. ∎

Lemma 5.5. *If $T_2/S = F_3$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

Proof. Recall that $(b_2, a_2) = (v_5, v_2)$, $s^* = v_1$, and $v_0 = v_3$. To establish the statement, by Lemma 3.4(ii), it suffices to prove that

(1) the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral.

Given an optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2) \backslash a_2\}$ for each $s_i \in S$. By Lemma 5.2(i) and (vi), we have

(2) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

(3) There exist precisely two vertices s_i's in S with $\varphi(s_i) \neq \emptyset$.

In view of (2) and the structure of F_3, there are at most two vertices s_i's in S with $\varphi(s_i) \neq \emptyset$. Suppose on the contrary that there exists precisely one vertex $s_i \in S$ with $\varphi(s_i) \neq \emptyset$. By Lemma

5.2(i), we may assume that $s_i = s_1$, the sink of $T[S]$. Let T' be obtained from T by reversing the direction of the arc v_4v_j for each j with $1 < j \le r$. Define the weight of each new arc to be zero. As $w(v_4v_j) = 0$ for each j with $1 < j \le r$ by Lemma 3.2(i), the optimal value of $\mathbb{D}(T', \boldsymbol{w})$ equals $\nu_w^*(T)$. Observe that T' is the 1-sum of two smaller Möbius-free tournaments T'_1 and T'_2 with the same hub b, where T'_2 arises from T_2 by deleting $S \backslash s_1$. Since $T'_2 = F_3$ and $\tau_w(T'_2 \backslash v_2) > 0$, statement (1) follows instantly from Lemma 4.3. So we may assume that (3) holds.

By (3) and Lemma 5.2(i), we may further assume that $\varphi(s_1) = \{v_5\}$ and $\varphi(s_2) = \{v_4\}$ for any optimal solution $\boldsymbol{y}$ to $\mathbb{D}(T, \boldsymbol{w})$.

In the remainder of our proof, we reserve $\boldsymbol{y}$ for an optimal solution to $\mathbb{D}(T, \boldsymbol{w})$ such that

(4) $y(\mathcal{C}_2)$ is maximized; and

(5) subject to (4), $(y(\mathcal{D}_q), y(\mathcal{D}_{q-1}), \dots, y(\mathcal{D}_3))$ is minimized lexicographically.

Let us make some observations about $\boldsymbol{y}$. By Lemma 5.2(v), we have

(6) $\mathcal{C}_2^y \subseteq \{v_5s_1v_3v_5, v_5s_1v_3v_4v_5, v_4s_2v_3v_4\}$.

In view of $\varphi(s_i)$ for $i = 1, 2$ and Lemma 5.2(iii), we obtain

(7) $w(v_5s_1) \ge z(v_5s_1) > 0$, $w(v_4s_2) \ge z(v_4s_2) > 0$, and $w(s_iv_3) = z(s_iv_3) > 0$ for $i = 1, 2$.

From Lemma 3.5(v) we see that

(8) if $y(v_5s_1v_3v_4v_5) > 0$, then v_3v_5 is saturated by $\boldsymbol{y}$ in T_2.

(9) If $w(v_3v_4) > 0$, then $y(v_4s_2v_3v_4)$ is a positive integer.

To justify this, observe that s_2v_3 is contained in some cycle $C \in \mathcal{C}_0^y$, for otherwise, s_2v_3 is saturated by $\boldsymbol{y}$ in T_2 and hence, by (6), we have $y(v_4s_2v_3v_4) = w(s_2v_3)$, which is a positive integer by (7). If C contains v_4s_2, then it also contains v_3v_5. By Lemma 3.5(iv), v_3v_4 is saturated by $\boldsymbol{y}$ in T_2. By (8), we have $y(v_5s_1v_3v_4v_5) = 0$. From (6) we deduce that $y(v_4s_2v_3v_4) = w(v_3v_4)$, which is a positive integer. So we assume that v_4s_2 is outside C. Furthermore, v_4s_2 is outside $\mathcal{C}_0^y$, because every cycle containing v_4s_2 passes through s_2v_3. If v_4s_2 is saturated by $\boldsymbol{y}$ in T_2, then $y(v_4s_2v_3v_4) = w(v_4s_2)$ by (6), as desired. So we assume that v_4s_2 is not saturated by $\boldsymbol{y}$ in T and that C contains v_3v_5. By Lemma 3.5(iii) and (iv), v_3v_4 is saturated by $\boldsymbol{y}$ in T_2. By (8), we have $y(v_5s_1v_3v_4v_5) = 0$. From (6) we see that $y(v_4s_2v_3v_4) = w(v_3v_4)$. Hence (9) holds.

By (9) and Lemma 3.2(iii), we may assume that $w(v_3v_4) = 0$. Let us show that

(10) $y(v_5s_1v_3v_5)$ is a positive integer.

If s_1v_3 is outside $\mathcal{C}_0^y$, then s_1v_3 is saturated by $\boldsymbol{y}$ in T_2. Thus $y(v_5s_1v_3v_5) = w(s_1v_3) > 0$. If s_1v_3 is contained in some cycle in $\mathcal{C}_0^y$, then, by Lemma 3.5(iv), v_5s_1 is saturated by $\boldsymbol{y}$ in T_2. So $y(v_5s_1v_3v_5) = w(v_5s_1) > 0$. Hence (10) holds in either case.

Using (10) and Lemma 3.2(iii), we conclude that the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, \boldsymbol{w})$ is integral, as described in (1) above. ∎

Lemma 5.6. *If $T_2/S \in \{F_4, G_2, G_3, G_4, G_5, G_6\}$, then $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution.*

The proof of this lemma goes along the same line as that of Lemma 5.5. To save space, we omit it here and refer the reader to the online appendix to our paper [6] (see Lemmas 5.7-5.9 and Lemmas 5.12-5.14).

With the aid of the above lemmas, we can now derive the desired total-dual integrality.

Proof of Theorem 5.1. By the hypothesis of this section, T is the 1-sum of two smaller strong Möbius-free tournaments T_1 and T_2 with properties (α) and (β). Since $T_2/S \in \mathcal{T}_3$, the statement follows instantly from Lemmas 5.3-5.6. ∎

6 Proof: Last Step

In the preceding two sections we have carried out a series of reduction operations, and finished the main body of the proof of Theorem 1.5. To complete the proof, we still need to consider two more cases. The following lemma is intended for case (*iii*) exhibited in Lemma 2.5.

Lemma 6.1. *Let $G = (V, A)$ be a digraph with a nonnegative integral weight $c(e)$ on each arc e, and let v be a vertex of G. If each positive cycle in G contains v, then $\mathbb{D}(G, \boldsymbol{c})$ has an integral optimal solution.*

Proof. Construct a flow network $N = (V', A')$ with vertex set $V' = (V \backslash v) \cup \{s, t\}$ as follows:
- for each arc $ab \in A$ with $a \neq v \neq b$, there is an arc $ab \in A'$ with capacity $c(ab)$;
- for each arc $va \in A$, there is an arc $sa \in A'$ with capacity $c(va)$; and
- for each arc $av \in A$, there is an arc $at \in A'$ with capacity $c(av)$.

Then there is a one-to-one correspondence between cycles containing v in G and s-t paths in N. So, by the max-flow min-cut theorem, $\mathbb{D}(G, \boldsymbol{c})$ has an integral optimal solution. ∎

Lemma 6.2. *Tournament G_1 is cycle Mengerian.*

For a computer-assisted proof of this lemma, see Appendix [6].

Proof of Theorem 1.5. Clearly, we may assume that T is strong, $T \neq C_3$, and $\tau_w(T) > 0$. Since F_1 can be obtained from G_1 by deleting vertex v_6 (see the labeling in Figure 4), from Lemma 6.2 we deduce that F_1 is also cycle Mengerian. So we may further assume that $F_1 \neq T \neq G_1$.

By Theorem 1.2, $\mathcal{T}_0 = \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$ is the list of all *i2s* Möbius-free tournaments. Hence

(1) if T is *i2s*, then $T \in \{F_0, F_2, F_3, F_4, G_2, G_3\} = \mathcal{T}_2 \backslash \{F_6\}$.

We claim that T can be expressed as a 1-sum of two strong Möbius-free tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2), such that one of the following three cases occurs:

(2) $\tau_w(T_2 \backslash a_2) > 0$ and $T_2 \in \mathcal{T}_2$;

(3) $\tau_w(T_2 \backslash a_2) > 0$ and there exists a vertex subset S of $T_2 \backslash \{a_2, b_2\}$ with $|S| \geq 2$, such that $T[S]$ is acyclic, $T_2/S \in \mathcal{T}_3$, and the vertex s^* arising from contracting S is a near-sink in T/S; and

(4) every positive cycle in T crosses the hub b of the 1-sum.

Indeed, if T is not *i2s*, then the statement follows from Lemma 2.5. It remains to consider the case when T is *i2s*. By (1), we have $T \in \mathcal{T}_2 \backslash \{F_6\}$. Since each tournament in $\mathcal{T}_2 \backslash \{F_6\}$ has a special arc, we may view T as a 1-sum of T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2), where T_1 is a triangle and $T_2 = T$. If $\tau_w(T_2 \backslash a_2) > 0$, then (2) holds. If $\tau_w(T_2 \backslash a_2) = 0$, then every positive cycle in T contains the hub of the 1-sum. So (4) occurs.

Applying Theorem 4.1, Theorem 5.1, and Lemma 6.1 to (2), (3), and (4), respectively, we conclude that $\mathbb{D}(T, \boldsymbol{w})$ has an integral optimal solution in any case. ∎

Proof of Theorem 1.4. Implication (*iii*) $\Rightarrow$ (*ii*) holds, because total-dual integrality implies primal integrality (see Edmonds-Giles theorem [11] stated in Section 1). Implication (*ii*) $\Rightarrow$ (*i*) is established in Lemma 2.1. Implication (*i*) $\Rightarrow$ (*iii*) follows instantly from Theorem 1.5. ∎

7 Concluding Remarks

In this paper we have characterized all tournaments with the min-max relation on packing and covering cycles. Our characterization yields a polynomial-time algorithm for the minimum-weight feedback arc set problem on cycle Mengerian tournaments. But this algorithm is based on the ellipsoid method for linear programming, and therefore very much unlike the typical combinatorial optimization procedures. It would be interesting to know whether it can be replaced by a strongly polynomial-time algorithm of a transparent combinatorial nature. In combinatorial optimization, there are some other min-max results that are obtained using the "structure-driven" approach. Despite availability of structural descriptions, combinatorial polynomial-time algorithms for the corresponding optimization problems have yet to be found, for instance, those on matroids with the max-flow min-cut property; see Seymour [20] for a characterization and Truemper [22] for efficient algorithms once again based on the ellipsoid method. Certainly, these types of problems deserve more research efforts.

References

[1] D. Applegate, W. Cook, and S. McCormick, Integral infeasibility and testing total dual integrality, *Oper. Res. Lett.* **10** (1991), 37-41.

[2] F. Barahona, J. Fonlupt, and A. Mahjoub, Compositions of graphs and polyhedra IV: Acyclic spanning subgraphs, *SIAM J. Discrete Math.* **7** (1994), 390-402.

[3] M. Cai, X. Deng, and W. Zang, A TDI system and its Application to approximation algorithms, in: *Proc.* 39^{th} *IEEE Symposium on Foundations of Computer Science* (FOCS), Palo Alto, CA, 1998, pp. 227-233.

[4] M. Cai, X. Deng, and W. Zang, An approximation algorithm for feedback vertex sets in tournaments, *SIAM J. Comput.* **30** (2001), 1993-2007.

[5] X. Chen, G. Ding, W. Zang, and Q. Zhao, Ranking tournaments with no errors I: Structural description, submitted.

[6] X. Chen, G. Ding, W. Zang, and Q. Zhao, Online appendix to "Ranking tournaments with no errors II: Minimax relation", and Appendix: Proof of Lemma 6.2. (see website http://www.math.lsu.edu/∼ding)

[7] G. Ding and P. Iverson, Internally 4-connected projective-planar graphs, *J. Combin. Theory Ser. B* **108** (2014), 123-138.

[8] G. Ding, L. Tan, and W. Zang, When is the matching polytope box-totally dual integral? *Math. Oper. Res.* **43** (2018), 64-99.

[9] G. Ding, Z. Xu, and W. Zang, Packing cycles in graphs, II, *J. Combin. Theory Ser. B* **87** (2003), 244-253.

[10] G. Ding and W. Zang, Packing cycles in graphs, *J. Combin. Theory Ser. B* **86** (2002), 381-407.

[11] J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs, in: *Ann. Discrete Math.* **1**, North-Holland, Amsterdam, 1977, pp. 185-204.

[12] J. Geelen and B. Guenin, Packing odd circuits in Eulerian graphs, *J. Combin. Theory Ser. B* **86** (2002), 280-295.

[13] M. Grötschel, L. Lovász, and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* **1** (1981), 169-197.

[14] B. Guenin, Circuit Mengerian directed graphs, in: *Integer Programming and Combinatorial Optimization* (Utrecht, 2001), Lecture Notes in Comput. Sci. 2081, pp. 185-195.

[15] B. Guenin, A short proof of Seymour's characterization of the matroids with the max-flow min-cut property, *J. Combin. Theory Ser. B* **86** (2002), 273-279.

[16] B. Guenin and R. Thomas, Packing directed circuits exactly, *Combinatorica* **31** (2011), 397-421.

[17] C. Lucchesi and D. Younger, A minimax theorem for directed graphs, *J. London Math. Soc.* **17** (1978), 369-374.

[18] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, New York, 1986.

[19] A. Schrijver, *Combinatorial Optimization – Polyhedra and Efficiency*, Springer-Verlag, Berlin, 2003.

[20] P. Seymour, The matroids with the max-flow min-cut property, *J. Combin. Theory Ser. B* **23** (1977), 189-222.

[21] P. Seymour, Packing circuits in eulerian digraphs, *Combinatorica* **16** (1996), 223-231.

[22] A. Truemper, Max-flow min-cut motroids: polynomial testing and polynomial algorithms for maximum flow and shortest routes, *Math. Oper. Res.* **12** (1987), 72-96.