**Online Appendix to**

**Ranking Tournaments with No Errors II: Minimax Relation**

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The purpose of this online appendix is to present proofs of Lemma 4.4 and Lemma 5.6 in the submitted version of this paper.

4 Basic Reductions

**Lemma 4.5.** If $T_2 = F_4$, then $\mathcal{D}(T, w)$ has an integral optimal solution.

**Proof.** It is routine to check that

- $\mathcal{C}_2 = \{v_1v_2v_3v_1, v_2v_3v_4v_2, v_1v_2v_3v_3, v_3v_4v_5v_3, v_1v_2v_3v_4v_1, v_1v_5v_2v_3v_1, v_1v_5v_3v_4v_1, v_2v_3v_4v_5v_2, v_1v_5v_2v_3v_1\}$ and

- $\mathcal{F}_2 = \{\{v_2v_3, v_3v_4\}, \{v_3v_4, v_3v_4\}, \{v_1v_2, v_1v_5, v_3v_4\}, \{v_2v_3, v_2v_3, v_4v_5\}, \{v_1v_2, v_2v_3, v_4v_5\}, \{v_1v_2, v_3v_4, v_5v_3\}, \{v_1v_2, v_4v_2, v_5v_3\}, \{v_2v_3, v_3v_4, v_4v_5\}, \{v_3v_1, v_4v_1, v_4v_2, v_5v_3\}, \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}\}$

We also have a computer verification of these results. So $|\mathcal{C}_2| = 9$ and $|\mathcal{F}_2| = 11$. Recall that $(b_2, a_2) = (v_5, v_6)$.

Let $y$ be an optimal solution to $\mathcal{D}(T, w)$ such that

1. $y(\mathcal{C}_2)$ is maximized;
2. subject to (1), $(y(D_q), y(D_{q-1}), \ldots, y(D_3))$ is minimized lexicographically;
3. subject to (1) and (2), $y(v_1v_5v_2v_3v_1) + y(v_1v_5v_3v_4v_1)$ is minimized;
4. subject to (1)-(3), $y(v_2v_3v_4v_5v_2)$ is minimized;
5. subject to (1)-(4), $y(v_1v_5v_2v_3v_1) + y(v_3v_4v_5v_3)$ is minimized; and
6. subject to (1)-(5), $y(v_1v_5v_3v_1)$ is minimized.

Let us make some simple observations about $y$.

(7) If $K \in \mathcal{F}_2$ satisfies $y(\mathcal{C}_2) = w(K)$, then $K$ is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

(8) If $y(v_1v_5v_2v_3v_4v_1) > 0$, then each arc in the set $\{v_1v_2, v_3v_1, v_4v_2, v_4v_5, v_5v_3\}$ is saturated by $y$ in $F_4$. Furthermore, $y(v_1v_2v_3v_1) = y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$.

To justify this, note that each arc in the given set is a chord of the cycle $v_1v_5v_2v_3v_4v_1$. So the first half follows instantly from Lemma 3.5(v). Let $\cup$ stand for the multisub sum. Then $v_1v_5v_2v_3v_4v_1 \cup v_1v_2v_3v_1 = v_1v_5v_2v_3v_1 \cup v_1v_2v_3v_4v_1$, $v_1v_5v_2v_3v_4v_1 \cup v_1v_5v_3v_1 = v_1v_5v_2v_3v_1 \cup v_1v_5v_3v_4v_1$, and $v_1v_5v_2v_3v_4v_1 \cup v_3v_4v_5v_3 = v_1v_5v_2v_3v_1 \cup v_3v_4v_5v_3$. Suppose on the contrary that $y(v_1v_2v_3v_1) > 0$. Let $\theta = \min\{y(v_1v_5v_2v_3v_1), y(v_1v_2v_3v_1)\}$ and let $y'$ be obtained from $y$ by replacing $y(v_1v_5v_2v_3v_4v_1), y(v_1v_2v_3v_1), y(v_1v_5v_3v_4v_1)$, and $y(v_1v_2v_3v_4v_1)$ with $y(v_1v_5v_2v_3v_1) - \theta$, $y(v_1v_2v_3v_1) - \theta$, $y(v_1v_5v_2v_3v_1) + \theta$, and $y(v_1v_2v_3v_4v_1) + \theta$. Then $y'$ is also an optimal solution to $\mathcal{D}(T, w)$. Since $y'(v_1v_5v_2v_3v_4v_1) < y(v_1v_5v_2v_3v_4v_1)$, the existence of $y'$ contradicts the assumption (2) on $y$. So $y(v_1v_2v_3v_1) = 0$. Similarly, $y(v_3v_4v_5v_3) = y(v_1v_5v_3v_1) = 0$.

(9) If $y(v_1v_5v_2v_3v_1) > 0$, then $v_1v_2$ and $v_3v_5$ are saturated by $y$ in $F_4$; so is $v_4v_5$ provided $y(v_1v_2v_3v_4v_1) > 0$. Furthermore, $y(v_3v_4v_5v_3) = 0$. 

1
To justify this, note that both $v_1v_2$ and $v_5v_3$ are chords of the cycle $v_1v_5v_2v_3v_1$, so they are saturated by $y$ in $F_4$ by Lemma 3.5(v). Since $v_1v_5v_2v_3v_1 \cup v_3v_4v_5v_2 = v_1v_5v_3v_1 \cup v_2v_3v_4v_5v_2$, from (3) we deduce that $y(v_3v_4v_5v_2) = 0$ (for a proof, see that of (8)).

Consider the case when $y(v_1v_2v_3v_4v_1) > 0$. If $v_4v_5$ is not saturated by $y$ in $T$, then the multiset sum of the cycles $v_1v_5v_2v_3v_1$, $v_1v_2v_3v_4v_1$, and the arc $v_4v_5$ contains two arc-disjoint cycles $v_1v_2v_3v_1$ and $v_2v_3v_4v_5v_2$; if $v_4v_5$ is saturated by $y$ in $T$ but contained in some cycle $C \in C''$, then the multiset sum of $v_1v_2v_3v_4v_1$, $v_1v_2v_3v_4v_1$, and $C$ contains three arc-disjoint cycles $v_1v_2v_3v_1$, $v_2v_3v_4v_5v_2$, and $C' = C[v_5, v_4] \cup \{v_2v_3v_1\}$. In either subcase we can obtain from $y$ an optimal solution $y'$ to $\mathcal{D}(T, w)$ that is better than $y$ by (2). So $v_4v_5$ is saturated by $y$ in $F_4$.

(10) If $y(v_1v_5v_2v_3v_1) > 0$, then both $v_3v_1$ and $v_4v_5$ are saturated by $y$ in $F_4$; so is $v_1v_2$ provided $y(v_1v_5v_2v_3v_1) > 0$, and so is $v_1v_2$ provided $y(v_2v_3v_4v_5v_2) > 0$. Furthermore, $y(v_1v_2v_3v_1) = 0$.

To justify this, note that both $v_3v_1$ and $v_4v_5$ are chords of the cycle $v_1v_5v_3v_4v_1$, so they are saturated by $y$ in $F_4$ by Lemma 3.5(v). Since $v_1v_5v_3v_4v_1 \cup v_1v_2v_3v_1 = v_1v_5v_3v_1 \cup v_2v_3v_4v_1$, from (3) we deduce that $y(v_1v_2v_3v_1) = 0$ (for a proof, see that of (8)).

Consider the case when $y(v_1v_5v_2v_3v_1) > 0$. If $v_4v_2$ is not saturated by $y$ in $T$, then the multiset sum of the cycles $v_1v_5v_2v_3v_1$, $v_1v_5v_3v_4v_1$, and the arc $v_4v_2$ contains arc-disjoint cycles $v_1v_5v_3v_1$ and $v_2v_3v_4v_2$; if $v_4v_2$ is saturated by $y$ in $T$ but contained in some cycle $C_1 \in C''$, then the multiset sum of $C_1$, $v_1v_5v_3v_1$, and $v_1v_5v_3v_4v_1$ contains three arc-disjoint cycles $v_1v_5v_3v_1$, $v_2v_3v_4v_2$, and $C'_1 = C_1[v_5, v_4] \cup \{v_1v_5, v_3v_1\}$. In either subcase we can obtain from $y$ an optimal solution $y'$ to $\mathcal{D}(T, w)$ that is better than $y$ by (2). So $v_4v_2$ is saturated by $y$ in $F_4$.

Next, consider the case when $y(v_2v_3v_4v_5v_2) > 0$. If $v_1v_2$ is not saturated by $y$ in $T$, then the multiset sum of the cycles $v_1v_5v_2v_3v_1$, $v_1v_5v_3v_4v_1$, and the arc $v_1v_2$ contains arc-disjoint cycles $v_3v_4v_5v_3$ and $v_1v_2v_3v_4v_1$; if $v_1v_2$ is saturated by $y$ in $T$ but contained in some cycle $C_2 \in C''$, then the multiset sum of $C_2$, $v_1v_5v_2v_3v_1$, and $v_1v_5v_3v_4v_1$ contains three arc-disjoint cycles $v_3v_4v_5v_3$, $v_1v_2v_3v_4v_1$, and $C'_2 = C_2[v_5, v_1] \cup \{v_1v_5\}$. In either subcase we can obtain from $y$ an optimal solution $y'$ to $\mathcal{D}(T, w)$ that is better than $y$ by (2). So $v_1v_2$ is saturated by $y$ in $F_4$.

(11) If $y(v_1v_2v_3v_4v_1) > 0$, then both $v_3v_1$ and $v_4v_2$ are saturated by $y$ in $F_4$; so is $v_4v_5$ provided $y(v_1v_5v_3v_1) > 0$.

The first half follows instantly from Lemma 3.5(v). Suppose $y(v_1v_5v_3v_1) > 0$. If $v_4v_5$ is not saturated by $y$ in $T$, then the multiset sum of the cycles $v_1v_5v_3v_1$, $v_1v_2v_3v_4v_1$, and the arc $v_4v_5$ contains arc-disjoint cycles $v_1v_2v_3v_1$ and $v_2v_3v_4v_5v_2$; if $v_4v_5$ is saturated by $y$ in $T$ but contained in some cycle $C \in C''$, then the multiset sum of $v_1v_2v_3v_4v_1$, $v_1v_5v_3v_1$, and $C$ contains three arc-disjoint cycles $v_1v_2v_3v_1$, $v_2v_3v_4v_5v_2$, and $C' = C[v_5, v_4] \cup \{v_1v_5, v_3v_1\}$. In either subcase we can obtain from $y$ an optimal solution $y'$ to $\mathcal{D}(T, w)$ that is better than $y$ by (2). So $v_4v_5$ is saturated by $y$ in $F_4$.

(12) If $y(v_2v_3v_4v_5v_2) > 0$, then both $v_4v_2$ and $v_5v_3$ are saturated by $y$ in $F_4$; so is $v_1v_2$ provided $y(v_1v_5v_3v_1) > 0$.

The first half follows instantly from Lemma 3.5(v). Suppose $y(v_1v_5v_3v_1) > 0$. If $v_1v_2$ is not saturated by $y$ in $T$, then the multiset sum of the cycles $v_1v_5v_3v_1$, $v_2v_3v_4v_5v_2$, and the arc $v_1v_2$ contains arc-disjoint cycles $v_1v_2v_3v_1$ and $v_2v_3v_4v_5v_2$; if $v_1v_2$ is saturated by $y$ in $T$ but contained in some cycle $C \in C''$, then the multiset sum of $C$, $v_2v_3v_4v_5v_2$, and $v_1v_5v_3v_1$ contains three arc-disjoint cycles $v_3v_4v_5v_3$, $v_1v_2v_3v_1$, and $C' = C[v_5, v_1] \cup \{v_1v_5\}$. In either subcase we can obtain from $y$ an optimal solution $y'$ to $\mathcal{D}(T, w)$ that is better than $y$ by (2). So $v_4v_5$ is saturated by $y$ in $F_4$.
from $y$ an optimal solution $y'$ to $\mathcal{D}(T, w)$ that is better than $y$ by (2). So $v_1v_2$ is saturated by $y$ in $F_4$.

**Claim 1.** $y(C_2) = \tau_w(F_4 \setminus v_5)$.

To justify this, observe that $v_2v_3$ is a special arc of $T$ and $v_2$ is a near-sink. By Lemma 3.4(iv), we may assume that $v_2v_3$ is saturated by $y$ in $T$. Depending on whether $v_2v_3$ is outside $C_0^y$, we distinguish between two cases.

**Case 1.1.** $v_2v_3$ is contained in some cycle in $C_0^y$.

Choose $C \in C_0^y$ that contains $v_2v_3$ and, subject to this, has the maximum number of arcs in $F_1 \setminus v_6$. We proceed by considering three subcases.

- **$C$ contains $v_1v_2$.** In this subcase, $C$ contains the path $P = v_1v_2v_3v_4v_5$. By Lemma 3.5(ii) and (iv), each arc in the set $K = \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}$ is saturated by $y$ in $F_4$. Hence, $y(C_2) = \tau_w(F_4 \setminus v_5)$.

- **$C$ contains $v_4v_2$.** In this subcase, $C$ contains the path $P = v_1v_2v_3v_4v_5$. By Lemma 3.5(ii) and (iv), each arc in the set $K = \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}$ is saturated by $y$ in $F_4$. Since no arc on $C$ (and hence on $P$) is saturated by $y$ in $F_4$, the multiset sum of $v_3v_4v_5$ and $C$ contains three arc-disjoint cycles $v_1v_2v_3v_4, v_3v_4v_5v_3$, and $C' = C[v_3, v_4] \cup \{v_1v_5\}$, from the optimality of $y$, we deduce that $\gamma(v_1v_5v_3v_4) = 0$. So $y(C_2) = w(K)$. By (7), $K$ is an MFAS and hence $y(C_2) = \tau_w(F_3 \setminus v_5)$.

- **$C$ contains neither $v_1v_2$ nor $v_4v_2$.** In this subcase, we may assume that both $v_1v_2$ and $v_4v_2$ are outside $C_0^y$, for otherwise, each cycle containing $v_1v_2$ or $v_4v_2$ passes through $v_2v_3$, and thus one of the preceding subcases occurs. Clearly, $C$ contains $v_3v_4$ or $v_3v_1$.

Assume first that $C$ contains $v_3v_4$. If $C$ contains $v_4v_1$, then it also contains $v_1v_5$. By Lemma 3.5(ii) and (iv), each arc in the set $K = \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}$ is saturated by $y$ in $F_4$. So $y(C_2) = w(K)$. By (7), $K$ is an MFAS and hence $y(C_2) = \tau_w(F_3 \setminus v_5)$. If $C$ does not contain $v_3v_1$, then $C$ contains $v_1v_5$. By Lemma 3.5(ii) and (iv), each arc in the set $\{v_4v_3, v_5v_3, v_5v_2\}$ is saturated by $y$ in $F_4$. If $v_1v_2$ is also saturated by $y$ in $F_4$, then $y(C_2) = w(K)$, where $K$ is as defined above. Again, $K$ is an MFAS and hence $y(C_2) = \tau_w(F_3 \setminus v_5)$. So we assume that $v_1v_2$ is not saturated by $y$ in $T$. Since $v_1v_2$ is outside $C_0^y$, so are $v_4v_1$ and $v_3v_4$. By Lemma 3.5(iii), both $v_4v_1$ and $v_3v_1$ are saturated by $y$ in $T$ and hence in $F_4$. Moreover, by (8)-(10), $y(v_1v_5v_2v_3v_4v_5), y(v_1v_5v_2v_3v_4v_1), y(v_1v_5v_3v_4v_1), y(v_1v_5v_3v_4v_1)$ are all 0. Since the multiset sum of the cycles $v_1v_5v_3v_4, C$, and the unsaturated arc $v_1v_2$ contains two arc-disjoint cycles $v_1v_2v_3v_4$ and $v_3v_4v_5$, $y(C_2)$ is contained in some cycle in $C_0^y$. By Lemma 3.5(vi), we have $y(v_1v_5v_3v_4v_1) = 0$. So $y(C_2) = w(J)$, where $J = \{v_3v_1, v_4v_1, v_4v_2, v_5v_2, v_5v_3\}$. By (7), $J$ is an MFAS and hence $y(C_2) = \tau_w(F_3 \setminus v_5)$.

Assume next that $C$ contains $v_3v_1$. Then $C$ contains $v_1v_5$. By Lemma 3.5(ii) and (iv), each arc in the set $\{v_1v_2, v_3v_2, v_5v_3\}$ is saturated by $y$ in $F_4$. If $v_1v_2$ is also saturated by $y$ in $F_4$, then $y(C_2) = w(K)$, where $K = \{v_1v_2, v_4v_2, v_5v_2, v_5v_3\}$. By (7), $K$ is an MFAS and hence $y(C_2) = \tau_w(F_3 \setminus v_5)$. So we assume that $v_1v_2$ is not saturated by $y$ in $F_4$ and hence in $T$ (recall that $v_4v_2$ is outside $C_0^y$). By Lemma 3.5(iv), $v_3v_4$ is outside $C_0^y$. By Lemma 3.5(iii), $v_3v_4$ is
saturated by $y$ in $T$ and hence in $F_4$. By (8) and (10)-(12), $y(v_1v_3v_2v_4v_1v_1)$, $y(v_1v_5v_2v_4v_1v_1)$, $y(v_3v_2v_3v_4v_2)$, and $y(v_3v_2v_3v_4v_2)$ are all 0. Since the multiset sum of the cycles $v_2v_3v_4v_2$, $C$, and the unsaturated arc $v_4v_2$ contains two arc-disjoint cycles $v_1v_5v_3v_1$ and $v_2v_3v_4v_2$, we have $y(v_3v_4v_5v_3) = 0$ by Lemma 3.5(vi). So $y(C_2) = w(J)$, where $J = \{v_1v_2, v_3v_4, v_5v_2, v_5v_3\}$. By (7), $K$ is an MFAS and hence $y(C_2) = \tau_w(F_3 \setminus v_2)$.

Case 1.2. $v_2v_3$ is outside $C_0^y$.

By the previous observation, $v_2v_3$ is saturated by $y$ in $F_3$ now. Note also that $v_5v_3$ is outside $C_0$. If $v_5v_3$ is saturated by $y$ in $T$, so is it in $F_4$, and hence $y(C_2) = w(K)$, where $K = \{v_2v_3, v_5v_3\}$.

By (7), $K$ is an MFAS and hence $y(C_2) = \tau_u(F_3 \setminus v_2)$. So we assume that $v_5v_3$ is unsaturated. By (8), (9), and (12), $y(v_1v_5v_2v_3v_4v_1)$, $y(v_1v_3v_2v_3v_4v_1)$, and $y(v_3v_2v_3v_4v_2)$ are all 0. Observe that both $v_3v_1$ and $v_3v_4$ are outside $C_0^y$, for otherwise, since each cycle passing through $v_3v_1$ or $v_3v_4$ contains $v_1v_5$ or $v_4v_5$, from Lemma 3.5(iv) we deduce that $v_5v_3$ is saturated, a contradiction. If both $v_3v_1$ and $v_3v_4$ are saturated by $y$ in $F_3$, then $y(C_2) = w(J)$, where $J = \{v_3v_1, v_3v_4\}$. By (7), $J$ is an MFAS and hence $y(C_2) = \tau_u(F_3 \setminus v_2)$. So we assume that (13) at most one of $v_3v_1$ and $v_3v_4$ is saturated by $y$ in $F_4$.

Since $C_0^y \neq \emptyset$, there is a cycle $C \in C_0^y$ passing through $v_4v_1$, or $v_1v_5$, or $v_4v_5$; subject to this, let $C$ be chosen to have the maximum number of arcs in $F_4 \setminus v_6$. We proceed by considering three subcases.

- $C$ contains both $v_4v_1$ and $v_1v_5$. In this subcase, since $v_5v_3$ is unsaturated, by Lemma 3.5(iii), $v_4v_1$ and $v_3v_4$ are both saturated by $y$ in $F_4$, a contradiction.

- $C$ contains $v_4v_1$ but not $v_1v_5$. In this subcase, from the choice of $C$, we see that $v_4v_1$ is outside $C_0^y$, because every cycle containing $v_4v_1$ passes through $v_1v_5$. Since $v_5v_3$ is unsaturated, Lemma 3.5(iii) implies that $v_3v_1$ is saturated by $y$ in $F_4$, and thus $v_3v_4$ is not saturated by $y$ in $F_4$ and hence in $T$ by (13). Once again, by Lemma 3.5(iii), $v_4v_1$ is saturated by $y$ in $F_4$, and $v_4v_3$ is outside $C_0^y$. Since both $v_5v_3$ and $v_3v_4$ are unsaturated, it follows from Lemma 3.5(i) that $v_4v_3$ is saturated by $y$ in $F_4$. If $v_4v_3$ is also saturated by $y$ in $F_3$, then $y(C_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_4v_2, v_5v_3\}$. By (7), $K$ is an MFAS and hence $y(C_2) = \tau_u(F_4 \setminus v_6)$. If $v_4v_3$ is not saturated by $y$ in $F_3$, then $y(v_1v_2v_3v_4v_1) = 0$ by (11). Moreover, since the multiset sum of the cycles $v_1v_2v_3v_4v_1$, and the unsaturated arcs $v_5v_3$, $v_3v_4$, and $v_4v_2$ contains two arc-disjoint cycles $v_3v_2v_3v_4v_2$ and $v_1v_5v_3v_1$, we have $y(v_1v_2v_3v_4v_1) = 0$ by Lemma 3.5(vi). Therefore, $y(C_2) = w(J)$, where $J = \{v_2v_3, v_3v_1, v_4v_1, v_4v_3\}$. By (7), $J$ is an MFAS and hence $y(C_2) = \tau_u(F_3 \setminus v_2)$.

- $C$ contains $v_1v_5$. In this subcase, we may assume that both $v_4v_1$ and $v_5v_3$ are outside $C_0^y$, otherwise one of the preceding subcases occurs. By Lemma 3.5(iii), $v_4v_1$ is saturated by $y$ in $T$ and hence in $F_4$, which together with (13) implies that $v_3v_1$ is not saturated by $y$ in $F_4$. Using (10) and (11), we deduce that $y(v_1v_5v_3v_4v_1v_1) = y(v_1v_2v_3v_4v_1) = 0$. Using Lemma 3.5(iii) and the triangle $v_1v_5v_3v_1$, we see that $v_1v_5$ is outside $C_0^y$. Using Lemma 3.5(i) and the triangle $v_1v_5v_3v_1$, we also deduce that $v_1v_5$ is saturated by $y$ in $T$ and hence in $F_4$. If $v_1v_5$ is also saturated by $y$ in $F_4$, then $y(C_2) = w(K)$, where $K = \{v_1v_2, v_1v_3, v_3v_4\}$. By (7), $K$ is an MFAS and hence $y(C_2) = \tau_u(F_4 \setminus v_6)$. So we assume that $v_1v_5$ is not saturated by $y$ in $F_4$ and hence in $T$, because $v_1v_5$ is outside $C_0^y$, by the hypothesis of the present case. Since the multiset sum of the cycles $C$, $v_2v_3v_4v_2$, and unsaturated arcs $v_5v_3$, $v_3v_1$, and $v_1v_5$ contains two arc-disjoint cycles $v_1v_2v_3v_1$ and $v_3v_4v_5v_3$, we have $y(v_1v_2v_3v_4v_2) = 0$ by Lemma 3.5(vi). It follows that $y(C_2) = w(J)$, where $J = \{v_1v_5, v_2v_3, v_4v_1\}$. By (7), $J$ is an MFAS and hence $y(C_2) = \tau_u(F_4 \setminus v_6)$. This completes the proof of Claim 1.
Claim 2. $y(C)$ is integral for all $C \in C_2$ or $\nu_\nu^w(T)$ is an integer.

To justify this, let $G_2 = F_2 \setminus \{v_1v_5, v_2v_3, v_4v_5, v_1v_2, v_1v_3, v_4v_2, v_4v_5\}$. From the proof of Claim 1, we see that $y(C_2) = w(K)$ for some $K \in G_2$. Observe that if $y(C_2) = w(J)$ for $J = \{v_1v_5, v_2v_3, v_4v_5\}$ or $\{v_1v_2, v_1v_3, v_4v_2, v_4v_5\}$, then both $v_1v_5$ and $v_4v_5$ are saturated by $y$ in $F_1$, so $C_0^y = \emptyset$ in this case, which has been excluded by Lemma 3.2(ii).

Let us make some further observations about $y$.

(14) $y(v_1v_5v_2v_3v_4v_1) = 0$.

Suppose on the contrary that $y(v_1v_5v_2v_3v_4v_1) > 0$. By (8), we have $y(v_1v_2v_3v_1) = y(v_3v_4v_5v_2) = y(v_1v_5v_3v_1) = 0$, and each arc in the set $\{v_1v_2, v_2v_3, v_4v_2, v_4v_5, v_5v_3\}$ is saturated by $y$ in $F_4$. So $y(C_2(v_1v_2)) = w(v_1v_2), y(C_2(v_3v_1)) = w(v_3v_1), y(C_2(v_4v_2)) = w(v_4v_2), y(C_2(v_4v_5)) = w(v_4v_5)$, and $y(C_2(v_5v_3)) = w(v_5v_3)$.

It follows that $y(v_1v_2v_3v_4v_1) = w(v_1v_2), y(v_1v_5v_2v_3v_1) = w(v_3v_1), y(v_2v_3v_4v_1) = w(v_4v_2), y(v_2v_3v_4v_5v_2) = w(v_4v_5), y(v_1v_5v_3v_4v_1) = w(v_3v_5)$. Given the above equations and (14), to prove that $y(C)$ is integral for all $C \in C_2$, it suffices to show that one of $y(v_1v_3v_4v_5v_1), y(v_1v_5v_2v_3v_1)$, and $y(v_1v_5v_3v_4v_1)$ is integral.

By Lemma 3.1 and Claim 1, each arc $e \in K$ satisfies $w(e) = y(C_2(e))$. Let us proceed by considering four subcases.

If $v_2v_3 \in K$, then $w(v_2v_3) = y(C_2(v_2v_3)) = y(v_2v_3v_4v_2) + y(v_1v_5v_3v_1)$ is integral.

If $v_3v_1 \in K$, then $w(v_3v_1) = y(C_2(v_3v_1)) = y(v_2v_3v_4v_2) + y(v_1v_5v_3v_1)$ is integral.

If $v_4v_1 \in K$, then $w(v_4v_1) = y(C_2(v_4v_1)) = y(v_2v_3v_4v_2) + y(v_1v_5v_3v_1)$ is integral.

If $v_4v_2 \in K$, then $w(v_4v_2) = y(C_2(v_4v_2)) = y(v_1v_5v_3v_1) + y(v_2v_3v_4v_2)$ is integral.

Since each $K \in G_2$ contains at least one arc in the set $\{v_2v_3, v_3v_4, v_4v_1, v_5v_2\}$, it follows that $y(C)$ is integral for all $C \in C_2$. So $y(v_1v_5v_2v_3v_1)$ is a positive integer, and hence $\nu_\nu^w(T)$ is an integer by Lemma 3.2(iii). Therefore we may assume that (15) holds.

Depending on what $K \in G_2$ is, we distinguish among three cases.

Case 2.1. $K = \{v_1v_5, v_2v_3, v_4v_4\}$.

In this case, by Lemma 3.1(i) and (iii), we have $y(v_2v_3v_4v_2) = y(v_1v_2v_3v_4v_1) = y(v_2v_3v_4v_1) = y(v_1v_5v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = 0$, and $y(C_2(c)) = y(C_2(c))$ for each $c \in K$, which together with (14) yields $w(v_1v_5) = y(C_2(v_1v_5)) = y(v_1v_5v_3v_1), w(v_2v_3) = y(C_2(v_2v_3)) = y(v_1v_5v_3v_1)$, and $w(v_3v_4) = y(C_2(v_3v_4)) = y(v_3v_5v_3v_5)$. So $y(C)$ is integral for all $C \in C_2$.

Case 2.2. $K = \{v_1v_2, v_2v_3, v_4v_2, v_5v_3\}$.

In this case, by Lemma 3.1(i) and (iii), we have $y(v_1v_5v_3v_4v_1) = y(v_3v_4v_5v_1) = y(v_1v_5v_3v_4v_1) = y(v_2v_3v_4v_5v_2) = 0$, which together with (14) yields $w(v_1v_2) = y(C_2(v_1v_2)) = y(v_1v_2v_3v_1)$,
$w(v_2v_4) = y(C_2(v_2v_4)) = y(v_2v_3v_4v_2)$, $w(v_5v_2) = y(C_2(v_5v_2)) = y(v_1v_5v_2v_3v_1)$, and $w(v_5v_3) = y(C_2(v_5v_3)) = y(v_1v_5v_3v_1)$. So $y(C)$ is integral for all $C \in C_2$.

**Case 2.3.** $K = \{v_2v_3, v_3v_1, v_4v_1, v_4v_3\}$.

In this case, by Lemma 3.1(i) and (iii), we have $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = y(v_1v_2v_3v_4v_1)$ and $w(v_2v_3) = 0$, which together with (14) yields $w(v_2v_3) = y(C_2(v_2v_3)) = y(v_2v_3v_4v_2)$, $w(v_3v_1) = y(C_2(v_3v_1)) = y(v_1v_5v_3v_1)$, $w(v_4v_1) = y(C_2(v_4v_1)) = y(v_1v_5v_3v_4v_1)$, and $w(v_4v_3) = y(C_2(v_4v_3)) = y(v_3v_4v_5v_3)$.

So $y(C)$ is integral for all $C \in C_2$.

**Case 2.4.** $K = \{v_3v_1, v_4v_1, v_2v_2, v_5v_2, v_5v_3\}$.

In this case, by Lemma 3.1(i) and (iii), we have $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = y(v_1v_5v_2v_4v_1) = 0$, which together with (14) yields $w(v_2v_3) = y(C_2(v_2v_3)) = y(v_2v_3v_4v_2)$, $w(v_3v_1) = y(C_2(v_3v_1)) = y(v_1v_5v_3v_1)$, $w(v_4v_1) = y(C_2(v_4v_1)) = y(v_1v_5v_3v_4v_1)$, and $w(v_4v_3) = y(C_2(v_4v_3)) = y(v_3v_4v_5v_3)$. So $y(C)$ is integral for all $C \in C_2$.

**Case 2.5.** $K = \{v_1v_2, v_1v_5, v_2v_3\}$.

In this case, by Lemma 3.1(i) and (iii), we have $y(v_1v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = y(v_1v_5v_2v_4v_1) = 0$ and $w(e) = y(C_2(e))$ for each $e \in K$, which together with (14) yields the following three equations:

$w(v_1v_2) = y(C_2(v_1v_2)) = y(v_1v_2v_3v_1)$;

$w(v_1v_5) = y(C_2(v_1v_5)) = y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1)$; and

$w(v_2v_4) = y(C_2(v_2v_4)) = y(v_2v_3v_4v_2) + y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_3)$. 

Depending on the value of $y(v_1v_2v_3v_1)$, we consider two subcases.

- $y(v_1v_2v_3v_1) = 0$. In this subcase, $y(v_1v_5v_3v_1) = w(v_1v_5)$. If $y(v_2v_3v_4v_5v_3) > 0$, then $w(v_2v_3) = y(C_2(v_2v_3)) = y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3)$ and $w(v_4v_2) = y(C_2(v_4v_2)) = y(v_2v_3v_4v_2)$ by (12). Thus both $y(v_3v_4v_5v_3)$ and $y(v_2v_3v_4v_5v_3)$ are integral, and hence $y(C)$ is integral for all $C \in C_2$. So we assume that $y(v_2v_3v_4v_5v_3) = 0$. Then $w(v_3v_1) = y(v_2v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_3)$. If $y(v_2v_3v_4v_2)$ is an integer, then $y(C)$ is integral for all $C \in C_2$. So we further assume that $y(v_2v_3v_4v_2)$ is not integral. Thus $y((v_3v_4v_5v_3) + y(v_3v_4v_5v_3)) = 1$. Since each arc in $K$ is saturated by $y$ in $F_4$, both $v_2v_3$ and $v_3v_2$ are outside $C_0^0$. Let $y'$ be obtained from $y$ by replacing $y(v_2v_3v_4v_2)$ and $y(v_3v_4v_5v_3)$ with $y(v_2v_3v_4v_2) + [y(v_3v_4v_5v_3)]$ and $[y(v_3v_4v_5v_3)]$ respectively. Then $y'$ is also an optimal solution to $\mathbb{D}(T, w)$. Since $y'(v_3v_4v_5v_3) > 0$, the existence of $y'$ contradicts the assumption (5) on $y$.

- $y(v_1v_2v_3v_1) > 0$. In this subcase, $y(v_2v_3v_4v_5v_3) = 0$ and $v_3v_3$ is saturated by $y$ in $F_4$ by (9). So $w(v_2v_3) = y(v_2v_3v_4v_2) + [y(v_2v_3v_4v_2)]$ and $w(v_3v_5) = y(v_1v_5v_3v_1)$. It follows that $y(v_1v_5v_3v_1) = w(v_1v_5) - w(v_3v_5)$. If $y(v_2v_3v_4v_2) = 0$, then $y(v_2v_3v_4v_2) = w(v_2v_3);$ otherwise, by (12), both $v_1v_5$ and $v_3v_2$ are saturated by $y$ in $F_4$. Thus $y(v_2v_3v_4) = w(v_2v_4)$ and $y(v_2v_3v_4v_5v_3) = w(v_2v_3) - w(v_2v_4)$. So $y(C)$ is integral for all $C \in C_2$.

**Case 2.6.** $K = \{v_3v_1, v_1v_4, v_1v_2, v_2v_3, v_5v_3\}$.

In this case, by Lemma 3.1(iii), we have $w(e) = y(C_2(e))$ for each $e \in K$, which together with (14) yields the following four equations:

$w(v_3v_1) = y(C_2(v_3v_1)) = y(v_1v_2v_3v_1) + y(v_1v_5v_3v_1) + y(v_1v_5v_2v_3v_1)$;

$w(v_1v_4) = y(C_2(v_4v_1)) = y(v_2v_3v_4v_2) + y(v_1v_5v_2v_3v_1)$;

$w(v_4v_2) = y(C_2(v_4v_2)) = y(v_2v_3v_4v_2);$ and

$w(v_4v_3) = y(C_2(v_4v_3)) = y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_3)$.

Depending on the values of $y(v_1v_2v_3v_1)$ and $y(v_1v_2v_3v_1)$, we consider three subcases.

- $y(v_1v_2v_3v_1) > 0$. In this subcase, by (10) and (15), we have $y(v_1v_2v_3v_1) = y(v_1v_2v_3v_1) = 0$. So $y(v_1v_2v_3v_1) = w(v_3v_1)$. If $y(v_2v_3v_4v_2) > 0$, then both $v_1v_2$ and $v_3v_3$ are saturated by $y$.
in $F_4$ by (10) and (12). So $w(v_1v_2) = y(C_2(v_1v_2)) = y(v_1v_2v_3v_4v_1)$ and $w(v_3v_4) = y(C_2(v_3v_4)) = y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_1) + y(v_1v_2v_3v_4v_1)$. Since $y(v_1v_2v_3v_4v_1) = w(v_1v_2) - y(v_1v_2v_3v_4v_1)$ and $y(v_2v_3v_4v_1) = w(v_1v_2) - y(v_1v_2v_3v_4v_1)$, it follows that $y(v_1v_2v_3v_4v_1) = y(v_3v_4v_1)$, and $y(v_2v_3v_4v_1)$ are all integral. So we assume that $y(v_2v_3v_4v_1) = w(v_1v_2)$. Since each arc in $K$ is saturated by $y$ in $F_4$, both $v_1v_2$ and $v_2v_3$ are outside $C_0^y$.

By Lemma 3.2(i), we may assume that $w(e) = [z(e)]$ for all arcs $e$ in $T$. Thus, from (3) we deduce that $y(v_1v_2v_3v_4v_1) = \min\{w(v_1v_2), w(v_2v_3) - w(v_4v_2)\}$ and $y(v_1v_2v_3v_4v_1) = w(v_1v_2) - y(v_1v_2v_3v_4v_1)$. Therefore $y(C)$ is integral for all $C \in C_2$.

- $y(v_1v_2v_3v_4v_1) > 0$. In this subcase, from (9) and (15), we deduce that $y(v_1v_2v_3v_4v_1) = y(v_1v_2v_3v_4v_1) = 0$, and that both $v_1v_2$ and $v_3v_4$ are saturated by $y$ in $F_4$. So $y(v_1v_2v_3v_4v_1) = w(v_1v_2) y(v_2v_3v_4v_1) = w(v_1v_2) y(v_2v_3v_4v_1) = y(v_1v_2v_3v_4v_1)$, and $w(v_3v_4) = y(C_2(v_3v_4)) = y(v_1v_2v_3v_4v_1)$. Thus $y(v_1v_2v_3v_4v_1) = w(v_1v_2) - w(v_4v_2)$ is integral, so is $y(v_1v_2v_3v_4v_1)$. Therefore $y(C)$ is integral for all $C \in C_2$.

- $y(v_1v_2v_3v_4v_1) = 0$. In this subcase, $y(v_1v_2v_3v_4v_1) = w(v_1v_2)$. Suppose $y(v_2v_3v_4v_1) > 0$. Then $v_2v_3$ is saturated by $y$ in $F_4$ by (12). So $w(v_2v_3) = y(C_2(v_2v_3)) = y(v_1v_2v_3v_4v_1) + y(v_1v_2v_3v_4v_1)$. If $y(v_1v_2v_3v_4v_1) > 0$, then $v_1v_2$ is saturated by $y$ in $F_4$ by (12). So $w(v_1v_2) = y(C_2(v_1v_2)) = y(v_1v_2v_3v_4v_1) + y(v_1v_2v_3v_4v_1)$. It follows that $y(v_1v_2v_3v_4v_1)$ and hence $y(C)$ is integral for any $C \in C_2$. If $y(v_1v_2v_3v_4v_1) = 0$, then $y(v_1v_2v_3v_4v_1) = y(v_3v_4)$, which implies that $y(C)$ is integral for any $C \in C_2$. So we assume that $y(v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_4v_1) = 0$. Observe that $y(v_1v_2v_3v_4v_1)$ is integral, for otherwise, let $y'$ be obtained from $y$ by replacing $y(v_1v_2v_3v_4v_1)$ and $y(v_1v_2v_3v_4v_1)$ with $y(v_1v_2v_3v_4v_1) + [y(v_1v_2v_3v_4v_1)]$ and $[y(v_1v_2v_3v_4v_1)]$, respectively. Since $v_1v_2$ and $v_2v_3$ are outside $C_0^y$, we see $y'$ is also an optimal solution to $D(T,u)$. Since $y'(v_1v_2v_3v_4v_1) < y(v_1v_2v_3v_4v_1)$, the existence of $y'$ contradicts the assumption (5) on $y$. From the above observation, it is easy to see that $y(C)$ is integral for any $C \in C_2$.

**Case 2.7.** $K = \{v_1v_2, v_3v_4, v_3v_4, v_3v_4, v_3v_4\}$.

In this case, by Lemma 3.1(iii), we have $w(e) = y(C_2(e))$ for each $e \in K$, which together with (14) yields the following four equations:

$w(v_1v_2) = y(C_2(v_1v_2)) = y(v_1v_2v_3v_4v_1)$; $w(v_2v_3) = y(C_2(v_2v_3)) = y(v_1v_2v_3v_4v_1)$; $w(v_3v_4) = y(C_2(v_3v_4)) = y(v_1v_2v_3v_4v_1)$; $w(v_4v_1) = y(C_2(v_4v_1)) = y(v_1v_2v_3v_4v_1)$.

Depending on the values of $y(v_1v_2v_3v_4v_1)$ and $y(v_1v_2v_3v_4v_1)$, we consider three subcases.

- $y(v_1v_2v_3v_4v_1) > 0$. In this subcase, by (10) and (15), $y(v_1v_2v_3v_4v_1) = y(v_1v_2v_3v_4v_1) = 0$ and both $v_1v_2$ and $v_4v_1$ are saturated by $y$ in $F_4$. So $w(v_1v_2v_3v_4v_1) = w(v_1v_2v_3v_4v_1) = w(v_1v_2v_3v_4v_1) = y(C_2(v_1v_2v_3v_4v_1)) = w(v_1v_2v_3v_4v_1) = y(C_2(v_1v_2v_3v_4v_1)) = w(v_1v_2v_3v_4v_1)$. Thus $y(v_1v_2v_3v_4v_1)$ is also integral.

- $y(v_1v_2v_3v_4v_1) = 0$. In this subcase, by (9) and (15), we have $y(v_1v_2v_3v_4v_1) = y(v_1v_2v_3v_4v_1) = 0$. So $y(v_1v_2v_3v_4v_1) = y(v_1v_2v_3v_4v_1)$, and both $v_1v_2$ and $v_4v_1$ are saturated by $y$ in $F_4$. So $w(v_1v_2v_3v_4v_1) = w(v_1v_2v_3v_4v_1) = w(v_1v_2v_3v_4v_1) = y(C_2(v_1v_2v_3v_4v_1)) = w(v_1v_2v_3v_4v_1)$. It follows that $y(C)$ is integral for all $C \in C_2$. So we assume that $y(v_1v_2v_3v_4v_1) = 0$. Then $y(v_1v_2v_3v_4v_1)$, $y(v_2v_3v_4v_1)$, and $y(v_1v_2v_3v_4v_1)$ are integral, and $y(v_1v_2v_3v_4v_1) + y(v_2v_3v_4v_1) = w(v_1v_2v_3v_4v_1)$. If $y(v_2v_3v_4v_1)$ is an integer, then $y(C)$ is integral for any $C \in C_2$. So we assume that $y(v_2v_3v_4v_1)$ is not integral. We propose to show that
(16) \( \nu_w^*(T) \) is an integer.

To justify this, let \( x \) be an optimal solution to \( P(T, w) \). By Lemma 3.2(iii), we may assume that \( w(v_1v_2) = w(v_4v_5) = 0 \). Thus \( y(C) = 0 \) for all \( C \in C_2 \setminus \{v_1v_5v_3v_1, v_2v_3v_4v_5v_2\} \). Observe that \( v_3v_4 \) is outside \( C_0 \), for otherwise, let \( D \) be a cycle in \( C_0 \) that contains \( v_3v_4 \). It is then easy to see that \( \nu_y^*(T) \) to \( D(T, w) \) can be obtained from \( y \) by modifying \( y(D) \), \( y(v_1v_5v_3v_1), \) and \( y(v_2v_3v_4v_5v_2) \) and by possibly rerouting \( D \), so that \( y'(v_1v_5v_3v_1) < y(v_1v_5v_3v_1) \), contradicting (3). Since \( y(v_2v_3v_4v_5v_2) < w(v_3v_4) \), we have \( x(v_3v_4) = 0 \) by Lemma 3.1(ii). Since both \( y(v_1v_5v_3v_1) \) and \( y(v_2v_3v_4v_5v_2) \) are positive, \( x(v_3v_1) + x(v_1v_3) = x(v_3v_4) + x(v_4v_5) \) by Lemma 3.1(i). So \( x(v_3v_5) = x(v_3v_1) + x(v_1v_3) \).

Let \( u \) be an optimal solution to \( P(T) \) by identifying \( v_3 \) and \( v_4 \); the resulting vertex is still denoted by \( v_4 \). Let \( w' \) be obtained from the restriction of \( w \) to \( A(T') \) by replacing \( w(uv_4) \) with \( w(uv_3) + w(uv_4) \) for each \( u \in V(T_2) \setminus \{b, a_1\} \). Note that \( T' \) is Möbius-free by Lemma 2.7, \( x \) corresponds to a feasible solution \( x' \) to \( P(T', w') \), and \( y \) corresponds to a feasible solution \( y' \) to \( P(T', w') \) with \( y'(v_1v_3v_4) = y'(v_2v_4v_5) = 0 \), both having the same objective value \( \nu_{w'}^*(T') \) as \( x \) and \( y \). So \( x' \) and \( y' \) are optimal solutions to \( P(T, w) \) and \( D(T, w) \), respectively. By Lemma 3.3, the optimal value \( \nu_w^*(T) \) of \( P(T', w') \) is integral. So (16) is established.

\[ \bullet \] \( y'(v_1v_5v_3v_1) = y'(v_1v_5v_3v_1) = 0 \). In this subcase, \( y(v_2v_3v_4v_5) \) and \( y(v_2v_3v_4v_5v_2) \) are integral. Assume first that \( y(v_1v_2v_3v_4v_1) > 0 \). Then, by (11), the arc \( v_3v_1 \) is saturated by \( y \) in \( F_4 \). So \( w(v_3v_1) = y(C_2(v_3v_1)) = y(v_1v_2v_3v_1) + y(v_1v_2v_3v_1). \) If \( y(v_1v_2v_3v_1) = 0 \), then \( y(v_3v_4v_5v_3) = w(v_3v_4) \). So \( y(C) \) is integral for any \( C \in C_2 \). If \( y(v_1v_2v_3v_4v_1) > 0 \), then \( v_3v_4 \) is saturated by \( y \) in \( F_4 \) by (11). Thus \( w(v_4v_3) = y(C_2(v_4v_3)) \). So \( y(v_3v_4v_5v_3) = w(v_3v_4) \), which is integral. It follows that \( y(v_3v_4v_5v_3) = w(v_3v_4) = w(v_3v_4) \). So \( y(C) \) is integral for any \( C \in C_2 \). Assume next that \( y(v_1v_2v_3v_4v_1) = 0 \). Then \( y(v_1v_2v_3v_1) \) is integral and \( y(v_1v_2v_3v_1) + y(v_3v_4v_5v_3) = w(v_3v_4) \). Clearly, we may assume that neither \( y(v_1v_3v_2v_1) \) nor \( y(v_3v_4v_5v_3) \) is integral, otherwise we are done. Similar to (16), we can show that

(17) \( \nu_w^*(T) \) is an integer.

The proof goes along the same line as that of (16). In fact, we only need to replace \( y(v_1v_5v_3v_1) \) and \( y(v_2v_3v_4v_5v_2) \) with \( y(v_1v_5v_3) \) and \( y(v_3v_4v_5v_3) \), respectively. So we omit the details here.

**Case 2.8.** \( K = \{v_2v_3, v_3v_4\} \).

In this case, by Lemma 3.1(iii), we have \( w(e) = y(C_2(e)) \) for each \( e \in K \), which together with (14) yields the following two equations:

\[ w(v_2v_3) = y(v_1v_2v_3v_1) + y(v_2v_3v_4v_5v_2) + y(v_2v_3v_4v_5v_2) + y(v_1v_5v_3v_1); \] and

\[ w(v_3v_4) = y(v_1v_2v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_2v_3v_1). \]

Since \( v_2v_3 \) is saturated by \( y \) in \( F_4 \), we have \( w(uv_2) = z(uv_2) = 0 \) for any \( u \in V(T_2) \setminus \{b, a_1\} \) in this case. Depending on the values of \( y(v_1v_5v_3v_1v_4) \) and \( y(v_1v_5v_3v_1v_4) \), we consider three
subcases.

- \( y(v_1v_2v_3v_4v_5) > 0 \). In this subcase, from (10) and (15) we deduce that \( y(v_1v_2v_3v_4v_5) = y(v_1v_5v_2v_3v_4v_1) = 0 \) and that both \( v_3v_1 \) and \( v_4v_5 \) are saturated by \( y \) in \( F_4 \). So \( y(v_3v_4v_5v_3) + y(v_2v_3v_4v_5v_2) = w(v_4v_5) \) and \( y(v_1v_5v_2v_3v_1) = w(v_5v_1) \). If \( y(v_2v_3v_4v_5v_2) > 0 \), then both \( v_1v_2 \) and \( v_4v_2 \) are saturated by \( y \) in \( F_4 \) by (10) and (12). Thus \( y(v_2v_3v_4v_2) = w(v_4v_2) \) and \( y(v_1v_2v_3v_4v_1) = w(v_2v_3) - w(v_4v_2) \). Moreover, \( y(v_2v_3v_4v_2) = w(v_4v_2) \) and \( y(v_1v_2v_3v_4v_1) = w(v_2v_3) - w(v_4v_2) \) if \( y(v_1v_2v_3v_4v_1) > 0 \), and \( y(v_2v_3v_4v_2) = w(v_2v_3) \) otherwise. Therefore \( y(C) \) is integral for all \( C \in C_2 \), no matter whether if \( y(v_1v_2v_3v_4v_5) > 0 \).

- \( y(v_1v_5v_2v_3v_1) > 0 \). In this subcase, by (9) and (15) we deduce that \( y(v_3v_4v_5v_3) = y(v_1v_5v_2v_3v_1) = 0 \) and that \( v_1v_2 \) is saturated by \( y \) in \( F_4 \). So \( y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1) = w(v_1v_2) \). If \( y(v_1v_2v_3v_1) > 0 \), then \( v_2v_1 \), \( v_4v_2 \), and \( v_4v_5 \) are saturated by \( y \) in \( F_4 \) by (9) and (11). So \( y(v_2v_3v_4v_2) = w(v_4v_2) \), \( y(v_2v_3v_4v_5v_2) = w(v_4v_5) \), and \( y(v_2v_3v_4v_1) = y(v_1v_5v_2v_3v_1) = w(v_3v_1) \). It follows that \( y(v_1v_2v_3v_4v_1) = y(v_1v_2v_3v_4v_1) + y(v_1v_5v_2v_3v_1) = w(v_2v_3) \). By Lemma 3.4(i), \( \mathbb{D}(T, w) \) has an integral optimal solution by Lemma 3.4(i).

- \( y(v_1v_5v_2v_3v_1) = y(v_1v_5v_2v_3v_1) = 0 \). In this subcase, depending on whether \( y(v_2v_3v_4v_5v_2) > 0 \), we distinguish between two subsubcases.

(a) We first assume that \( y(v_2v_3v_4v_5v_2) > 0 \). Now, in view of (12), \( v_1v_2 \) is saturated by \( y \) in \( F_4 \), which yields \( w(v_4v_2) = y(v_2v_3v_4v_2) \). If \( y(v_1v_5v_2v_3v_1) > 0 \), then \( v_2v_1 \) is saturated by \( y \) in \( F_4 \). So \( y(v_1v_2v_3v_1) + y(v_1v_2v_3v_4v_1) = w(v_1v_2) \) and \( y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_1v_2) \). Thus \( w(v_1v_2) = [z(v_1v_2)] = z(v_1v_2) \) for \( i = 1, 4, 5 \). By Lemma 3.4(i), \( \mathbb{D}(T, w) \) has an integral optimal solution. So we assume that \( y(v_1v_2v_3v_1) = 0 \). If \( y(v_1v_2v_3v_1) = 0 \), then \( y(v_1v_2v_3v_1) + y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_1v_2) \). Since \( y \) satisfies (1), we have \( y(v_1v_2v_3v_1) = \min\{w(v_1v_2), w(v_3v_1)\} \) and \( y(v_2v_3v_4v_5v_2) = w(v_2v_3) - w(v_1v_2) = y(v_1v_2v_3v_4v_1) + y(v_1v_2v_3v_4v_1) - w(v_1v_2) \). Assume \( y(v_1v_2v_3v_1) \) is not integral. Then \( y[v_1v_2v_3v_4v_1] + y[v_2v_3v_4v_5v_2] = 1 \). We propose to show that

\[ (18) v_4v_1 \text{ is saturated by } y \text{ in } F_4. \]

Suppose the contrary. If \( v_4v_1 \) is not saturated by \( y \) in \( T \), we set \( \theta = \min\{w(v_4v_1) - z(v_1v_4), y[v_2v_3v_4v_5v_2]\} \), and let \( y' \) arise from \( y \) by replacing \( y(v_1v_2v_3v_4v_1) \) and \( y(v_2v_3v_4v_5v_2) \) with \( y'(v_1v_2v_3v_4v_1) + \theta \) and \( y'(v_2v_3v_4v_5v_2) - \theta \), respectively. Since \( v_1v_2 \) is outside \( C_0 \), \( y' \) is also an optimal solution to \( \mathbb{D}(T, w) \), contradicting (4). If \( v_4v_1 \) is saturated by \( y \) in \( T \) but contained in a cycle \( C \in C_0 \), let \( C' = C[v_5, v_4] \cup \{v_4v_5\} \) and \( \sigma = \min\{y(C), y(v_2v_3v_4v_5v_2)\} \), and let \( y' \) be obtained from \( y \) by replacing \( y(v_1v_2v_3v_4v_1) \), \( y(v_2v_3v_4v_5v_2) \), \( y'(C) \), and \( y'(C') \) with \( y(v_1v_2v_3v_4v_1) + \sigma \), \( y(v_2v_3v_4v_5v_2) - \sigma \), \( y(C) - \sigma \), and \( y(C') + \sigma \), respectively. Then \( y' \) is also an optimal solution to \( \mathbb{D}(T, w) \), contradicting (4) again. So (18) is established.
By (18), we have \( y(v_1 v_2 v_3 v_4 v_1) = w(v_1 v_1) \). It follows that \( y(C) \) is integral for all \( C \in \mathcal{C}_2 \).

(b) We next assume that \( y(v_1 v_2 v_3 v_4 v_5) = 0 \). If \( y(v_1 v_2 v_3 v_4 v_5) > 0 \), then \( v_4 v_5 \) is saturated by \( y \) in \( F_3 \) by (11). So \( y(v_2 v_3 v_4 v_5) = w(v_2 v_2) \) and \( y(v_1 v_2 v_3 v_4 v_5) = w(v_2 v_3) - w(v_4 v_5) \). Thus \( w(v_1 v_2) = z(v_1 v_2) \) for \( i = 1, 4, 5 \). By Lemma 3.4(i), \( \mathbb{D}(T, w) \) has an integral optimal solution. So we assume that \( y(v_1 v_2 v_3 v_4 v_5) = 0 \). Then \( y(v_1 v_2 v_3 v_1) + y(v_2 v_3 v_4 v_5) = w(v_2 v_3) \) and \( y(v_1 v_5 v_3 v_1) + y(v_3 v_4 v_5 v_3) = w(v_5 v_3) \). If \( y(v_1 v_2 v_3 v_1) \) is integral, then \( w(v_2 v_3) = z(v_2 v_3) \) for \( i = 1, 4, 5 \). Hence, by Lemma 3.4(i), \( \mathbb{D}(T, w) \) has an integral optimal solution. So we assume that \( y(v_1 v_2 v_3 v_1) \) is not integral. We propose to show that

\[(19) \nu^*_w(T) \text{ is an integer.}\]

To justify this, let \( x \) be an optimal solution to \( \mathbb{P}(T, w) \). Since \( 0 < y(v_1 v_2 v_3 v_1) < w(v_1 v_2) \) and \( 0 < y(v_2 v_3 v_4 v_5) < w(v_4 v_2) \), by Lemma 3.1(i) and (ii), we have \( x(v_1 v_2) = x(v_4 v_2) = 0 \) and \( x(v_3 v_1) = x(v_3 v_4) \).

Let us show that \( x(v_1 v_3) = x(v_4 v_5) \). If both \( y(v_1 v_3 v_3 v_1) \) and \( y(v_3 v_4 v_5 v_3) \) are positive, then, by Lemma 3.1(i), we have \( x(v_1 v_3 v_1) = x(v_3 v_4 v_5 v_3) = 1 \), which implies \( x(v_1 v_3) = x(v_4 v_5) \), as desired. If one of \( y(v_1 v_3 v_3 v_1) \) and \( y(v_3 v_4 v_5 v_3) \) is zero, then the other equals \( w(v_5 v_3) \). By Lemma 3.2(ii), we may assume that \( w(v_5 v_3) = 0 \). Since \( v_2 v_3 \) is saturated by \( y \) in \( F_3 \), both \( v_1 v_2 \) and \( v_2 v_3 \) are outside \( \mathcal{C}_0^3 \). If \( v_3 v_4 \) is also outside \( \mathcal{C}_0^3 \), let \( y' \) be obtained from \( y \) by replacing \( y(v_3 v_4 v_5 v_3) \) and \( y(v_1 v_3 v_3 v_1) \) with \( y(v_3 v_4 v_5 v_3) + [y(v_1 v_3 v_3 v_1)] \) and \( [y(v_1 v_3 v_3 v_1)] \), respectively, then \( y' \) is an optimal solution to \( \mathbb{D}(T, w) \). Since \( y'(v_3 v_4 v_5 v_3) \) is a positive integer, \( \mathbb{D}(T, w) \) has an integral optimal solution by Lemma 3.2(iii). So we may assume that \( v_3 v_4 \) is contained in some cycle in \( \mathcal{C}_0^3 \), the same holds for \( v_3 v_1 \). Let \( C_1 \) and \( C_2 \) be two cycles in \( \mathcal{C}_0^3 \) passing through \( v_3 v_1 \) and \( v_3 v_4 \), respectively. By Lemma 3.1(iii), we have \( x(v_3 v_1) + x(v_1 v_3) = x(v_3 v_4) + x(v_4 v_5) \). Thus \( x(v_1 v_3) = x(v_4 v_5) \) also holds.

Similarly, we can prove that \( x(uv_3) = x(uv_4) \) for each vertex \( u \in V(T_1) \setminus \{b, a_1\} \), where \( b \) is the hub of the 1-sum. Let \( T' = (V', A') \) be the digraph obtained from \( T \) by identifying \( v_1 \) and \( v_4 \); the resulting vertex is still denoted by \( v_1 \). Let \( w' \) be the restriction of \( w \) to \( A' \). Then \( x' \) corresponds to a feasible solution \( x' \) to \( \mathbb{P}(T', w') \) with \( x'(v_1 v_3) = x(v_4 v_1) + x(v_1 v_3) = x(v_4 v_5) \), by Lemma 3.1(iii), and \( y' \) corresponds to a feasible solution \( y' \) to \( \mathbb{D}(T', w') \); both having the same objective value \( \nu^*_w(T) \) as \( \mathbb{P}(T, w) \) and \( \mathbb{D}(T, w) \). By the LP-duality theorem, \( x' \) and \( y' \) are optimal solutions to \( \mathbb{P}(T', w') \) and \( \mathbb{D}(T', w') \), respectively. By Lemma 3.3, \( \mathbb{D}(T', w') \) has an integral optimal solution. So \( \nu^*_w(T) \) is an integer. This proves (19).

**Case 2.9.** \( K = \{v_3 v_1, v_4 v_3\} \).

In this case, by Lemma 3.1(iii), we have \( w(e) = y(C_2(e)) \) for each \( e \in K \), which together with (14) yields the following two equations:

\[
w(v_3 v_1) = y(v_1 v_2 v_3 v_1) + y(v_1 v_3 v_3 v_1) + y(v_1 v_3 v_3 v_1); \quad \text{and} \quad w(v_3 v_4) = y(v_2 v_3 v_4 v_2) + y(v_2 v_3 v_4 v_3) + y(v_1 v_2 v_3 v_4 v_1) + y(v_2 v_3 v_4 v_5 v_2) + y(v_1 v_3 v_4 v_1).
\]

Since each \( e \in K \) is saturated by \( y \) in \( F_4 \), we have \( w(uv_i) = z(uv_i) = 0 \) for \( i = 2, 3 \) and all \( u \in V(T_1) \setminus \{b, a_1\} \), where \( b \) is the hub of the 1-sum. Depending on the values of \( y(v_1 v_3 v_4 v_1) \) and \( y(v_1 v_3 v_4 v_3) \), we consider three subcases.

- \( y(v_1 v_3 v_3 v_1) > 0 \). In this subcase, from (9) and (15) we deduce that \( y(v_3 v_4 v_5 v_3) = y(v_1 v_3 v_3 v_1) = 0 \) and that \( v_1 v_2 \) and \( v_3 v_3 \) are saturated by \( y \) in \( F_3 \). So \( w(v_1 v_2) = y(v_1 v_2 v_3 v_1) + y(v_1 v_2 v_3 v_1) \) and \( w(v_2 v_3) = y(v_1 v_2 v_3 v_1) \). If \( y(v_1 v_2 v_3 v_1) > 0 \), then both \( v_2 v_3 \) and \( v_3 v_4 \) are saturated by \( y \) in \( F_3 \) by (9) and (11). Thus \( y(v_2 v_3 v_4 v_2) = w(v_2 v_4) \) and \( y(v_2 v_3 v_4 v_2) = w(v_2 v_4) \). It follows that \( y(C) \) is integral for all \( C \in \mathcal{C}_2 \). So we assume that \( y(v_1 v_2 v_3 v_4 v_1) = 0 \). If
y(v_{2v3v4v5v2}) > 0$, then $v_4v_2$ is saturated by $y$ in $F_4$ by (12), which implies that $y(v_{2v3v4v5v2}) = w(v_4v_2)$; if $y(v_{2v3v4v5v2}) = 0$, then $y(v_{2v3v4v2}) = w(v_3v_4)$. So $y(C)$ is integral for all $C \in C_2$, regardless of the value of $y(v_{2v3v4v5v2})$.

- $y(v_{1v2v3v4v1}) > 0$. In this subcase, from (10) and (15) we deduce that $y(v_{1v2v3v1}) = y(v_{1v2v3v2}) = 0$ and that $v_4v_5$ is saturated by $y$ in $F_4$. So $w(v_3v_1) = y(v_{1v2v3v1})$ and $w(v_4v_5) = y(v_{3v4v5v3}) + y(v_{2v3v4v5v2})$. If $y(v_{2v3v4v5v2}) > 0$, then $v_1v_2$, $v_4v_2$, and $v_5v_3$ are all saturated by $y$ in $F_3$ by (10) and (12). So $y(v_{1v2v3v4v1}) = w(v_1v_2)$, $y(v_{2v3v4v2}) = w(v_4v_2)$, and $y(v_{3v4v5v3}) + y(v_{1v2v3v4v1}) = w(v_3v_4) - y(v_{1v2v3v1})$. It follows that $y(C)$ is integral for all $C \in C_2$. So we assume that $y(v_{2v3v4v5v2}) = 0$. Then $y(v_{3v4v5v3}) = w(v_3v_4)$. If $y(v_{1v2v3v4v1}) > 0$, then $v_4v_2$ is saturated by $y$ in $F_4$ by (11). So $w(v_2v_3) = y(v_{1v2v3v1}) + y(v_{1v2v3v2})$. Moreover, if $y(v_{1v2v3v4v1}) > 0$, then $v_4v_2$ is saturated by $y$ in $F_4$ by (11), which yields one more equation $w(v_4v_2) = y(v_{3v4v5v3}) + y(v_{2v3v4v5v2})$. Hence $y(C)$ is integral for all $C \in C_2$, no matter whether $y(v_{1v2v3v4v1}) = 0$. So we assume that $y(v_{1v2v3v4v1}) = 0$. Then $y(v_1v_2v_3v_4v_1) = w(v_3v_1)$, $y(v_{3v4v5v3}) = w(v_3v_4)$ and $y(v_{1v2v3v4v1}) + y(v_{2v3v4v5v2}) = w(v_3v_4) - w(v_4v_2) - w(v_5v_3)$. If $y(v_{1v2v3v4v1})$ is integral, then $y(C)$ is integral for all $C \in C_2$. So we assume $y(v_{1v2v3v4v1})$ is integral. Similar to (18), we can prove that $v_1v_2$ is saturated by $y$ in $F_4$. Then $y(v_{1v2v3v4v1}) = w(v_3v_1)$, a contradiction.

(b) We next assume that $y(v_{2v3v4v5v2}) = 0$. Suppose $y(v_{1v2v3v4v1}) = 0$. Then $y(v_{1v2v3v4v1}) + y(v_{1v2v3v2}) + y(v_{3v4v5v3}) = w(v_3v_4)$. If neither $y(v_{1v2v3v1})$ nor $y(v_{3v4v5v3})$ is integral, then neither $y(v_{1v2v3v1})$ nor $y(v_{2v3v4v2})$ is integral. Similar to (19), we can show that $\nu'_w(T)$ is an integer. So we may assume that $y(v_{1v2v3v1})$ or $y(v_{3v4v5v3})$ is integral. Observe that both of them are integral, for otherwise, let $y'$ be obtained from $y$ by replacing $y(v_{1v2v3v1})$ and $y(v_{1v5v3v1})$ with $y(v_{1v2v3v1}) + [y(v_{1v2v3v1})]$ and $[y(v_{1v2v3v1})]$, respectively. Since $v_1v_2$, $v_2v_3$, and $v_4v_2$ are all outside $C_0$, $y'$ is an optimal solution to $D(T, w)$, with $y'(v_{1v1v5v3v1}) < y(v_{1v1v5v3v1})$, contradicting (5).

Suppose $y(v_{1v2v3v4v1}) > 0$. Then $y(v_{2v3v4v5v2}) = w(v_4v_2)$. If $y(v_{1v2v3v1}) > 0$, then $v_4v_5$ is saturated by $y$ in $F_4$ by (11), which implies $y(v_{1v2v3v5v1}) = w(v_4v_5)$, $y(v_{1v2v3v4v1}) = w(v_4v_2) - w(v_4v_2)$, and $y(v_{2v3v4v1}) + y(v_{1v5v3v1}) = w(v_3v_1)$. If $y(v_{1v5v3v1})$ is not integral, let $y'$ be obtained from $y$ by replacing $y(v_{1v2v3v1})$ and $y(v_{1v5v3v1})$ with $y(v_{1v2v3v1}) + [y(v_{1v2v3v1})]$ and $[y(v_{1v5v3v1})]$, respectively. Since both $v_1v_2$ and $v_2v_3$ are outside $C_0$, $y'$ is an optimal solution to $D(T, w)$, with $y'(v_{1v1v5v3v1}) < y(v_{1v1v5v3v1})$, contradicting (5). So $y(v_{1v5v3v1})$ is integral and hence is zero by Lemma 3.2(iii). It follows that $y(v_{1v2v3v1}) = w(v_3v_1)$ and $y(v_{1v2v3v4v1}) + y(v_{3v4v5v3}) = w(v_3v_4) - w(v_2v_3)$. If $y(v_{2v3v4v5v2})$ is not integral, then $y(C)$ is integral for all $C \in C_2$. So we assume that $y(v_{3v4v5v3})$ is not integral. Let us show that
(20) \( \nu_w^*(T) \) is an integer.

By Lemma 3.2(iii), we may assume that \( w(v_3v_1) = w(v_4v_2) = 0 \). Recall that \( w(v_5v_2) = z(v_5v_2) = 0 \) and \( w(u_i) = z(u_i) = 0 \) for \( i = 2, 3 \) and all \( u \in V(T_1) \setminus \{b, a_1\} \). So we may assume that \( x(uw_2) = x(uw_3) \). Let \( T' = (V', A') \) be the digraph obtained from \( T \) by identifying \( v_2 \) and \( v_3 \); the resulting vertex is still denoted by \( v_3 \), and let \( \mathbf{w}' \) be the restriction of \( \mathbf{w} \) to \( A' \). Then \( \mathbf{x} \) corresponds to a feasible solution \( \mathbf{x}' \) to \( P(T', \mathbf{w}') \), and \( \mathbf{y} \) corresponds to a feasible solution \( \mathbf{y}' \) to \( D(T', \mathbf{w}') \); both having the same objective value \( \nu_w^*(T) \) as \( P(T, \mathbf{w}) \) and \( D(T, \mathbf{w}) \). By the LP-duality theorem, \( \mathbf{x}' \) and \( \mathbf{y}' \) are optimal solutions to \( P(T', \mathbf{w}') \) and \( D(T', \mathbf{w}') \), respectively. By Lemma 3.3, \( D(T', \mathbf{w}') \) has an integer optimal solution. So \( \nu_w^*(T) \) is an integer. This proves (20) and hence Claim 2.

Since \( \tau_w(F_4 \setminus v_6) > 0 \), from Claim 2, Lemma 3.2(iii) and Lemma 3.4(ii) we deduce that \( D(T, \mathbf{w}) \) has an integral optimal solution. This completes the proof of Lemma 4.4.

Lemma 4.6. If \( T_2 = G_2 \), then \( D(T, \mathbf{w}) \) has an integer optimal solution.

Proof. It is routine to check that
- \( \mathcal{C}_2 = \{v_1v_2v_4v_1, v_1v_2v_3v_1, v_1v_6v_2v_4v_1, v_1v_6v_3v_4v_1, v_1v_6v_3v_1v_2, v_1v_6v_3v_4v_1\} \) and
- \( \mathcal{F}_2 = \{\{v_1v_6, v_1v_2\}, \{v_1v_6, v_2v_4\}, \{v_1v_6, v_4v_1\}, \{v_3v_1, v_4v_1\}, \{v_4v_1, v_6v_3\}, \{v_2v_4, v_6v_3, v_6v_4\}, \{v_2v_4, v_3v_1, v_3v_4, v_6v_1\}, \{v_1v_2, v_5v_1, v_5v_2, v_5v_4, v_6v_2, v_6v_4\}\} \)

We also have a computer verification of these results. So \( |\mathcal{C}_2| = 6 \) and \( |\mathcal{F}_2| = 9 \). Recall that \( (b_2, a_2) = (v_4, v_5) \).

Let \( \mathbf{y} \) be an optimal solution to \( D(T, \mathbf{w}) \) such that
- (1) \( y(C_2) \) is maximized;
- (2) subject to (1), \((y(D_q), y(D_{q-1}), \ldots, y(D_3))\) is minimized lexicographically;
- (3) subject to (1) and (2), \( y(v_1v_6v_3v_2v_4v_1) \) is maximized; and
- (4) subject to (1)-(3), \( y(v_1v_6v_4v_1) \) is minimized.

Let us make some simple observations about \( \mathbf{y} \).

(5) If \( K \in \mathcal{F}_2 \) satisfies \( y(C_2) = w(K) \), then \( K \) is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

The three statements below follow instantly from Lemma 3.5(v).

(6) If \( y(v_1v_6v_3v_2v_4v_1) > 0 \), then each arc in the set \( \{v_1v_2, v_3v_1, v_3v_4, v_6v_2, v_6v_4\} \) is saturated by \( \mathbf{y} \) in \( G_2 \).

(7) If \( y(v_1v_6v_3v_2v_4v_1) > 0 \), then both \( v_3v_1 \) and \( v_6v_4 \) are saturated by \( \mathbf{y} \) in \( G_2 \).

(8) If \( y(v_1v_6v_3v_2v_4v_1) > 0 \), then both \( v_2v_4 \) and \( v_6v_4 \) are saturated by \( \mathbf{y} \) in \( G_2 \).

Claim 1. \( y(C_2) = \tau_w(G_2 \setminus v_5) \).

To justify this, observe that if both \( v_1v_2 \) and \( v_1v_6 \) are saturated by \( \mathbf{y} \) in \( G_2 \), then \( y(C_2) = w(K) \), where \( K = \{v_1v_2, v_1v_6\} \); if both \( v_3v_1 \) and \( v_4v_1 \) are saturated by \( \mathbf{y} \) in \( G_2 \), then \( y(C_2) = w(K) \), where \( K = \{v_3v_1, v_4v_1\} \). By (5), \( K \) is an MFAS and hence \( y(C_2) = \tau_w(G_2 \setminus v_5) \) in either case. So we assume that

(9) at most one of \( v_1v_2 \) and \( v_1v_6 \) is saturated by \( \mathbf{y} \) in \( G_2 \). The same holds for \( v_3v_1 \) and \( v_4v_1 \).

As \( v_2v_4 \) is a special arc of \( T \) and \( v_2 \) is a near-sink, by Lemma 3.4(iv), we may assume that \( v_2v_4 \) is saturated by \( \mathbf{y} \) in \( T \). Depending on whether \( v_2v_4 \) is outside \( C_0^y \), we distinguish between two cases.

Case 1.1. \( v_2v_4 \) is contained by some cycle in \( C_0^y \).
In this case, we proceed by considering two subcases.

- $v_3v_1$ is saturated by $y$ in $G_2$. In this subcase, by (9), $v_3v_1$ is not saturated by $y$ in $G_2$ and hence in $T$, because $v_4v_1$ is outside $C_0$. By the hypothesis of the present case and Lemma 3.5(iii), $v_1v_2$ is saturated by $y$ in $T$. Observe that $v_1v_2$ is outside $C_0^y$, for otherwise, a cycle $C \in C_0^y$ containing $v_1v_2$ must pass through $v_2v_4$. Thus, by Lemma 3.5(iv), $v_2v_1$ is saturated by $y$ in $G_2$, a contradiction. It follows that $v_1v_2$ is saturated by $y$ in $G_2$. So, by (9), $v_1v_6$ is not saturated by $y$ in $G_2$. If $v_1v_6$ is contained in some cycle $C \in C_0^y$, applying Lemma 3.5(iv) to the cycle $C[v_1, v_4] \cup \{v_1v_2, v_2v_3, v_3v_4, v_4v_6, v_6v_4\}$, by the hypothesis of the present case and by Lemma 3.5(iii) and (iv), we further deduce that $v_1v_6$ is saturated by $y$ in $G_2$. If $v_1v_6$ is also saturated by $y$ in $G_2$, then $y(C_2) = w(K)$, where $K = \{v_1v_2, v_2v_3, v_3v_4, v_4v_6\}$. By (5), $K$ is an MFAS and thus $y(C_2) = \tau_w(G_2 \setminus v_5)$. If $v_1v_6$ is saturated by $y$ in $T$ then, by Lemma 3.5(iii), $v_3v_2$ is saturated by $y$ in $G_2$ and $v_3v_4$ is outside $C_0^y$. By Lemma 3.5(i), we further deduce that $v_3v_4$ is saturated by $y$ in $G_2$. Thus $y(C_2) = w(J)$, where $J = \{v_1v_2, v_2v_3, v_3v_4, v_4v_6, v_6v_4\}$. By (5), $J$ is an MFAS and thus $y(C_2) = \tau_w(G_2 \setminus v_5)$.

- $v_3v_1$ is not saturated by $y$ in $G_2$. In this subcase, we have $y(v_1v_2v_3v_4v_1) = y(v_1v_2v_3v_4v_1) = 0$ by (6) and (7). Assume first that $v_1v_2$ is saturated by $y$ in $G_2$. Then $v_1v_6$ is not saturated by $y$ in $G_2$ by (9). Thus $v_1v_6$ is saturated by $y$ in $G_2$ by Lemma 3.5(iii) and (iv). If $v_1v_6$ is also saturated by $y$ in $G_2$, then $y(C_2) = w(K)$, where $K = \{v_1v_2, v_2v_3, v_3v_4, v_4v_6\}$; otherwise, both $v_3v_2$ and $v_3v_4$ are saturated by $y$ in $G_2$ by Lemma 3.5(iii) and (iv). So $y(C_2) = w(K)$, where $K = \{v_1v_2, v_2v_3, v_3v_4, v_4v_6\}$. By (5), $K$ is an MFAS in either subcase, and thus $y(C_2) = \tau_w(G_2 \setminus v_5)$.

Assume next that $v_1v_2$ is not saturated by $y$ in $G_2$. By (8), we have $y(v_1v_2v_3v_4v_1) = 0$. By the hypothesis of the present case and by Lemma 3.5(iii) and (iv), $v_4v_1$ is saturated by $y$ in $G_2$. If $v_4v_1$ is also saturated by $y$ in $G_2$, then $y(C_2) = w(K)$, where $K = \{v_1v_4, v_4v_1\}$; otherwise, both $v_3v_2$ and $v_3v_4$ are saturated by $y$ in $G_2$ by Lemma 3.5(iii) and (iv). So $y(C_2) = w(K)$, where $K = \{v_1v_2, v_2v_3, v_3v_4, v_4v_6\}$. By (5), $K$ is an MFAS in either subcase, and thus $y(C_2) = \tau_w(G_2 \setminus v_5)$.

Let us show that $v_3v_1$ is outside $C_0^y$, because every cycle containing $v_3v_2$ in $C_0^y$ must pass through $v_1v_2$. So neither $v_1v_2$ nor $v_3v_1$ is saturated by $y$ in $T$.
on $y$. Let $D \in C_0^y$ be a cycle containing $v_2v_4$. Then the multiset sum of $D$, $v_1v_6v_4v_1$, and the unsaturated arcs $v_6v_3$, $v_3v_1$, and $v_1v_2$ contains two arc-disjoint cycles $v_1v_2v_4v_1$ and $v_1v_6v_3v_1$. Thus, by Lemma 3.5(vi), we obtain $y(v_1v_6v_4v_1) = 0$; this contradiction proves (10).

From (10), we deduce that $y(C_2) = w(K)$, where $K = \{v_1v_6, v_4v_1\}$. So, by (5), $K$ is an MFAS and thus $y(C_2) = \tau_w(G_2,v_5)$.

**Case 1.2.** $v_2v_4$ is outside $C_0^y$.

In this case, $v_2v_4$ is saturated by $y$ in $G_2$. So $v_1v_2, v_3v_2,$ and $v_6v_2$ are all outside $C_0^y$. Assume first that $v_1v_6$ is saturated by $y$ in $G_2$. Then $v_1v_2$ is not saturated by $y$ by (9). By (6) and (8), we have $y(v_1v_6v_3v_2v_4v_1) = y(v_1v_6v_2v_4v_1) = 0$ and hence $y(C_2) = w(K)$, where $K = \{v_1v_6, v_2v_4\}$. It follows from (5) that $K$ is an MFAS and thus $y(C_2) = \tau_w(G_2,v_5)$. Assume next that $v_1v_6$ is not saturated by $y$ in $G_2$. If $v_4v_1$ is not saturated by $y$ in $T$, then $v_6v_4$ is outside $C_0^y$ by Lemma 3.5(iii). So $v_3v_4$ is contained in some cycle in $C_0^y$ because $C_0^y \neq \emptyset$. Using Lemma 3.5(iii), we deduce that both $v_6v_3$ and $v_6v_4$ are saturated by $y$ in $G_2$. Using (6), we obtain $y(v_1v_6v_3v_2v_4v_1) = 0$. Thus $y(C_2) = w(K)$, where $K = \{v_2v_4, v_6v_3, v_6v_4\}$. If $v_4v_1$ is saturated by $y$ in $T$, then so is it in $G_2$ because $v_4v_1$ is outside $C_0^y$. By (9), $v_3v_1$ is not saturated by $y$ in $G_2$. By Lemma 3.5(iii), $v_6v_3$ is saturated by $y$ in $G_2$. By (6) and (7), we have $y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$. Hence $y(C_2) = w(K)$, where $K = \{v_1v_1, v_6v_3\}$. In either subcase, $K$ is an MFAS by (5) and thus $y(C_2) = \tau_w(G_2,v_5)$. This proves Claim 1.

**Claim 2.** $y(C)$ is integral for all $C \in C_2$ or $v_\nu^*(T)$ is an integer.

To justify this, we may assume that

(11) $y(v_1v_6v_3v_2v_4v_1) = 0$.

Otherwise, by (6), we have $w(e) = y(C_2(e))$ for each $e$ in the set $\{v_1v_2, v_3v_1, v_3v_4, v_6v_2, v_6v_4\}$. So $y(v_1v_2v_4v_1) = w(v_1v_2), y(v_1v_6v_3v_1) = w(v_3v_1), y(v_1v_6v_3v_4v_1) = w(v_3v_4), y(v_1v_6v_2v_4v_1) = w(v_6v_2), and y(v_1v_6v_4v_1) = w(v_6v_4). By Claim 1, y(C_2) is an integer, so it is y(v_1v_6v_3v_2v_4v_1). Hence y(C) is integral for all C in C_2.

By Claim 1, $y(C_2) = w(K)$ for some $K \in F_2$. Depending on what $K$ is, we distinguish among nine cases.

**Case 2.1.** $K = \{v_1v_2, v_3v_1, v_3v_4, v_6v_2, v_6v_4\}$.

In this case, by Lemma 3.1 (iii), we have $w(e) = y(C_2(e))$ for each $e \in K$. It follows instantly that $y(C)$ is integral for all $C \in C_2$.

**Case 2.2.** $K = \{v_1v_6, v_4v_1\}$.

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_4v_1) = y(v_1v_6v_2v_4v_1) = y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_2v_4v_1) = 0$. By Lemma 3.1 (iii), we further obtain $w(e) = y(C_2(e))$ for each $e \in K$. It follows that $y(v_1v_4v_1) = w(v_4v_1)$ and $y(v_1v_6v_3v_1) = w(v_1v_6)$. Therefore $y(C)$ is integral for all $C \in C_2$.

**Case 2.3.** $K = \{v_1v_2, v_6v_2, v_6v_3, v_6v_4\}$.

In this case, by Lemma 3.1 (iii), we have $w(e) = y(C_2(e))$ for each $e \in K$, which together with (11) yields the following equations: $y(v_1v_6v_3v_2v_4v_1) = y(v_1v_6v_2v_4v_1) = w(v_6v_2), y(v_1v_6v_4v_1) = w(v_6v_4), and y(v_1v_6v_3v_1) + y(v_1v_6v_3v_4v_1) = w(v_6v_3)$. Note that if $y(v_1v_6v_3v_4v_1) > 0$, we have one more equation $y(v_1v_6v_3v_1) = w(v_3v_1)$ by (7). Hence $y(C)$ is integral for all $C \in C_2$, no matter whether $y(v_1v_6v_3v_4v_1) = 0$.

**Case 2.4.** $K = \{v_2v_4, v_6v_3, v_6v_4\}$.

In this case, by Lemma 3.1 (iii), we have $w(e) = y(C_2(e))$ for each $e \in K$, which together
with (11) yields the following equations: \( y(v_1v_2v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_2v_4), \ y(v_1v_6v_3v_1) + y(v_1v_6v_3v_4v_1) = w(v_6v_4), \) and \( y(v_1v_6v_3v_1) = w(v_6v_4). \) Note that if \( y(v_1v_6v_2v_4v_1) > 0, \) we have one more equation \( y(v_1v_2v_4v_1) = w(v_1v_2) \) by (8); if \( y(v_1v_6v_2v_4v_1) > 0, \) we have one more equation \( y(v_1v_6v_3v_1) = w(v_3v_1) \) by (7). Hence \( y(C) \) is integral for all \( C \in \mathcal{C}_2 \) in any subcase.

**Case 2.5.** \( K = \{v_2v_4, v_2v_1, v_3v_4, v_6v_4\}. \)

In this case, by Lemma 3.1 (iii), we have \( w(e) = y(C_2(e)) \) for each \( e \in K, \) which together with (11) yields the following equations: \( y(v_1v_2v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_2v_4), \ y(v_1v_6v_3v_1) = w(v_6v_4), \) \( y(v_1v_6v_3v_4v_1) = w(v_3v_4), \) and \( y(v_1v_6v_2v_4v_1) = w(v_6v_4). \) Note that if \( y(v_1v_6v_2v_4v_1) > 0, \) we have one more equation \( y(v_1v_2v_4v_1) = w(v_1v_2) \) by (8). Hence \( y(C) \) is integral for all \( C \in \mathcal{C}_2, \) no matter whether \( y(v_1v_6v_2v_4v_1) = 0. \)

**Case 2.6.** \( K = \{v_1v_6, v_2v_3\}. \)

In this case, by Lemma 3.1 (i), we have \( y(v_1v_6v_2v_4v_1) = 0. \) By Lemma 3.1 (iii), we obtain \( w(e) = y(C_2(e)) \) for each \( e \in K, \) which together with (11) yields the following equations: \( y(v_1v_2v_4v_1) = w(v_2v_4) \) and \( y(v_1v_6v_3v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6). \) Moreover, in this case \( v_1v_2, v_3v_2, \) and \( v_6v_2 \) are all outside \( \mathcal{C}_0^y, \) and \( w(uv_2) = z(uv_2) = 0 \) for any \( u \in V(T_1) \setminus \{b, a_1\}, \) where \( b \) is the hub of the 1-sum. Examining the cycles in \( \mathcal{C}_2, \) we see that \( z(v_2v_3) = z(v_2v_6) = 0 \) and so \( w(uv_2) = z(uv_2) \) for \( i = 1, 3, 6. \) Thus \( \mathcal{D}(T, w) \) has an integral optimal solution by Lemma 3.4(i).

**Case 2.7.** \( K = \{v_4v_1, v_6v_3\}. \)

In this case, by Lemma 3.1 (i) and (iii), we have \( y(v_1v_6v_3v_4v_1) = 0, \) \( y(v_1v_6v_3v_1) = w(v_6v_3), \) and \( y(v_1v_2v_4v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_2v_4v_1) = w(v_4v_1). \) Lemma 3.2(iii) allows us to assume that \( w(v_6v_3) = 0. \) If \( y(v_1v_6v_2v_4v_1) > 0, \) then both \( v_1v_2 \) and \( v_6v_2 \) are saturated by \( y \) in \( G_2 \) by (8). So \( y(v_1v_2v_4v_1) = w(v_1v_2) \) and \( y(v_1v_6v_4v_1) = w(v_6v_4). \) Hence \( y(C) \) is integral for all \( C \in \mathcal{C}_2; \) the same holds if \( y(v_1v_6v_2v_4v_1) = 0 \) and \( y(v_1v_2v_4v_1) \) is integral. So we assume that \( y(v_1v_6v_2v_4v_1) = 0 \) and \( y(v_1v_2v_4v_1) \) is not integral. Observe that \( v_1v_2 \) is outside \( \mathcal{C}_0^y, \) for otherwise, let \( C \) be a cycle in \( \mathcal{C}_0^y \) containing \( v_1v_2, \) let \( C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_4\}, \) and set \( \theta = \min\{y(C), y(v_1v_6v_4v_1)\}. \) Let \( y' \) be obtained from \( y \) by replacing \( y(v_1v_6v_4v_1), y(v_1v_2v_4v_1), y(C), \) and \( y(C') \) with \( y(v_1v_6v_4v_1) - \theta, \) \( y(v_1v_2v_4v_1) + \theta, \) \( y(C - \theta), \) and \( y(C') + \theta, \) respectively. Then \( y' \) is also an optimal solution to \( \mathcal{D}(T, w) \). Since \( y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1), \) the existence of \( y' \) contradicts the assumption (4) on \( y. \) Similarly, we can prove that \( v_6v_2 \) is outside \( \mathcal{C}_0^y. \) Examining cycles in \( \mathcal{C}_2, \) we see that \( w(v_6v_2) = z(v_6v_2) = 0. \) Now we propose to show that

\[
(12) \quad v^*_6(T) \text{ is an integer.}
\]

To justify this, let \( x \) be an optimal solution to \( \mathcal{P}(T, w). \) Since both \( y(v_1v_2v_4v_1) \) and \( y(v_1v_6v_4v_1) \) are positive, we have \( x(v_1v_2) + x(v_4v_2) = x(v_1v_6) + x(v_6v_4) \) by Lemma 3.1(i). Since \( y(v_1v_6v_2v_4v_1) < w(v_1v_2), \) we have \( x(v_1v_2) = 0 \) by Lemma 3.1(ii). So \( x(v_2v_4) = x(v_1v_6) + x(v_6v_4). \) If each of \( v_3v_1 \) and \( v_3v_2 \) is contained in some cycle in \( \mathcal{C}_0^y, \) then \( x(v_3v_1) = x(v_3v_2) \) by Lemma 3.1(iv). If one of \( v_3v_1 \) and \( v_3v_2 \) is outside \( \mathcal{C}_0^y, \) say \( v_3v_1, \) then we may assume that \( w(v_3v_1) = 0 \) and \( x(v_3v_1) = x(v_3v_2). \) Similarly, we can prove that \( x(uv_2) = z(uv_2) \) for each \( u \in V(T_1) \setminus \{a_1, b\}. \)

Let \( T' = (V', A') \) be obtained from \( T \) by deleting vertex \( v_2, \) let \( w' \) be obtained from the restriction of \( w \) to \( A' \) by defining \( w'(uv_1) = w(uv_1) + w(uv_2) \) for \( u = v_3 \) or \( u \in V(T_1) \setminus \{b, a_1\} \) and \( w'(v_1v_2) = w(v_1v_2) + w(v_2v_4) \) for \( i, j = (1, 6) \) or \( (6, 4). \) Let \( x' \) be the restriction of \( x \) to \( A' \) and let \( y' \) be obtained from \( y \) as follows: for each cycle \( C \) passing through the path \( uv_2v_4 \) with \( u \in (V(T_1) \setminus \{a_1, b\}) \cup \{v_3\}, \) let \( C' \) be the cycle arising from \( C \) by replacing \( uv_2v_4 \) with \( uv_1v_6v_4, \) and set \( y'(C') = y(C) + y(C') \) and \( y'(v_1v_6v_4v_1) = y(v_1v_6v_4v_1) + y(v_1v_2v_4v_1). \) From
the LP-duality theorem, we see that \( \mathbf{x}' \) and \( \mathbf{y}' \) are optimal solutions to \( \mathbb{P}(T', \mathbf{w}') \) and \( \mathbb{D}(T', \mathbf{w}') \) respectively, both having the same value \( \nu_\mathbb{w}^*(T) \) as \( \mathbf{x} \) and \( \mathbf{y} \). Hence \( \nu_\mathbb{w}^*(T) \) is an integer by the hypothesis of Theorem 1.5.

**Case 2.8.** \( K = \{v_1v_6, v_1v_2\} \).

In this case, by Lemma 3.1 (iii), we have \( w(e) = y(C_2(e)) \) for each \( e \in K \), which together with (11) yields the following equations: \( y(v_1v_2v_4v_1) = w(v_1v_2) \) and \( y(v_1v_6v_3v_1) + y(v_1v_6v_4v_1) + y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6) \). Moreover, \( v_3v_1 \) is outside \( C_0^y \). Depending on whether \( y(v_1v_6v_2v_4v_1) = 0 \), we consider two subcases.

**Case 1.** 

\( y(v_1v_6v_3v_4v_1) = 0 \). In this subcase, we first assume that \( y(v_1v_6v_2v_4v_1) > 0 \). Then \( y(v_1v_6v_2v_4v_1) = w(v_6v_4) \) by (8). Thus \( y(v_1v_6v_3v_1) + y(v_1v_6v_2v_4v_1) = w(v_1v_6) - w(v_6v_4) \). Let us show that \( y(v_1v_6v_3v_1) \) is integral. Suppose not. If \( v_6v_3 \) is outside \( C_0^y \), let \( \mathbf{y}' \) be obtained from \( \mathbf{y} \) by replacing \( y(v_1v_6v_3v_1) \) with \( y(v_1v_6v_3v_1) + [y(v_1v_6v_2v_4v_1)] \), respectively; if \( v_6v_3 \) is contained in some cycle \( C \) in \( C_0^y \), set \( \theta = \min\{y(C), y(v_1v_6v_2v_4v_1)\} \), and let \( \mathbf{y}' \) be obtained from \( \mathbf{y} \) by replacing \( y(v_1v_6v_3v_1) \), \( y(v_1v_6v_2v_4v_1) \), \( y(C) \), and \( y(C') \) with \( y(v_1v_6v_3v_1) + \theta, y(v_1v_6v_2v_4v_1) - \theta, y(C) - \theta, \) and \( y(C') + \theta \), respectively. In both subcases, \( \mathbf{y}' \) is an optimal solution to \( \mathbb{D}(T, \mathbf{w}) \) with \( y(v_1v_6v_2v_4v_1) < y(v_1v_6v_3v_1) \), contradicting (2). We next assume that \( y(v_1v_6v_2v_4v_1) = 0 \). The proof of this subcase is similar to that in the preceding one (with \( y(v_1v_6v_3v_1) \)) in place of \( y(v_1v_6v_2v_4v_1) \).

Thus we reach a contradiction to (4).

**Case 2.** 

\( y(v_1v_6v_3v_4v_1) > 0 \). In this subcase, by (7), both \( v_3v_1 \) and \( v_6v_4 \) are saturated by \( \mathbf{y} \) in \( G_2 \). So \( y(v_1v_6v_3v_1) = w(v_3v_1), y(v_1v_6v_4v_1) = w(v_6v_4), \) and \( y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_1) = w(v_1v_6) - w(v_3v_1) - w(v_6v_4) \). If \( y(v_1v_6v_2v_4v_1) \) is integral, then \( y(C) \) is integral for all \( C \in E_2 \). So we assume that \( y(v_1v_6v_2v_4v_1) \) is not integral. Then \( [y(v_1v_6v_2v_4v_1)] + [y(v_1v_6v_3v_1)] = 1 \). Observe that \( v_4v_2 \) is outside \( C_0^y \), for otherwise, let \( C \) be a cycle in \( C_0^y \) containing \( v_6v_2 \), let \( C' = C[v_4, v_6] \cup \{v_6v_3, v_4v_1\} \), let \( \theta = \min\{y(C), y(v_1v_6v_2v_4v_1)\} \), and let \( \mathbf{y}' \) be obtained from \( \mathbf{y} \) by replacing \( y(v_1v_6v_3v_1) \) with \( y(v_1v_6v_3v_1) + \theta, y(v_1v_6v_2v_4v_1) - \theta, y(C) \) and \( y(C') + \theta \), respectively. Then \( \mathbf{y}' \) is an optimal solution to \( \mathbb{D}(T, \mathbf{w}) \) with \( y(v_1v_6v_2v_4v_1) < y(v_1v_6v_3v_1) \), contradicting (2). Similarly, we can show that \( v_3v_2 \) is also outside \( C_0^y \). Thus \( w(v_3v_2) = z(v_3v_2) = 0 \). By Lemma 3.2(iii), we may assume that \( w(v_1v_2), w(v_3v_1), \) and \( w(v_6v_4) \) are all 0. We propose to show that

\[ (13) \nu_\mathbb{w}^*(T) \text{ is an integer.} \]

To justify this, let \( \mathbf{x} \) be an optimal solution to \( \mathbb{P}(T, \mathbf{w}) \). Since \( y(v_1v_6v_2v_4v_1) > 0 \) and \( y(v_1v_6v_3v_4v_1) > 0 \), from Lemma 3.1(i) we deduce that \( x(v_3v_2) + x(v_2v_4) = x(v_6v_3) + x(v_3v_4) \). Since \( y(v_1v_6v_2v_4v_1) = w(v_6v_2) \), we have \( x(v_6v_2) = 0 \) by Lemma 3.1(ii). It follows that \( x(v_2v_4) = x(v_6v_3) + x(v_3v_4) \). Since \( w(v_6v_4) = 0 \) and \( v_6v_2 \) is outside \( C_0^y \), \( x(v_6v_4) = x(v_6v_2) \) for each \( u \in V(T_1) \backslash \{b, a_1\} \). Let \( T' = (V', A') \) be the tournament obtained from \( T \) by deleting vertex \( v_2 \), let \( \mathbf{w}' \) be obtained from the restriction of \( \mathbf{w} \) to \( A' \) by replacing \( w(uv_6) = w(uv_2) \) for each \( u \in V(T_1) \backslash \{b, a_1\} \) and replacing \( w(v_1v_j) \) with \( w(v_1v_j) + w(v_2v_4) \) for \( (i, j) = (6, 3) \) or \( (3, 4) \). Let \( \mathbf{x}' \) be the restriction of \( \mathbf{w}' \) to \( A' \) and let \( \mathbf{y}' \) be obtained from \( \mathbf{y} \) as follows: for each cycle \( C \) passing through \( uv_2v_4 \) with \( u \in V(T_1) \backslash \{b, a_1\} \), let \( C' \) be the cycle arising from \( C \) by replacing \( uv_2v_4 \) with \( uv_6v_3v_4 \), and set \( y'(C') = y(C') + y(C) \) and \( y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1) + y(v_1v_6v_3v_4v_1) \). From the LP-duality theorem, we deduce that \( \mathbf{x}' \) and \( \mathbf{y}' \) are optimal solutions to \( \mathbb{P}(T', \mathbf{w}') \) and \( \mathbb{D}(T', \mathbf{w}') \), respectively, both having the same value \( \nu_\mathbb{w}^*(T) \) as \( \mathbf{x} \) and \( \mathbf{y} \). Hence \( \nu_\mathbb{w}^*(T) \) is an integer by the hypothesis of Theorem 1.5.
Lemma 4.7.

If \( w(e) = y(C_{2}(e)) \) for each \( e \in K \), which together with (11) yields the following equations: \( y(v_{1}v_{6}v_{3}v_{1}) = w(v_{3}v_{1}) \) and \( y(v_{1}v_{2}v_{4}v_{1}) + y(v_{1}v_{6}v_{4}v_{1}) + y(v_{1}v_{6}v_{4}v_{1}) + y(v_{1}v_{6}v_{3}v_{1}v_{1}) = w(v_{4}v_{1}) \). Assume first that \( y(v_{1}v_{6}v_{2}v_{4}v_{1}) = 0 \).

If \( y(v_{1}v_{6}v_{3}v_{4}v_{1}) > 0 \), then \( v_{6}v_{4} \) is saturated by \( y \) in \( G_{2} \). So \( y(v_{1}v_{6}v_{4}v_{1}) = w(v_{6}v_{4}) \) and hence \( y(v_{1}v_{2}v_{4}v_{1}) + y(v_{1}v_{6}v_{3}v_{4}v_{1}) = w(v_{4}v_{1}) - w(v_{6}v_{4}); \) if \( y(v_{1}v_{6}v_{3}v_{4}v_{1}) = 0 \), then \( y(v_{1}v_{2}v_{4}v_{1}) + y(v_{1}v_{6}v_{4}v_{1}) = w(v_{4}v_{1}) \). If \( y(v_{1}v_{2}v_{4}v_{1}) \) is an integer, then \( y(C) \) is integral for all \( C \in C_{2} \). So we assume that \( y(v_{1}v_{2}v_{4}v_{1}) \) is not integral. Then we can prove that both \( v_{6}v_{2} \) and \( v_{1}v_{2} \) are outside \( C_{0}^{0} \) and that \( \nu_{u}^{0}(T) \) is an integer. The proof is the same as that of (12) (with \( y(v_{1}v_{6}v_{4}v_{1}) \) in place of \( y(v_{1}v_{6}v_{3}v_{4}v_{1}) \) when \( y(v_{1}v_{6}v_{3}v_{4}v_{1}) > 0 \)), so we omit the details here.

Assume next that \( y(v_{1}v_{6}v_{2}v_{4}v_{1}) > 0 \). Then both \( v_{1}v_{2} \) and \( v_{6}v_{4} \) are saturated by \( y \) in \( G_{2} \). So \( y(v_{1}v_{2}v_{4}v_{1}) = w(v_{1}v_{2}), y(v_{1}v_{6}v_{4}v_{1}) = w(v_{6}v_{4}), \) and \( y(v_{1}v_{6}v_{2}v_{4}v_{1}) + y(v_{1}v_{6}v_{3}v_{4}v_{1}) = w(v_{4}v_{1}) - w(v_{1}v_{2}) - w(v_{6}v_{4}). \) If \( y(v_{1}v_{6}v_{3}v_{4}v_{1}) \) is an integer, then \( y(C) \) is integral for all \( C \in C_{2} \).

So we assume that \( y(v_{1}v_{6}v_{3}v_{4}v_{1}) \) is not integral. Then we can prove that both \( v_{6}v_{2} \) and \( v_{3}v_{2} \) are outside \( C_{0}^{0} \) and that \( \nu_{u}^{0}(T) \) is an integer. The proof is the same as that of (13), so we omit the details here. Thus Claim 2 is established.

Since \( \tau_{u}(F_{3} \backslash v_{6}) = 0 \), from Claim 2, Lemma 3.2(iii) and Lemma 3.4(ii) we deduce that \( D(T, w) \) has an integral optimal solution. This completes the proof of Lemma 4.5.

Lemma 4.7. If \( T_{2} = G_{3} \), then \( D(T, w) \) has an integral optimal solution.

Proof. It is routine to check that

\begin{itemize}
  \item \( C_{2} = \{v_{1}v_{2}v_{4}v_{1}, v_{1}v_{6}v_{3}v_{1}, v_{2}v_{4}v_{6}v_{2}, v_{3}v_{4}v_{6}v_{3}, v_{1}v_{6}v_{2}v_{4}v_{1}, v_{1}v_{6}v_{3}v_{4}v_{1}, v_{2}v_{4}v_{6}v_{3}v_{2}, v_{1}v_{6}v_{3}v_{2}v_{4}v_{1}, \}
  \item \( F_{2} = \{v_{2}v_{4}, v_{6}v_{3}, \{v_{1}v_{2}, v_{1}v_{6}, v_{4}v_{6} \}, \{v_{1}v_{2}, v_{1}v_{6}, v_{3}v_{4} \}, \{v_{1}v_{2}, v_{1}v_{6}, v_{3}v_{4} \}, \{v_{1}v_{2}, v_{1}v_{6}, v_{4}v_{6} \}, \{v_{1}v_{2}, v_{1}v_{6}, v_{4}v_{6} \}, \{v_{1}v_{2}, v_{1}v_{6}, v_{4}v_{6} \}, \{v_{1}v_{2}, v_{1}v_{6}, v_{4}v_{6} \}, \{v_{1}v_{2}, v_{1}v_{6}, v_{4}v_{6} \}, \{v_{1}v_{2}, v_{1}v_{6}, v_{4}v_{6} \}, \}
\end{itemize}

We also have a computer verification of these results. So \( |C_{2}| = 9 \) and \( |F_{2}| = 13 \). Recall that \( (b_{2}, a_{2}) = (v_{4}, v_{5}). \)

Let \( y \) be an optimal solution to \( D(T, w) \) such that

\begin{enumerate}
  \item \( y(C_{2}) \) is maximized;
  \item subject to (1), \( y(D_{0}), y(D_{q-1}), \ldots, y(D_{q}) \) is minimized lexicographically;
  \item subject to (1) and (2), \( y(v_{1}v_{6}v_{3}v_{4}v_{1}) \) is minimized; and
  \item subject to (1)-(3), \( y(v_{1}v_{2}v_{4}v_{1}) + y(v_{3}v_{4}v_{6}v_{3}) \) is minimized;
\end{enumerate}

Let us make some simple observations about \( y \).

(5) If \( K \in F_{2} \) satisfies \( y(C_{2}) = w(K) \), then \( K \) is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

(6) If \( y(v_{1}v_{2}v_{4}v_{6}v_{3}v_{1}) > 0 \), then each arc in the set \( \{v_{1}v_{6}, v_{3}v_{2}, v_{3}v_{4}, v_{4}v_{1}, v_{6}v_{2}\} \) is saturated by \( y \) in \( G_{3} \). Furthermore, \( y(v_{1}v_{6}v_{2}v_{4}v_{1}) = y(v_{1}v_{6}v_{3}v_{4}v_{1}) = y(v_{1}v_{6}v_{3}v_{2}v_{4}v_{1}) = 0 \).

To justify this, note that each arc in the given set is a chord of the cycle \( v_{1}v_{2}v_{4}v_{6}v_{3}v_{1} \). So the first half follows instantly from Lemma 3.5(v). Once again let \( w \) stand for the multiset sum. Then \( v_{1}v_{2}v_{4}v_{6}v_{3}v_{1} \cup v_{1}v_{6}v_{2}v_{4}v_{1} = v_{1}v_{2}v_{4}v_{1} \cup v_{1}v_{6}v_{2}v_{4}v_{1} \cup v_{1}v_{6}v_{2}v_{4}v_{1} = v_{1}v_{2}v_{4}v_{1} \cup v_{1}v_{6}v_{2}v_{4}v_{1} \cup v_{1}v_{6}v_{2}v_{4}v_{1}, \) and \( v_{1}v_{2}v_{4}v_{6}v_{3}v_{1} \cup v_{1}v_{6}v_{2}v_{4}v_{1} = v_{1}v_{2}v_{4}v_{1} \cup v_{1}v_{6}v_{2}v_{4}v_{1} \cup v_{1}v_{6}v_{2}v_{4}v_{1}. \) It follows from the optimality of \( y \) that \( y(v_{1}v_{6}v_{2}v_{4}v_{1}) = y(v_{1}v_{6}v_{3}v_{4}v_{1}) = y(v_{1}v_{6}v_{3}v_{2}v_{4}v_{1}) = 0 \).
(7) If \( y(v_1v_3v_4v_6v_2v_4v_1) > 0 \), then each arc in the set \( \{v_1v_2, v_3v_1, v_3v_4, v_4v_6, v_6v_2\} \) is saturated by \( y \) in \( G_3 \). Furthermore, \( y(v_2v_4v_6v_2) = y(v_3v_4v_6v_3) = 0 \).

To justify this, note that each arc in the given set is a chord of the cycle \( v_1v_2v_4v_6v_3v_1 \). So the first half follows instantly from Lemma 3.5(v). Observe that \( v_1v_6v_3v_2v_4v_1 \equiv v_3v_4v_6v_3 = v_1v_6v_3v_4v_1 \cup v_2v_4v_6v_3 \) and \( v_1v_6v_3v_2v_4v_1 \equiv v_2v_4v_6v_2 = v_1v_6v_3v_4v_1 \cup v_2v_4v_6v_3 \). Since \( y \) satisfies (2), it is clear that \( y(v_2v_4v_6v_2) = y(v_3v_4v_6v_3) = 0 \).

(8) If \( y(v_1v_3v_4v_6v_1) > 0 \), then both \( v_3v_1 \) and \( v_4v_6 \) are saturated by \( y \) in \( G_3 \); so is \( v_1v_2 \) if \( y(v_2v_4v_6v_2v_2) > 0 \). Furthermore, \( y(v_2v_4v_6v_2) = 0 \).

To justify this, note that both \( v_3v_1 \) and \( v_4v_6 \) are chords of the cycle \( v_1v_2v_4v_6v_3v_1 \), so they are saturated by \( y \) in \( G_3 \) by Lemma 3.5(v). Suppose \( y(v_2v_4v_6v_3v_1) = 0 \). If \( v_1v_2 \) is not saturated by \( y \) in \( T \), then \( v_1v_6v_3v_4v_1 \equiv v_2v_4v_6v_3 \) \( \cup \{v_1v_2\} = v_1v_6v_3v_4v_1 \equiv v_2v_4v_6v_3 \) \( \cup \{v_1v_2\} \); if \( v_1v_2 \) is saturated by \( y \) in \( T \) but contained in some cycle \( C \in C_0^y \), then the multiset sum of \( C \), \( v_1v_6v_3v_4v_1 \), and \( v_2v_4v_6v_3 \) contains \( \{v_1v_2, v_3v_4, v_6v_3, v_2v_4v_6v_3\} \). Thus we can obtain an optimal solution \( y \) to \( \mathcal{D}(T, w) \) that contradicts the assumption (3) on \( y \). Moreover, since \( v_1v_6v_3v_4v_1 \equiv v_2v_4v_6v_3 \equiv v_1v_6v_3v_4v_1 \), it follows from (3) that \( y(v_2v_4v_6v_2) = 0 \).

(9) If \( y(v_1v_6v_3v_4v_1) = 0 \), then both \( v_1v_2 \) and \( v_4v_6 \) are saturated by \( y \) in \( G_3 \); so is \( v_3v_1 \) if \( y(v_2v_4v_6v_3v_2) = 0 \) or \( y(v_2v_4v_6v_3v_2) > 0 \).

The first half follows instantly from Lemma 3.5(v). To prove the second half, assume the contrary. If \( v_3v_1 \) is not saturated by \( y \) in \( T \), then \( v_3v_1v_6v_3v_4v_1 \equiv v_1v_6v_3v_4v_1 \equiv \{v_3v_1\} = v_3v_1v_6v_3v_4v_1 \equiv v_1v_6v_3v_4v_1 \), and \( v_2v_4v_6v_3v_2 \equiv v_1v_6v_3v_4v_1 \equiv \{v_3v_1\} = v_2v_4v_6v_3v_2 \equiv v_1v_6v_3v_4v_1 \); if \( v_3v_1 \) is saturated by \( y \) in \( T \) but contained in some cycle \( C \in C_0^y \), then the multiset sum of \( C \), \( v_1v_6v_3v_4v_1 \), and \( v_2v_4v_6v_3 \) (resp. \( v_2v_4v_6v_3v_2 \)) contains arc-disjoint cycles \( v_2v_4v_6v_3v_2 \), \( v_1v_6v_3v_4v_1 \), and \( C' = C[v_4, v_3] \cup \{v_3v_4\} \) (resp. \( C' = C[v_4, v_3] \cup \{v_3v_2, v_2v_4\} \)). Since \( y \) satisfies (2), we have \( y(v_3v_4v_6v_3) = y(v_2v_4v_6v_3v_2) = 0 \), a contradiction.

(10) If \( y(v_2v_4v_6v_3v_2) = 0 \), then both \( v_3v_4 \) and \( v_6v_2 \) are saturated by \( y \) in \( G_3 \) by Lemma 3.5(v).

(11) If \( v_1v_6 \) is contained in a cycle in \( C_0^y \), then both \( v_4v_6 \) and \( v_6v_2 \) are saturated by \( y \) in \( G_3 \).

Since both \( C[v_1, v_4] \cup \{v_4v_1\} \) and \( C[v_6, v_4] \cup \{v_4v_6\} \) are cycles in \( C_2 \), the statement follows instantly from Lemma 3.5(iv).

(12) If \( v_6v_3 \) is contained in a cycle in \( C_0^y \), then \( v_4v_6 \) is saturated by \( y \) in \( G_3 \); so is \( v_1v_6 \) or \( v_4v_1 \).

The first half follows instantly from Lemma 3.5(iv). To prove the second half, we may assume, by (11), that \( v_1v_6 \) is outside \( C_0^y \). Let \( C \) be a cycle in \( C_0^y \) containing \( v_6v_3 \). Then both \( C[v_6, v_4] \cup \{v_4v_6\} \) and \( C[v_6, v_4] \cup \{v_4v_1, v_1v_6\} \) are cycles in \( C_2 \). Thus, by Lemma 3.5(iv), \( v_4v_6 \) and at least one of \( v_1v_6 \) and \( v_4v_1 \) are saturated by \( y \) in \( G_3 \).

**Claim 1.** \( y(C_2) = \tau_w(G_3 \setminus v_5) \).

To justify this, observe that \( v_2v_4 \) is a special arc of \( T \) and \( v_2 \) is a near-sink. By Lemma 3.4(iv), we may assume that \( v_2v_4 \) is saturated by \( y \) in \( T \). Let \( G_2 = \{v_1v_2, v_3v_1, v_3v_4, v_2v_4, v_3v_1, v_3v_4\}, \{v_1v_2, v_3v_1, v_6v_3\}, \{v_1v_2, v_3v_1, v_6v_3\} \). Then \( \mathcal{G}_2 \subset \mathcal{F}_2 \). Observe that

(13) if \( y(v_1v_2v_4v_6v_3v_1) = 0 \), then for each \( K \in \mathcal{G}_2 \), not all arcs in \( K \) are saturated by \( y \) in \( G_3 \).

Suppose the contrary: all arcs in \( K \) are saturated by \( y \) in \( G_3 \). Examining cycles in \( C_2 \), we see that \( y(C_2) = w(K) \). By (5), \( K \) is an MFAS and hence \( y(C_2) = \tau_w(G_3 \setminus v_5) \). So we may assume
that (13) holds.

Depending on whether \(v_2v_4\) is outside \(C_0^y\), we distinguish between two cases.

**Case 1.1.** \(v_2v_4\) is contained in some cycle in \(C_0^y\).

We proceed by considering four subcases.

- Neither \(v_4v_1\) nor \(v_4v_6\) is saturated by \(y\) in \(G_3\). In this subcase, by Lemma 3.5(iii) and (iv), both \(v_4v_2\) and \(v_4v_2\) are saturated by \(y\) in \(G_3\). By (6)-(9), \(y(v_1v_2v_4v_6v_3v_4v_1), y(v_1v_6v_3v_4v_1),\) and \(y(v_2v_4v_2v_4v_1)\) are all zero. By (12) and (13), \(v_6v_3\) is outside \(C_0^y\) and not saturated by \(y\). By Lemma 3.5(iii), both \(v_2v_3\) and \(v_2v_4\) are saturated by \(y\) in \(G_3\). By Lemma 3.5(i) and (iii), at least one of \(v_1v_6\) and \(v_3v_1\) is saturated by \(y\) in \(G_3\). Thus \(y(C_2) = w(K)\), where \(K\) is \(\{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\}\) or \(\{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2\}\). By (5), \(K\) is an MFAS and hence \(y(C_2) = \tau_y(G_3\setminus v_5)\).

- \(v_4v_6\) is saturated by \(y\) in \(G_3\) while \(v_4v_1\) is not. In this subcase, by Lemma 3.5(iii), \(v_4v_2\) is saturated by \(y\) in \(G_3\). By (6), we have \(y(v_1v_2v_4v_6v_3v_4v_1) = 0\). By (11) and (13), \(v_1v_6\) is outside \(C_0^y\) and not saturated by \(y\). By Lemma 3.5(i) and (iii), \(v_6v_3\) is saturated by \(y\) in \(G_3\). So, by (12) and (13), \(v_6v_3\) is outside \(C_0^y\) and not saturated by \(y\). It follows from Lemma 3.5(i) and (iii) that \(v_2v_3\) and \(v_3v_4\) are all saturated by \(y\) in \(G_3\). Thus \(y(C_2) = w(K)\), where \(K\) is \(\{v_1v_2, v_3v_1, v_3v_2, v_3v_4, v_6v_2\}\). By (5), \(K\) is an MFAS and hence \(y(C_2) = \tau_y(G_3\setminus v_5)\).

- \(v_4v_1\) is saturated by \(y\) in \(G_3\) while \(v_4v_6\) is not. In this subcase, by Lemma 3.5(iii), \(v_4v_3\) is saturated by \(y\) in \(G_3\). By (7)-(9), \(y(v_1v_6v_3v_4v_1), y(v_1v_6v_3v_4v_1),\) and \(y(v_1v_6v_2v_4v_1)\) are all zero. By (12), \(v_6v_3\) is outside \(C_0^y\). Furthermore, we may assume that \(v_6v_3\) is not saturated by \(y\), for otherwise \(y(C_2) = w(K)\), where \(K\) is \(\{v_1v_1, v_6v_2, v_6v_3\}\). Then, by Lemma 3.5(iii) and (iv), both \(v_3v_2\) and \(v_3v_4\) are saturated by \(y\) in \(G_3\). If \(v_3v_1\) is also saturated by \(y\) in \(G_3\), then \(y(C_2) = w(K)\), where \(K\) is \(\{v_3v_1, v_3v_2, v_3v_4, v_4v_1, v_6v_2\}\); otherwise, by Lemma 3.5(i) and (iii), both \(v_2v_3\) and \(v_1v_6\) are saturated by \(y\) in \(G_3\). So \(y(C_2) = w(J)\), where \(J\) is \(\{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\}\).

- Both \(v_4v_1\) and \(v_4v_6\) are saturated by \(y\) in \(G_3\). In this subcase, if \(y(v_1v_2v_4v_6v_3v_1) > 0\), then \(v_1v_6\) is saturated by \(y\) in \(G_3\) and \(y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1) = 0\) by (6). Thus \(y(C_2) = w(K)\), where \(K\) is \(\{v_1v_1, v_6v_6, v_1v_6\}\). So we assume that \(y(v_1v_2v_4v_6v_3v_1) = 0\). Then \(v_3v_1\) is not saturated by \(y\) in \(G_3\) by (13). Thus \(y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1) = 0\) by (7) and (8). If \(y(v_1v_6v_2v_4v_1) > 0\), then \(v_2v_3\) is saturated by \(y\) in \(G_3\) and \(y(v_3v_4v_6v_3) = y(v_2v_4v_6v_3v_2) = 0\) by (9). By (13), \(v_6v_3\) is not saturated by \(y\) in \(G_3\). Hence, by Lemma 3.5(iii), \(v_6v_3\) is saturated by \(y\) in \(G_3\). Therefore, \(y(C_2) = w(K)\), where \(K\) is \(\{v_1v_1, v_6v_6, v_6v_3\}\). So we may assume that \(y(v_1v_6v_2v_4v_1) = 0\) and that \(v_6v_3\) is not saturated by \(y\) in \(G_3\), for otherwise \(y(C_2) = w(K)\), where \(K\) is \(\{v_1v_1, v_6v_6, v_1v_6\}\). Thus, by Lemma 3.5(iii) and (iv), \(v_6v_3\) is saturated by \(y\) in \(G_3\). We may further assume that \(v_4v_2\) is not saturated by \(y\) in \(G_3\), for otherwise, \(y(C_2) = w(J)\), where \(J\) is \(\{v_4v_1, v_6v_6, v_6v_3\}\). Then \(y(v_2v_4v_6v_3v_2) = 0\) by (10). We propose to show that

\[(14)\ y(v_3v_4v_6v_3) = 0.\]

Assume the contrary: \(y(v_3v_4v_6v_3) > 0\). Since neither \(v_1v_6\) nor \(v_3v_1\) is saturated by \(y\) in \(G_3\), we distinguish among four subcases.

(a) Neither \(v_1v_6\) nor \(v_3v_1\) is saturated by \(y\) in \(T\). In this subcase, set \(\theta = \min\{w(v_1v_6) - z(v_1v_6), w(v_3v_1) - z(v_3v_1), y(v_1v_3v_4v_6v_3)\}\). Let \(y'\) be obtained from \(y\) by replacing \(y(v_3v_4v_6v_3)\) and \(y(v_1v_6v_3v_1)\) with \(y(v_3v_4v_6v_3) - \theta\) and \(y(v_1v_6v_3v_1) + \theta\), respectively. Then \(y'\) is also an optimal solution to \(\mathcal{D}(T, w)\) with \(y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)\), contradicting (4).

(b) \(v_3v_1\) is not saturated by \(y\) in \(T\) and \(v_1v_6\) is contained in some cycle \(C_1 \in C_0^y\). In this
subsubcase, since \( v_6v_3 \) is saturated by \( y \) in \( G_3 \), cycle \( C_1 \) contains the path \( v_6v_2v_4 \). Thus the multiset sum of \( C_1, v_3v_4v_6v_3, \) and \( v_3v_1 \) contains two arc-disjoint cycles \( v_2v_4v_6v_2 \) and \( v_1v_6v_3v_1 \). By Lemma 3.5(iv), we have \( y(v_3v_4v_6v_3) = 0 \), a contradiction.

(c) \( v_1v_6 \) is not saturated by \( y \) in \( T \) and \( v_3v_1 \) is contained in some cycle \( C_2 \in C'_0 \). In this subcase, it is clear that \( C_2 \) contains the path \( v_1v_2v_4 \). Observe that the multiset sum of \( C_2, v_3v_4v_6v_3, \) and the unsaturated \( v_1v_6 \) contains two arc-disjoint cycles \( v_1v_6v_3v_1 \) and \( C''_2 = C_2[v_4, v_3] \cup \{v_2v_4\} \). Set \( \theta = \min \{y(C_2), y(v_3v_4v_6v_3), w(v_1v_6) - z(v_1v_6)\} \). Let \( y' \) be obtained from \( y \) by replacing \( y(C_2), y(v_3v_4v_6v_3), y(v_1v_6v_3v_1) \), and \( y(C''_2) \) with \( y(C_2) - \theta, y(v_3v_4v_6v_3) - \theta, y(v_1v_6v_3v_1) + \theta, \) and \( y(C''_2) + \theta \), respectively. Then \( y' \) is also an optimal solution to \( D(T, w) \) with \( y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3) \), contradicting (4).

(d) \( v_1v_6 \) and \( v_3v_1 \) are contained in some cycles \( C_1 \) and \( C_2 \) in \( C'_0 \), respectively. In this subcase, if \( v_3v_1 \) is also on \( C_1 \), then the multiset sum of \( C_1 \) and \( v_3v_4v_6v_3 \) contains arc-disjoint cycles \( v_1v_6v_2v_1, v_2v_4v_6v_2, \) and \( C'_1 = C_1[v_4, v_3] \cup \{v_3v_4\} \). From the optimality of \( y \), we deduce that \( y(v_3v_4v_6v_3) = 0 \). If \( v_3v_1 \) is outside \( C_1 \), then the multiset sum of \( C_1, C_2, \) and \( v_3v_4v_6v_3 \) contains arc-disjoint cycles \( v_1v_6v_3v_1, v_2v_4v_6v_2, \) \( C'_1 = C_1[v_4, v_1] \cup \{v_2v_2, v_2v_4\} \), and \( C''_2 = C_2[v_4, v_3] \cup \{v_3v_4\} \). From the optimality of \( y \), we again deduce that \( y(v_3v_4v_6v_3) = 0 \).

By (14), we have \( y(C_2) = w(K), \) where \( K = \{v_1v_6, v_4v_6, v_6v_3\} \). So \( K \) is an MFAS by (5) and hence \( y(C_2) = \tau_w(G_3 \setminus v_5) \).

Case 1.2. \( v_2v_4 \) is outside \( C'_0 \).

In this case, \( v_2v_4 \) is saturated by \( y \) in \( G_3 \), and \( v_1v_2, v_3v_2, \) and \( v_6v_2 \) are all outside \( C'_0 \). Since \( C'_0 \neq \emptyset \), there exists a cycle \( C \in C'_0 \) containing \( v_2v_4 \). From (6), (7), and (10), we see that \( y(v_1v_2v_4v_6v_3v_1), y(v_1v_6v_3v_2v_4v_1), \) and \( y(v_2v_4v_6v_3v_2) \) are all zero. If \( v_6v_3 \) is also saturated by \( y \) in \( G_3 \), then \( y(C_2) = w(K), \) where \( K = \{v_2v_4, v_6v_3\} \). So we assume that \( v_6v_3 \) is not saturated by \( y \) in \( G_3 \). By Lemma 3.5(iii) and (iv), \( v_4v_6 \) is saturated by \( y \) in \( G_3 \).

Assume first that \( v_1v_6 \) is not saturated by \( y \) in \( G_3 \). Then, by Lemma 3.5(iii) and (iv), \( v_1v_6 \) is saturated by \( y \) in \( G_3 \). By (13), \( v_1v_6 \) is not saturated by \( y \) in \( G_3 \) and hence in \( T \). By (9), \( y(v_1v_6v_2v_4v_1) = 0 \). If \( v_6v_3 \) is not saturated by \( y \) in \( T \), then the multiset sum of \( C, v_2v_4v_6v_2, \) and the unsaturated arcs \( v_3v_2, v_4v_1, \) and \( v_1v_6 \) contains two arc-disjoint cycles \( v_1v_2v_4v_1 \) and \( v_2v_4v_6v_3 \); if \( v_6v_3 \) is saturated by \( y \) in \( T \) but contained in some cycle \( C \in C'_0 \), then the multiset sum of \( C, v_2v_4v_6v_2, \) and the unsaturated arcs \( v_3v_2, v_4v_1, \) and \( v_1v_6 \) contains two arc-disjoint cycles \( v_1v_2v_4v_1 \) and \( v_2v_4v_6v_3 \). By Lemma 3.5(vi), we have \( y(v_2v_4v_6v_3) = 0 \) in either subcase. So \( y(C_2) = w(K), \) where \( K = \{v_1v_6, v_4v_6, v_6v_3\} \).

Assume next that \( v_1v_6 \) is saturated by \( y \) in \( G_3 \). Then, by (13), \( v_3v_1 \) and at least one of \( v_1v_6 \) and \( v_1v_6v_3 \) is not saturated by \( y \) in \( G_3 \). By Lemma 3.5(iii) and (iv), both \( v_3v_1 \) and \( v_1v_6 \) are outside \( C'_0 \); using Lemma 3.5(i) and (iii), we further deduce that \( v_1v_6 \) is saturated by \( y \) in \( G_3 \). Thus, by (13), \( v_1v_6 \) is not saturated by \( y \) in \( G_3 \). It follows from (8) and (9) that \( y(v_1v_6v_3v_4v_1) = y(v_1v_6v_2v_4v_1) = 0 \). Therefore \( y(C_2) = w(K), \) where \( K = \{v_1v_6, v_4v_6, v_6v_3\} \). So \( K \) is an MFAS by (5) and hence \( y(C_2) = \tau_w(G_3 \setminus v_5) \). This proves Claim 1.

Claim 2. \( y(C) \) is integral for all \( C \in C_2 \) or \( \nu_w^*(T) \) is an integer.

To justify this, we may assume that

(15) \( y(v_1v_2v_3v_4v_6v_3v_1) = y(v_1v_6v_3v_4v_6v_3v_1) = 0 \).

Assume the contrary: \( y(v_1v_2v_3v_4v_6v_3v_1) = 0 \). Then, from (6) we deduce that \( y(v_1v_6v_3v_4v_6v_3v_1) = y(v_1v_6v_3v_4v_6v_3v_1) = 0 \) and that each arc in the set \( \{v_1v_6, v_3v_2, v_3v_4, v_4v_1, v_6v_2\} \) is
saturated by \( y \) in \( G_3 \). So \( y(v_1v_2v_3v_1) = w(v_1v_1), y(v_1v_4v_5v_3) = w(v_3v_4), y(v_2v_4v_5v_2) = w(v_5v_6), \) and \( y(v_2v_4v_5v_3v_1) = w(v_3v_2) \). By Claim 1, \( y(C_2) \) is an integer; so is \( y(v_1v_2v_4v_5v_3v_1) \). Thus Lemma 3.2(iii) allows us to assume that \( y(v_1v_2v_4v_5v_3v_1) = 0 \).

If \( y(v_1v_6v_7v_2v_4v_1) > 0 \), then from (7) we deduce that \( y(v_2v_4v_5v_2) = y(v_3v_2v_4v_5v_3v_1) = 0 \) and that each arc in the set \( \{v_1v_2, v_3v_1, v_4v_1, v_6v_5, v_5v_2\} \) is saturated by \( y \) in \( G_3 \). So \( y(v_1v_2v_4v_5v_3v_1) = w(v_1v_2), y(v_1v_6v_7v_2v_4v_1) = w(v_1v_2), y(v_1v_6v_7v_2v_4v_1) = w(v_3v_4), y(v_1v_6v_7v_2v_4v_1) = w(v_5v_6), \) and \( y(v_2v_4v_5v_2v_4v_1) = w(v_4v_5) \). By Claim 1, \( y(C_2) \) is an integer; so is \( y(v_1v_6v_7v_2v_4v_1) \). Thus Lemma 3.2(iii) allows us to further assume that \( y(v_1v_6v_7v_2v_4v_1) = 0 \).

By Claim 1, \( y(C_2) = w(K) \) for some \( K \in \mathcal{F}_2 \). Depending on what \( K \) is, we distinguish among 13 cases.

**Case 2.1.** \( K = \{v_1v_6, v_2v_4, v_4v_6\} \).
In this case, by Lemma 3.1 (i), we have \( y(v_2v_4v_5v_2) = y(v_1v_6v_7v_2v_4v_1) = y(v_1v_6v_7v_2v_4v_1) = 0 \). By Lemma 3.1 (iii), we obtain \( w(e) = y(C_2(e)) \) for each \( e \in K \), which together with (15) yields the following equations: \( y(v_1v_6v_7v_2v_4v_1) + y(v_1v_6v_7v_2v_4v_1) = w(v_1v_6), y(v_1v_6v_7v_2v_4v_1) = w(v_2v_4), y(v_1v_6v_7v_2v_4v_1) = w(v_4v_5) \). If \( y(v_1v_6v_7v_2v_4v_1) > 0 \), then by (8) we have one more equation \( y(v_1v_6v_7v_2v_4v_1) = w(v_3v_4) \). So \( y(C) \) is integral for any \( C \in C_2 \), no matter whether \( y(v_1v_6v_7v_2v_4v_1) = 0 \).

**Case 2.2.** \( K = \{v_2v_1, v_4v_6, v_6v_3\} \).
In this case, by Lemma 3.1 (i), we have \( y(v_2v_1v_4v_6) = y(v_1v_6v_7v_2v_4v_1) = y(v_1v_6v_7v_2v_4v_1) = 0 \). By Lemma 3.1 (iii), we obtain \( w(e) = y(C_2(e)) \) for each \( e \in K \), which together with (15) yields the following equations: \( y(v_1v_6v_7v_2v_4v_1) + y(v_1v_6v_7v_2v_4v_1) = w(v_1v_6), y(v_1v_6v_7v_2v_4v_1) = w(v_2v_4), y(v_1v_6v_7v_2v_4v_1) = w(v_4v_5) \). Observe that if \( y(v_1v_6v_7v_2v_4v_1) > 0 \), then by (8) we have \( y(v_1v_6v_7v_2v_4v_1) = w(v_1v_6) - w(v_4v_5) \) and \( y(v_1v_6v_7v_2v_4v_1) = 0 \) if \( y(v_1v_6v_7v_2v_4v_1) > 0 \), then by (9) we have \( y(v_1v_6v_7v_2v_4v_1) = w(v_1v_6) - w(v_4v_5) \). So \( y(C) \) is integral for any \( C \in C_2 \), no matter whether \( y(v_1v_6v_7v_2v_4v_1) \) or \( y(v_1v_6v_7v_2v_4v_1) \) is zero.

**Case 2.4.** \( K = \{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\} \).
In this case, by Lemma 3.1 (i), we have \( y(v_1v_6v_7v_2v_4v_1) = y(v_1v_6v_7v_2v_4v_1) = 0 \). By Lemma 3.1 (iii), we obtain \( w(e) = y(C_2(e)) \) for each \( e \in K \), which together with (15) yields the following equations: \( y(v_1v_6v_7v_2v_4v_1) = w(v_1v_2), y(v_1v_6v_7v_2v_4v_1) = w(v_1v_6), y(v_2v_4v_5v_2v_4v_1) = w(v_4v_5), y(v_2v_4v_5v_2v_4v_1) = w(v_3v_4), \) and \( y(v_2v_4v_5v_2v_4v_1) = w(v_6v_2) \). Hence \( y(C) \) is integral for all \( C \in C_2 \).

**Case 2.5.** \( K = \{v_1v_2, v_1v_6, v_3v_2, v_3v_4, v_6v_2\} \).
In this case, by Lemma 3.1 (i), we have \( y(v_1v_6v_7v_2v_4v_1) = 0 \). By Lemma 3.1 (iii), we obtain \( w(e) = y(C_2(e)) \) for each \( e \in K \), which together with (15) yields the following equations: \( y(v_1v_6v_7v_2v_4v_1) = w(v_1v_2), y(v_1v_6v_7v_2v_4v_1) = w(v_1v_6), y(v_2v_4v_5v_2v_4v_1) = w(v_4v_5), \) and \( y(v_2v_4v_5v_2v_4v_1) = w(v_6v_2) \). Observe that if \( y(v_1v_6v_7v_2v_4v_1) > 0 \), then by (9) we have \( y(v_1v_6v_7v_2v_4v_1) = w(v_1v_6) - w(v_3v_4) \). So \( y(C) \) is integral for all \( C \in C_2 \), no matter
whether $y(v_1v_6v_2v_4v_1)$ is zero.

**Case 2.6.** $K = \{v_1v_6, v_2v_4, v_3v_1\}$

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which together with (15) yields the following equations:

\[ y(v_1v_6v_3v_1) = w(v_1v_6), \quad y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) + y(v_2v_4v_6v_3) = w(v_2v_4), \quad y(v_3v_4v_6v_3) = w(v_3v_4). \]

If $y(v_2v_4v_6v_3) > 0$, then $y(v_2v_4v_6v_2) = w(v_6v_2)$ by (10). Since $v_4v_1$ and $v_1v_2$ are outside $C_0^\prime$ and disc $y$ satisfies (2), it is easy to see that $y(v_1v_2v_4v_1) = \min\{w(v_1v_2), w(v_4v_1)\}$. So $y(C)$ is integral for all $C \in C_2$. Thus we may assume that $y(v_2v_4v_6v_3) = 0$. Since both $v_4v_6$ and $v_6v_2$ are outside $C_0^\prime$, by (4) we have $y(v_2v_4v_6v_2) = \min\{w(v_6v_2), w(v_4v_6) - w(v_3v_4)\}$. It follows that $y(C)$ is integral for all $C \in C_2$.

**Case 2.7.** $K = \{v_2v_4, v_3v_1, v_4v_3\}$

In this case, by Lemma 3.1 (iii), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which together with (15) yields the following equations:

\[ y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) + y(v_1v_6v_2v_2) + y(v_2v_4v_6v_3) = w(v_2v_4), \quad y(v_1v_6v_3v_1) = w(v_3v_1), \quad y(v_3v_4v_6v_3) = y(v_1v_6v_3v_1) = w(v_3v_1). \]

Assume first that $y(v_1v_6v_3v_1) > 0$. Then, by (8), we have $y(v_2v_4v_6v_2) = 0$ and $y(v_2v_4v_6v_3) + y(v_2v_4v_6v_3) = w(v_2v_4)$. If $y(v_2v_4v_6v_3) > 0$, then, by (8) and (10), we obtain $y(v_1v_6v_2v_1) = w(v_1v_2)$ and $y(v_1v_6v_2v_1) = w(v_1v_2)$; if $y(v_2v_4v_6v_3) = 0$ and $y(v_1v_6v_2v_1) > 0$, then, by (9), we get $y(v_1v_2v_4v_1) = w(v_1v_2), y(v_1v_2v_4v_1) = w(v_2v_4) - w(v_4v_2)$, and $y(v_3v_4v_6v_3) = w(v_4v_2)$. If $y(v_1v_6v_2v_1) = y(v_2v_4v_6v_3) = 0$, then $y(v_2v_4v_6v_2) = w(v_2v_4)$, and $y(v_3v_4v_6v_3) = w(v_4v_2)$. Thus $y(C)$ is integral for all $C \in C_2$ in any subcase.

Assume next that $y(v_1v_6v_3v_1) = 0$. If $y(v_1v_6v_2v_1) > 0$, then, by (9), we have $y(v_1v_2v_4v_1) = w(v_1v_2)$ and $y(v_2v_4v_6v_3) = w(v_2v_4) - y(v_3v_4v_6v_3) = w(v_2v_4) - w(v_3v_4)$, and so $y(v_1v_6v_2v_1) = w(v_1v_2) + y(v_2v_4v_6v_3) = \min\{w(v_1v_2), w(v_4v_2), w(v_4v_2)\}.$ Observe that if $y(v_2v_4v_6v_3) = 0$, then we have one more equation $y(v_2v_4v_6v_2) + y(v_1v_6v_2v_1) = w(v_4v_2)$ by (10). Thus $y(C)$ is integral for all $C \in C_2$, no matter whether $y(v_2v_4v_6v_3) = 0$. So we assume that $y(v_1v_6v_2v_1) = 0$. If $y(v_2v_4v_6v_3) > 0$, then $y(v_2v_4v_6v_2) = w(v_4v_2)$ and $y(v_1v_6v_2v_1) + y(v_2v_4v_6v_3) = w(v_2v_4) - w(v_4v_2)$; if $y(v_2v_4v_6v_3) = 0$, then $y(v_1v_6v_2v_1) + y(v_2v_4v_6v_2) = w(v_2v_4)$. Since $y$ satisfies (2) and (4) and since $v_1v_2, v_4v_6, v_1v_2, v_4v_6$ are all outside $C_0^\prime$, if $y(v_2v_4v_6v_3) > 0$, then $y(v_1v_6v_2v_1) = \min\{w(v_1v_2), w(v_1v_2)\}$ or $y(v_2v_4v_6v_2) = min\{w(v_4v_2), w(v_2v_4v_6v_3), w(v_6v_2)\}$, regardless of the value of $y(v_2v_4v_6v_3)$. Hence $y(C)$ is integral for all $C \in C_2$.

**Case 2.8.** $K = \{v_1v_2, v_4v_6, v_3v_1\}$

In this case, by Lemma 3.1 (iii), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which together with (15) yields the following equations:

\[ y(v_1v_2v_4v_1) = w(v_1v_2), y(v_2v_4v_6v_2) + y(v_1v_6v_3v_1) = w(v_4v_2), \quad y(v_1v_5v_3v_1) + y(v_1v_5v_3v_1) + y(v_2v_4v_6v_2) + y(v_2v_4v_6v_3) = w(v_2v_4). \]

Depending on the value of $y(v_1v_5v_3v_1)$, we consider two subcases.

- $y(v_1v_5v_3v_1) = 0$. In this case, by (8), we have $y(v_2v_4v_6v_2) = 0, y(v_1v_6v_3v_1) = w(v_3v_1)$, and $y(v_2v_4v_6v_3) = y(v_2v_4v_6v_3) = w(v_4v_2)$. So $y(v_1v_6v_3v_1) = w(v_1v_6v_3v_1) - w(v_4v_2)$. Observe that if $y(v_2v_4v_6v_3) > 0$, then we have one more equation $y(v_2v_4v_6v_3) = w(v_3v_4) - y(v_2v_4v_6v_3)$; by (10). So $y(C)$ is integral for all $C \in C_2$, no matter whether $y(v_2v_4v_6v_3) = 0$.

- $y(v_1v_6v_3v_1) = 0$. In this case, assume first that $y(v_1v_6v_3v_1) > 0$. If $y(v_2v_4v_6v_3) > 0$ or $y(v_2v_4v_6v_3) > 0$, then, by (9), we have $y(v_1v_6v_3v_1) = w(v_3v_1), y(v_2v_4v_6v_3) = w(v_4v_2) - w(v_3v_1)$, and $y(v_2v_4v_6v_3) = w(v_4v_2) = w(v_4v_2) - w(v_3v_1) - w(v_4v_2)$. If $y(v_2v_4v_6v_3) > 0$, then $y(v_2v_4v_6v_3) = w(v_3v_4)$; by (10). Thus $y(C)$ is integral for all $C \in C_2$.
$w(v_1v_6)$. So $y(C)$ is integral for all $C \in C_2$ in any subcase. Assume next that $y(v_1v_6v_2v_1v_4) = 0$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_2v_4v_6v_3) = w(v_2v_4)$ by (10) and $y(v_1v_6v_3v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3) - w(v_3v_4)$; if $y(v_2v_4v_6v_3v_2) = 0$, then $y(v_1v_6v_3v_1) + y(v_3v_1v_6v_3) = w(v_6v_3)$. Note that both $v_3v_1$ and $v_1v_6$ are outside $C_0'$. As $y$ satisfies (2) and (4), we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_1v_6), w(v_3v_1)\}$, no matter whether $y(v_2v_4v_6v_3v_2) > 0$. Hence $y(C)$ is integral for all $C \in C_2$.

**Case 2.9.** $K = \{v_4v_1, v_6v_2, v_6v_3\}$.

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_2v_1v_4) = y(v_1v_6v_3v_1v_4) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which together with (15) yields the following equations:

$y(v_1v_2v_4v_1) = w(v_1v_4)$,

$y(v_2v_4v_6v_2) = w(v_6v_2)$,

$y(v_1v_6v_3v_1) + y(v_3v_1v_6v_3) = w(v_6v_3)$. If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_3v_1v_6v_3) = w(v_3v_1)$ by (10), so

$y(v_1v_6v_3v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3) - w(v_3v_1)$; if $y(v_2v_4v_6v_3v_2) = 0$, then $y(v_1v_6v_3v_1) + y(v_2v_4v_6v_3v_2) = w(v_6v_3)$. Clearly, $v_1v_6$ is outside $C_0'$. We propose to show that

(16) $y(v_1v_6v_3v_1) = 0$.

Suppose on the contrary that $y(v_1v_6v_3v_1) = 0$ is not integral. If $v_3v_1$ is outside $C_0'$, then from (2) and (4) we deduce that $y(v_1v_6v_3v_1) = 0$ and $\min\{w(v_3v_1), w(v_1v_6)\}$, a contradiction. So we assume that $v_3v_1$ is contained in some cycle $C$ in $C_0'$. Then $C$ contains the path $v_1v_2v_4v_1$. Set $C' = C[v_4, v_3] \cup \{v_2, v_2v_4\}$ if $y(v_2v_4v_6v_3v_2) > 0$ and $C' = C[v_4, v_3] \cup \{v_2, v_2v_4\}$ otherwise, and set

$\theta = \min\{y(v_2v_4v_6v_3v_2), y(C)\}$ if $y(v_2v_4v_6v_3v_2) > 0$ and $\theta = \min\{y(v_3v_1v_6v_3), y(C)\}$ otherwise. Let $y'$ be obtained from $y$ by replacing $y(v_2v_4v_6v_3v_2)$ (resp. $y(v_3v_1v_6v_3)$, $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_2v_4v_6v_3v_2) - \theta$ (resp. $y(v_3v_1v_6v_3) - \theta$, $y(v_1v_6v_3v_1) + \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $y'(v_2v_4v_6v_3v_2) < y(v_2v_4v_6v_3v_2)$ or $y'(v_3v_1v_6v_3) < y(v_3v_1v_6v_3)$, contradicting (2) and (4). So (16) is established.

From (16) it follows that $y(C)$ is integral for all $C \in C_2$.

**Case 2.10.** $K = \{v_2v_4, v_6v_3\}$.

In this case, by Lemma 3.1 (i), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which together with (15) yields the following equations:

$y(v_1v_2v_4v_1) + y(v_2v_4v_6v_2) + y(v_1v_6v_3v_1) = w(v_1v_2v_4v_1) + y(v_1v_6v_3v_1) + y(v_3v_1v_6v_3) = w(v_6v_3)$. It follows that all arcs in $G_3\backslash v_5$ are outside $C_0'$ except possibly $v_3v_1$. If $y(v_1v_6v_3v_1) > 0$, then, by (8), we have $y(v_2v_4v_6v_2) = 0$, $y(v_1v_6v_3v_1) = w(v_1v_2v_4v_1)$, and $y(v_3v_1v_6v_3) = w(v_3v_1v_6v_3)$; if $y(v_1v_6v_3v_1) = 0$, then we have one more equation $y(v_1v_6v_3v_1) = w(v_1v_6v_3v_1)$. Thus $y(C)$ is integral for all $C \in C_2$, no matter whether $y(v_1v_6v_3v_1) = 0$. So we assume that $y(v_1v_6v_3v_1) = 0$.

If $y(v_1v_2v_4v_1) > 0$, then, by (9), we obtain $y(v_1v_2v_4v_1) = w(v_1v_2v_4v_1)$ and $y(v_1v_6v_3v_1) = w(v_6v_3)$. Furthermore, $y(v_1v_6v_3v_1) = 0$ if $y(v_3v_1v_6v_3) > 0$ and $y(v_1v_6v_3v_1) = w(v_6v_3)$ otherwise. Hence $y(C)$ is integral for all $C \in C_2$, no matter whether $y(v_3v_1v_6v_3) = 0$. So we may assume that $y(v_1v_6v_3v_1v_4) = 0$.

If $y(v_3v_1v_6v_3) = 0$, then $y(v_1v_6v_3v_1) = w(v_6v_3)$. Recall that both $v_3v_1$ and $v_6v_3$ are outside $C_0'$. If $y(v_1v_6v_3v_1v_4) > 0$, then from (4) we deduce that $y(v_2v_4v_6v_2) = \min\{w(v_1v_6), w(v_6v_3)\}$. Hence $y(C)$ is integral for all $C \in C_2$, no matter whether $y(v_1v_2v_4v_1) > 0$. It remains to consider the subcase when $y(v_3v_1v_6v_3) > 0$. Since both $v_3v_1$ and $v_6v_3$ are outside $C_0'$, from (4) we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_1v_6), w(v_1v_6)\}$. If $y(v_1v_6v_3v_1) = 0$, then $y(v_2v_4v_6v_2) = w(v_2v_4v_1)$; otherwise, by (4), at least one of $v_6v_3$ and $v_6v_3$ is saturated by $y$ in $G_3$. It follows that $y(v_2v_4v_6v_2) = \min\{w(v_6v_3), w(v_6v_3) - y(v_3v_1v_6v_3)\}$. Hence $y(C)$ is integral for all $C \in C_2$, no
matter whether \( y(v_1v_2v_4v_1) = 0 \).

**Case 2.11.** \( K = \{ v_3v_1, v_4v_1, v_4v_6 \} \).

In this case, by Lemma 3.1 (iii), we obtain \( w(e) = y(C_2(e)) \) for each \( e \in K \), which together with (15) yields the following equations: 
\[
\begin{align*}
y(v_1v_5v_3v_1) &= w(v_3v_1), \\
y(v_1v_5v_4v_3) &= w(v_5v_4v_3)v_1 + y(v_1v_5v_4v_3) = w(v_1v_4v_3), \\
y(v_2v_5v_4v_2) &= w(v_2v_5v_4v_2) + y(v_2v_5v_4v_2) + y(v_2v_4v_3v_2) = w(v_2v_4v_3v_2).
\end{align*}
\]
Depending on the value of \( y(v_1v_5v_3v_1) \), we consider two subcases.

1. \( y(v_1v_5v_3v_1) > 0 \). In this subcase, \( y(v_2v_5v_4v_3) = 0 \) by (8). If \( y(v_2v_5v_4v_3) = 0 \), then, by (8) and (10), we have \( y(v_1v_5v_3v_1) = w(v_1v_5v_3v_1) = w(v_2v_5v_4v_3) \). Hence \( y(C) \) is integral for all \( C \in C_2 \). So we assume that \( y(v_2v_5v_4v_3) = 0 \). Then \( y(v_5v_3v_1) = w(v_5v_3v_1) \). Depending on the value of \( y(v_1v_5v_3v_1) \), we distinguish between two subsubcases.

   a. \( y(v_1v_5v_3v_1) > 0 \). By (9), \( y(v_1v_5v_3v_1) = w(v_1v_5v_3v_1) + y(v_1v_5v_3v_1) = w(v_1v_5v_3v_1) - w(v_1v_5v_3v_1) \).

   If \( y(v_1v_5v_3v_1) \) is integral, then \( y(C) \) is integral for all \( C \in C_2 \). So we assume that \( y(v_1v_5v_3v_1) \) is not integral. By Lemma 3.2(iii), we may assume that \( w(v_3v_1), w(v_1v_5v_3v_1), \) and \( w(v_5v_3) \) are all zero. Observe that \( v_6v_2 \) is outside \( C_0^0 \), for otherwise, let \( C \) be a cycle in \( C_0^0 \). Then \( C \) passes through \( v_2v_4 \). Let \( C' = C[v_4, v_6] \). For each vertex \( u \) in \( V(T_i) \), we obtain \( w(v_5v_3) = w(v_5v_3) \). Let \( T' = (V', A') \) be obtained from \( T' \) by deleting vertex \( v_2 \), and let \( u \) be obtained from the restriction of \( w \) to \( A' \) by replacing \( w(v_5v_3) = w(v_5v_3) \). Since \( v_6v_2 \) is outside \( C_0^0 \), for each vertex \( u \) in \( V(T_i) \), we obtain \( w(v_5v_3) = w(v_5v_3) \). Let \( T' = (V', A) \) be obtained from \( T \) by deleting vertex \( v_2 \), and let \( u' \) be obtained from the restriction of \( w \) to \( A' \) by replacing \( w(v_5v_3) = w(v_5v_3) \).

   b. \( y(v_1v_5v_3v_1) = 0 \). Then \( y(v_1v_5v_3v_1) = y(v_1v_5v_3v_1) + y(v_1v_5v_3v_1) = w(v_1v_5v_3v_1) \).

   If \( y(v_1v_5v_3v_1) \) is integral, then \( y(C) \) is integral for all \( C \in C_2 \). So we assume that \( y(v_1v_5v_3v_1) \) is not integral. Observe that \( v_6v_2 \) is outside \( C_0^0 \), for otherwise, let \( C \) be a cycle in \( C_0^0 \). Then \( C \) passes through \( v_2v_4 \). Since the multiset sum of \( C \) and \( v_1v_5v_3v_1 \) contains \( v_1v_2 \). Thus \( w(v_1v_2) = z(v_1v_2) = 0 \). If \( v_6v_2 \) and \( v_3v_2 \) are outside \( C_0^0 \). Then \( w(v_1v_2) = z(v_1v_2) = 0 \). If \( v_6v_2 \) and \( v_3v_2 \) are outside \( C_0^0 \). Thus \( w(v_1v_2) = z(v_1v_2) = 0 \).
$y(v_1v_6v_3v_4v_1)$ are positive, by Lemma 3.1(i) we have $x(v_1v_2) + x(v_2v_4) = x(v_1v_6) + x(v_6v_3) + x(v_3v_4)$. Since $y(v_1v_2v_4v_1) < w(v_1v_2)$, by Lemma 3.1(ii) we obtain $x(v_1v_2) = 0$, which implies that $x(v_2v_4) = x(v_1v_6) + x(v_6v_3) + x(v_3v_4)$. Since $v_1v_2$ is outside $C_0^y$, for each vertex $u \in V(T_1) \setminus \{b, a_1\}$, we obtain $x(uv_1) = x(uv_2)$. Let $T' = \langle V', A' \rangle$ be obtained from $T$ by deleting vertex $v_2$, and let $w'$ be the restriction of $w$ to $A'$ by replacing $w(uv_1)$ with $w(uv_1) + w(v_2v_4)$ for each $u \in V(T_1) \setminus \{b, a_1\}$ and replacing $w(v_i)$ with $w(v_i) + w(v_2v_4)$ for $(i, j) = (1, 6), (6, 3), \text{ and } (3, 4)$. Let $x'$ be the restriction of $x$ to $A'$ and let $y'$ be defined from $y$ as follows: for each cycle $C$ passing through $uv_2v_3$ with $u \in V(T_1) \setminus \{b, a_1\}$, let $C'$ be obtained from $C$ by replacing $uv_2v_3$ with $uv_1v_6v_3v_4$, and set $y'(C') = y(C) + y(C')$ and $y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1) + y(v_1v_6v_3v_4v_1)$. Then $x'$ and $y'$ are optimal solutions to $F(T', w')$ and $D(T', w')$, respectively, with the same value $\nu_w^p(T)$ as $x$ and $y$. Hence $\nu_w^p(T)$ is an integer by the hypothesis of Theorem 1.5. This proves (18).

- $y(v_1v_6v_3v_4v_1) = 0$. In this subcase, $y(v_1v_2v_4v_1) = w(v_1v_2)$. By (9), if $y(v_1v_6v_2v_4v_1) > 0$, then $y(v_1v_6v_2v_4v_1) = w(v_1v_2) - w(v_1v_2)$; otherwise, $y(v_1v_2v_4v_1) = w(v_4v_1)$. If $y(v_2v_4v_6v_3v_2) > 0$, then, by (10), we have $y(v_3v_4v_6v_3) = w(v_3v_4), \ y(v_2v_4v_6v_3) = w(v_4v_6) - y(v_2v_4v_6v_3)$, and $y(v_2v_4v_6v_3) = w(v_4v_6) - w(v_4v_6) - y(v_2v_4v_6v_3)$. Hence $y(C)$ is integer for all $C \in C_2$. So we assume that $y(v_2v_4v_6v_3) = 0$. Thus $y(v_2v_4v_6v_3) + y(v_3v_4v_6v_3) = w(v_4v_6)$. If $y(v_2v_4v_6v_3)$ is integral, then $y(C)$ is integral for all $C \in C_2$. So we further assume that $y(v_2v_4v_6v_3)$ is not integral. By Lemma 3.2(iii), we may assume that $w(v_3v_4) = w(v_4v_6) = 0$. Observe that $v_6v_2$ is outside $C_0^y$, for otherwise, let $C$ be a cycle in $C_0^y$ containing $v_6v_2$. Then $C$ passes through $v_2v_4$. Let $C = C[v_2v_3, C \setminus \{v_2v_3, v_3v_4\}$, let $\theta = \min\{y(C), y(v_3v_4v_6v_3)\}$, and let $y'$ be obtained from $y$ by replacing $(v_3v_4v_6v_3), y(v_2v_4v_6v_2), y(C)$, and $y(C')$ with $(v_3v_4v_6v_3), y(v_2v_4v_6v_2) + \theta, y(v_2v_4v_6v_2) + \theta, y(C) + \theta$, and $y(C') + \theta$, respectively. Then $y'$ is also an optimal solution to $D(T, w)$ with $y'(v_3v_4v_6v_3) < y(v_3v_4v_6v_3)$, contradicting (4). Similarly, we can show that $v_3v_4$ is outside $C_0^y$. So $w(v_3v_4) = z(v_3v_4) = 0$. Moreover, $\nu_w^p(T)$ is an integer; the proof is the same as that of (17) (with $y(v_2v_4v_6v_3)$ and $y(v_3v_4v_6v_3)$ in place of $y(v_1v_6v_2v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$, respectively), so we omit the details here.

**Case 2.12.** $K = \{v_1v_6, v_4v_1, v_4v_6\}$.

In this case, by Lemma 3.1 (i), we have $y(v_1v_6v_2v_4v_1) = y(v_1v_6v_3v_4v_1) = 0$. By Lemma 3.1 (iii), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_6v_3v_4v_1) = w(v_1v_6), y(v_1v_6v_3v_1) = w(v_1v_6), y(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_1) = w(v_1v_6), y(v_2v_4v_6v_3) + y(v_2v_4v_6v_3) = w(v_4v_6)$. If $y(v_2v_4v_6v_3) > 0$, then, by (10), we have $y(v_2v_4v_6v_3) = w(v_4v_6) - y(v_2v_4v_6v_3)$. So $y(C)$ is integral for all $C \in C_2$. It remains to assume that $y(v_2v_4v_6v_3) = 0$. Then $y(v_2v_4v_6v_3) + y(v_2v_4v_6v_3) = w(v_4v_6)$. If $y(v_2v_4v_6v_3)$ is integral, then $y(C)$ is integral for all $C \in C_2$. So we further assume that $y(v_2v_4v_6v_3)$ is not integral. Then we can prove that $\nu_w^p(T)$ is an integer; the proof is the same as that of (17), so we omit the details here.

**Case 2.13.** $K = \{v_1v_2, v_1v_6, v_4v_6\}$.

In this case, by Lemma 3.1 (iii), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which together with (15) yields the following equations: $y(v_1v_2v_4v_1) = w(v_1v_2), y(v_1v_6v_3v_1) + y(v_1v_6v_3v_1) = y(v_1v_6v_3v_1) = w(v_1v_6), y(v_1v_6v_3v_1) + y(v_1v_6v_3v_1) = w(v_1v_6), y(v_2v_4v_6v_3) + y(v_2v_4v_6v_3) + y(v_2v_4v_6v_3) = w(v_4v_6)$. Clearly, $v_2v_1$ is outside $C_0^y$. Depending on the value of $y(v_1v_6v_3v_4v_1)$, we consider two subcases.

- $y(v_1v_6v_3v_4v_1) > 0$. In this subcase, $y(v_1v_6v_3v_4v_1) = 0$ and $y(v_1v_6v_3v_1) = w(v_1v_6)$ by (8). If $y(v_2v_4v_6v_3v_2) > 0$, then $y(v_1v_6v_2v_4v_1) = w(v_6v_2)$ and $y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) = w(v_3v_4)$
Suppose on the contrary that there exists precisely one vertex $v_1$. Then $y(v_1v_2v_4v_6v_3) = y(v_1v_6v_2v_4v_1) + y(v_1v_6v_3v_1) = w(v_1v_6)$. If $y(v_1v_6v_2v_4v_1)$ is integral, then $y(C)$ is integral for all $C \subseteq C_2$. So we further assume that $y(v_1v_6v_2v_4v_1)$ is not integral. Then we can prove that $\nu_w^*(T)$ is an integer; the proof is the same as that of (17), so we omit the details here.

- $y(v_1v_6v_3v_1) = 0$. In this subcase, $y(v_1v_6v_3v_1) + y(v_1v_6v_2v_4v_1) = w(v_1v_6)$. If $y(v_1v_6v_2v_4v_1) > 0$, then $y(v_1v_6v_3v_1) = 0$, and $y(v_1v_6v_3v_1) = w(v_1v_6v_2v_4v_1)$ by (10). Observe that if $y(v_1v_6v_2v_4v_1) > 0$, then we have one more equation $y(v_1v_6v_3v_1) = w(v_1v_6v_2v_4v_1)$ by (9). So $y(C)$ is integral for all $C \subseteq C_2$, and we may assume that $y(v_1v_6v_2v_4v_1) = 0$. We proceed by considering two subcases.

(a) Assume first that $y(v_3v_4v_6v_3) = 0$. Then $y(v_3v_4v_6v_3) = w(v_3v_4v_6)$. If $y(v_1v_6v_3v_1)$ is integral, then so is $y(C)$ for all $C \subseteq C_2$. Thus we assume that $y(v_1v_6v_3v_1)$ is not integral. If $v_3v_6$ is outside $C_0$, then it follows from (4) that $y(v_1v_6v_3v_1) + y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_6v_3)\}$; this contradiction implies that $v_3v_6$ is contained in a cycle $C$ in $C_0$. Let $C' = C[v_3, v_6] \cup \{v_6v_2, v_2v_1\}$. Let $\theta = \min\{y(v_1v_6v_2v_4v_1), y(v_1v_6v_3v_1), y(C), y(C')\}$, and let $y'$ be obtained from $y$ by replacing $y(v_1v_6v_2v_4v_1)$, $y(v_1v_6v_3v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_2v_4v_1) - \theta$, $y(v_1v_6v_3v_1) + \theta$, and $y(C') + \theta$, respectively. Then $y'$ is also an optimal solution to $\mathbb{D}(T, w)$ with $y'(v_1v_6v_2v_4v_1) < y(v_1v_6v_2v_4v_1)$, contradicting (2).

(b) Assume next that $y(v_3v_4v_6v_3) > 0$. If $y(v_1v_6v_2v_4v_1) > 0$, then $y(v_1v_6v_3v_1) = w(v_3v_1)$ and $y(v_1v_6v_2v_4v_1) = w(v_1v_6) - w(v_2v_4v_1)$ by (9); otherwise, $y(v_1v_6v_3v_1) = w(v_1v_6)$. If $y(v_3v_4v_6v_3)$ is integral, then so is $y(C)$ for all $C \subseteq C_2$. Thus we assume that $y(v_3v_4v_6v_3)$ is not integral. Let us prove that

(19) $\nu_w^*(T)$ is an integer.

By Lemma 3.2(iii), we may assume that $w(v_1v_2) = w(v_1v_6) = 0$. Let $T' = (V', A')$ be obtained from $T$ by deleting $v_1$, and let $w'$ be the restriction of $w$ to $A'$. It is routine to check that $\mathbb{D}(T', w')$ has the same optimal value $\nu_w^*(T)$ as $\mathbb{D}(T, w)$. Hence $\nu_w^*(T)$ is an integer by the hypothesis of Theorem 1.5. This proves (19) and hence Claim 2.

Since $\tau_w(G_3 \setminus v_5) > 0$, from Claim 2, Lemma 3.2(iii) and Lemma 3.4(ii) we deduce that $\mathbb{D}(T, w)$ has an integral optimal solution. This completes the proof of Lemma 4.6.

5 Composite Reductions

Lemma 5.7. If $T_2/S = F_4$, then $\mathbb{D}(T, w)$ has an integral optimal solution.

Proof. Recall that $(b_2, a_2) = (v_5, v_6)$, $s^* = v_2$, and $v_0 = v_3$. To establish the statement, by Lemma 3.4(ii), it suffices to prove that

(1) the optimal value $\nu_w^*(T)$ of $\mathbb{D}(T, w)$ is integral.

Given an optimal solution $y$ to $\mathbb{D}(T, w)$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2) \setminus a_2\}$ for each $s_i \in S$. By Lemma 5.2(i) and (vi), we have

(2) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

(3) There exist at least two and at most three vertices $s_i$'s in $S$ with $\varphi(s_i) \neq \emptyset$.

In view of (2) and the structure of $F_4$, there are at most three vertices $s_i$'s in $S$ with $\varphi(s_i) \neq \emptyset$. Suppose on the contrary that there exists precisely one vertex $s_i \in S$ with $\varphi(s_i) \neq \emptyset$. Then (1)
follows immediately from Lemma 4.4; the argument can be found in that of (3) in the proof of Lemma 5.5.

Lemma 5.2(i) allows us to assume that

(4) if \( \varphi(s_i) \neq \emptyset \), then \( i \in \{1, 2, 3\} \).

Let \( t \) be the subscript in \( \{1, 2, 3\} \) with \( v_5 \in \varphi(s_i) \), if any. By (2), \( t \) is well defined. In the remainder of our proof, we reserve \( y \) for an optimal solution to \( \mathbb{D}(T, w) \) such that

(5) \( y(C_2) \) is maximized;
(6) subject to (5), \( (y(D_q), y(D_{q-1}), \ldots, y(D_3)) \) is minimized lexicographically; and
(7) subject to (5) and (6), \( y(v_1v_5s_1v_3v_1) + y(v_1v_5v_3v_4v_1) \) is minimized.

Let us make a few observations about \( y \) before proceeding.

(8) If \( y(v_1v_5s_1v_3v_4v_1) > 0 \) for some \( i \in \{1, 2, 3\} \), then each arc in the set \( \{v_1s_i, v_3v_1, v_4s_i, v_4v_5, v_5v_3\} \) is saturated by \( y \) in \( T_2 \). Furthermore, \( y(v_1s_jv_3v_1) = y(v_1v_4v_5v_3) = y(v_1v_5v_3v_1) = 0 \) for any \( j \in \{1, 2, 3\}\) \( \setminus \{i\} \).

To justify this, note that each arc in the given set is a chord of the cycle \( v_1v_5s_iv_3v_4v_1 \). So the first half follows instantly from Lemma 3.5(v). Once again let \( \cup \) stand for the multiset sum. Then \( v_1v_5s_1v_3v_4v_1 \cup v_1s_1v_3v_1 = v_1v_5s_1v_3v_1 \cup v_1s_1v_3v_4v_1 = v_1v_5s_1v_3v_4v_1 \cup v_1v_5v_3v_1 = v_1v_5v_3v_1 \cup v_1v_5s_1v_3v_4v_1, \) and \( v_1v_5s_1v_3v_4v_1 \cup v_3v_4v_5v_3 = v_1v_5v_3v_1 \cup v_5s_1v_3v_4v_5. \) Since \( y \) satisfies (6), we deduce that \( y(v_1s_jv_3v_1) = y(v_1v_4v_5v_3) = y(v_1v_5v_3v_1) = 0 \).

(9) If \( y(v_1v_5s_1v_3v_1) > 0 \) for some \( i \in \{1, 2, 3\} \), then both \( v_1s_i \) and \( v_3v_1 \) are saturated by \( y \) in \( T_2 \); so are \( v_4s_i \) and \( v_4v_5 \) if \( y(v_1s_jv_3v_1) > 0 \). Furthermore, \( y(v_3v_4v_5v_3) = 0 \).

Since both \( v_1s_i \) and \( v_3v_1 \) are chords of the cycle \( v_1v_5s_iv_3v_1 \), the first half follows instantly from Lemma 3.5(v). To establish the second half, observe that \( v_1v_5s_1v_3v_1 \cup v_3v_4v_5v_3 = v_1v_5v_3v_1 \cup v_3v_4v_5v_3. \) Hence \( y(v_3v_4v_5v_3) = 0 \) by (7). Suppose \( y(v_1s_jv_3v_4v_1) > 0 \). Since the multiset sum of the cycles \( v_1v_5s_1v_3v_1, v_1s_1v_3v_4v_1, \) and the arc \( v_4v_5 \) (resp. \( v_4s_i \)) contains arc-disjoint cycles \( v_1s_1v_3v_1 \) and \( v_3v_4v_5v_3 \) (resp. \( v_4s_iv_4v_5 \)), from (7) we deduce that both \( v_4s_i \) and \( v_4v_5 \) are saturated by \( y \) in \( T_2 \).

(10) If \( y(v_1v_5s_1v_3v_1) > 0 \) then both \( v_3v_1 \) and \( v_4v_5 \) are saturated by \( y \) in \( T_2 \). Furthermore, \( y(v_1s_jv_3v_1) = 0 \) for any \( i \in \{1, 2, 3\} \).

Since both \( v_3v_1 \) and \( v_4v_5 \) are chords of the cycle \( v_1v_5v_3v_4v_1 \), the first half follows instantly from Lemma 3.5(v). To establish the second half, observe that \( v_1v_5v_3v_4v_1 \cup v_1s_1v_3v_1 = v_1v_5v_3v_1 \cup v_1s_1v_3v_1. \) Since \( y \) satisfies (7), we have \( y(v_1s_jv_3v_1) = 0 \).

The following two statements can be seen from Lemma 3.5(v).

(11) If \( y(v_1s_jv_3v_4v_1) > 0 \), then both \( v_3v_1 \) and \( v_4s_i \) are saturated by \( y \) in \( T_2 \), for \( i \in \{1, 2, 3\} \).

(12) If \( y(v_5s_jv_3v_4v_3) > 0 \), then both \( v_4s_i \) and \( v_5v_3 \) are saturated by \( y \) in \( T_2 \), for \( i \in \{1, 2, 3\} \).

We proceed by considering two cases, depending on whether \( \varphi(s_k) = \{v_4\} \) for some \( k \in \{1, 2, 3\} \) (see (4)).

Case 1. \( \varphi(s_k) = \{v_4\} \) for some \( k \in \{1, 2, 3\} \).

By Lemma 5.2(i), we may assume that \( k = 1 \); that is, \( \varphi(s_1) = \{v_4\} \). Let \( i \) and \( j \) be the subscripts in \( \{2, 3\} \), if any (possibly \( i = j \)), such that \( v_5 \in \varphi(s_i) \) and \( v_1 \in \varphi(s_j) \). Then

(13) \( C_2^j \subseteq \{v_4s_1v_3v_4, v_1s_1v_3v_1, v_1s_1v_3v_4v_1, v_1v_5s_1v_3v_1, v_5s_1v_3v_4v_5, v_1v_5s_1v_3v_4v_1, v_1v_5v_3v_1 \} \).

We propose to show that

(14) if \( w(v_3v_4) > 0 \), then \( y(v_4s_1v_3v_4) \) is a positive integer.
For this purpose, note that \( z(s_1v_3) = w(s_1v_3) > 0 \) by Lemma 5.2(iii). If \( s_1v_3 \) is outside \( C_0^o \), then \( y(v_1s_1v_3v_4) = w(s_1v_3) > 0 \). So we assume that \( s_1v_3 \) is contained in some cycle \( C \in C_0^o \). If \( C \) contains \( v_4s_1 \), then \( v_3v_4 \) is saturated by \( y \) in \( T_2 \) by Lemma 3.5(iii). Moreover, the multiset sum of \( C \) and each cycle in the set \( \{ v_1s_1v_3v_4, v_5s_1v_3v_4v_5, v_1v_5s_1v_3v_4v_5, v_3v_4v_5v_3, v_1v_5v_3v_4v_5 \} \) contains the cycle \( v_4s_1v_3v_4, a \) cycle in \( \{ v_1s_1v_3v_4, v_5s_1v_3v_4v_5, v_1v_5s_1v_3v_4v_5 \} \), and a cycle \( C' \in C_0^o \) that are arc-disjoint, where \( C' = C[v_5, v_4] \cup \{ v_4v_3 \} \) or \( C[v_5, v_4] \cup \{ v_4v_1, v_1v_5 \} \). From the optimality of \( y \), we thus deduce that \( y(v_1s_1v_3v_4), y(v_5s_1v_3v_4v_5), y(v_1v_5s_1v_3v_4v_5), y(v_3v_4v_5v_3), \) and \( y(v_1v_5v_3v_4v_5) \) are all zero. Hence \( y(v_4s_1v_3v_4) = w(v_3v_4) > 0 \). So we assume that \( C \) does not contain \( v_4s_1 \).

Furthermore, \( v_4s_1 \) is outside \( C_0^o \), because every cycle using \( v_4s_1 \) passes through \( s_1v_3 \). Note that \( v_3v_4 \) is not saturated by \( y \) in \( T \), for otherwise \( y(v_4s_1v_3v_4) = w(v_4s_1v_3v_4) > 0 \), as desired. By Lemma 3.5(vii), \( v_3v_4 \) is saturated by \( y \) in \( T_2 \) and \( C \) contains \( v_3v_1 \). It follows from (8), (10) and (11) that \( y(v_1v_5s_1v_3v_4v_5), y(v_1v_5v_3v_1v_4v_5) \) and \( y(v_1s_1v_3v_4v_5) \) are all zero. As the multiset sum of \( C \), each of \( v_5s_1v_3v_4v_5 \) and \( v_3v_4v_5v_3 \), and the unsaturated arc \( v_4s_1 \) contains arc-disjoint cycles \( v_4s_1v_3v_4v_5 \) and one of \( v_1v_5s_1v_3v_4v_5 \) and \( v_1v_5v_3v_1v_4v_5 \), both \( y(v_5s_1v_3v_4v_5) \) and \( y(v_3v_4v_5v_3) \) are zero by Lemma 3.5(vi).

So \( y(v_3v_4v_5v_3) = w(v_3v_4) > 0 \). This proves (14).

By (14) and Lemma 3.2(iii), we may assume that \( w(v_3v_4) = 0 \). It follows that \( w(v_3v_4) \geq z(v_3v_1) > 0 \), for otherwise, \( \tau_w(T_2, v_2) = w(v_3v_1) + w(v_3v_4) = 0 \), contradicting (a). Since \( z(v_3v_1) > 0 \) and \( w(v_3v_4) = 0 \), the arc \( v_4s_1 \) is contained in some cycle in \( C_0^o \). From the proof of (14) we see that

(15) \( y(v_1s_1v_3v_4v_5), y(v_5s_1v_3v_4v_5), y(v_1v_5s_1v_3v_4v_5), y(v_3v_4v_5v_3), \) and \( y(v_1v_5v_3v_1v_4v_5) \) are all zero.

(16) If \( w(v_1s_j) \geq z(v_1s_j) > 0 \), then \( y(v_1s_jv_3v_1) \) is a positive integer.

To justify this, note that \( z(s_jv_3) = w(s_jv_3) > 0 \) by Lemma 5.2(iii). Assume first that \( s_jv_3 \) is outside \( C_0^o \). If \( i \neq j \), then \( y(v_1s_jv_3v_1) = w(s_jv_3) > 0 \). So we assume that \( i = j \). Then \( y(v_1s_jv_3v_1) = y(v_1v_5s_1v_3v_1) = w(s_jv_3) \). If \( y(v_1v_5s_1v_3v_1) > 0 \), then \( s_1v_3 \) is saturated by \( y \) in \( T_2 \) by (9). Thus \( y(v_1v_5s_1v_3v_1) = w(s_1v_3) \). Next assume that \( i = j \) is contained in some cycle \( C \in C_0^o \). Since \( w(v_3v_4) = 0 \), cycle \( C \) contains \( v_3v_1 \). It follows that \( v_1s_j \) is saturated by \( y \) in \( T_2 \). So \( y(v_1v_5s_1v_3v_1) = w(v_1s_j) > 0 \) and hence (16) is established.

By (16) and Lemma 3.2(iii), we may assume that \( w(v_1s_j) = 0 \) by (3), we have \( z(v_5s_1) > 0 \) and \( \varphi(s_1) = \{ v_5 \} \). By (13)-(16), we obtain

(17) \( C_0^o \subseteq \{ v_5v_3s_1v_3v_1, v_1v_5v_3v_1 \} \).

(18) \( y(v_1v_5s_1v_3v_1) \) is a positive integer.

To justify this, note that \( z(s_1v_3) = w(s_1v_3) > 0 \) by Lemma 5.2(iii). If \( s_1v_3 \) is outside \( C_0^o \), then \( y(v_1v_5s_1v_3v_1) = w(s_1v_3) > 0 \) by (17), as desired. So we assume that \( s_1v_3 \) is contained in some cycle \( C \in C_0^o \). Applying Lemma 3.5(iii) to the cycle \( v_1v_5s_1v_3v_1 \), we deduce that \( (v_5, s_1) \) is saturated by \( y \) in \( T_2 \). So \( y(v_1v_5s_1v_3v_1) = w(v_5s_1) > 0 \) and hence (18) holds.

By (18) and Lemma 3.2(iii), \( \mathbb{D}(T, w) \) has an integral optimal solution, which implies (1).

**Case 2.** \( \varphi(s_k) \neq \{ v_1 \} \) for any \( k \in \{ 1, 2, 3 \} \).

By (3), the hypothesis of the present case, and Lemma 5.2(i), we may assume that \( v_1 \in \varphi(s_1) \) and \( v_5 \in \varphi(s_2) \). Then

(19) \( C_0^o \subseteq \{ v_1v_3v_4v_1, v_1v_5v_3v_1v_3v_4v_1, v_5v_2v_3v_1v_3v_4v_1, v_1v_5s_1v_3v_4v_1, v_1v_5v_3v_4, v_4s_1v_3v_4v_5 \} \).

By Lemma 5.2(iii), we have

(20) if \( v_4 \in \varphi(s_1) \), then \( z(v_4s_3-i) = 0 \) and \( y(v_4s_3-i, v_3v_4) = 0 \) for \( i = 1, 2 \).
Claim 1. \( y(C_2) = \tau_w(T_2 \setminus a_2) \).

To justify this, observe that

(21) if \( K \) is an FAS of \( T_2 \setminus a_2 \) such that \( y(C_2) = w(K) \), then \( K \) is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

In view of Lemma 5.2(iii), we distinguish among three subcases, depending on whether \( s_i v_3 \) is contained in a cycle in \( C'_0 \).

Subcase 1.1. Both \( s_1 v_3 \) and \( s_2 v_3 \) are outside \( C'_0 \). In this subcase, \( s_i v_3 \) is saturated by \( y \) in \( T_2 \) for \( i = 1, 2 \). If \( v_3 v_5 \) is also saturated by \( y \) in \( T_2 \), then \( y(C_2) = w(K) \), where \( K = \{ v_3 v_5, s_1 v_3, s_2 v_3 \} \). Since \( K \) is an FAS of \( T_2 \setminus a_2 \), it is an MFAS by (21) and hence \( y(C_2) = \tau_w(T_2 \setminus a_2) \). So we assume that \( v_3 v_5 \) is not saturated by \( y \) in \( T_2 \).

(22) Both \( v_3 v_1 \) and \( v_3 v_4 \) are outside \( C'_0 \). Furthermore, at least one of them is not saturated by \( y \) in \( T_2 \).

Indeed, the first half follows directly from Lemma 3.5(iii). To justify the second half, assume the contrary. Then \( y(C_2) = w(K) \), where \( K = \{ v_3 v_1, v_3 v_4 \} \). Thus \( K \) is an MFAS of \( T_2 \setminus a_2 \) by (21) and hence \( y(C_2) = \tau_w(T_2 \setminus a_2) \).

By (22), (8), (9), and (12), we have

(23) \( y(v_1 v_5 s_2 v_3 v_1), y(v_5 v_2 s_2 v_3 v_5), \) and \( y(v_1 v_5 s_2 v_3 v_4 v_1) \) are all zero.

Since \( C'_0 \neq \emptyset \), some cycle \( C \in C'_0 \) contains \( v_1 v_5 \) or \( v_4 v_5 \). Thus there are two possibilities to consider.

- \( C \) contains \( v_1 v_5 \). Now by (22) and Lemma 3.5(iii), \( v_3 v_1 \) is saturated by \( y \) in \( T_2 \) and hence \( v_3 v_4 \) is not saturated by \( y \) in \( T_2 \). It follows from Lemma 3.5(i) and (iii) that both \( v_4 v_1 \) and \( v_4 v_5 \) are saturated by \( y \) in \( T_2 \). If \( z(v_4 s_1) = w(v_4 s_1) \) for \( i = 1, 2 \), then \( y(C_2) = w(K) \), where \( K = \{ v_4 v_1, v_4 v_4, v_4 v_1, v_4 s_1, v_4 s_2 \} \). Thus \( K \) is an MFAS of \( T_2 \setminus a_2 \) by (21) and hence \( y(C_2) = \tau_w(T_2 \setminus a_2) \). So we assume that \( 0 < z(v_4 s_1) < w(v_4 s_1) \) for \( i = 1 \) or \( 2 \). Then \( z(v_4 s_3 - i) = w(v_4 s_3 - i) = 0 \) by (2). If \( i = 2 \), then \( y(C_2) = w(K) \), where \( K = \{ v_3 v_1, v_4 v_1, v_4 v_5, v_4 s_1, s_2 v_3 \} \), and hence \( y(C_2) = \tau_w(T_2 \setminus a_2) \). If \( i = 1 \), then \( y(v_1 s_1 v_3 v_1) = 0 \) by (11). Since the multiset sum of the cycles \( v_1 s_1 v_3 v_1, C \), and the unsaturated arcs \( \{ v_4 s_1, v_5 v_3, v_3 v_4 \} \) contains arc-disjoint cycles \( v_1 s_1 v_3 v_1 \) and \( v_3 v_1 v_5 v_3 \), we have \( y(v_1 s_1 v_3 v_1) = 0 \) by Lemma 3.5(vi). Thus \( y(C_2) = w(K) \), where \( K = \{ v_3 v_1, v_4 v_1, v_4 v_5, s_1 v_3, v_4 s_2 \} \). It follows that \( y(C_2) = \tau_w(T_2 \setminus a_2) \).

- \( C \) contains \( v_4 v_5 \). Now by (22) and Lemma 3.5(iii), \( v_3 v_4 \) is saturated by \( y \) in \( T_2 \) and hence \( v_4 v_5 \) is not saturated by \( y \) in \( T_2 \). It follows from Lemma 3.5(i) and (iii) that \( v_1 v_1 \) is saturated by \( y \) in \( T_2 \). By (10) and (11), we have \( y(v_1 v_5 v_3 v_4 v_1) = y(v_1 s_1 v_3 v_4 v_1) = 0 \). If \( v_1 s_1 \) is saturated by \( y \) in \( T_2 \), then \( y(C_2) = w(K) \), where \( K = \{ v_1 v_5, v_3 v_4, v_1 s_1 \} \). Thus \( y(C_2) = \tau_w(T_2 \setminus a_2) \). So we assume that \( v_1 s_1 \) is not saturated by \( y \) in \( T_2 \) and hence not in \( T \) by (22). Since the multiset sum of the cycles \( C, v_1 s_1 v_3 v_4, \) and the unsaturated arcs \( \{ v_3 v_1, v_5 v_3, v_3 s_1 \} \) contains arc-disjoint cycles \( v_1 s_1 v_3 v_1 \) and \( v_3 v_1 v_5 v_3 \), we have \( y(v_1 s_1 v_3 v_1) = 0 \) by Lemma 3.5(vi). So \( y(C_2) = w(K) \), where \( K = \{ v_1 v_5, v_3 v_1, s_1 v_3 \} \). It follows that \( y(C_2) = \tau_w(T_2 \setminus a_2) \).

Subcase 1.2. \( s_1 v_3 \) is contained in some cycle \( C \in C'_0 \); subject to this, we choose \( C \) so that it contains as many edges in \( T_2 \setminus a_2 \) as possible.

Assume first that \( C \) contains \( v_1 s_1 \). Then \( C \) contains the path \( v_1 s_1 v_3 v_4 v_5 \). By Lemma 3.5(iii), each arc in the set \( \{ v_3 v_1, v_4 v_1, v_4 s_1, v_5 v_3 \} \) is saturated by \( y \) in \( T_2 \). By (2), (8) and (10), we have \( y(v_1 v_5 s_2 v_3 v_4 v_1) = y(v_1 v_5 v_3 v_4 v_1) = 0 \). Since the multiset sum of \( C \) and one of \( v_1 v_5 v_3 v_1 \) and \( v_1 v_5 s_2 v_3 v_1 \) contains arc-disjoint cycles \( v_3 v_4 v_5 v_3, C' = [v_5, v_1] \cup \{ v_1 v_5 \}, \) and one of \( v_1 s_1 v_3 v_1 \) and \( v_5 s_2 v_3 v_4 v_5 \), from the optimality of \( y \) we deduce that \( y(v_1 v_5 s_2 v_3 v_1) = y(v_1 v_5 s_2 v_3 v_1) = 0 \). If
$s_2v_3$ is outside $C'_0$, then $s_2v_3$ is saturated by $y$ in $T_2$ by Lemma 5.2(iii). So $y(C_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_5s_1, s_2v_3, v_5v_3\}$. Hence $y(C_2) = r_w(T_2 \setminus \{v_2\})$. So we assume that $s_2v_3$ is contained in some cycle in $C'_0$. Since $v_3v_1$ is saturated by $y$ in $T_2$, every cycle in $C'_0$ containing $s_2v_3$ passes through $v_3v_4$. By Lemma 3.5(iii), both $v_4s_2$ and $v_5s_2$ are saturated by $y$ in $T_2$. Thus $y(C_2) = w(K)$, where $K = \{v_3v_1, v_4v_1, v_5s_1, s_2v_3, v_5v_3\}$. It follows that $y(C_2) = r_w(T_2 \setminus \{a_2\})$.

Assume next that $v_1s_1$ is not on $C$. Then we may further assume that $v_1s_1$ is outside $C'_0$. We proceed by considering three subcases.

- **$C$ contains $v_3v_1$.** Now $v_1s_1$ and $v_5v_3$ are saturated by $y$ in $T_2$ by Lemma 3.5(iii). Hence $y(v_1v_5s_2v_3v_4v_1) = y(v_1v_5v_3v_4v_1) = y(v_1s_1v_5v_4v_1) = 0$ by (8), (10) and (11). If $v_4s_1$ is not saturated by $y$ in $T_2$, then $v_3v_4$ is saturated by $y$ in $T_2$ by Lemma 3.5(iii). Moreover, for each $D \in \{v_3v_4v_5, v_5s_2v_3v_4\}$, if $v_4s_1$ is on $C$, then the multiset sum of $C$ and $D$ contains arc-disjoint cycles $v_4s_1v_3v_4$. $C' = \{v_5, v_4 \} \cup \{v_4v_5\}$, and one of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$; if $v_4s_1$ is not saturated by $y$ in $T_2$, then the multiset sum of $C$, $D$ and the arc $v_4s_1$ contains $v_4s_1v_3v_4$ and one of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$ that are arc-disjoint. It follows from the optimality of $y$ or Lemma 3.5(iv) that $y(v_3v_4v_5v_3) = y(v_5s_2v_3v_4v_5) = 0$. So $y(C_2) = w(K)$ if $s_2v_3$ is contained in some cycle in $C'_0$ and $y(C_2) = w(J)$ otherwise, where $K = \{v_1s_1, v_4v_1, v_5s_2, v_4s_2\}$ and $J = \{v_1s_1, v_3v_4, v_5v_3, v_5s_2\}$. Hence $y(C_2) = r_w(T_2 \setminus \{a_2\})$. So we assume that $v_4s_1$ is saturated by $y$ in $T_2$. If $s_2v_3$ is outside $C'_0$, then $y(C_2) = w(K)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$, which implies that $y(C_2) = r_w(T_2 \setminus \{a_2\})$. So we further assume that $s_2v_3$ is contained in some cycle in $C'_0$. By Lemma 3.5(iii), $v_5s_2$ is saturated by $y$ in $T_2$. If $v_5s_2$ is also saturated by $y$ in $T_2$, then $y(C_2) = w(K)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$; otherwise, $v_5s_2$ is saturated by $y$ in $T_2$, and $w(v_1s_1) = 0$. Similar to the case when $v_1s_1$ is not saturated by $y$ in $T_2$, we can show that $y(v_3v_4v_5v_3) = y(v_5s_2v_3v_4v_5) = 0$. Thus $y(C_2) = w(J)$, where $J = \{v_1s_1, v_3v_4, v_5v_3, v_5s_2\}$. Therefore $y(C_2) = r_w(T_2 \setminus \{a_2\})$ in either situation.

- **$C$ contains both $v_3v_4$ and $v_4v_1$.** Now $v_1s_1$, $v_4s_1$ and $v_5v_3$ are saturated by $y$ in $T_2$ by Lemma 3.5(iii). If $s_2v_3$ is outside $C'_0$, then $y(C_2) = w(K)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, s_2v_3\}$; otherwise, $v_5s_2$ and $v_4s_2$ are saturated by $y$ in $T_2$ by Lemma 3.5(iii). So $y(C_2) = w(J)$, where $J = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$. Therefore $y(C_2) = r_w(T_2 \setminus \{a_2\})$ in either situation.

- **$C$ contains both $v_3v_4$ and $v_4v_5$.** Now $v_4s_1$ and $v_5v_3$ are saturated by $y$ in $T_2$ by Lemma 3.5(iii) and $y(v_1v_5v_3v_4v_1) = y(v_1v_5s_2v_3v_4v_1) = 0$ by (8) and (10). If $v_1s_1$ is also saturated by $y$ in $T_2$, then $y(C_2) = w(K)$ or $w(J)$, where $K = \{v_1s_1, v_4s_1, v_5v_3, s_2v_3\}$ and $J = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$; otherwise, both $v_1v_4$ and $v_5v_3$ are saturated by $y$ in $T_2$, and every cycle in $C'_0$ containing $s_2v_3$ traverses $v_3v_4v_5$. Since the multiset sum of $C$, each of $v_1v_5v_3v_1$ and $v_1v_5s_2v_3v_1$, and the unsaturated arc $v_1s_1$ contains $v_1s_1v_3v_2v_3$ and one of $v_3v_4v_5v_3$ and $v_5s_2v_3v_4v_5$ that are arc-disjoint, we have $y(v_1v_5v_3v_1) = y(v_1v_5s_2v_3v_1) = 0$ by Lemma 3.5(iv). So $y(C_2) = w(K)$ if $s_2v_3$ is outside $C'_0$ and $y(C_2) = w(J)$ otherwise, where $K = \{v_1v_3, v_4v_1, v_5s_1, v_5v_3, s_2v_3\}$ and $J = \{v_1s_1, v_4s_1, v_5v_3, v_5s_2, v_4s_2\}$. Therefore $y(C_2) = r_w(T_2 \setminus \{a_2\})$ in either situation.

**Subcase 1.3.** $s_2v_3$ is contained in some cycle $C \in C'_0$ and $s_1v_3$ is saturated by $y$ in $T_2$. In this subcase, both $v_5s_2$ and $v_5v_3$ are saturated by $y$ in $T_2$ by Lemma 3.5(iii). If $v_4s_2$ is also saturated by $y$ in $T_2$, then $y(C_2) = w(K)$, where $K = \{s_1v_3, v_5v_3, v_4s_2, v_5s_2\}$; otherwise, $z(v_4s_2) > 0$ and $w(v_1s_1) = z(v_4s_1) = 0$ by Lemma 5.2(vii). In this case $C$ contains $v_3v_1$, so $v_3v_1$ is saturated by $y$ in $T_2$ by Lemma 3.5(iii). By (8) and (10)-(12), we have $y(v_1v_5s_2v_3v_1)$, $y(v_1v_5v_3v_4v_1)$, $y(v_1s_1v_3v_4v_1)$, and $y(v_5s_2v_3v_4v_5)$ are all zero. Since the multiset sum of the cycles $C$, $v_3v_4v_5v_3$, and the unsaturated arc $v_4s_2$ contains arc-disjoint cycles $v_4s_2v_3v_4$ and
Claim 2. $y(C)$ is a positive integer for some $C \in C_2$ or $v_w^n(T)$ is an integer.

To justify this, note that $y(C_2) = w(K)$ for some MFAS $K$ of $T_2 \backslash a_2$ by Claim 1. Depending on what $K$ is, we distinguish among eight cases.

Subcase 2.1. $K$ is one of \{ $v_1v_5, v_2v_4, v_1s_1$, $v_1s_1, v_3v_4, v_5s_2, v_5v_3$, $v_1v_5, v_3v_4, s_1v_3$, $v_1s_1, v_3v_4, s_2v_3, v_5v_3$ \}.

In this case, by Lemma 3.1(i), we have $y(C) = 0$ for some cycles $C$ listed in (19). By Lemma 3.1(iii), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) = w(v_1v_1)$, $y(v_1v_1v_3v_3v_1) = w(v_1v_1)$, $y(v_1s_1v_3v_1) = w(v_1s_1)$, $y(v_4s_2v_3v_4) + y(v_5s_2v_3v_4v_5) = w(s_2v_3)$, $y(v_5s_2v_3v_4v_5) = w(v_5v_3)$. If $y(v_5s_2v_3v_4v_5) = 0$, then $y(v_5s_2v_3v_4) = w(s_2v_3) > 0$ by Lemma 5.2(iii). If $y(v_5s_2v_3v_4v_5) > 0$, then $v_4s_2$ is saturated by $y$ in $T_2$ by Lemma 3.5(iii). So $w(v_4s_2) = (C_2(v_4s_2))$. It follows that $y(v_4s_2v_3v_4) = w(v_4s_2)$, and hence $y(v_5s_2v_3v_4v_5)$ is a positive integer.

Subcase 2.2. $K = \{ v_3v_1, v_4v_1, v_4s_1, s_2v_3, v_5v_3 \}$.

In this case, by Lemma 3.1(i), we have $y(C) = 0$ for some cycles $C$ listed in (19). By Lemma 3.1(iii), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) = w(v_1v_1)$, $y(v_1s_1v_3v_4v_1) = w(v_1v_1)$, $y(v_4s_1v_3v_1) = w(v_4s_1)$, $y(v_4s_2v_3v_4) + y(v_5s_2v_3v_4v_5) = w(s_2v_3)$, $y(v_5s_2v_3v_4v_5) = w(v_5v_3)$. Since $v_5 \in \varphi(s_2)$, we have $w(v_5s_2) > 0$. So $y(v_5s_2v_3v_4v_5)$ is a positive integer.

Subcase 2.4. $K = \{ v_3v_1, v_4v_1, v_4v_5, v_4s_1, s_2v_3, v_5v_3 \}$ or \{ $v_3v_1, v_4v_1, v_4v_5, v_1s_1, s_2v_3, v_5v_3$ \}.

In this case, by Lemma 3.1(i), we have $y(C) = 0$ for some cycles $C$ listed in (19). By Lemma 3.1(iii), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) = w(s_2v_3) > 0$ or $y(v_1s_1v_3v_1) = w(s_2v_3) > 0$ by Lemma 5.2(iii).

Subcase 2.5. $K = \{ v_1s_1, v_4s_1, s_2v_3, v_5v_3 \}$ or \{ $v_1s_1, v_4s_1, v_4s_2, v_5s_2, v_5v_3$ \}.

We only consider the subcase when $K = \{ v_1s_1, v_1s_1, s_2v_3, v_5v_3 \}$, as the other subcase can be justified likewise.

By Lemma 3.1(i), we have $y(C) = 0$ for some cycles $C$ listed in (19). By Lemma 3.1(iii), we obtain $w(e) = y(C_2(e))$ for each $e \in K$, which, together with (19), implies that $y(v_1s_1v_3v_1) + y(v_1s_1v_3v_3v_1) = w(v_1s_1)$, $y(v_1s_1v_3v_1) + y(v_1s_1v_3v_3v_1) = w(v_1s_1)$, $y(v_1s_1v_3v_1) + y(v_1s_1v_3v_3v_1) = w(v_1s_1)$, and $y(v_4s_2v_3v_4) + y(v_1s_1v_3v_3v_1) + y(v_5s_2v_3v_4v_5) + y(v_1s_1v_3v_3v_1) = w(s_2v_3)$. We may assume that $y(v_1s_1v_3v_1) + y(v_1s_1v_3v_1) = w(v_1s_1v_3v_1) = 0$, for otherwise, by (8) or (10), we have $y(v_1s_1v_3v_1) = 0$ and hence $y(v_1s_1v_3v_1) = w(v_1s_1) > 0$.

If $y(v_1s_1v_3v_1) = 0$, then $y(v_1s_1v_3v_3v_1) + y(v_1s_1v_3v_3v_1) = w(s_2v_3)$. Observe that $y(v_1s_1v_3v_3v_1) > 0$, for otherwise, $y(v_1s_1v_3v_3v_1) = w(s_2v_3) > 0$. By (6), we obtain $y(v_1s_1v_3v_3v_1) = w(s_2v_3)$ or $w(v_1s_1v_3v_3v_1) = 0$. Thus, by
(9), both \(v_4s_2\) and \(v_4v_5\) are saturated by \(y\) in \(T_2\). It follows that \(y(v_4s_2v_3v_4) = w(v_4s_2)\) and 
\(y(v_5s_2v_3v_4v_5) = w(v_5v_4)\). So \(y(v_1v_5s_2v_3v_4) = w(s_2v_3) - y(v_2v_3v_4) - y(v_5s_2v_3v_4v_5)\). Since \(w(s_2v_3) > 0\), at least one of \(y(v_4s_2v_3v_4)\), \(y(v_5s_2v_3v_4v_5)\), and \(y(v_1v_5s_2v_3v_4)\) is a positive integer.

**Subcase 2.6.** \(K = \{s_1v_3, s_2v_4, v_3s_2, s_3v_3\}\) or \(\{s_1v_3, s_2v_3, v_5v_3\}\).

We only consider the subcase when \(K = \{s_1v_3, s_2v_3, v_5v_3\}\), as the other subcase can be justified likewise.

By Lemma 3.1(iii), we obtain \(w(e) = y(C_2(e))\) for each \(e \in K\), which, together with (19), implies that \(y(v_4s_1v_3v_4) + y(v_1s_1v_3v_1) + y(v_1s_1v_3v_4) = w(s_1v_3)\), \(y(v_1v_5v_3v_1) + y(v_3v_4v_5v_3) + y(v_1v_5v_3v_4v_5v_3) = w(v_3v_4)\), and \(y(v_1s_2v_3v_4) + y(v_1s_2v_3v_4v_5) + y(v_1v_5s_2v_3v_4v_5) = w(s_2v_3)\).

We may assume that \(y(v_1v_5s_2v_3v_4v_5) = y(v_1v_5v_3v_4v_1) = 0\), for otherwise, by (8) or (10), we have \(y(v_1s_1v_3v_1) = 0\) and hence \(y(v_1s_1v_3v_4) + y(v_1s_1v_3v_4v_1) = w(v_1s_1) > 0\), which together with (6) implies that \(y(v_4s_1v_3v_4) = w(s_1v_3)\) or \(w(v_4s_1)\), so \(y(v_1s_1v_3v_4v_1) = w(s_1v_3) - y(v_4s_1v_3v_4)\). Since \(w(s_1v_3) > 0\), at least one of \(y(v_1s_1v_3v_4)\) and \(y(v_1s_1v_3v_4v_1)\) is a positive integer.

If \(y(v_1v_5s_2v_3v_4v_5) = 0\), then \(y(v_5s_2v_3v_4v_5) = w(s_2v_3)\) or \(w(v_2s_2)\), which together with (6) implies that \(y(v_4s_1v_3v_4) = w(s_2v_3)\) or \(y(v_5s_2v_3v_4v_5) = w(s_2v_3) - y(v_4s_2v_3v_4v_5)\). Since \(w(s_2v_3) > 0\), at least one of \(y(v_4s_1v_3v_4)\) and \(y(v_5s_2v_3v_4v_5)\) is a positive integer. So we assume that \(y(v_1v_5s_2v_3v_4v_5) > 0\). Thus, by (9), we have \(y(v_1v_5v_3v_4) = w(v_4s_2)\).

If \(y(v_1s_2v_3v_4) = 0\), then \(y(v_4s_4v_3v_4) = w(s_2v_3)\) or \(w(v_4s_2)\), so \(y(v_1s_2v_3v_4v_5) = w(s_2v_3) - y(v_4s_2v_3v_4v_5)\). Since \(w(s_2v_3) > 0\), at least one of \(y(v_4s_2v_3v_4)\) and \(y(v_5s_2v_3v_4v_5)\) is a positive integer. So we further assume that \(y(v_4s_1v_3v_4v_1) = 0\). Then 
\[y(v_1s_1v_3v_4v_1) + y(v_4s_1v_3v_4) = w(s_1v_3)\]. If \(y(v_4s_1v_3v_4) = 0\), then \(y(v_1s_1v_3v_4) = w(s_1v_3) > 0\).

We propose to show that 
\[\nu_w^*(T)\] is an integer.

To justify this, let \(x\) be an optimal solution to \(P(T, w)\). Since \(0 < y(v_1s_1v_3v_1) < w(v_1s_1)\) and \(0 < y(v_4s_1v_3v_4) < w(v_4s_1)\), by Lemma 3.1(i) and (ii), we have \(x(v_1s_1) = x(v_4s_1) = 0\) and \(x(v_1s_1v_3v_1) = x(v_4s_1v_3v_4) = 1\), which implies \(x(v_3v_1) = x(v_3v_4)\). Furthermore, since \(y(v_1v_5s_2v_3v_4v_5) > 0\) and \(y(v_5s_2v_3v_4v_5) > 0\), we have \(x(v_1v_5v_3v_4v_5) = x(v_5s_2v_3v_4v_5) = 1\), which implies \(x(v_1v_5v_3v_4v_5) = x(v_1v_5v_3v_4v_5) = x(v_1v_5v_3v_4v_5) = x(v_1v_5v_3v_4v_5) = x(v_1v_5v_3v_4v_5)\). Thus \(x(v_1v_5v_3v_4v_5) = x(v_1v_5v_3v_4v_5)\). Similarly, for each vertex \(u \in V_1(V_2(T_2) \setminus a_2)\), we deduce that \(x(uv_1) = x(uv_4)\). Let \(T' = (V', A')\) be obtained from \(T\) by identifying \(v_1\) and \(v_4\); the resulting vertex is still denoted by \(v_1\). Let \(w'\) be obtained from the restriction of \(w\) by setting \(w'(v_1v_5) = w(v_1v_5) + w(v_5v_4)\), \(w'(v_3v_4) = w(v_3v_4) + w(v_3v_4)\), \(w'(v_3s_i) = w(v_3s_i) + w(v_3s_i)\) for \(1 \leq i \leq r\), and \(w'(uv_1) = w(uv_1) + w(uv_4)\) for each \(u \in V_1(V_2(T_2) \setminus a_2)\). By the LP-duality theorem, \(x\) and \(y\) naturally correspond to solutions to \(P(T', w')\) and \(D(T', w')\) respectively with the same optimal value \(\nu_w^*(T)\). From the hypothesis of Theorem 1.5, we deduce that \(\nu_w^*(T)\) is an integer. This proves (24).

**Subcase 2.7.** \(K = \{v_3v_1, v_4v_1, v_4v_5, v_4s_1, v_4s_2\}\).

In this case, by Lemma 3.1(iii), we obtain \(w(e) = y(C_2(e))\) for each \(e \in K\), which, to-
gether with (19), implies that \( y(v_is_1v_3v_4) = w(v_is_1) \) for \( i = 1, 2, \) \( y(v_is_1v_3v_4) = w(v_is_1) \), and \( y(v_is_4v_3v_4) = w(v_is_4) \). We may assume that \( w(v_is_1) = 0 \) for \( i = 1, 2, \) for otherwise, \( y(v_is_1v_3v_4) \) or \( y(v_is_4v_3v_4) \) is a positive integer. Note that both \( s_is_1v_3 \) and \( s_is_2v_3 \) are outside \( C^0_H \). So \( s_is_1v_3 \) is saturated by \( y \) in \( T_2 \) for \( i = 1, 2, \) and hence \( y(v_is_1v_3v_4) = w(s_is_1v_3) \) and \( y(v_is_4v_3v_4) = w(s_is_4v_3) \). If \( y(v_is_1v_3v_4) > 0 \) or \( y(v_is_4v_3v_4) > 0 \), then \( y(v_is_1v_3v_4) = w(s_is_1v_3) > 0 \) by (8) or (10). So we assume that \( y(v_is_1v_3v_4) = y(v_is_4v_3v_4) = 0 \). Then \( y(v_is_1v_3v_4) = w(v_is_1v_3v_4) \). Since \( w(s_is_1v_3) > 0 \), at least one of \( y(v_is_1v_3v_4) \) and \( y(v_is_4v_3v_4) \) is a positive integer.

Subcase 2.8. \( K = \{ v_3v_1, v_3v_4 \} \).

In this case, by Lemma 3.1(iii), we obtain \( w(e) = y(C_2(e)) \) for each \( e \in K \), which, together with (19), implies that \( y(v_is_1v_3v_4) = y(v_is_1v_3v_4) = w(v_is_1v_3v_4) \) for each \( e \in K \). Hence \( y(v_is_1v_3v_4) = w(s_is_1v_3) \) and \( y(v_is_4v_3v_4) = w(s_is_4v_3) = w(s_is_1v_3) - y(v_is_1v_3v_4) \). Since \( w(s_is_1v_3) > 0 \), at least one of \( y(v_is_1v_3v_4) \) and \( y(v_is_4v_3v_4) \) is a positive integer. So we assume that \( y(v_is_1v_3v_4) = y(v_is_4v_3v_4) = 0 \). If \( y(v_is_1v_3v_4) = 0 \), then either \( y(v_is_1v_3v_4) = w(s_is_1v_3) \) or \( y(v_is_4v_3v_4) = w(s_is_4v_3) \) by (12), so \( y(v_is_1v_3v_4) = w(s_is_3) - y(v_is_1v_3v_4) \). Since \( w(s_is_3) > 0 \), at least one of \( y(v_is_1v_3v_4) \) and \( y(v_is_4v_3v_4) \) is a positive integer.

Suppose \( y(v_is_1v_3v_4) > 0 \). Then \( y(v_is_1v_3v_4) = w(v_is_1v_3v_4) \) by (9). If \( y(v_is_1v_3v_4) > 0 \), then \( y(v_is_4v_3v_4) = w(v_is_4v_3v_4) \), and \( y(v_is_4v_3v_4) = w(v_is_4v_3v_4) \) by (11). It follows that \( y(v_is_1v_3v_4) = w(s_is_1v_3v_4) = y(v_is_1v_3v_4) \). Since \( w(s_is_3) > 0 \), at least one of \( y(v_is_1v_3v_4) \), \( y(v_is_4v_3v_4) \), and \( y(v_is_4v_3v_4) \) is a positive integer. So we assume that \( y(v_is_1v_3v_4) = 0 \). If \( y(v_is_2v_3v_4) = 0 \), then \( y(v_is_2v_3v_4) = w(v_is_2v_3v_4) \). By Lemma 5.2(vii), at most one of \( w(v_is_1v_3v_4) \) and \( w(v_is_2v_3v_4) \) is nonzero. Thus either \( y(v_is_1v_3v_4) = 0 \) or \( y(v_is_2v_3v_4) = 0 \), and hence either \( y(v_is_1v_3v_4) = w(s_is_1v_3v_4) \) or \( y(v_is_2v_3v_4) = w(s_is_2v_3v_4) \). So we further assume that \( y(v_is_2v_3v_4) > 0 \). If \( y(v_is_1v_3v_4) \) or \( y(v_is_2v_3v_4) \) is an integer, then accordingly \( y(v_is_1v_3v_4) \) or \( y(v_is_2v_3v_4) \) is an integer. Since \( w(s_is_3) > 0 \) for \( i = 1, 2, \) at least one of \( y(v_is_1v_3v_4) \), \( y(v_is_2v_3v_4) \), \( y(v_is_3v_2v_3v_4) \), \( y(v_is_4v_3v_4) \), \( y(v_is_4v_3v_4) \), and \( y(v_is_4v_3v_4) \) is a positive integer, as claimed.

It remains to consider the subcase when neither \( y(v_is_1v_3v_4) \) nor \( y(v_is_1v_3v_4) \) is an integer. Now we can prove that \( \nu_i^a(T) \) is an integer. Since the proof is the same as that of (24), we omit the details here.

Combining the above subcases, we see that Claim 2 holds. Hence, by Lemma 3.2(iii), the optimal value \( \nu_i^a(T) \) of \( D(T, w) \) is integral, as described in (1) above.

Lemma 5.8. If \( T_2/S = G_2 \), then \( D(T, w) \) has an integral optimal solution.

Proof. Recall that \( (b_2, a_2) = (v_4, v_5), \) \( s^* = v_2, \) and \( v_0 = v_4. \) To establish the statement, by Lemma 3.4(ii), it suffices to prove that

1. the optimal value \( \nu_i^a(T) \) of \( D(T, w) \) is integral.

33
Given an optimal solution $y$ to $\mathbb{D}(T, w)$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2) \setminus a_2\}$ for each $s_i \in S$. By Lemma 5.2(i) and (vi), we have
(2) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.
(3) There exist at least two and at most three vertices $s_i$s in $S$ with $\varphi(s_i) \neq \emptyset$.

In view of (2) and the structure of $G_2$, there are at most three vertices $s_i$s in $S$ with $\varphi(s_i) \neq \emptyset$. Suppose on the contrary that there exists precisely one vertex $s_i \in S$ with $\varphi(s_i) \neq \emptyset$. Then (1) follows immediately from Lemma 4.5; the argument can be found in that of (3) in the proof of Lemma 5.5.

Lemma 5.2(i) allows us to assume that
(4) if $\varphi(s_i) \neq \emptyset$, then $i \in \{1, 2, 3\}$.

In the remainder of our proof, we reserve $y$ for an optimal solution to $\mathbb{D}(T, w)$ such that
(5) $y(C_2)$ is maximized;
(6) subject to (5), $(y(D_q), y(D_{q-1}), \ldots, y(D_4))$ is minimized lexicographically;
(7) subject to (5) and (6), $y(\{v_1v_0v_3v_4v_1\})$ is minimized; and
(8) subject to (5)-(7), $y(\{v_1v_0v_3v_4v_1\})$ is minimized.

Let us make some observations about $y$ before proceeding.
(9) If $K$ is an FAS of $T_2 \setminus a_2$ such that $y(C_2) = w(K)$, then $K$ is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

The statements below follow instantly from Lemma 3.5(v).
(10) If $y(v_1v_0v_3v_4v_1) > 0$, then both $v_3v_1$ and $v_0v_4$ are saturated by $y$ in $T_2$.
(11) If $y(v_1v_0v_3s_iv_4v_1) > 0$ for some $i \in \{1, 2, 3\}$, then both $v_1s_i$ and $v_0v_4$ are saturated by $y$ in $T_2$.
(12) If $y(v_1v_0v_3s_iv_3v_4v_1) > 0$ for some $i \in \{1, 2, 3\}$, then each arc in the set $\{v_3v_1, v_3v_4, v_0v_4, v_1s_i, v_0v_4\}$ is saturated by $y$ in $T_2$.

**Claim 1.** $y(C_2) = \tau_w(T_2 \setminus a_2)$.

To justify this, we may assume that
(13) at most one of $v_3v_1$ and $v_4v_1$ is saturated by $y$ in $T_2$, for otherwise, $y(C_2) = w(K)$, where $K = \{v_3v_1, v_4v_1\}$. Since $K$ is an FAS of $T_2 \setminus a_2$, it is an MFAS by (9) and hence $y(C_2) = \tau_w(T_2 \setminus a_2)$.

We proceed by considering two cases, depending on whether $v_1 \in \varphi(s_i)$ for some $i$.

**Case 1.** $v_1 \notin \varphi(s_i)$ for any $i \in \{1, 2, 3\}$.

By (2), (3) and Lemma 5.2(i), we may assume that $\varphi(s_1) = \{v_6\}$ and $\varphi(s_2) = \{v_3\}$. Thus
(14) $C_2^y \subseteq \{v_1v_0v_3v_1, v_1v_0v_3v_4v_1, v_1v_0v_3v_4v_1, v_1v_6s_1v_4v_1, v_1v_6v_3s_2v_4v_1\}$.

By Lemma 5.2(iii), $z(s_iv_4) = w(z_iv_4) > 0$. If $s_iv_4$ is outside $C_2^y$ for $i = 1$ or $2$, then $s_iv_4$ is saturated by $y$ in $T_2$. In view of (14), we have $y(v_1v_6s_1v_4v_1) = w(s_1v_4) > 0$ or $y(v_1v_6v_3s_2v_4v_1) = w(s_2v_4) > 0$, and hence (1) follows from Lemma 3.2(iii). Similarly, if $v_6s_1$ or $v_3s_2$ is saturated by $y$ in $T_2$, then $y(v_1v_6s_1v_4v_1) = w(v_6s_1) > 0$ or $y(v_1v_6v_3s_2v_4v_1) = w(v_3s_2) > 0$, and hence (1) follows from Lemma 3.2(iii). So we assume that
(15) $s_iv_4$ is contained in some cycle in $C_2^y$ for $i = 1$ and $2$. Furthermore, neither $v_6s_1$ nor $v_3s_2$ is saturated by $y$ in $T_2$.

By (15) and Lemma 3.5(iii), at least one of $v_1v_6$ and $v_4v_1$ is saturated by $y$ in $T_2$. If $v_1v_6$ is saturated by $y$ in $T_2$, then $y(C_2) = w(v_1v_6)$. By (9), $\{v_1v_6\}$ is an MFAS of $T_2 \setminus a_2$ and hence $y(C_2) = \tau_w(T_2 \setminus a_2)$. If $v_4v_1$ is saturated by $y$ in $T_2$, then $v_3v_1$ is not saturated by $y$ in $T_2$.

34
by (13). So, by Lemma 3.5(vi), \( v_6v_3 \) is saturated by \( y \) in \( T_2 \) and, by (10) and (12), we have \( y(v_1v_6v_3v_2v_1) = y(v_1v_6v_3v_4v_1) = 0 \). Thus \( y(C_2) = w(K) \), where \( K = \{v_1v_6, v_6v_3\} \). Since \( K \) is an FAS of \( T_2 \setminus a_2 \), it is an MFAS by (9) and hence \( y(C_2) = \tau_w(T_2 \setminus a_2) \).

**Case 1.2.** \( v_1 \in \varphi(s_i) \) for some \( i \in \{1, 2, 3\} \).

By (2), (3) and Lemma 5.2(i), we may assume that \( v_1 \in \varphi(s_1), v_6 \in \varphi(s_i), \) and \( v_3 \in \varphi(s_j) \), with \( \{1 \neq i, j \} \subseteq \{1, 2, 3\} \). Furthermore,

\[ C_2'^{y} \subseteq \{v_1v_6v_3v_1, v_1v_6v_4v_1, v_1v_6s_3v_4v_1, v_1s_1v_4v_1, v_1v_6s_3v_3v_1, v_1v_6v_3s_3v_4v_1\} \]

We may further assume that \( s_1v_4 \) is contained in some cycle in \( C_0'^{y} \) and \( v_1s_1 \) is not saturated by \( y \) in \( T_2 \), for otherwise, \( y(v_1s_1v_4v_1) = w(s_1v_4) > 0 \) or \( y(v_1s_1v_4v_1) = w(s_1v_4) > 0 \). Hence (1) follows instantly from Lemma 3.2(iii). It follows from Lemma 3.5(vii) that \( v_1v_6 \) is saturated by \( y \) in \( T_2 \) and hence, by (13), \( v_6v_3 \) is not saturated by \( y \) in \( T_2 \). By (10) and (12), we obtain \( y(v_1v_6s_3v_4v_1) = y(v_1v_6v_3v_4v_1) = 0 \). If \( v_6v_3 \) is saturated by \( y \) in \( T_2 \), then \( y(C_2) = w(K) \), where \( K = \{v_3v_1, v_6v_3\} \). Since \( K \) is an FAS of \( T_2 \setminus a_2 \), it is an MFAS by (9) and hence \( y(C_2) = \tau_w(T_2 \setminus a_2) \). So we assume that \( v_6v_3 \) is not saturated by \( y \) in \( T_2 \). Thus, by Lemma 3.5(vii), \( v_1v_6 \) is saturated by \( y \) in \( T_2 \). We propose to show that

\[ y(v_1v_6v_3v_1) = y(v_1v_6s_3v_4v_1) = 0 \]

Assume the contrary: \( y(v_1v_6v_3v_1) > 0 \) or \( y(v_1v_6s_3v_4v_1) > 0 \). Then \( v_1s_1 \) is outside \( C_0'^{y} \), for otherwise, let \( C \) be a cycle in \( C_0'^{y} \) containing \( v_1s_1 \). Then the multiset sum of the cycles \( C \) and \( v_1v_6v_4v_1 \) (resp. \( v_1v_6s_3v_4v_1 \)) contains arc-disjoint cycles \( v_1s_1v_4v_1 \) and \( C' = C[v_4, v_1] \cup \{v_1v_6, v_6v_4\} \) (resp. \( C' = C[v_4, v_1] \cup \{v_1v_6, v_6s_3, s_4v_1\} \)). Set \( \theta = \min \{y(v_1v_6v_3v_1), y(C)\} \) (resp. \( \min \{y(v_1v_6s_3v_4v_1), y(C)\} \)). Let \( y' \) be obtained from \( y \) by replacing \( y(v_1v_6v_4v_1) \) (resp. \( y(v_1v_6s_3v_4v_1) \)), \( y(v_1v_2v_4v_1), y(C) \), and \( y(C') \) with \( y(v_1v_6v_4v_1) - \theta \) (resp. \( y(v_1v_6s_3v_4v_1) - \theta \)), \( y(v_1v_2v_4v_1) + \theta \), \( y(C) - \theta \), and \( y(C') + \theta \), respectively. It is easy to see that \( y' \) is an optimal solution to \( \mathbb{D}(T, w) \) with \( y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1) \) or \( y'(v_1v_6s_3v_4v_1) < y(v_1v_6s_3v_4v_1) \), contradicting (8) or (6). Since \( v_1v_6 \) is saturated by \( y \) in \( T_2 \), every cycle in \( C_0'^{y} \) containing \( v_3v_1 \) passes through \( v_1s_1 \). Thus \( v_3v_1 \) is outside \( C_0'^{y} \), and neither \( v_1s_1 \) nor \( v_3v_1 \) is saturated by \( y \) in \( T \).

Observe that \( v_6v_3 \) is outside \( C_0'^{y} \), for otherwise, let \( C \) be a cycle in \( C_0'^{y} \) containing \( v_6v_3 \). Then the multiset sum of the cycles \( C \), \( v_1v_6v_4v_1 \) (resp. \( v_1v_6s_3v_4v_1 \)), and the unsaturated arc \( v_3v_1 \) contain arc-disjoint cycles \( v_1v_6v_3v_1 \) and \( C' = C[v_4, v_6] \cup \{v_6v_4\} \) (resp. \( C' = C[v_4, v_6] \cup \{v_6s_3, s_4v_1\} \)). Set \( \theta = \min \{y(v_1v_6v_3v_1), y(C), w(v_6v_4) - z(v_6v_3)\} \) (resp. \( \theta = \min \{y(v_1v_6s_3v_4v_1), y(C), w(v_3v_1) - z(v_3v_4)\} \)). Let \( y' \) be obtained from \( y \) by replacing \( y(v_1v_6v_4v_1) \) (resp. \( y(v_1v_6s_3v_4v_1) \)), \( y(v_1v_2v_4v_1), y(C) \), and \( y(C') \) with \( y(v_1v_6v_4v_1) - \theta \) (resp. \( y(v_1v_6s_3v_4v_1) - \theta \)), \( y(v_1v_2v_4v_1) + \theta \), \( y(C) - \theta \), and \( y(C') + \theta \), respectively. It is easy to see that \( y' \) is an optimal solution to \( \mathbb{D}(T, w) \) with \( y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1) \) or \( y'(v_1v_6s_3v_4v_1) < y(v_1v_6s_3v_4v_1) \), contradicting (8) or (6). Hence \( v_6v_3 \) is not saturated by \( y \) in \( T \).

Let \( C \) be a cycle in \( C_0'^{y} \) containing \( s_1v_4 \). Then the multiset sum of the cycles \( C \), each of the cycles \( v_1v_6v_4v_1 \) and \( v_1v_6s_3v_4v_1 \), and the unsaturated arcs \( v_3v_1, v_5v_1, \) and \( v_1s_1 \) contains arc-disjoint cycles \( v_1s_1v_4v_1 \) and \( v_1v_6v_4v_1 \). So, by Lemma 3.5(vi), we have \( y(v_1v_6v_4v_1) = y(v_1v_6s_3v_4v_1) = 0 \); this contradiction establishes (17).

Using (17), we obtain \( y(C_2) = w(K) \), where \( K = \{v_1v_6, v_4v_1\} \). Since \( K \) is an FAS of \( T_2 \setminus a_2 \), it is an MFAS by (9) and hence \( y(C_2) = \tau_w(T_2 \setminus a_2) \). This proves Claim 1.

The above proof yields the following statement, which will be used later.

(18) If Case 1.1 occurs, then every MFAS comes from \( \{v_3v_1, v_4v_1\}, \{v_1v_6, v_4v_1\} \). If Case 1.2 occurs, then every MFAS comes from \( \{v_3v_1, v_4v_1\}, \{v_1v_6, v_4v_1\}, \{v_4v_1, v_6v_3\} \).
Claim 2. \( y(C) \) is a positive integer for some \( C \in C_2 \) or \( \nu'_w(T) \) is an integer.

To justify this, we first show that

(19) if \( v_3 \in \varphi(s_i) \) for \( i \in \{1, 2, 3\} \), then \( y(v_1v_6v_3s_iv_1v_1) = 0 \).

Assume the contrary: \( y(v_1v_6v_3s_iv_1v_1) > 0 \). Then \( y(v_1v_6v_3v_1v_1) = w(v_3v_1) \), \( y(v_1v_6v_3s_iv_1) = w(v_3v_4) \), and \( y(v_1v_6v_4v_1) = w(v_6v_4) \) by (12). So Lemma 3.2(iii) allows us to assume that \( w(v_3v_1) = w(v_3v_4) = w(v_6v_4) = 0 \). Let \( j \) and \( k \) be subscripts in \( \{1, 2, 3\} \), if any, such that \( v_6 \in \varphi(s_j) \) and \( v_1 \in \varphi(s_k) \). If both \( v_1s_kv_4v_1 \) and \( v_1v_4sv_4v_1 \) are integral, then, by Claim 1, \( y(v_1v_6v_3sv_4v_1) \) is a positive integer, so Claim 2 holds. Thus we may assume that \( y(v_1s_kv_4v_1) \) or \( y(v_1v_6v_3sv_4v_1) \) is not integral. Then, by (11) and Lemma 3.2(iii), we have \( j, k \neq i \). Furthermore, both \( v_1s_k \) and \( v_6s_j \) are outside \( C_0^n \); for otherwise, we can construct an optimal solution \( y' \) to \( \mathbb{D}(T, u) \) with \( y'(v_1v_6v_3sv_4v_1) < y(v_1v_6v_3sv_4v_1) \), contradicting (6).

Consider first the case when \( y(v_1v_6v_3sv_4v_1) \) is not integral. If \( j = k \) and \( y(v_1s_kv_4v_1) > 0 \), then \( y(v_1s_kv_4v_1) = w(v_1s_k) > 0 \) by (11), so Claim 2 holds. Thus we may assume that \( j \neq k \) if \( y(v_1s_kv_4v_1) > 0 \). Let us show that \( \nu'_w(T) \) is an integer.

For this purpose, let \( x \) be an optimal solution to \( \mathbb{P}(T, u) \). Since both \( y(v_1v_6s_jv_4v_1) \) and \( y(v_1v_6v_3sv_4v_1) \) are positive, \( x(v_1v_6s_jv_4v_1) = x(v_1v_6v_3sv_4v_1) = 1 \) by Lemma 3.1(i). By Lemma 5.2(iv), \( x(v_6s_j) = x(v_3s_j) = 0 \). It follows that \( x(s_jv_4) = x(v_3v_4) = x(s_i) \). If \( v_6v_3 \) is outside \( C_0^n \), then \( x(v_6v_3) = 0 \) by Lemma 3.1(ii), because \( z(v_6v_3) = y(v_1v_6v_3sv_4v_1) = w(v_6v_3) \); thus \( x(s_i) = x(v_6v_3) \), contradicting Lemma 5.2(iv). So we assume that \( v_6v_3 \) is contained in some cycle in \( C_0^n \). Since \( w(v_6v_3) = w(v_6v_4) = 0 \) and \( (v_6, s_j) \) is outside \( C_0^n \), for any \( u \in \nabla(V(T_2) \setminus a_2) \), if a cycle in \( C_0^n \) contains \( w_6 \), then it passes through \( v_6v_3s_i \). Moreover, if a cycle in \( C_0^n \) contains \( s_ju \), then it passes through \( s_i \).

By Lemma 3.1(iv), we obtain \( x(uw_6) = x(v_6v_3) + x(s_i) = x(s_i) = x(v_6v_3) \). Clearly, we may assume that this equality holds in any other situation. Let \( T' = (V', A') \) be obtained from \( T \) by deleting vertex \( s_j \), and let \( w' \) be obtained from the restriction of \( w \) to \( A' \) by replacing \( w(e) + w(s_jv_4) \) for each \( e \in \{v_6v_3, v_5s_i, s_iu_4\} \) and replacing \( w(uw_6) \) with \( w(uw_6) + w(s_j) \) for each \( u \in \nabla(V(T_2) \setminus a_2) \). Let \( x' \) be the restriction of \( x \) to \( A' \) and let \( y' \) be obtained by \( y \) as follows:

set \( y'(v_1v_6v_3sv_4v_1) = y(v_1v_6sv_4v_1) + y(v_1v_6v_3sv_4v_1) \); for each \( C' \in C_0^n \) passing through \( u_6v_3v_4 \) for any \( u \in V(V(T_2) \setminus a_2) \), let \( C' \) be the cycle arising from \( C \) by replacing the path \( u_6v_3v_4 \) with the path \( u_6v_3v_4u_3 \), and set \( y'(C') = y(C') \). From the LP-duality theorem, we see that \( x' \) and \( y' \) are optimal solutions to \( \mathbb{P}(T, w') \) and \( \mathbb{D}(T, w') \), respectively, with the same value \( \nu'_w(T) \) as \( x \) and \( y \). By the hypothesis of Theorem 1.5, \( \nu'_w(T) \) is an integer.

In the other case when \( y(v_1v_6v_3sv_4v_1) = 0 \) and \( y(v_1s_kv_4v_1) \) is not integral, the proof goes along the same line, so we omit the details here.

By Claim 1, \( y(C_2) = w(K) \) for some FAS \( K \) of \( T_2 \setminus a_2 \) as described in (18). Recall that

(20) in Case 1.1, we have \( v_i \notin \varphi(s_i) \) for any \( i \in \{1, 2, 3\} \), \( \varphi(s_1) = \{v_6\} \), and \( \varphi(s_2) = \{v_3\} \); in Case 1.2, we have \( v_i \in \varphi(s_1) \), \( v_6 \in \varphi(s_i) \), and \( v_3 \in \varphi(s_j) \), with \( \{1\} \neq \{i, j\} \subset \{1, 2, 3\} \).

Depending on what \( K \) is, we distinguish among four cases.

Case 2.1. \( K = \{v_4v_1, v_6v_3\} \) in Case 1.1 or \( K = \{v_1v_6, v_4v_1\} \) in Case 1.2.

Consider first the subcase when \( K = \{v_4v_1, v_6v_3\} \) in Case 1.1. Now \( y(v_1v_6v_3v_1) = w(v_6v_3) \) and \( y(v_1v_6v_3v_4) + y(v_1v_6v_3v_4v_1) = w(v_4v_1) \) (see (20)). If \( y(v_1v_6v_3s_1v_1) = 0 \), then \( y(v_1v_6v_4v_1) = w(v_4v_1) \). If \( y(v_1v_6s_1v_4v_1) > 0 \), then \( y(v_1v_6v_1v_4) = w(v_4v_1) \) by (11), and hence \( y(v_1v_6v_3s_1v_4v_1) = w(v_4v_1) - w(v_4v_1) \). By the hypothesis of the present section, \( w(K) = \tau_w(T_2 \setminus a_2) > 0 \). So at least
one of $y(v_1v_6v_3v_1), y(v_1v_6v_4v_1)$, and $y(v_1v_6s_1v_4v_1)$ is a positive integer.

Next consider the subcase when $K = \{v_6, v_1v_1\}$ in Case 1.2. Now $y(v_1v_6s_1v_4v_1) = w(a^2)$ and $y(v_1v_6v_3v_1) = w(v_1v_6)$. So at least one of $y(v_1v_6s_1v_4v_1)$ and $y(v_1v_6v_3v_1)$ is a positive integer.

**Case 2.2.** $K = \{v_1v_6\}$ or $\{v_6, v_1v_1\}$ in Case 1.1.

We only consider the subcase when $K = \{v_1v_6\}$, as the proof in the other subcase goes along the same line. Now $y(v_1v_6v_3v_1) + y(v_1v_6s_1v_4v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$, and $v_3v_1$ is outside $C_0^y$.

Observe that $y(v_1v_6v_3v_4v_1) > 0$, for otherwise, if $y(v_1v_6s_1v_4v_1) > 0$, then $y(v_1v_6v_4v_1) = w(v_6v_1)$ by (11), and hence $y(v_1v_6v_3v_1) + y(v_1v_6s_1v_4v_1) = w(v_1v_6) - w(v_6v_1)$; if $y(v_1v_6s_1v_4v_1) = 0$, then $y(v_1v_6v_3v_1) + y(v_1v_6s_1v_4v_1) = w(v_1v_6)$. Let us show that $y(v_1v_6v_3v_1) = w(v_1v_6)$ is integral. Assume first that $y(v_1v_6s_1v_4v_1) > 0$. If $v_6v_3$ is outside $C_0^y$, let $y'$ be obtained from $y$ by replacing $y(v_1v_6v_3v_1)$ and $y(v_1v_6s_1v_4v_1)$ with $y(v_1v_6v_3v_1) + y(v_1v_6s_1v_4v_1)$ and $y(v_1v_6v_3v_1)$, respectively; if $v_6v_3$ is contained in a cycle $C \in C_0^y$, set $\theta = \min\{y(C), y(v_1v_6s_1v_4v_1)\}$ and $C' = C[v_1v_6, v_6] \cup \{v_6s_1, v_4v_1\}$, and let $y'$ be obtained from $y$ by replacing $y(v_1v_6s_1v_4v_1)$, $y(v_1v_6s_1v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_1) + \theta$, $y(v_1v_6s_1v_4v_1) - \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $y'$ is also an optimal solution to $D(T, w)$ with $y'(v_1v_6s_1v_4v_1) > y'(v_1v_6v_3v_1)$ while $y'(v_1v_6s_1v_4v_1) < y'(v_1v_6s_1v_4v_1)$ is in either situation, so $y'$ is a better choice than $y$ (see (6)), a contradiction. Assume next that $y(v_1v_6s_1v_4v_1) = 0$. Imitating the above proof, with $y(v_1v_6v_3v_1) in place of $y(v_1v_6s_1v_4v_1)$, we can reach a contradiction to (8).

Since $y(v_1v_6v_3v_4v_1) > 0$, by (10), we have $y(v_1v_6v_3v_1) = w(v_6v_1)$ and $y(v_1v_6v_4v_1) = w(v_6v_1)$; so Lemma 3.2(iii) allows us to assume that $w(v_6v_1) = w(v_6v_1)$ is 0. Thus the previous equality concerning $w(v_6v_1)$ becomes $y(v_1v_6v_3v_1) + y(v_1v_6v_3v_4v_1) = w(v_1v_6)$. So we may assume that neither $y(v_1v_6v_3v_1)$ nor $y(v_1v_6v_3v_4v_1)$ is integral, for otherwise, at least one of them is a positive integer. Observe that $v_6s_1$ is outside $C_0^y$, for otherwise, let $C$ be a cycle in $C_0^y$ that contains $v_6s_1$, let $C' = C[v_6, v_6] \cup \{v_6v_3, v_3v_4\}$, and let $\theta = \min\{y(C), y(v_1v_6v_3v_4v_1)\}$. Let $y'$ be obtained from $y$ by replacing $y(v_1v_6s_1v_4v_1)$, $y(v_1v_6v_3v_4v_1)$, $y(C)$, and $y(C')$ with $y(v_1v_6v_3v_1) + \theta$, $y(v_1v_6s_1v_4v_1) - \theta$, $y(C) - \theta$, and $y(C') + \theta$, respectively. Then $y'$ is also an optimal solution to $D(T, w)$ with $y'(v_1v_6v_3v_4v_1) < y'(v_1v_6v_3v_1)$, contradicting (7).

We propose to show that $\nu_w(T)$ is an integer. For this purpose, let $x$ be an optimal solution to $P(T, w)$. Since both $y(v_1v_6s_1v_4v_1)$ and $y(v_1v_6v_3v_4v_1)$ are positive, $x(v_1v_6s_1v_4v_1) = y(v_1v_6s_1v_4v_1) = x(v_1v_6v_3v_4v_1)$ by Lemma 3.2(i). Since $y(v_1v_6v_3v_4v_1) = w(v_6s_1)$, we have $x(v_6s_1) = 0$ by Lemma 3.2(ii). Thus $x(s_1v_4) = x(v_6v_3) + x(v_6v_4)$. Since $w(v_6v_3) = 0$, for any $u \in V \setminus (V(T_2) \cup T_2)$, if a cycle in $C_0^y$ contains $uv_6$, then it passes through $v_6v_3$ or $v_6v_4$. Moreover, if a cycle in $C_0^y$ contains $uv_6$, then it passes through $s_1v_4$. By Lemma 3.1(iv), we obtain $x(u_6b) + x(v_6v_3) + x(v_6v_4) = x(u_6s_1) + x(s_1v_4)$ or $x(u_6b) + x(v_6v_3) + x(v_6s_1) + x(s_1v_4) = x(u_6s_1) + x(s_1v_4)$. Hence $x(u_6b) = x(u_6s_1)$. Clearly, we may assume that this equality holds in any other situation. Let $T = (V', A')$ be obtained from $T$ by deleting vertex $s_1$, and let $w'$ be obtained from the restriction of $w$ to $A'$ by replacing $w(e)$ with $w(e) + w(s_1v_4)$ for $e = v_6v_3$ and $v_6v_4$ and replacing $w(uv_6)$ with $w(uv_6) + w(uv_6)$ for any $u \in V \setminus (V(T_2) \cup T_2)$. Let $x'$ be the restriction of $x$ to $A'$ and let $y'$ be obtained from $y$ as follows: set $y'(v_1v_6v_3v_4v_1) = y(v_1v_6s_1v_4v_1) + y(v_1v_6v_3v_4v_1)$; for each $C \in C_0^y$ passing through $uv_6$ for any $u \in V \setminus (V(T_2) \cup T_2)$, let $C'$ be the cycle arising from $C$ by replacing the path $uv_6$ with the path $uv_6v_3v_4$, and set $y'(C') = y(C') + y(C')$. From the LP-duality theorem, we see that $x'$ and $y'$ are optimal solutions to $P(T', w')$ and $D(T', w')$, respectively, with the same value $\nu_w(T)$ as $x$ and $y$. By the hypothesis of Theorem 1.5, $\nu_w(T)$
is an integer.

**Case 2.3.** \( K = \{v_1v_3, v_0v_3\} \) in Case 1.2.

In this case, \( y(v_1v_3v_3v_1) = w(v_0v_3) \) and \( y(v_1s_1v_4v_1) + y(v_1v_3v_3v_1) + y(v_1v_6v_4v_1) = w(v_4v_1) \) (see (20)). By Lemma 3.2(iii), we may assume that \( w(v_0v_3) = 0 \). Let us show that

\[ (21) \quad y(v_1v_6s_1v_4v_1) = 0. \]

Assume the contrary. Then, by (11), we have \( y(v_1v_6v_4v_1) = w(v_6v_4) \), and \( v_1s_1 \) is saturated by \( y \) in \( T_2 \). Lemma 3.2(iii) allows us to assume that \( w(v_0v_3) = 0 \) and that \( y(v_1v_6s_1v_4v_1) \) is not integral. It follows from (6) and Lemma 3.5(v) that \( i \neq 1 \) and \( v_1s_1 \) is outside \( C_0^0 \). We propose to prove that \( \nu^*_w(T) \) is an integer.

For this purpose, let \( x \) be an optimal solution to \( \mathbb{P}(T, w) \). Since both \( y(v_1s_1v_4v_1) \) and \( y(v_1v_6s_1v_4v_1) \) are positive, by Lemma 3.1(i), we have \( x(v_1s_1v_4v_1) = x(v_1v_6s_1v_4v_1) = 1 \). Since \( y(v_1s_1v_4v_1) < w(v_1s_1) \), by Lemma 3.1(ii), we obtain \( x(v_1s_1) = 0 \), so \( x(s_1v_4) = x(v_1v_4) + x(v_6s_1) + x(s_1v_4) \). If \( v_1v_6 \) is outside \( C_0^0 \), then \( x(v_1v_6) = 0 \), because \( z(v_1v_6) = y(v_1v_6s_1v_4v_1) < w(v_1v_6) \).

By Lemma 5.2(vi), \( x(v_1s_1) = x(v_6s_1) = 0 \). Hence, \( x(s_1v_4) = x(s_4v_4) \), contradicting Lemma 5.2(iv). So we assume that \( v_1v_6 \) is contained in some cycle in \( C_0^0 \). Since \( w(v_0v_3) = w(v_6v_4) = 0 \), for any \( u \in V \setminus (V(T_2) \setminus a_2) \), if a cycle in \( C_0^0 \) contains \( wu_1 \), then it passes through \( v_1v_6s_1v_4v_1 \).

Moreover, if a cycle in \( C_0^0 \) contains \( us_1 \), then it passes through \( s_1v_4 \). By Lemma 3.1(iv), we obtain \( x(uw_1) + x(v_1v_6) + x(v_6s_1) + x(s_1v_4) = x(us_1) + x(s_1v_4) \). Hence \( x(uw_1) = x(us_1) \). Clearly, we may assume that this equality holds in any other situation. Let \( T' = (V', A') \) be obtained from \( T \) by deleting vertex \( s_1 \), and let \( w' \) be obtained from the restriction of \( w \) to \( A' \) by replacing \( w(e) \) with \( w(e) + w(s_1v_1) \) for \( e \in \{v_1v_6, v_6s_1, s_1v_4\} \) and replacing \( w(uw_1) \) with \( w(uw_1) + w(us_1) \) for any \( u \in V \setminus (V(T_2) \setminus a_2) \). Let \( x' \) be the restriction of \( x \) to \( A' \), and let \( y' \) be obtained from \( y \) as follows:

\[ y'(v_1v_6v_4v_1) = y(v_1s_1v_4v_1) + y(v_1v_6s_1v_4v_1); \quad \text{for each } C \in C_0^0 \text{ passing through } us_1, \text{ let } C' \text{ arise from } C \text{ by replacing the path } us_1v_4 \text{ with the path } uw_1v_6s_1v_4, \text{ and set } y'(C') = y(C') + y(C). \]

From the LP-duality theorem, we see that \( x' \) and \( y' \) are optimal solutions to \( \mathbb{P}(T', w') \) and \( \mathbb{D}(T', w) \), respectively, with the same value \( \nu^*_w(T) \) as \( x \) and \( y \). By the hypothesis of Theorem 1.5, \( \nu^*_w(T) \) is an integer. So we may assume that \((21)\) holds.

By (21), the equality concerning \( w(v_4v_1) \) becomes \( y(v_1s_1v_4v_1) + y(v_1v_6v_4v_1) = w(v_4v_1) \). As \( w(v_3v_1) = w(K) = \tau_v(T_2 \setminus a_2) > 0 \), neither \( y(v_1s_1v_4v_1) \) nor \( y(v_1v_6v_4v_1) \) is integral. Observe that \( v_1s_1 \) is outside \( C_0^0 \), for otherwise, let \( C \) be a cycle containing \( v_1s_1 \) in \( C_0^0 \), let \( C' = C[v_4, v_1] \cup \{v_1v_6, v_4v_1\} \), and let \( \theta = \min\{y(C), y(v_1v_6v_4v_1)\} \). Let \( y' \) be obtained from \( y \) by replacing \( y(v_1s_1v_4v_1), y(v_1v_6v_4v_1), y(C), \) and \( y(C') \) with \( y(v_1s_1v_4v_1) + \theta, y(v_1v_6v_4v_1) - \theta, y(C) - \theta, \) and \( y(C') - \theta \), respectively. Then \( y' \) is also an optimal solution to \( \mathbb{T}(T, w) \) with \( \langle y'(v_1v_6v_4v_1) < y(v_1v_6v_4v_1) \rangle \), contradicting (8). Moreover, \( i \neq 1 \), for otherwise, it can be shown similarly that \( v_1s_1 \) is outside \( C_0^0 \), which implies \( z(v_6s_1) = 0 \), contradicting that \( v_6 \in \varphi(s_1) \). Let us show that

\[ (22) \quad \nu^*_w(T) \text{ is an integer.} \]

For this purpose, let \( x \) be an optimal solution to \( \mathbb{P}(T, w) \). Since both \( y(v_1s_1v_4v_1) \) and \( y(v_1v_6v_4v_1) \) are positive, we have \( x(v_1s_1v_4v_1) = x(v_1v_6v_4v_1) = 1 \) by Lemma 3.1(i). By (16) and Lemma 3.2(iii), we have \( y(v_1s_1v_4v_1) < w(v_1s_1) \) and hence \( x(v_1s_1) = 0 \). So \( x(s_1v_4) = x(v_1v_6) + x(v_6v_4) \). Note that if a cycle in \( C_0^0 \) contains \( us_1 \), then it passes through \( s_1v_4 \). For any \( u \in V \setminus (V(T_2) \setminus a_2) \), if there exists a cycle \( C \in C_0^0 \) containing \( uw_1 \) and passing through \( v_1v_6v_4v_1 \), then by Lemma 3.1(iv), we obtain \( x(uw_1) + x(v_1v_6) + x(v_6v_4) = x(us_1) + x(s_1v_4) \), and hence \( x(uw_1) = x(us_1) \). Otherwise, since \( w(v_0v_3) = 0 \), if a cycle in \( C_0^0 \) contains \( uw_1 \), then it passes through \( v_1v_6s_1v_4 \). By Lemma 3.1(i) and (iv), we have \( x(v_6v_4) \geq x(v_6s_1) + x(s_1v_4) \) and
Lemma 5.9.

If remainder of our proof, we reserve each \( s \) with \( P \) set \( y \) contained in the argument of (21), we omit the routine details here. 

(7) subject to (5) and (6), \( y \) is saturated by \( u \) in \( V \). Let \( (3) \) There exist at least two and at most three vertices \( \phi \). The proof of (22), it can be shown that \( \nu^*_w(T) \) is an integer. So we assume that at least one of \( y(v_1v_6v_3v_1) \) and \( y(v_1v_6s_1v_4v_1) \) is positive. By (10) or (11), \( v_6v_4 \) is saturated by \( y \) in \( T_2 \), and hence \( y(v_1v_6v_4v_1) = w(v_6v_4) \). By Lemma 3.2(iii), we may assume that \( w(v_6v_4) = 0 \). If neither \( y(v_1v_6s_1v_4v_1) \) nor \( y(v_1v_6v_3v_1) \) is integral then, imitating the proof in Case 2.2, it can be shown that \( \nu^*_w(T) \) is an integer. It remains to consider the subcase when precisely one of them is positive. Now it can be shown that \( \nu^*_w(T) \) is an integer. Since the proof is the same as that contained in the argument of (21), we omit the routine details here.

Combining the above four cases, we see that Claim 2 holds. Hence, by Lemma 3.2(iii), the optimal value \( \nu^*_w(T) \) of \( \mathcal{D}(T, w) \) is integral, as described in (1) above.

Lemma 5.9. If \( T_2/S = G_3 \), then \( \mathcal{D}(T, w) \) has an integral optimal solution.

Proof. Recall that \((b_2, a_2) = (v_4, v_5)\), \( s^* = v_2 \), and \( v_0 = v_4 \). To establish the statement, by Lemma 3.2(iii) and Lemma 3.4(ii), it suffices to prove that

(1) \( y(C) \) is a positive integer for some \( C \in C_2 \) or the optimal value \( \nu^*_w(T) \) of \( \mathcal{D}(T, w) \) is an integer.

Given an optimal solution \( y \) to \( \mathcal{D}(T, w) \), set \( \varphi(s_i) = \{ u : z(us_i) > 0 \} \) for \( u \in V(T_2) \setminus \{a_2\} \) for each \( s_i \in S \). By Lemma 5.2 (i) and (vi), we have

(2) \( \varphi(s_i) \cap \varphi(s_j) = \emptyset \) whenever \( i \neq j \).

(3) There exist at least two and at most three vertices \( s_i \)’s in \( S \) with \( \varphi(s_i) \neq \emptyset \). (The statement is exactly the same as (3) in the proof of Lemma 5.7.)

Lemma 5.2(ii) allows us to assume that

(4) if \( \varphi(s_i) \neq \emptyset \), then \( i \in \{1, 2, 3\} \).

Let \( t \) be the subscript in \( \{1, 2, 3\} \) with \( v_1 \in \varphi(s_i) \), if any. By (2), \( t \) is well defined. In the remainder of our proof, we reserve \( y \) for an optimal solution to \( \mathcal{D}(T, w) \) such that

(5) \( y(C_2) \) is maximized;

(6) subject to (5), \( y(D_0), y(D_{q-1}), \ldots, y(D_3) \) is minimized lexicographically;

(7) subject to (5) and (6), \( y(v_1v_6v_3v_1) \) is minimized; and

(8) subject to (5)-(7), \( y(v_1s_1v_4v_1) + y(v_3v_4v_6v_3) \) is minimized.
Let us make some observations about $y$ before proceeding.

(9) If $K$ is an FAS of $T_2 \setminus a_2$ such that $y(C_2) = w(K)$, then $K$ is an MFAS. (The statement is exactly the same as (4) in the proof of Lemma 4.3.)

The statements below follow instantly from Lemma 3.5(v) and the choice of $y$.

(10) If $y(v_1 v_5 v_3 v_4 v_1) > 0$, then both $v_3 v_1$ and $v_4 v_6$ are saturated by $y$ in $T_2$. Furthermore, for any $i \in \{1, 2, 3\}$, we have $y(v_i s_i v_4 v_6) = 0$ if $y(v_3 s_i v_4 v_6 v_3) > 0$, then $v_i s_i$ is saturated by $y$ in $T_2$.

(11) If $y(v_1 v_5 s_i v_4 v_1) > 0$ for some $i \in \{1, 2, 3\}$, then both $v_1 s_i$ and $v_4 v_6$ are saturated by $y$ in $T_2$. Furthermore, if $y(v_3 v_4 v_6 v_3) > 0$, then $v_3 v_1$ is saturated by $y$ in $T_2$; for any $1 \leq j \neq i \leq 3$, if $y(v_3 s_j v_4 v_6 v_3) > 0$, then both $v_3 v_1$ and $v_1 s_j$ are saturated by $y$ in $T_2$.

(12) If $y(v_3 s_i v_4 v_6 v_3) > 0$ for some $i \in \{1, 2, 3\}$, then both $v_3 v_4$ and $v_6 s_i$ are saturated by $y$ in $T_2$.

(13) If $v_1 \in \varphi(s_i)$ for some $i \in \{1, 2, 3\}$, then $y(v_1 s_i v_4 v_6 v_3 v_1) = 0$.

Assume the contrary: $y(v_1 s_i v_4 v_6 v_3 v_1) > 0$. Then $v_1 v_6$, $v_3 v_4$, and $v_4 v_1$ are saturated by $y$ in $T_2$ by Lemma 3.5(v). Let $j$ and $k$ be subscripts in $\{1, 2, 3\}$, if any, such that $v_3 \in \varphi(s_j)$ and $v_6 \in \varphi(s_k)$ (possibly $j = k$). As before, let $\cup$ denote the multiset sum. Then $v_1 s_i v_4 v_6 v_3 v_1 \cup v_1 v_6 v_3 v_1 \cup v_1 v_6 v_3 v_1 \cup v_3 v_4 v_6 v_3 \cup v_3 s_i v_4 v_6 v_3 \cup v_6 v_3 s_i v_4 v_1 \cup v_1 v_6 v_3 v_1 \cup v_1 v_6 v_3 v_1 \cup v_3 v_4 v_6 v_3 \cup v_3 s_i v_4 v_6 v_3$. Thus, from the optimality of $y$, we deduce that $y(v_1 v_6 v_3 v_4 v_1)$, $y(v_1 v_6 v_3 s_i v_4 v_1)$, and $y(v_1 v_6 v_3 s_j v_4 v_1)$ are all zero. So $y(v_1 v_6 v_3 v_1) = w(v_1 v_6)$, $y(v_1 s_i v_4 v_1) = w(v_1 v_4)$, and $y(v_3 v_4 v_6 v_3) = w(v_3 v_4)$. Clearly, we may assume that $w(v_1 v_6) = w(v_1 v_4) = w(v_3 v_4) = 0$, otherwise (1) holds. By (3), we have $\{j, k\} \neq \{i\}$. Let us show that one of $y(v_6 s_i v_4 v_6)$, $y(v_3 s_i v_4 v_6)$, and $y(v_1 s_i v_4 v_6 v_3 v_1)$ is a positive integer or $\nu^*_{\chi}(T)$ is an integer. We proceed by considering two cases.

• $k$ exists and $i \neq k$. In this case, observe first that $v_6 s_k$ is not saturated by $y$ in $T_2$, for otherwise, $y(v_6 s_k v_4 v_6) = w(v_6 s_k) > 0$ and hence (1) holds. Next, $v_6 v_3$ is not saturated by $y$ in $T_2$, for otherwise, if $k \neq j$, then $y(v_6 s_k v_4 v_6) = w(v_6 s_k) > 0$; if $k = j$, then $y(v_6 s_k v_4 v_6) + y(v_3 s_k v_4 v_6) = w(v_6 s_k) > 0$, and $y(v_6 s_k v_4 v_6) = w(v_6 s_k) > 0$ by Lemma 3.5(v) provided $y(v_3 s_k v_4 v_6) > 0$. So $y(v_6 s_k v_4 v_6)$ is a positive integer, and hence (1) also holds. Moreover, both $v_6 s_k$ and $v_6 s_j$ are outside $C^*_k$, for otherwise, let $C_1$ (resp. $C_2$) be a cycle in $C^*_k$ containing $v_6 s_k$ (resp. $v_3 s_j$). Since $C_1 \cup \{v_1 s_i v_4 v_6 v_3 v_1 = v_6 s_k v_4 v_6 \cup C'_1\}$ and $C_2 \cup \{v_1 s_i v_4 v_6 v_3 v_1 = v_3 s_j v_4 v_6 v_3 \cup C'_2\}$, where $C'_1 = C[v_4, v_6] \cup \{v_3 v_4, v_3 v_1, v_1 s_i, v_4 v_1\}$ and $C'_2 = C[v_4, v_3] \cup \{v_3 v_1, v_1 s_i, v_4 v_1\}$, by Lemma 3.5(vii), we have $y(C_i) = 0$ for $i = 1, 2$, a contradiction. It follows that $v_6 s_k$ is not saturated by $y$ in $T_2$, and $s_i v_4$ is contained in some cycle in $C^*_k$. By Lemma 3.5(vii), $v_6 v_3$ is saturated by $y$ in $T_2$, so $y(v_1 s_i v_4 v_6 v_3 v_1) + y(v_6 s_k v_4 v_6 v_3) = w(v_6 v_3)$. If $j = k$ and $y(v_3 s_j v_4 v_6 v_3) > 0$, then $v_6 s_k$ is saturated by $y$ in $T_2$ by Lemma 3.5(v), a contradiction. So either $j \neq k$ or $j = k$ and $y(v_3 s_k v_4 v_6 v_3) = 0$. Since $w(v_6 s_k) > 0$ and $v_6 s_k$ is outside $C^*_k$, we have $y(v_6 s_k v_4 v_6 v_3 v_1) > 0$. Assume $y(v_6 s_k v_4 v_6 v_3 v_1)$ is not integral. Let us show that $\nu^*_{\chi}(T)$ is an integer.

For this purpose, let $x$ be an optimal solution to $P(T, w)$. Since both $y(v_6 s_k v_4 v_6)$ and $y(v_1 s_i v_4 v_6 v_3 v_1)$ are positive, by Lemma 3.1(i), we have $x(v_6 s_k v_4 v_6) = x(v_1 s_i v_4 v_6 v_3 v_1) = 1$. By Lemma 3.1(ii), we obtain $x(v_6 s_k) = 0$. Hence $x(s_i v_4) + x(v_6 s_k) + x(v_4 v_1) + x(v_1 s_i) + x(s_i v_4) = 1 + 0 + 0 + 0 + 0 = 1$. Since $w(v_3 v_4) = 0$ and $v_6 s_k$ is outside $C^*_k$, for any $u \in V \setminus (V(T_2) \setminus a_2)$, if a cycle in $C^*_k$ contains $u v_6$, then it passes through $v_6 v_3 s_i v_4 v_1$. Moreover, if a cycle in $C^*_k$ contains $u s_k$, then it passes through $v_6 s_k$. By Lemma 3.1(iv), we obtain $x(u v_6) + x(v_6 v_3) + x(v_4 v_1) + x(v_1 s_i) + x(s_i v_4) + x(u s_k) + x(s_k v_4) = 0$. Hence $x(v_6 v_3) = x(u s_k)$. Clearly, we may assume that this equality holds in any
other situation. Let $T' = (V', A')$ be obtained from $T$ by deleting $s_k$, and let $w'$ be obtained from the restriction of $w$ to $A'$ by replacing $w(e)$ with $w(e) + w(v_4s_k)$ for $e \in \{v_4v_3, v_3v_1, v_1s_1, s_1v_4\}$ and replacing $w(uw_b)$ with $w(uw_b) + w(us_k)$ for any $u \in V \setminus (V(T_2) \setminus \{a_2\})$. Let $x'$ be the restriction of $x$ to $A'$, and let $y'$ be obtained from $y$ as follows: set $y'(v_1s_1v_6v_3v_1) = y(v_1s_1v_4v_3v_1) + y(v_6s_kv_4v_6)$; for each $C \in C_0'$ passing through $us_kv_4$, let $C'$ arise from $C$ by replacing the path $us_kv_4$ with the path $uv_kv_3v_1s_1v_4$, and set $y'(C') = y(C) + y(C)$. From the LP-duality theorem, we see that $x'$ and $y'$ are optimal solutions to $P(T', w')$ and $D(T', w')$, respectively, with the same value $\nu_w^*(T)$ as $x$ and $y$. By the hypothesis of Theorem 1.5, $\nu_w^*(T)$ is an integer.

• Either $k$ does not exist or $i = k$. In this case, by (3), we see that $j$ exists; that is, $v_3 \in \varphi(s_j)$. Similar to the above case, we can show that either $y(v_3s_jv_4v_3v_1)$ is a positive integer or $\nu_w^*(T)$ is an integer. Since the proof goes along the same line (with $v_3s_j$ and $y(v_3s_jv_4v_3v_1)$ in place of $v_6s_k$ and $y(v_6s_kv_4v_6)$, respectively), we omit the details here. Hence we may assume that (13) holds.

(14) If $v_3 \in \varphi(s_j)$ for some $j \in \{1, 2, 3\}$, then $y(v_1v_6v_3s_jv_4v_1) = 0$.

Assume the contrary: $y(v_1v_6v_3s_jv_4v_1) > 0$. Then $v_3v_1, v_3v_4$, and $v_4v_6$ are saturated by $y$ in $T_2$ by Lemma 3.5(v). Let $i$ and $k$ be subscripts in $\{1, 2, 3\}$, if any, such that $v_1 \in \varphi(s_i)$ and $v_4 \in \varphi(s_k)$ (possibly $i = k$). Since $v_1v_6v_3s_jv_4v_1 \cup v_3v_1v_6v_3 = v_1v_6v_3v_1 \cup v_3s_jv_6v_3$, and $v_1v_6v_3s_jv_1 \cup v_6s_kv_4v_6 = v_1v_6s_kv_1 \cup v_3s_jv_6v_3$, from the optimality of $y$, we deduce that $y(v_2v_4v_6v_3) = y(v_6s_kv_4v_6) = 0$. So $y(v_1v_6v_3v_1) = w(v_3v_1), y(v_1v_6v_3v_4v_1) = w(v_4v_3)$, and $y(v_3s_jv_4v_3v_1) = w(v_4v_6)$. Clearly, we may assume that $w(v_3v_1) = w(v_4v_3) = w(v_4v_6) = 0$, otherwise (1) holds. By (3), we have $\{i, k\} \neq \{j\}$. Let us show that one of $y(v_1s_iv_4v_3), y(v_1v_6s_kv_4v_1)$, and $y(v_1v_6v_3s_jv_4v_1)$ is a positive integer or $\nu_w^*(T)$ is an integer. We proceed by considering two cases.

• $i$ exists and $i \neq j$. In this case, observe first that $v_1s_i$ is not saturated by $y$ in $T_2$, for otherwise, $y(v_1s_iv_4v_1) = w(v_1s_i) > 0$ and hence (1) holds. Next, $s_i v_4$ is not saturated by $y$ in $T_2$, for otherwise, if $i \neq k$, then $y(v_1s_iv_4v_1) = w(s_i v_4) > 0$; if $i = k$, then $y(v_1s_iv_4v_1) + y(v_1v_6s_kv_4) = w(s_i v_4) > 0$, and $y(v_1s_iv_4v_1) = w(v_1s_i) > 0$ by Lemma 3.5(v) provided $y(v_1v_6s_kv_4v_1) > 0$. So $y(v_1s_iv_4v_1)$ is a positive integer, and hence (1) also holds. Moreover, both $v_1s_i$ and $v_6s_k$ are outside $C_0'$, for otherwise, let $C_1$ (resp. $C_2$) be a cycle in $C_0'$ containing $v_1s_i$ (resp. $v_6s_k$). Since $C_1 \cup v_1v_6v_3s_jv_4v_1 = v_1v_6v_3v_1 \cup C_1'$ and $C_2 \cup v_1v_6v_3s_jv_4v_1 = v_1v_6s_kv_1 \cup C_2'$, where $C_1' = C_1[v_4, v_1] \cup \{v_1v_6, v_3v_3, v_3s_j, s_1v_4\}$ and $C_2' = C_2[v_4, v_6] \cup \{v_6v_3, v_3s_j, s_1v_4\}$, by Lemma 3.5(viii), we have $y(C_1) = 0$ for $i = 1, 2$, a contradiction. It follows that $v_1s_i$ is not saturated by $y$ in $T$ and $s_i v_4$ is contained in some cycle in $C_0'$. By Lemma 3.5(viii), $v_4v_3$ is saturated by $y$ in $T_2$, so $y(v_1s_iv_4v_1) + y(v_1v_6s_kv_4v_1) + y(v_1v_6v_3s_jv_4v_1) = w(v_4v_3)$. If $i = k$ and $y(v_1v_6s_kv_4v_1) > 0$, then $v_1s_i$ is saturated by $y$ in $T_2$ by Lemma 3.5(v), a contradiction. So either $i \neq k$ or $i = k$ and $y(v_1v_6s_kv_4v_1) = 0$. Since $w(v_1s_i) > 0$ and $v_1s_i$ is outside $C_0'$, we have $y(v_1v_6s_kv_4v_1) > 0$. Assume $y(v_1v_6s_kv_4v_1)$ is not an integer.

For this purpose, let $x$ be an optimal solution to $P(T, w)$. Since both $y(v_1s_iv_4v_3v_1)$ and $y(v_1v_6v_3s_jv_4v_1)$ are positive, by Lemma 3.1(i), we have $x(v_1s_iv_4v_1) = y(v_1v_6v_3s_jv_4v_1) = 1$. By Lemma 3.1(ii), we obtain $x(v_1s_i) = 0$. Hence $x(s_i v_4) = x(v_1v_6) + x(v_6v_3) + x(v_3s_j) + x(s_jv_4)$. Since $w(v_3v_1) = w(v_4v_3) = 0$, for any $u \in V \setminus (V(T_2) \setminus \{a_2\})$, if a cycle in $C_0'$ contains $uv_k$, then it passes through $v_6s_kv_4$. Moreover, if a cycle in $C_0'$ contains $us_i$, then it passes through $s_i v_4$. By Lemma 3.1(iv), we obtain $x(v_1s_i) + x(v_1v_6) + x(v_6v_3) + x(v_3s_j) + x(s_jv_4) = x(us_i) + x(s_jv_4)$. Hence $x(uv_1) = x(us_i)$. Clearly, we may assume that this equality holds in any other situation.

41
Let $T' = (V', A')$ be obtained from $T$ by deleting $s_i$, and let $w'$ be obtained from the restriction of $w$ to $A'$ by replacing $w(e)$ with $w(e) + w(v_{is_i})$ for $e \in \{v_{1s_i}, v_{6s_i}, v_{2s_i}, s_is_i, v_{3s_i}, v_{4s_i}\}$ and replacing $w(uv_1)$ with $w(uv_1) + w(uv_1)$ for any $u \in V \setminus (V(T_2) \setminus a_2)$. Let $x'$ be the restriction of $x$ to $A'$ and let $y'$ be obtained from $y$ as follows: set $y'(v_{1v_6v_3s_is_1v_4}) = y(v_{1v_6v_3s_is_1v_4} + y(v_{1s_is_1v_4})$; for each $C \in C_0^y$ passing through $uv_1$, let $C'$ be obtained from $C$ by replacing the path $uv_1$ with the path $w(uv_1v_6v_3s_is_1v_4)$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $x'$ and $y'$ are optimal solutions to $\mathbb{P}(T', w')$ and $\mathbb{D}(T', w')$, respectively, with the same value $\nu_{w'}(T)$ as $x$ and $y$. By the hypothesis of Theorem 1.5, $\nu_{w'}(T)$ is an integer.

- Either $i$ does not exist or $i = j$. In this case, by (3), we see that $k$ exists; that is, $v_k \in \varphi(s_k)$.

Similar to the above case, we can show that either $y(v_{1v_6s_4v_1v_4})$ is a positive integer or $\nu_{w'}(T)$ is an integer. Since the proof goes along the same line (with $v_6s_k$ and $y(v_{1v_6s_4v_1v_4})$ in place of $v_1s_1$ and $y(v_1s_1v_4)$, respectively), we omit the details here. Hence we may assume that (14) holds.

We proceed by considering two cases, depending on whether $\varphi(s_i) = \{v_1\}$ for some $i$.

**Case 1.** $\varphi(s_i) = \{v_1\}$ for some $i \in \{1, 2, 3\}$.

By Lemma 5.2(i), we may assume that $\varphi(s_1) = \{v_1\}$. Let $j$ and $k$ be subscripts in $\{1, 2, 3\}$, if such, that $v_j \in \varphi(s_j)$ and $v_k \in \varphi(s_k)$ (possibly $j = k$). By (13) and (14), we have

$$(15) C_2^y \subseteq \{v_{1v_6v_3v_4}, v_{1v_6s_4v_1v_4}, v_3v_4v_{1v_6v_3}, v_3v_1v_4, v_6s_4v_1v_4, v_{1v_6s_4v_1v_4}, v_6v_3v_4v_1\}.$$  

Observe that neither $s_1v_4$ nor $v_1s_1$ is saturated by $y$ in $T_2$, for otherwise, $y(v_1s_1v_4v_1) = w(s_1v_4)$ or $w(v_1s_1)$; both of them are positive, so (1) holds. By Lemma 5.2(iii), $z(s_1v_4) = w(z_1v_4) > 0$. Thus there exists a cycle $C \in C_0^y$ containing $s_1v_4$; subject to this, $C$ is chosen to contain $v_1s_1$ if possible. If $v_1s_1$ is outside $C$, then $v_1s_1$ is not saturated by $y$ in $T$. By Lemma 3.5(vii), $v_1s_1$ is saturated by $y$ in $T_2$ and hence $y(v_1s_1v_4v_1) + y(v_1v_6s_4v_1v_4) + y(v_1v_6s_4v_1v_4) = w(v_1s_1v_4v_1)$.

$$(16)$$  

If $w(v_1s_1v_4v_1) > 0$, then either $y(v_1s_1v_4v_1)$ is a positive integer or $\nu_{w'}(T)$ is an integer.

To justify this, assume $y(v_1s_1v_4v_1) = 1$ by Lemma 3.1(iii), we have $x(v_1s_1v_4v_1) = x(v_1v_6v_3v_4v_1) = 1$. By Lemma 3.1(ii), we obtain $x(v_1s_1v_4v_1) = 0$, because $v_1s_1$ is not saturated by $y$. It follows that $x(s_1v_4v_1) = x(v_1v_6v_3v_4v_1) + x(v_6v_3v_4) + x(v_3v_4)$.

Observe that there is no cycle $D$ in $C_0^y$ that contains the path $v_1v_6s_4v_1$, for otherwise, let $\theta = \min\{y(D), y(v_1v_6v_3v_4v_1)\}$, let $D' = D[v_1v_4, v_1] \setminus \{v_1v_6, v_6v_3, v_3v_4\}$, and let $y'$ be obtained from $y$ by replacing $y(D)$, $y(D')$, $y(v_1v_6v_3v_4v_1)$, and $y(v_1v_6s_4v_1v_4)$ with $y(D) - \theta$, $y(D') - \theta$, $y(v_1v_6v_3v_4v_1) - \theta$, and $y(v_1v_6s_4v_1v_4) + \theta$, respectively. Then $y'$ is also an optimal solution to $\mathbb{P}(T, w)$ with $y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1)$, contradicting (7). For any $u \in V \setminus (V(T_2) \setminus a_2)$, if a cycle in $C_0^y$ contains $v_1s_1$, then it passes through $v_6v_3v_4v_1$. Moreover, if a cycle in $C_0^y$ contains $s_1v_4$, then it passes through $s_1v_4$. By Lemma 3.1(iv), we obtain $x(uv_1)$ +
\[ x(v_1v_b) + x(v_6v_3) + x(v_3v_4) = x(u_1) + x(s_1v_4). \] Hence \( x(uv_1) = x(us_1). \) Clearly, we may assume that this equality holds in any other situation. Let \( T' = (V', A') \) be obtained from \( T \) by deleting \( s_1 \), and let \( u' \) be obtained from the restriction of \( u \) to \( A' \) by replacing \( w(e) \) with \( w(e) + w(s_1v_4) \) for \( e \in \{ v_1v_6, v_6v_3, v_3v_4 \} \) and replacing \( w(uv_1) \) with \( w(uv_1) + w(us_1) \) for any \( u \in V \setminus (V(T_2) \setminus a_2) \). Let \( x' \) be the restriction of \( x \) to \( A' \), and let \( y' \) be obtained from \( y \) as follows: set \( y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1) + y(v_1s_1v_4v_1); \) for each \( C \in C_0' \) passing through \( us_1v_4 \), let \( C' \) be obtained from \( C \) by replacing the path \( us_1v_4 \) with the path \( uv_1v_6v_3v_4 \), and set \( y'(C') = y(C') + y(C). \) From the LP-duality theorem, we see that \( x' \) and \( y' \) are optimal solutions to \( \mathcal{P}(T', u') \) and \( \mathcal{D}(T', u') \), respectively, with the same value \( \nu_u'(T) \) as \( x \) and \( y \). By the hypothesis of Theorem 1.5, \( \nu_u'(T) \) is an integer. So (16) follows.

By (16) and Lemma 3.2(iii), we may assume that \( u'(v_4v_1) = 0 \) hereafter.

(17) If \( k \) exists (so \( v_6 \in \varphi(s_k) \)) and \( u'(v_4v_6) > 0 \), then either \( y'(v_6s_kv_4v_6) \) is a positive integer or \( \nu_u'(T) \) is an integer.

To justify this, observe first that \( v_6s_k \) is not saturated by \( y \) in \( T_2 \), for otherwise, \( y(v_6s_kv_4v_6) = w(v_6v_6) > 0 \), so (17) holds. Next, \( s_kv_4 \) is not saturated by \( y \) in \( T_2 \), for otherwise, if \( j \neq k \), then \( y'(v_6s_kv_4v_6) = w(s_kv_4) > 0 \); if \( j = k \), then \( y(v_6s_kv_4v_6) + y(v_3s_kv_4v_6) = w(s_kv_4) \), and \( y(v_6s_kv_4v_6) = w(s_kv_4) > 0 \) by Lemma 3.5(v) provided \( y(v_3s_kv_4v_6v_3) > 0 \), so (17) also holds. By Lemma 5.2(iii), \( s_kv_4 \) is saturated by \( y \) in \( T \), so \( s_kv_4 \) is contained in some cycle \( C \in C_0' \), subject to this, \( C \) is chosen to contain \( v_6s_k \) if possible. Clearly, if \( v_6s_k \) is not on \( C \), then \( v_6s_k \) is not saturated by \( y \) in \( T \). By Lemma 3.5(vii), \( v_6v_6 \) is saturated by \( y \) in \( T_2 \), and hence \( y(v_6s_kv_4v_6) + y(v_3s_kv_4v_6v_3) + y(v_3s_kv_4v_6v_3) = w(v_6v_6) \).

Assume \( y'(v_6s_kv_4v_6) \) is a positive integer. Then at least one of \( y'(v_3v_4v_6v_3) \) and \( y'(v_3s_kv_4v_6v_3) \) is positive, say the former. Note that \( v_6s_k \) is outside \( C_0' \), for otherwise, let \( D \) be a cycle in \( C_0' \) containing \( v_6s_k \). Set \( D' = D[v_4, v_6] \cup \{ v_6v_3, v_3v_4 \} \) and \( \theta = \min\{y'(v_3v_4v_6v_3), y(C)\} \). Let \( y' \) be obtained from \( y \) by replacing \( y'(v_3v_4v_6v_3), y(v_6s_kv_4v_6), y(C), \) and \( y'(C') \) with \( y(v_3v_4v_6v_3) - \theta, y(v_6s_kv_4v_6) + \theta, y(C) - \theta, \) and \( y'(C') + \theta \), respectively. Then \( y' \) is also an optimal solution to \( \mathcal{D}(T, u) \) with \( y'(v_3v_4v_6v_3) < y'(v_3v_4v_6v_3) \), contradicting (8). Since \( w'(v_6s_k) > 0 \), we have \( y'(v_6s_kv_4v_6) > 0 \). As \( y'(v_6s_kv_4v_6) \) is not integral, \( y'(v_3v_4v_6v_3) \) or \( y'(v_3s_kv_4v_6v_3) \) is not integral. If \( y'(v_3s_kv_4v_6v_3) > 0 \), then \( v_3v_4 \) is saturated by \( y \) in \( T_2 \) by Lemma 3.5(v), so \( y(v_3v_4v_6v_3) = w(v_3v_4) \). Hence we may assume that exactly one of \( y(v_3v_4v_6v_3) \) and \( y(v_3s_kv_4v_6v_3) \) is positive. Let us show that \( \nu_u'(T) \) is an integer.

We only consider the case when \( y'(v_3v_4v_6v_3) \) is not integral, because the proof in the other case when \( y'(v_3v_4v_6v_3) = 0 \) and \( y'(v_3s_kv_4v_6v_3) > 0 \) goes along the same line.

Let \( x' \) be an optimal solution to \( \mathcal{P}(T, u) \). Since both \( y'(v_6s_kv_4v_6) \) and \( y'(v_3v_4v_6v_3) \) are positive, we have \( x(v_6s_kv_4v_6) = x(v_3v_4v_6v_3) = 1 \) by Lemma 3.1(i). Since \( v_6s_k \) is not saturated by \( y \) in \( T \), we obtain \( x(v_6s_kv_4v_6) = 0 \) by Lemma 3.1(ii). It follows that \( x(v_6s_kv_4v_6) = x(v_6v_6) + x(v_3v_4v_6v_3j) \). For any \( u \in V \setminus (V(T_2) \setminus a_2) \), if a cycle in \( C_0' \) contains \( u_6v_6 \), then it passes through \( v_6v_3v_4v_6 \). Moreover, if a cycle in \( C_0' \) contains \( u_6v_6 \), then it passes through \( s_kv_4v_6v_3v_4 \) to \( v_3v_4v_6v_3v_4 \). By Lemma 3.1(iv), we obtain \( x(uv_6) + x(v_6v_3) + x(v_3v_4v_6v_3v_4) = x(u_6v_6) \). Hence \( x(uv_6) = x(u_6v_6) \). Clearly, we may assume that this equality holds in any other situation. Let \( T' = (V', A') \) be obtained from \( T \) by deleting \( s_k \), and let \( u' \) be obtained from the restriction of \( u \) to \( A' \) by replacing \( w(e) \) with \( w(e) + w(s_kv_4v_6v_3) \) for \( e = v_6v_3v_4v_6 \) and \( v_3v_4v_6v_3v_4 \), and replacing \( w(uv_6) \) with \( w(uv_6) + w(u_6v_6) \) for any \( u \in V \setminus (V(T_2) \setminus a_2) \). Let \( x' \) be the restriction of \( x \) to \( A' \), and let \( y' \) be obtained from \( y \) as follows: set \( y'(v_3v_4v_6v_3v_4) = y'(v_3v_4v_6v_3v_4) + y(v_6s_kv_4v_6v_6) \); for each \( C \in C_0' \) passing through \( u_6v_6 \), let \( C' \) be the cycle arising
from $C$ by replacing the path $u_s v_4$ with the path $u w v_3 v_4$, and set $y'(C') = y(C') + y(C)$. From the LP-duality theorem, we see that $x'$ and $y'$ are optimal solutions to $\mathbb{P}(T', w')$ and $\mathbb{D}(T', w')$, respectively, with the same value $\nu_w'(T)$ as $x$ and $y$. By the hypothesis of Theorem 1.5, $\nu_w'(T)$ is an integer. So (17) holds.

By (17) and Lemma 3.2(iii), we may assume that if $w(v_4 v_6) > 0$, then $k$ does not exist, and hence $j$ exists (so $v_3 \in \varphi(s_j)$) by (3).

(18) If $w(v_4 v_6) > 0$, then at least one of $y(v_1 v_6 v_3 v_1), y(v_3 v_4 v_6 v_3)$, and $y(v_3 s_j v_4 v_6 v_3)$ is a positive integer.

To justify this, note that neither $s_j v_4$ nor $v_3 s_j$ is saturated by $y$ in $T_2$, for otherwise, $y(v_3 s_j v_4 v_6 v_3) = w(s_j v_4)$ or $w(v_3 s_j)$; both of them are positive, so (18) holds. By Lemma 5.2(iii), $s_j v_4$ is saturated by $y$ in $T$, so $s_j v_4$ is contained in a cycle $C \in C_0$; subject to this, $C$ is chosen to contain $v_3 s_j$ if possible. Clearly, if $v_3 s_j$ is not on $C$, then $v_3 s_j$ is not saturated by $y$ in $T$. By Lemma 3.5(iii), at least one of $v_4 v_6$ and $v_6 v_3$ is saturated by $y$ in $T_2$. Furthermore, by Lemma 3.5(iv), if $v_6 v_3$ is contained in some cycle in $C_0$, then $v_4 v_6$ is saturated by $y$ in $T_2$. If $v_4 v_6$ is saturated by $y$ in $T_2$, then $y(v_3 v_4 v_6 v_3) + y(v_3 s_j v_4 v_6 v_3) = w(v_4 v_6)$, and $y(v_3 v_4 v_6 v_3) = w(v_4 v_6)$ by Lemma 3.5(v) provided $y(v_3 s_j v_4 v_6 v_3) > 0$. So at least one of $y(v_3 v_4 v_6 v_3)$ and $y(v_3 s_j v_4 v_6 v_3)$ is a positive integer, and hence (18) holds. Thus we may assume that $v_4 v_6$ is not saturated by $y$ in $T_2$, which implies that $v_6 v_3$ saturated by $y$ in $T_2$. It follows that $y(v_1 v_6 v_3 v_1) + y(v_3 v_4 v_6 v_3) + y(v_3 s_j v_4 v_6 v_3) = w(v_6 v_3)$. If $w(v_6 v_3) = 0$, then $K = \{v_1 v_6, v_6 v_3, v_6 s_j\}$ is an FAS of $T$ with total weight zero, so $\tau_w(T_2 \backslash a_2) = 0$, contradicting the hypothesis (a) of this section. Therefore $w(v_6 v_3) > 0$. If $y(v_3 s_j v_4 v_6 v_3) > 0$, then $y(v_3 v_4 v_6 v_3) = w(v_3 v_4)$ by (15) and Lemma 3.5(v). So we may further assume that exactly one of $y(v_3 v_4 v_6 v_3)$ and $y(v_3 s_j v_4 v_6 v_3)$ is positive, and thus $y(v_1 v_6 v_3 v_1) > 0$.

Let us show that $y(v_1 v_6 v_3 v_1)$ is an integer. Suppose not. Then $y(v_3 v_4 v_6 v_3)$ or $y(v_3 s_j v_4 v_6 v_3)$ is not integral, say the former (the proof in the other case goes along the same line). Since $v_6 v_3$ is saturated by $y$ in $T_2$ and $w(v_6 v_3) = 0$, the arc $v_1 v_6$ is outside $C_0$. If $v_3 v_4$ is also outside $C_0$, let $y'$ be obtained from $y$ by replacing $y(v_3 v_4 v_6 v_3)$ or $y(v_1 v_6 v_3 v_1)$ with $y(v_3 v_4 v_6 v_3) - \theta$ and $y(v_1 v_6 v_3 v_1) + \theta$, respectively, where $\theta = \min(w(v_1 v_6), -z(v_1 v_6), w(v_1 v_6) - z(v_1 v_6), y(v_3 v_4 v_6 v_3))$: if $v_3 v_4$ is contained in some cycle $C \in C_0$, let $y'$ be obtained from $y$ by replacing $y(v_3 v_4 v_6 v_3)$ and $y(v_1 v_6 v_3 v_1)$ with $y(v_3 v_4 v_6 v_3) - \sigma$, $y(v_1 v_6 v_3 v_1) + \sigma$, $y(C) - \sigma$, $y(C') + \sigma$, respectively, where $C' = C[v_4, v_3] \cup \{v_3 v_1\}$ and $\sigma = \min(w(v_1 v_6) - z(v_1 v_6), y(C), y(y(v_3 v_4 v_6 v_3)))$. It is easy to see that in either situation $y'$ is also an optimal solution to $\mathbb{D}(T, w)$ with $y'(v_3 v_4 v_6 v_3) < y(v_3 v_4 v_6 v_3)$, contradicting (8). This proves (18).

By (16)-(18), we may assume that $w(v_4 v_1) = w(v_4 v_6) = 0$. Since each of $\{v_4 v_1, v_4 v_6, v_1 v_6\}, \{v_4 v_1, v_4 v_6, v_3 v_1\}$ and $\{v_4 v_1, v_4 v_6, v_3 v_1\}$ is a minimal FAS of $T_2 \backslash a_2$,

$$\epsilon = \min\{w(v_1 v_6), w(v_6 v_3), w(v_3 v_1)\} > 0$$

by the hypothesis (a) of this section. By Lemma 3.5(vii), we obtain $y(v_1 v_6 v_3 v_1) = \epsilon > 0$. Thus (1) is established in the present case.

**Case 2.** $\varphi(s_i) \neq \{v_i\}$ for any $i = 1, 2, 3$.

By the hypothesis of the present case, we may assume that $v_6 \in \varphi(s_1)$, $v_3 \in \varphi(s_2)$, and $v_1 \in \varphi(s_i)$ for $i = 1$ or 2. By (13) and (14), we have

(19) $C_2 \subseteq \{v_1 v_6 v_3 v_1, v_3 v_4 v_6 v_3, v_1 v_6 v_3 v_4 v_1, v_6 v_3 v_4 v_6, v_1 v_6 v_3 v_4 v_1, v_3 s_j v_4 v_6 v_3, v_1 s_j v_4 v_1, v_1 s_j v_4 v_1\}$
and $y(v_1 s_i v_i v_1) = 0$ for $i = 1$ or 2.

Claim 1. $y(C_2) = \tau_w(T_2 \setminus a_2)$.

To justify this, note that $z(s_i v_i) = w(s_i v_i) > 0$ for $i = 1$ and 2 by Lemma 5.2(iii). Depending on the saturation of $s_1 v_1$ and $s_2 v_1$, we distinguish among three subcases.

Subcase 1.1. $s_1 v_4$ is contained in some cycle $C \in C'_0$. In this subcase, $v_4 v_6$ is saturated by $y$ in $T_2$, for otherwise, $v_4 v_6$ is not saturated by $y$ in $T$, because it is outside $C'_0$. By Lemma 3.5(iii), $v_6 s_1$ is saturated by $y$ in $T_2$. By (11), we have $y(v_1 v_6 s_1 v_4 v_1) = 0$, which together with (19) implies $y(v_6 s_1 v_4 v_6) = w(v_6 s_1) > 0$, so (1) holds. Clearly, $v_4 v_1$ is outside $C'_0$. We proceed by considering two subsubcases.

Assume first that $v_4 v_1$ is not saturated by $y$ in $T_2$ (and hence in $T$). Then, by Lemma 3.5(iii), $v_1 s_1$ and at least one of $v_1 s_2$ and $s_2 v_4$ are saturated by $y$ in $T_2$. Furthermore, $v_1 s_2$ is outside $C'_0$. If $v_1 s_2$ is not saturated by $y$ in $T$, then $y(v_3 s_2 v_4 v_6 v_3) = 0$, for otherwise, let $y'$ be obtained from $y$ by replacing $y(v_3 s_2 v_4 v_6 v_3)$ with $y(v_1 s_2 v_4 v_1) + \theta$ and $y(v_3 s_2 v_4 v_6 v_3) - \theta$, where $\theta = \min\{w(v_4 v_1) - z(v_1 v_4), w(v_1 s_2) - z(v_1 s_2), y(v_3 s_2 v_4 v_6 v_3)\} > 0$. Then $y'$ is also an optimal solution to $D(T, w)$, contradicting (6). It follows from (19) that $y(v_1 s_2 v_4 v_1) = w(s_2 v_4) > 0$, so (1) holds. Thus we may assume that $v_1 s_2$ is saturated by $y$ in $T_2$. If $v_1 v_6$ is saturated by $y$ in $T_2$, then $y(C_2) = w(K)$, where $K = \{v_1 v_6, v_6 s_1, v_1 s_1, v_1 s_2\}$. By (9), $K$ is an MFAS of $T_2 \setminus a_2$ and hence $y(C_2) = \tau_w(T_2 \setminus a_2)$. By Lemma 3.5(iii), $v_6 s_1$ is outside $C'_0$, for otherwise, $v_6 v_1$ would be saturated by $y$ in $T_2$, a contradiction. So we may assume that $v_1 v_6$ is not saturated by $y$ in $T$. By Lemma 3.5(iii), $v_6 s_1$ is saturated by $y$ in $T_2$. If $v_6 v_3$ is also saturated by $y$ in $T_2$, then $y(C_2) = w(K)$, where $K = \{v_6 v_3, v_6 s_1, v_1 s_1, v_1 s_2\}$. So we assume that $v_6 v_3$ is not saturated by $y$ in $T_2$. By Lemma 3.5(iii), $v_6 v_3$ is outside $C'_0$. Furthermore, $v_3 v_1, v_3 s_2$, and $v_3 v_4$ are all saturated by $y$ in $T_2$. So $y(C_2) = w(J)$, where $J = \{v_3 v_1, v_3 v_4, v_6 s_1, v_1 s_1, v_1 s_2, v_3 s_2\}$. By (9), $J$ is an MFAS of $T_2 \setminus a_2$ and hence $y(C_2) = \tau_w(T_2 \setminus a_2)$.

Next assume that $v_4 v_1$ is saturated by $y$ in $T_2$. We may assume that $v_3 v_1$ is not saturated by $y$ in $T_2$, for otherwise, $y(C_2) = w(K)$, where $K = \{v_3 v_1, v_4 v_1, v_4 v_6\}$. By (9), $K$ is an MFAS of $T_2 \setminus a_2$ and hence $y(C_2) = \tau_w(T_2 \setminus a_2)$. Thus, by (10), we have $y(v_1 v_6 v_3 v_4 v_1) = 0$. If $y(v_1 v_6 s_1 v_4 v_1) = 0$ and $v_1 v_6$ is saturated by $y$ in $T_2$, then $y(C_2) = w(K)$, where $K = \{v_1 v_6, v_6 v_3, v_1 s_1, v_1 s_2\}$. So we may assume that $y(v_1 v_6 s_1 v_4 v_1) > 0$ or $v_1 v_6$ is not saturated by $y$ in $T_2$. Consider the situation when $y(v_1 v_6 s_1 v_4 v_1) > 0$. Now, by (11), $v_1 s_1$ is saturated by $y$ in $T_2$, and $y(v_3 v_4 v_6 v_3) = y(v_3 s_2 v_4 v_6 v_3) = 0$. Moreover, at least one of $v_1 s_2$ and $s_2 v_4$ is saturated by $y$ in $T_2$ (otherwise, $y(v_1 s_2 v_4 v_1)$ can be made larger). If $v_1 v_6$ is saturated by $y$ in $T_2$, then $y(C_2) = w(K)$, where $K = \{v_1 v_6, v_6 v_3, v_1 s_1, v_1 s_2, v_3 v_4\}$ or $y(v_1 v_6 v_3 v_4 v_1, v_1 s_1, v_1 s_2, v_3 v_4)$; if $v_1 v_6$ is not saturated by $y$ in $T_2$, then $v_1 v_6$ is saturated by $y$ in $T_2$ and hence $y(C_2) = \tau_w(T_2 \setminus a_2)$. So we may assume that $y(v_1 v_6 s_1 v_4 v_1) = 0$ and $v_1 v_6$ is not saturated by $y$ in $T_2$. By Lemma 3.5(vii), $v_6 v_3$ is saturated by $y$ in $T_2$. If $v_6 s_1$ is also saturated by $y$ in $T_2$, then $y(C_2) = w(K)$, where $K = \{v_4 v_1, v_6 s_1, v_6 v_3\}$. So we further assume that $v_6 s_1$ is not saturated by $y$ in $T_2$. We propose to show that

(20) $y(v_3 v_4 v_6 v_3) = y(v_3 s_2 v_4 v_6 v_3) = 0$.

We only prove that $y(v_3 s_2 v_4 v_6 v_3) = 0$, as the proof of the other equality $y(v_3 v_4 v_6 v_3) = 0$ goes along the same line. Assume the contrary: $y(v_3 s_2 v_4 v_6 v_3) > 0$. Depending on the saturation of $v_1 v_6$ and $v_3 v_1$, we consider several possibilities.

• Both $v_1 v_6$ and $v_3 v_1$ are not saturated by $y$ in $T$. Define $\theta = \min\{w(v_1 v_6) - z(v_1 v_6), w(v_3 v_1) - z(v_3 v_1), y(v_3 s_2 v_4 v_6 v_3)\}$. Then $\theta > 0$. Let $y'$ be obtained from $y$ by replacing $y(v_3 s_2 v_4 v_6 v_3)$ and
Dyv and the unsaturated arcs y that C one of C. Since v6v3 is saturated by y in T2, cycle C passes through v6v1v4. Thus the multiset sum of the cycles C, v3v2v4v6 and the unsaturated arc v3f1 contains arc-disjoint cycles v6v1v4v6 and v1v6v3v1. From Lemma 3.5(vi) we deduce that y(v3s2v4v6v3) = 0, a contradiction.

\bullet v1v6 is not saturated by y in T and v3v1 is contained in some cycle D ∈ C′0. It is clear that D passes through v1v6v4 for i = 1 or 2. Furthermore, the multiset sum of D, v3s2v4v6v3, and the unsaturated arc v1v6 contains arc-disjoint cycles v1v6v3v1 and D′ = D[v4, v3] ∪ {v3s2, s2v4}. Define \( \theta = \min \{y(D), y(v3s2v4v6v3), w(v1v6) - z(v1v6)\} \). Let y′ be obtained from y by replacing y(D), y(D′), y(v3s2v4v6v3), and y(v1v6v3v1) with y(D) − \( \theta \), y(D′) + \( \theta \), y(v3s2v4v6v3) − \( \theta \), and y(v1v6v3v1) + \( \theta \), respectively. Then y′ is also an optimal solution to \( \mathbb{D}(T, w) \) with y′(v3s2v4v6v3) < y(v3s2v4v6v3), contradicting (6).

\bullet v1v6 and v3v1 are contained in some cycles C and D in C′0, respectively. If v3v1 is on C, then the multiset sum of C and v3s2v4v6v3 contains arc-disjoint cycles v1v6v3v1, v6v1v4v6, and C′ = C[v4, v3] ∪ {v3s2, s2v4}; if v3v1 is outside C, then the multiset sum of C, D, and v3s2v4v6v3 contains arc-disjoint cycles v1v6v3v1, v6v1v4v6, C′ = C[v4, v3] ∪ {v1s1, v1s2} for i = 1 or 2, and D′ = D[v4, v3] ∪ {v3s2, s2v4}. In either situation from the optimality of y we deduce that y(v3s2v4v6v3) = 0.

Combining the above observations, we see that (20) holds. Thus y(C2) = w(K), where K = {s1v4v1, v4v6, v6v3}. By (9), K is an MFAS of T2 \ A2 and hence y(C2) = \( \tau_w(T2 \ A2) \).

**Subcase 1.2.** s1v4 is saturated by y in T2 and s2v4 is contained in some cycle C ∈ C′0, subject to this, C is chosen to contain v3s2 if possible. In this subcase, observe first that both v1s1 and v6s1 are outside C′0. Next, v3s2 is not saturated by y in T2, for otherwise, y(v3s2v4v6v3) = w(v3s2) > 0, so (1) holds. If both v3v6 and v1s2 are saturated by y in T2, then y(C2) = w(K), where K = {s1v4v1, v1s2, v6v3}. By (9), K is an MFAS of T2 \ A2 and hence y(C2) = \( \tau_w(T2 \ A2) \). We proceed by considering two subsubcases.

(a) v6v3 is not saturated by y in T2. Now v4v6 is saturated by y in T2 by Lemma 3.5(iii).

Assume first that v1v3 is not saturated by y in T. Then both v1v6 and v1s2 are saturated by y in T2 by Lemma 3.5(iii). If v1s1 is also saturated by y in T2, then y(C2) = w(K), where K = {v1v4, v4v6, v1s1, v1s2}; otherwise, v1s1 is not saturated by y in T. By (11), we have y(v1v6v1v4v1) = 0. Let us show that

\( \text{(21) } y(v6v1v4v1) = 0. \)

Indeed, if v6v3 is not saturated by y in T, then the multiset sum of the cycles C, v6v1v4v6, and the unsaturated arcs v1v1, v1s1, and v6v3 (or v3s2 if it is outside C) contains arc-disjoint cycles v1v6v1v4 and v3s2v4v6v3. Thus, by Lemma 3.5(vi), we have y(v6v1v4v6v3) = 0. If v6v3 is contained in some cycle C ∈ C′0, then C contains v3v4 or v3s2. Thus the multiset sum of cycles C, v6v1v4v6, and the unsaturated arcs v1v1 and v1s1 contains arc-disjoint cycles v1v6v1v4 and one of v3s2v4v6v3 and v3s2v4v6v3. Thus, by Lemma 3.5(vi), we have y(v6v1v4v6v3) = 0. This proves (21).

It follows from (19) and (21) that y(v1v6v1v4v1) = w(s1v4) > 0, so (1) holds. Thus we may assume that v1v4 is saturated by y in T (and hence in T2). Then we may further assume that v3v1 is not saturated by y in T2, for otherwise, y(C2) = w(K), where K = {v4v1, v4v6, v1v4}. Thus y(C2) = \( \tau_w(T2 \ A2) \). By Lemma 3.5(vii), v1v6 is saturated by y in T2 and hence, by (10),

46
we have \( y(v_1v_6v_3v_4v_1) = 0 \). Let us show that

\[(22) \ y(v_1v_6v_3v_4v_1) = 0.\]

To justify this, we consider four possibilities, depending on the saturation of \( v_6v_3 \) and \( v_3v_1 \).

- Both \( v_6v_3 \) and \( v_3v_1 \) are saturated by \( y \) in \( T \). Now define \( \theta = \min\{ w(v_6v_3) - z(v_6v_3), w(v_3v_1) - z(v_3v_1) \} \). Then \( \theta > 0 \). Let \( y' \) be obtained from \( y \) by replacing \( y(v_1v_6v_3v_1) \) and \( y(v_1v_6v_3v_4v_1) \) with \( \theta + y(v_1v_6v_3v_1) + \theta \) and \( y(v_1v_6v_3v_4v_1) - \theta \), respectively. Then \( y' \) is also an optimal solution to \( D(T, w) \) with \( y'(v_1v_6s_1v_4v_1) < y(v_1v_6s_1v_4v_1) \), contradicting (6).

- \( v_3v_1 \) is not saturated by \( y \) in \( T \) and \( v_6v_3 \) is contained in some cycle \( C \in C_0^y \). Now the multiset sum of the cycles \( C, v_1v_6s_1v_4v_1 \), and the unsaturated arc \( v_3v_1 \) contains arc-disjoint cycles \( v_1v_6v_3v_1 \) and \( C' = C[v_4, v_6] \cup \{ v_6s_1, s_1v_4 \} \). Define \( \theta = \min\{ w(v_3v_1) - z(v_3v_1) \} \). Then \( \theta > 0 \). Let \( y' \) be obtained from \( y \) by replacing \( y(v_1v_6s_1v_4v_1) \), \( y(v_1v_6v_3v_1) \), \( y(C) \), and \( y(C') \) with \( y(v_1v_6s_1v_4v_1) - \theta \), \( y(v_1v_6v_3v_1) + \theta \), \( y(C) - \theta \), and \( y(C') + \theta \), respectively. Then \( y' \) is also an optimal solution to \( D(T, w) \) with \( y'(v_1v_6s_1v_4v_1) < y(v_1v_6s_1v_4v_1) \), contradicting (6).

- \( v_6v_3 \) is not saturated by \( y \) in \( T \) and \( v_3v_1 \) is contained in some cycle \( D \in C_0^y \). Now \( D \) passes through \( v_1s_2v_4 \). Since the multiset sum of the cycles \( D, v_1v_6s_1v_4v_1 \), and the unsaturated arc \( v_6v_3 \) contains arc-disjoint cycles \( v_1v_6v_3v_1 \) and \( v_1s_2v_4 \), by Lemma 3.5(vi), we have \( y(v_1v_6s_1v_4v_1) = 0 \), a contradiction.

- \( v_6v_3 \) and \( v_3v_1 \) are contained in some cycles \( C \) and \( D \in C_0^y \), respectively. Now if \( v_3v_1 \) is on \( C \), then the multiset sum of the cycles \( C, v_1v_6s_1v_4v_1 \), and the unsaturated arc \( v_3v_1 \) contains arc-disjoint cycles \( v_1v_6v_3v_1 \), \( v_1s_2v_4 \), and \( C' = C[v_4, v_6] \cup \{ v_6s_1, s_1v_4 \} \); otherwise, the multiset sum of the cycles \( C, D \), and \( v_1v_6s_1v_4v_1 \) contains arc-disjoint cycles \( v_1v_6v_3v_1 \), \( v_1s_2v_4 \), and \( C' = C[v_4, v_6] \cup \{ v_6s_1, s_1v_4 \} \). Then \( y(C) = \tau(w(T), v_6v_3) = 0 \). In each situation from the optimality of \( y \) we deduce that \( y(v_1v_6s_1v_4v_1) = 0 \).

Combining the above observations, we see that (22) holds. Thus \( y(C_2) = w(K) \), where \( K = \{ v_4v_1, v_4v_6, v_6v_6 \} \). By (9), \( K \) is an MFAS of \( T_2 \backslash a_2 \) and hence \( y(C_2) = \tau(w(T_2 \backslash a_2)) \).

(b) \( v_4v_3 \) is saturated by \( y \) in \( T_2 \). Now \( v_1s_2 \) is not saturated by \( y \) in \( T_2 \). By Lemma 3.5(vii), \( v_4v_3 \) is saturated by \( y \) in \( T_2 \). Since \( z(v_1s_2) > 0 \), by Lemma 5.2(vii), we have \( z(v_1s_2) = 0 \). Furthermore, we may assume that \( y(v_1v_6v_3v_4v_1) = 0 \). Otherwise, both \( v_3v_1 \) and \( v_6v_3 \) are saturated by \( y \) in \( T_2 \) by (10). Hence \( y(C_2) = w(K) \), where \( K = \{ v_4v_1, v_4v_6, v_6v_3 \} \). If \( y(v_1v_6v_3v_4v_1) = 0 \), then \( y(C_2) = w(K) \), where \( K = \{ v_4v_1, v_6v_3, s_1v_4 \} \); if \( y(v_1v_6v_3v_4v_1) > 0 \) then, by (11), \( v_6v_3 \) is saturated by \( y \) in \( T_2 \) and either \( v_3v_1 \) is saturated by \( y \) in \( T_2 \) or \( y(v_3v_4v_1v_6) = y(v_3v_4v_1v_6) = 0 \). Thus \( y(C_2) = w(J) \), where \( J = \{ v_1v_4, v_4v_6, v_3v_1 \} \). Therefore \( y(C_2) = \tau(w(T_2 \backslash a_2)) \).

**Subcase 1.3.** \( s_1v_4 \) is saturated by \( y \) in \( T_2 \) for \( i = 1 \) and 2. In this subcase, since \( C_0^y \neq \emptyset \), \( v_3v_4 \) is contained in some cycle in \( C_0^y \). By (12), we have \( y(v_3v_2v_6v_3) = 0 \). Thus \( y(v_1s_2v_4v_1) = w(s_2v_4) > 0 \) and (10). This completes the proof of Claim 1.

**Claim 2.** \( y(C) \) is a positive integer for some \( C \in C_0^y \) or \( v_6^o(T) \) is an integer.

To justify this, note that \( y(C_2) = w(K) \) for some MFAS \( K \) of \( T_2 \backslash a_2 \) by Claim 1. From the proof of Claim 1, we see that \( K \) has ten possibilities. So we proceed by considering them accordingly.

**Subcase 2.1.** \( K \) is one of \( \{ v_1v_4, v_4v_6, v_1s_1, s_2v_4 \} \), \( \{ v_1v_4, v_6v_3, v_6s_1 \} \), and \( \{ v_4v_1, v_6v_3, s_1v_4 \} \).

In this subcase, by (15) and (19), we have \( y(v_1v_6v_3v_1) = w(s_2v_4) > 0 \) if \( K = \{ v_4v_1, v_6v_3, s_1v_4 \} \) and \( y(v_6s_1v_4v_6) = w(v_6s_1v_4v_6) = 0 \) if \( K = \{ v_4v_1, v_6v_3, s_1v_4 \} \), as desired.
Subcase 2.2. $K = \{v_3v_1, v_3v_4, v_6s_1, v_1s_1, v_1s_2, v_3s_2\}$.

In this subcase, by (15) and (19), we have $y(v_6s_1v_4v_6) + y(v_1v_6s_1v_4v_1) = w(v_6s_1) > 0$ and $y(v_3v_4v_6v_3) + y(v_1v_6v_3v_1v_1) = w(v_3v_4)$. So we may assume that $y(v_1v_6s_1v_4v_1) > 0$, for otherwise, $y(v_6s_1v_4v_6) = w(v_6s_1) > 0$. It follows from Lemma 3.5(v) that $v_3v_4$ is saturated by $y$ in $T_2$. If $y(v_1v_6v_3v_4v_1) > 0$, then $y(v_6s_1v_4v_6) = 0$ by (10), and hence $y(v_1v_6s_1v_4v_1) = w(v_6s_1) > 0$; if $y(v_1v_6v_3v_4v_1) = 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ and so $y(v_6s_1v_4v_6) = w(v_6v_4) - y(v_3v_4v_6v_3)$.

Since $w(s_1v_4) > 0$, at least one of $y(v_6s_1v_4v_6)$ and $y(v_1v_6s_1v_4v_1)$ is a positive integer.

Subcase 2.3. $K = \{v_6v_3, v_6s_1, v_1s_1, v_1s_2\}$ or $\{v_6v_3, s_1v_4, v_1s_2\}$.

In this subcase, we only consider the situation when $K = \{v_6v_3, s_1v_4, v_1s_2\}$, as the proof in the other situation goes along the same line.

Given the arcs in $K$, we have $y(v_1s_2v_4v_1) = w(v_1s_2)$, $y(v_1s_1v_4v_1) + y(v_6s_1v_4v_6) + y(v_1v_6s_1v_4v_1) = w(s_1v_4) > 0$, and $y(v_1v_6v_3v_1) + y(v_3v_4v_6v_3) + y(v_1v_6v_3v_4v_1) + y(v_3s_2v_4v_6v_3) = w(v_6v_3)$. If $y(v_1v_6v_3v_4v_1) > 0$, then $y(v_6s_1v_4v_6) = 0$ by (10). Thus $y(v_1s_1v_4v_1) + y(v_1v_6s_1v_4v_1) = w(s_1v_4)$.

If $y(v_1v_6s_1v_4v_1) = 0$, then more equality $y(v_1s_1v_4v_1) = w(v_1s_1)$ holds by (11). Since $w(s_1v_4) > 0$, at least one of $y(v_1s_1v_4v_1)$ and $y(v_1v_6s_1v_4v_1)$ is a positive integer, So we assume that $y(v_1v_6s_1v_4v_1) = 0$ in the following discussion.

Assume first that $y(v_1v_6s_1v_4v_1) > 0$. Then $y(v_1s_1v_4v_1) = w(v_1s_1)$ and $y(v_6s_1v_4v_6) + y(v_3v_4v_6v_3) + y(v_3s_2v_4v_6v_3) = w(v_3v_4)$ by (11). If $y(v_3v_4v_6v_3) = y(v_3s_2v_4v_6v_3) = 0$, then $y(v_6s_1v_4v_6) = w(v_6v_4)$, and hence $y(v_1v_6s_1v_4v_1) = w(s_1v_4) - y(v_1s_1v_4v_1) - y(v_6s_1v_4v_6)$. Since $w(s_1v_4) > 0$, at least one of $y(v_1s_1v_4v_1)$, $y(v_6s_1v_4v_6)$, and $y(v_1v_6s_1v_4v_1)$ is a positive integer. So we assume that $y(v_3v_4v_6v_3)$ or $y(v_3s_2v_4v_6v_3)$ is positive. By (11), we have $y(v_1v_6v_3v_1) = w(v_3v_1)$; by (12), one more equality $y(v_3v_4v_6v_3) = w(v_3v_4)$ holds if $y(v_3s_2v_4v_6v_3) > 0$. Thus $y(v_6s_1v_4v_6)$, $y(v_1v_6s_1v_4v_1)$, $y(v_3v_4v_6v_3)$, and $y(v_3s_2v_4v_6v_3)$ are all integers.

Assume next that $y(v_1v_6s_1v_4v_1) = 0$. Then $y(v_1s_1v_4v_1) + y(v_6s_1v_4v_6)$ by (12), so $y(v_1v_6v_3v_1) + y(v_3s_2v_4v_6v_3) = w(v_6v_3) - w(v_3v_4)$; if $y(v_3s_2v_4v_6v_3) = 0$, then $y(v_1v_6v_3v_1) + y(v_6v_3v_4v_6) = w(v_6v_3)$. Since both $v_6v_3$ and $v_3v_4$ are outside $C_0^R$, from the choice of $y$, we deduce that $y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_1v_6v_3)\}$. This implies that in either situation $y(v_3s_2v_4v_6v_3)$ and $y(v_3s_2v_4v_6v_3)$ are integers. On the other hand, since both $v_6v_3$ and $v_6s_1$ are outside $C_0^R$, by (8), we obtain $y(v_6s_1v_4v_6) = \min\{w(v_6s_1), w(v_4v_6) - y(v_6v_3v_4v_6) - y(v_3s_2v_4v_6v_3)\}$, which is also an integer. Since $w(s_1v_4) > 0$, at least one of $y(v_1s_1v_4v_1)$ and $y(v_6s_1v_4v_6)$ is a positive integer.

Subcase 2.4. $K = \{v_1v_6, v_6v_4, v_4v_1\}$.

In this subcase, we have $y(v_1v_6v_3v_1) = w(v_1v_6), y(v_1v_6v_4v_1) + y(v_1v_6v_4v_1) = w(v_4v_1), and y(v_3v_4v_6v_3) + y(v_6s_1v_4v_6) + y(v_3s_2v_4v_6v_3) = w(v_6v_4)$. By Lemma 3.2(iii) and Lemma 5.2(vi), we may assume that $w(v_1v_6) = y(v_1v_6) = 0$ and thus $w(v_4v_1) = w(K) > 0$. If $y(v_3s_2v_4v_6v_3) > 0$, then $y(v_3v_4v_6v_3) = w(v_3v_4)$ by (12), and thus we may assume that $w(v_3v_4) = 0$. Hence $y(v_3v_4v_6v_3) + y(v_6s_1v_4v_6) = w(v_3v_4)$ or $y(v_6s_1v_4v_6) + y(v_3s_2v_4v_6v_3) = w(v_6v_3)$. If $y(v_6s_1v_4v_6)$ is an integer, then one of $y(v_3v_4v_6v_3), y(v_6s_1v_4v_6),$ and $y(v_3s_2v_4v_6v_3)$ is a positive integer. So we assume that $y(v_6s_1v_4v_6)$ is not integral. Then we can prove that $\nu_u(T)$ is an integer; for a proof, see the argument of the same statement contained in the proof of (17) (with $y(v_6s_1v_4v_6)$ in place of $y(v_6s_1v_4v_6)$).

Subcase 2.5. $K = \{v_1v_6, v_4v_6, v_1s_1, v_1s_2\}$.

In this subcase, we have $y(v_1v_6v_3v_1) = w(v_1s_1), y(v_1s_2v_4v_1) = w(v_1s_2), y(v_1v_6v_3v_1) + y(v_1v_6v_3v_1) + y(v_1v_6v_3v_1) + y(v_1v_6v_3v_1) = w(v_1v_6), and y(v_3v_4v_6v_3) + y(v_6s_1v_4v_6) + y(v_3s_2v_4v_6v_3) =
By Lemma 2.2(iii), we may assume that \( w(v_1v_k) = w(v_1v_{s_2}) = 0 \).

Assume first that \( y(v_1v_3v_4v_1) > 0 \). Then \( y(v_1v_1v_3v_1) = 0 \) and \( y(v_1v_3v_4v_1) = w(v_3v_1) \) by (10). So \( y(v_3v_4v_6v_3) + y(v_3v_2v_4v_6v_3) = w(v_4v_6) \). By (12), one more equality \( y(v_3v_4v_6v_3) = w(v_3v_4) \) holds if \( y(v_3v_4v_6v_3) > 0 \). So both \( y(v_3v_4v_6v_3) \) and \( y(v_3v_2v_4v_6v_3) \) are integers. By Lemma 2.2(iii), we may assume that \( w(v_3v_1) \) and \( w(v_4v_6) \) are both zero. Thus \( y(v_1v_3v_4v_1) = y(v_1v_6s_1v_4v_1) = 0 \). By Lemma 2.2(iii), we may assume that neither \( y(v_1v_6s_1v_4v_1) \) nor \( y(v_1v_6s_1v_4v_1) \) is integral. Observe that \( v_8s_1 \) is outside \( C_0^g \), for otherwise, let \( C \in C_0^g \) be a cycle containing \( v_6s_1 \). Then \( C \) contains \( s_1v_4 \). Let \( C' = C[v_4, v_6] \cup \{ v_6v_3, v_3v_4 \} \) and \( \theta = \min\{y(C), y(v_1v_6v_3v_4)\} \). Let \( \gamma' \) be obtained from \( \gamma \) by replacing \( y(v_1v_6v_3v_4) \), \( y(v_1v_6s_1v_4v_1) \), \( y(C) \), and \( y(C') \) with \( y(v_1v_6v_3v_4v_1) - \theta, y(v_1v_6s_1v_4v_1) + \theta, y(C) - \theta, \) and \( y(C') + \theta \), respectively. Then \( \gamma' \) is also an optimal solution to \( D(T, w) \) with \( y'(v_1v_6v_3v_4v_1) < y(v_1v_6v_3v_4v_1) \), contradicting (7). Let us show that \( \nu_0^*(T) \) is an integer.

For this purpose, let \( x \) be an optimal solution to \( \mathcal{P}(T, w) \). Since both \( y(v_1v_6s_1v_4v_1) \) and \( y(v_1v_6v_3v_4v_1) \) are positive, we have \( x(v_1v_6s_1v_4v_1) = x(v_1v_6v_3v_4v_1) = 1 \) by Lemma 3.1(i). So \( x(v_6s_1) - x(s_1v_4) = x(v_6s_1) = x(v_6s_1) \). Since \( y(v_1v_6s_1v_4v_1) < w(v_6s_1) \), by Lemma 3.1(ii), we have \( x(v_6s_1) = 0 \), which implies \( x(s_1v_4) = x(v_6s_1) + x(v_3v_4) \). For any \( u \in \{ V(T_2) \setminus s_1 \} \), if a cycle in \( C_0^g \) contains \( u \), then it passes through \( v_6s_1v_4 \). Moreover, if a cycle in \( C_0^g \) contains \( u \), then it passes through \( v_6s_1v_4 \). By Lemma 3.1(iv), we obtain \( x(uw_6) + x(v_6uw_6) + x(v_3v_4) = x(uw_1) + x(v_3v_4) \). Hence \( x(uw_6) = x(uw_1) \). Clearly, we may assume that this equality holds in any other situation. Let \( T' = (V', A') \) be obtained from \( T \) by deleting vertex \( s_1 \), and let \( w' \) be obtained from the restriction of \( w \) to \( A' \) by setting \( w'(uw_6) = w(uw_6) + w(uw_1) \) for any \( u \in \{ V(T) \setminus s_1 \} \). Let \( x' \) be the restriction of \( x \) to \( A' \) and let \( \gamma' \) be obtained from \( \gamma \) as follows: for each cycle \( C \) passing through \( u \) with \( u \in \{ V(T_2) \setminus s_1 \} \), let \( C' \) arise from \( C \) by replacing the path \( u \) with \( w' \). Set \( y'(C') = y(C') \) and \( y'(v_1v_6v_3v_4v_1) = y(v_1v_6v_3v_4v_1) + y(v_1v_6s_1v_4v_1) \). It is easy to see that \( x' \) and \( \gamma' \) are optimal solutions to \( \mathcal{P}(T', w') \) and \( D(T', w') \), respectively, with the same value \( \nu_0^*(T) \) as \( x \) and \( \gamma \). By the hypothesis of Theorem 1.5, \( \nu_0^*(T) \) is an integer.

Assume next that \( y(v_1v_6v_3v_4v_1) = 0 \). Then both \( y(v_1v_6v_3v_4v_1) \) and \( y(v_1v_6s_1v_4v_1) \) are integers, for otherwise, neither of them is integral, because their sum is \( w(v_1v_k) \). If \( y(v_3v_2v_4v_6v_3) \) or \( y(v_3v_2v_4v_6v_3) \) is positive, then \( y(v_1v_6v_3v_1) = w(v_3v_1) \) by (11), a contradiction. So \( y(v_3v_2v_4v_6v_3) = y(v_3v_2v_4v_6v_3) = 0 \). Since \( v_1v_k \) is saturated by \( y \) in \( T_2 \), the arc \( v_3v_4 \) is outside \( C_0^g \). If \( v_3v_4 \) is saturated by \( y \) in \( T_2 \), then \( y(v_1v_6v_3v_1) = w(v_3v_1); \) this contradiction implies that \( v_3v_4 \) is not saturated by \( y \) in \( T_2 \) (and hence in \( T \) if \( v_3v_4 \) is outside \( C_0^g \), then from the choice of \( y \) we see that \( y(v_1v_6v_3v_1) = \min\{w(v_3v_1), w(v_3v_1)\} \), a contradiction again. So we assume that \( v_3v_4 \) is contained in some cycle \( C \in C_0^g \). Define \( \theta = \min\{w(v_3v_1) - z(v_3v_1), y(C), y(v_1v_6v_3v_1)\} \). Let \( C' = C[v_4, v_6] \cup \{ v_6s_1, s_1v_4 \} \), and let \( \gamma' \) be obtained from \( \gamma \) by replacing \( y(v_1v_6v_3v_1) \), \( y(v_1v_6s_1v_4v_1) \), \( y(C) \), and \( y(C') \) with \( y(v_1v_6v_3v_1) + \theta, y(v_1v_6s_1v_4v_1) - \theta, y(C) - \theta, \) and \( y(C') + \theta \), respectively. Then \( \gamma' \) is also an optimal solution to \( D(T, w) \) with \( y'(v_1v_6v_3v_1) < y(v_1v_6v_3v_1) \), contradicting (6). By Lemma 2.2(iii), we may assume that \( w(v_3v_1) = 0 \). Thus \( z(v_3v_1) = w(v_3v_1) = 0 \); the remainder of the proof is exactly the same as that in the preceding subcase.

**Subcase 2.6.** \( K = \{ v_4v_1, v_4v_6, v_5v_3 \} \).

In this subcase, we have \( y(v_1v_6v_3v_1) = w(v_6v_3), y(v_6s_1v_4v_1) = w(v_4v_6), \) and \( y(v_1v_6s_1v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_2v_4) \). Since \( w(K) = \tau_0(T_2 \setminus a_2) > 0 \), we have \( w(v_2v_4) > 0 \). By Lemma 5.2(vi), \( y(v_1v_6v_3v_1) \) or \( y(v_1v_6s_1v_4v_1) \) is zero. By Lemma 3.2(ii), we may assume that \( w(v_5v_3) = w(v_4v_6) = 0 \) and \( y(v_1v_6s_1v_4v_1) > 0 \). So \( y(v_1v_6s_1v_4v_1) = w(v_1s_1) \) by (11). By
Lemma 3.2(iii), we may further assume that \( w(v_1s_1) = 0 \). Thus \( y(v_1s_2v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_1v_4) \), and hence neither \( y(v_1s_2v_4v_1) \) nor \( y(v_1v_6s_1v_4v_1) \) is integral. Observe that \( v_1s_2 \) is outside \( C^y_0 \), for otherwise, let \( C \in C^y_0 \) be a cycle containing \( v_1s_2 \). Then \( C \) contains \( s_2v_4 \). Let \( C' = C[v_4, v_1] \cup \{ v_1v_6, v_6s_1, s_1v_4 \} \) and \( \theta = \min\{ y(C), y(v_1v_6s_1v_4v_1) \} \). Let \( y' \) be obtained from \( y \) by replacing \( y(v_1s_2v_4v_1), y(v_1v_6s_1v_4v_1), y(C) \), and \( y(C') \) with \( y(v_1s_2v_4v_1) + \theta \), \( y(v_1v_6s_1v_4v_1) - \theta \), \( y(C) - \theta \), and \( y(C') + \theta \), respectively. Then \( y' \) is also an optimal solution to \( \mathbb{D}(T, w) \) with \( y'(v_1v_6s_1v_4v_1) < y(v_1v_6s_1v_4v_1) \), contradicting (6). Furthermore, since \( w(v_1s_1) = 0 \), the arc \( v_3v_1 \) is also outside \( C^y_0 \). Thus \( w(v_3v_1) = z(v_3v_1) = 0 \). Let us show that \( \nu_w^*(T) \) is an integer.

For this purpose, let \( x \) be an optimal solution to \( \mathbb{P}(T, w) \). Since both \( y(v_1s_2v_4v_1) \) and \( y(v_1v_6s_1v_4v_1) \) are positive, we have \( x(v_1s_2v_4v_1) = x(v_1v_6s_1v_4v_1) = 1 \) by Lemma 3.1(i). Since \( y(v_1s_2v_4v_1) < w(v_1s_2) \), we have \( x(v_1s_2) = 0 \) by Lemma 3.1(ii). It follows that \( x(s_2v_4) = x(v_1s_2) + x(v_6s_1) + x(s_1v_4) = x(v_6s_2) + x(s_2v_4) \). Hence \( x(w(v_1)) = x(u_0) \). Clearly, we may assume that this equality holds in any other situation. Let \( T' = (V', A') \) be obtained from \( T \) by deleting \( s_2 \), and let \( w' \) be the restriction of \( w \) to \( A' \) by replacing \( w(e) \) with \( w(e) + w(s_2v_4) \) for \( e \in \{ v_1v_6, v_6s_1, s_1v_4 \} \), replacing \( w(v_1) \) with \( w(v_1) + w(u_0) \) for any \( u \in V \setminus \{ V(T_2) \} \), and replacing \( w(v_3v_1) \) with \( w(v_3v_1) + w(v_3v_2) \). Let \( x' \) be obtained from \( x \) by setting \( x(v_3v_1) = x(v_3v_2) \). Since \( w(v_3v_1) = 0 \) and \( w'(v_3v_1) = w(v_3v_2) \), we have \( (w')^T x' = w^T x \). Let \( y' \) be obtained from \( y \) as follows: set \( y'(v_1v_6s_1v_4v_1) = y(v_1v_6s_1v_4v_1) + y(v_1s_2v_4v_1) \); for each \( C \in C^y_0 \) passing through \( us_2v_4 \), let \( C' \) arise from \( C \) by replacing the path \( us_2v_4 \) with the path \( uv_1v_6s_1v_4 \), and set \( y'(C') = y(C') + y(C) \). From the LP-duality theorem, we see that \( x' \) and \( y' \) are optimal solutions to \( \mathbb{P}(T', w') \) and \( \mathbb{D}(T', w') \), with the same value \( \nu_w^*(T) \) as \( x \) and \( y \). By the hypothesis of Theorem 1.5, \( \nu_w^*(T) \) is an integer.

Subcase 2.7. \( K = \{ v_4v_1, v_4v_6, v_3v_1 \} \).

In this subcase, we have \( y(v_1v_6v_3v_1) = w(v_3v_1), y(v_1s_1v_4v_1) + y(v_1s_2v_4v_1) + y(v_6s_1v_4v_1) = w(v_4v_1), \) and \( y(v_6s_1v_4v_1) + y(v_3s_2v_4v_6v_3) = w(v_6v_3) \). By Lemma 3.2(iii), we may assume that \( w(v_3v_1) = 0 \).

Assume first that \( y(v_1v_6v_3v_1) > 0 \). Then \( y(v_6s_1v_4v_1) = 0 \). If \( y(v_3s_2v_4v_6v_3) > 0 \), then \( y(v_3s_2v_4v_6v_3) = w(v_3v_4) \) by (12); otherwise, \( y(v_3s_2v_4v_6v_3) = w(v_4v_3) \). Both \( y(v_3s_2v_4v_6v_3) \) and \( y(v_3s_2v_4v_6v_3) \) are integers in either situation. Thus we may assume that \( w(v_3v_4) = 0 \). The remainder of the proof is exactly the same as that of (16).

Assume next that \( y(v_1v_6v_3v_1) = 0 \). Consider first the subsubcase when \( w(v_3v_1) = 0 \). Then \( w(v_4v_6) = w(K) > 0 \). If \( y(v_3s_2v_4v_6v_3) > 0 \), then \( y(v_3s_2v_4v_6v_3) = w(v_4v_3) \); if \( y(v_3s_2v_4v_6v_3) < 0 \), then \( y(v_6s_1v_4v_1) + y(v_3s_2v_4v_6v_3) = w(v_6v_3) \). It can be shown that \( \nu_w^*(T) \) is an integer; for a proof, see the argument of the same statement contained in the proof of (17).

Consider next the subsubcase when \( w(v_3v_1) > 0 \). Observe that \( y(v_1v_6v_3v_1) > 0 \) and \( y(v_3s_2v_4v_6v_3) = 0 \), for otherwise, since \( w(v_1s_1)w(v_1s_2) = 0 \) by Lemma 5.2(vi), at most one of \( y(v_1s_1v_4v_1) \) and \( y(v_1s_2v_4v_1) \) is positive. Hence, if \( y(v_1v_6s_1v_4v_1) = 0 \), then \( y(v_1v_6s_1v_4v_1) = w(v_2v_1) \); or \( y(v_1s_2v_4v_1) = w(v_4v_3) \); if \( y(v_1v_6s_1v_4v_1) > 0 \) and \( y(v_3s_2v_4v_6v_3) > 0 \), then, by (11), we have \( y(v_1s_1v_4v_1) = w(v_1s_1), y(v_1s_2v_4v_1) = w(v_1s_2) \). So \( y(v_1v_6s_1v_4v_1) = w(v_1v_4) - w(v_1s_1) - w(v_1s_2) \). By Lemma 3.2(iii), we see that \( \nu_w^*(T) \) is an integer. The preceding observation together
with (11) implies that $y(v_1s_1v_4v_1) = w(v_1s_1)$, $y(v_1s_2v_4v_1) + y(v_1v_6s_1v_4v_1) = w(v_4v_1) - w(v_1s_1)$, and $y(v_6s_1v_4v_6) + y(v_3v_4v_6v_3) = w(v_4v_6)$. Lemma 3.2(iii) allows us to assume that $w(v_1s_1) = 0$ and that neither $y(v_1s_2v_4v_1)$ nor $y(v_1v_6s_1v_4v_1)$ is integral.

It can then be shown that $v_1s_2$ is outside $C_0^y$ and $\nu_w^*(T)$ is an integer; for a proof, see the argument of the same statement contained in the preceding case.

Combining the above seven subcases, we see that Claim 2 holds. Hence, by Lemma 3.2(iii), the optimal value $\nu_w^*(T)$ of $D(T, \mathbf{w})$ is integral, as described in (1) above.

To establish the corresponding lemmas for the cases when $T_2/S \in \{G_4, G_5, G_6\}$, we need some further preparations.

**Lemma 5.10.** If $T_2/S \in \{G_5, G_6\}$, then we may assume that $\min\{w(v_1v_3), w(v_3v_4), w(v_4v_1)\} = 0$.

**Proof.** Let $\theta = \min\{w(v_1v_3), w(v_3v_4), w(v_4v_1)\}$ and $C_0 = v_1v_3v_4v_1$. Assume the contrary: $\theta > 0$. Let $\mathbf{y}$ be an optimal solution to $D(T, \mathbf{w})$ such that

1. $y(C_2)$ is maximized; and
2. subject to (1), $(y(D_q), y(D_{q-1}), \ldots, y(D_4))$ is minimized lexicographically.

Let $C_2' = C_2 \setminus C_0$. Note that every cycle in $C_2'$ passes through $b$. By Lemma 3.5(vii), at least one of $v_1v_3$, $v_3v_4$, and $v_4v_1$ is saturated by $\mathbf{y}$ in $T_2$, say $v_1v_3$ (by symmetry). Thus $w(v_1v_3) = \theta$. We proceed to show that

3. there is no cycle $C \in C_2'$ with $y(C) > 0$ passing through $v_1v_3$.

Assume the contrary: $v_1v_3$ is contained in some cycle $C_1 \subseteq C_2'$ with $y(C_1) > 0$. Clearly, $|C_1| \geq 4$. If neither $v_3v_4$ nor $v_4v_1$ is saturated by $\mathbf{y}$ in $T$, then $\theta_1 = \min\{w(v_3v_4) - z(v_3v_4), w(v_4v_1) - z(v_4v_1)\} > 0$. Let $\mathbf{y}'$ be obtained from $\mathbf{y}$ by replacing $y(C_1)$ and $y(C_0)$ with $y(C_1) - \theta_1$ and $y(C_0) + \theta_1$, respectively. Then $\mathbf{y}'$ is an optimal solution to $D(T, \mathbf{w})$ with $y'(C_1) < y(C_1)$, contradicting (2). Thus at least one of $v_3v_4$ and $v_4v_1$ is saturated by $\mathbf{y}$ in $T$. We proceed by considering two cases.

- Both $v_3v_4$ and $v_4v_1$ are saturated by $\mathbf{y}$ in $T$. In this case, let $C_2 \in C_0^y \cup C_2'$ be a cycle containing $v_3v_4$ with $y(C_2) > 0$; subject to this, $C_2$ is chosen to contain $v_4v_1$, if possible. If $v_4v_1$ is on $C_2$, then the multiset sum of $C_1$ and $C_2$ contains three arc-disjoint cycles $C_0$, $C_0' = \{bv_1\} \cup C_2[v_1, b]$ and $C_2' = C_2[b, v_3] \cup C_1[v_3, b]$. Define $\epsilon = \min\{y(C_1), y(C_2)\}$. Let $\mathbf{y}'$ be obtained from $\mathbf{y}$ by replacing $y(C_0)$ with $y(C_0) + \epsilon$, and replacing $y(C_1)$ and $y(C_0')$ with $y(C_1) - \epsilon$ and $y(C_1') + \epsilon$, respectively, for $i = 1, 2$. Then $\mathbf{y}'$ is an optimal solution to $D(T, \mathbf{w})$ with $(\mathbf{y}')^T \mathbf{1} = y^T \mathbf{1} + \epsilon$, a contradiction. If $v_4v_1$ is outside $C_2$, then there exists a cycle $C_3 \subseteq C_2 \cup C_2'$ containing $v_4v_1$ with $y(C_3) > 0$. Observe that the multiset sum of $C_1$, $C_2$, and $C_3$ contains four arc-disjoint cycles $C_0$, $C_0' = \{bv_1\} \cup C_3[v_1, b]$, $C_2' = C_2[b, v_3] \cup C_1[v_3, b]$, and $C_3' = C_3[b, v_4] \cup C_2[v_4, b]$. Define $\epsilon = \min_{1 \leq i \leq 3} y(C_i)$. Let $\mathbf{y}'$ be obtained from $\mathbf{y}$ by replacing $y(C_0)$ with $y(C_0) + \epsilon$, and replacing $y(C_1)$ and $y(C_0')$ with $y(C_1') - \epsilon$ and $y(C_1') + \epsilon$, respectively, for $1 \leq i \leq 3$. Then $\mathbf{y}'$ is an optimal solution to $D(T, \mathbf{w})$ with $(\mathbf{y}')^T \mathbf{1} = y^T \mathbf{1} + \epsilon$, a contradiction again.

- Exactly one of $v_3v_4$ and $v_4v_1$ is saturated by $\mathbf{y}$ in $T$. In this case, by symmetry, we may assume that $v_3v_4$ is saturated while $v_4v_1$ is not. Let $C_2 \in C_0^y \cup C_2'$ be a cycle containing $v_3v_4$ with $y(C_2) > 0$. Then the multiset sum of $C_1$, $C_2$, and the unsaturated arc $v_4v_1$ contains two arc-disjoint cycles $C_0$ and $C_0' = C_2[b, v_3] \cup C_1[v_3, b]$. Clearly, $C_0' \subseteq C_2'$ if $C_0 \subseteq C_2'$. Define $\epsilon = \min\{y(C_1), y(C_2), w(v_4v_1) - z(v_4v_1)\}$. Let $\mathbf{y}'$ be obtained from $\mathbf{y}$ by replacing $y(C_0)$ with $y(C_0) + \epsilon$ and replacing $y(C_1)$ and $y(C_0')$ with $y(C_1) - \epsilon$ and $y(C_1') + \epsilon$, respectively, for $1 \leq i \leq 3$. Then $\mathbf{y}'$ is an optimal solution to $D(T, \mathbf{w})$ with $(\mathbf{y}')^T \mathbf{1} = y^T \mathbf{1} + \epsilon$, a contradiction again.
\( \epsilon \), replacing \( y(C_1) \) with \( y(C_1) - \epsilon \), and replacing \( y(C_2) \) and \( y(C_2') \) with \( y(C_2) - \epsilon \) and \( y(C_2') + \epsilon \), respectively. Then \( y' \) is an optimal solution to \( \mathbb{D}(T, w) \) with \( y'(C_1) < y(C_1) \), contradicting (2).

Combining the above two cases, we see that (3) holds. So \( y(C_0) = \theta > 0 \), and hence \( \mathbb{D}(T, w) \) has an integral optimal solution by Lemma 3.2(iii). This proves the lemma.

Let \( Q = V(T_2) \setminus (S \cup \{b_2, a_2\}) \). Then \( Q = \{v_2, v_3\} \) if \( T_2/S = G_4 \), \( Q = \{v_1, v_3, v_4\} \) if \( T_2/S = G_5 \), and \( Q = \{v_1, v_2, v_3, v_4\} \) if \( T_2/S = G_6 \). Moreover, \( v_1v_3v_4v_1 \) is the unique cycle in \( T[Q] \) when \( T_2/S = G_5 \) or \( G_6 \). Let \( T' = T \) if \( T_2/S = G_4 \), and let \( T' \) be obtained from \( T \) by reversing precisely one arc \( e \) on \( v_1v_3v_4v_1 \) with \( w(e) = 0 \) (see Lemma 5.9) so that \( T[Q] \) is acyclic if \( T_2/S = G_5 \) and \( G_6 \). From Lemma 2.3 we see that \( T' \) is also Möbius-free. Note that every integral optimal solution to \( \mathbb{D}(T, w) \) naturally corresponds to an integral optimal solution to \( \mathbb{D}(T', w) \) with the same value, and vice versa. So we shall not make effort to distinguish between \( \mathbb{D}(T, w) \) and \( \mathbb{D}(T', w) \). Let us label the vertices in \( Q \) as \( q_1, q_2, \ldots, q_t \) such that \( q_jq_i \) is an arc in \( T' \) for \( 1 \leq i < j \leq t \), where \( t = |Q| \).

**Lemma 5.11.** Suppose \( T_2/S \in \{G_4, G_5, G_6\} \). Let \( x \) and \( y \) be optimal solutions to \( P(T, w) \) and \( \mathbb{D}(T, w) \), respectively. Then we may assume that the following statements hold:

(i) For each \( q_i \in Q \), there exists exactly one \( s_k \in S \) such that \( z(q_is_k) > 0 \);

(ii) \( z(q_jq_i) = w(q_jq_i) = 0 \) for \( 1 \leq i < j \leq t \), where \( t = |Q| \);

(iii) If \( z(q_is_k)z(q_js_k) > 0 \) for some \( 1 \leq i < j \leq t \) and \( s_k \in S \), then \( x(q_is_k) \neq x(q_js_k) \).

**Proof.** As remarked above the lemma, we may simply treat \( T, P(T, w), \) and \( \mathbb{D}(T, w) \) as \( T' \) and \( P(T', w) \), and \( \mathbb{D}(T', w) \), respectively, in our proof.

(i) By Lemma 5.2(vi), for each vertex \( q_i \in Q \), there exists at most one \( s_k \in S \) with \( z(q_is_k) > 0 \). Assume on the contrary that \( z(q_is_k) = 0 \) for all \( s_k \in S \). Then no cycle in \( C^0 \) passes through \( q_i \). Let \( G = T' \setminus q_i \) and let \( w' \) be the restriction of \( w \) to the arcs of \( G \). By the hypothesis of Theorem 1.5, \( \mathbb{D}(G, w') \) has an integral optimal solution, and so does \( \mathbb{D}(T', w) \). Hence we assume that (i) holds.

(ii) Assume the contrary: \( z(q_jq_i) > 0 \); subject to this, \( j + i \) is minimized. If there exists exactly one \( s_k \in S \) such that \( z(q_is_k)z(q_js_k) > 0 \), then the proof is the same as that of Lemma 5.2(i) (with \( s_k, q_i, \) and \( q_j \) in place of \( v_0, s_i, \) and \( s_j \), respectively), so we omit the details here. In view of Lemma 5.2(i), we may assume that \( z(q_is_1)z(q_js_2) > 0 \). We proceed by considering two cases.

**Case 1.** \( x(q_jq_i) = 0 \). In this case, we may assume that \( x(uq_j) = x(uq_i) \) for any \( u \in V \setminus (S \cup Q) \). Indeed, if \( z(uq_j)z(uq_i) > 0 \), then Lemma 3.1(iv) implies \( x(uq_j) = x(uq_i) \); if \( z(uq_j)z(uq_i) = 0 \), then \( w(us')w(us') = 0 \) by Lemma 3.2(i). Thus we may modify \( x(uq_j) \) and \( x(uq_i) \) so that they become equal. Let \( T' = (V', A') \) be obtained from \( T \) by identifying \( q_i \) with \( q_j \); we still use \( q_i \) to denote the resulting vertex. Let \( w' \) be obtained from the restriction of \( w \) to \( A' \) by replacing \( w(uq_j) \) with \( w(uq_j) + w(uq_i) \) for any \( u \in V \setminus (S \cup Q) \). Let \( x' \) and \( y' \) be the projections of \( x \) and \( y \) onto \( T' \), respectively. From the LP-duality theorem, it is easy to see that \( x' \) and \( y' \) are optimal solutions to \( P(T, w') \) and \( \mathbb{D}(T, w') \), respectively, with the same value as \( x \) and \( y \). By the hypothesis of Theorem 1.5, \( \nu(T') \) is an integer. It follows from Lemma 3.4(ii) that \( \mathbb{D}(T, w) \) has an integral optimal solution.

**Case 2.** \( x(q_jq_i) > 0 \). In this case, \( z(q_jq_i) = w(q_jq_i) > 0 \) by Lemma 3.1(iii). Let \( C_1 \) and \( C_2 \) be two cycles in \( C^0 \) that passes through \( q_jq_i \) and \( q_jq_2 \), respectively. Clearly, both \( C_1 \) and \( C_2 \) pass
Lemma 5.2(i) allows us to assume that $y = x(q_j q_i) + x(q_i s_1) + x(s_1 b) = x(q_j s_2) + x(s_2 b)$. Let $w'$ be obtained from $w$ by replacing $w(e_1)$ with $w(e_1) + w(q_j q_i)$ for $e_1 = q_j s_2$ and $s_2 b$ and replacing $w(e_2)$ with $w(e_2) - w(q_j q_i)$ for $e_2 = q_j q_i$, $q_i s_1$, and $s_1 b$. Let $x' = x$, and let $y'$ be obtained from $y$ as follows: for each cycle passing through $q_j q_i$, let $C'$ be the cycle arising from $C$ by replacing the path $q_j q_i b$ with $q_j s_2 b$. From the LP-duality theorem, we see that $x'$ and $y'$ are optimal solutions to $P(T, w')$ and $D(T, w')$, respectively, with the same value $\nu^*_y(T)$ as $x$ and $y$. Since $w'(A) < w(A)$, by the hypothesis of Theorem 1.5, $\nu^*_y(T)$ is an integer. It follows from Lemma 3.5(ii) that $D(T, w)$ has an integral optimal solution.

Combining the above two cases, we may assume that $z(q_j q_i) = 0$.

(iii) Since the proof is the same as that of Lemma 5.2(iv) (with $s_k$, $q_i$, and $q_j$ in place of $v_0$, $s_i$, and $s_j$, respectively), we omit the routine details here.

Lemma 5.12. If $T_2/S = G_4$, then $D(T, w)$ has an integral optimal solution.

Proof. Recall that $(b_2, a_2) = (v_1, v_3)$, $s^* = v_4$, and $Q = \{v_2, v_3\}$. Given an optimal solution $y$ to $D(T, w)$, set $\varphi(s_i) = \{u : z(us_i) > 0 \text{ for } u \in V(T_2) \setminus a_2\}$ for each $s_i \in S$. By Lemma 5.2(i) and (vi), we have

1. $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

From (1) and Lemma 5.10(i), we see that

2. there exists at least one and at most two vertices $s_i$’s in $S$ with $\varphi(s_i) \neq \emptyset$.

Lemma 5.2(i) allows us to assume that

3. if $\varphi(s_i) \neq \emptyset$, then $i \in \{1, 2\}$.

By Lemma 5.10(ii), we obtain

4. $w(v_2 v_3) = z(v_2 v_3) = 0$.

In the remainder of our proof, we reserve $y$ for an optimal solution to $D(T, w)$ such that

5. $y(C_2)$ is maximized; and

6. subject to (5), $(y(D_q), y(D_{q-1}), \ldots, y(D_3))$ is minimized lexicographically.

Claim. $y(C)$ is integral for some $C \in C^y_2$.

To justify this, we distinguish between two cases.

Case 1. $\varphi(s_i) = \{v_2\}$ for $i = 1$ or 2.

In this case, by Lemma 5.2(i) and Lemma 5.10(i), we may assume that $\varphi(s_1) = \{v_2\}$ and $\varphi(s_2) = \{v_3\}$. By (4), we obtain

7. $C^y_2 \subseteq \{v_1 v_2 s_1 v_1, v_1 v_3 s_2 v_1\}$.

From Lemma 3.5(vii), we deduce that $y(v_1 v_2 s_1 v_1) = \min\{w(v_1 v_2), w(v_2 s_1), w(s_1 v_1)\}$ and $y(v_1 v_3 s_2 v_1) = \min\{w(v_1 v_3), w(v_3 s_2), w(s_2 v_1)\}$. If both $y(v_1 v_2 s_1 v_1)$ and $y(v_1 v_3 s_2 v_1)$ are zero, then $\tau_w(T_2 \setminus a_2) = \min\{w(v_1 v_2), w(v_2 s_1), w(s_1 v_1)\} + \min\{w(v_1 v_3), w(v_3 s_2), w(s_2 v_1)\} = 0$, contradicting (a). Therefore, $y(v_1 v_2 s_1 v_1)$ or $y(v_1 v_3 s_2 v_1)$ is a positive integer.

Case 2. $\varphi(s_i) \neq \{v_2\}$.

In this case, Lemma 5.10(i), (2) and (3) allow us to assume that $\varphi(s_1) = \{v_2, v_3\}$. By (4), we have

8. $C^y_2 \subseteq \{v_1 v_2 s_1 v_1, v_1 v_3 s_2 v_1\}$.

By Lemma 5.2(iii), we also obtain $z(s_1 v_1) = w(s_1 v_1) > 0$. Assume first that $s_1 v_1$ is outside $C^y_2$. Then both $v_2 s_1$ and $v_3 s_1$ are outside $C^y_2$, and $s_1 v_1$ is saturated by $y$ in $T_2$. So $y(v_1 v_2 s_1 v_1) + y(v_1 v_3 s_1 v_1) = w(s_1 v_1) > 0$. Observe that both $y(v_1 v_2 s_1 v_1)$ and $y(v_1 v_3 s_1 v_1)$ are integral, for
otherwise, \(0 < y(v_1v_is1v_i) < w(v_is1)\) for \(i = 2, 3\), by Lemma 3.1(i) and (ii), we have \(x(v_is1) = x(v_is1) = 0\), contradicting Lemma 5.9(iii). Hence \(y(v_is1v_1v_1)\) or \(y(v_is1v_1v_1)\) is a positive integer.

Assume next that \(s_is1\) is contained in some cycle \(C \in \mathcal{C}_0\). From Lemma 3.5(vii), we see that \(y(v_is1v_1v_1) = \min\{w(v_is1v_1w(v_is1v_1))\} \text{ for } i = 2, 3\). If \(y(v_is1v_1v_1) = 0\) for \(i = 2, 3\), then \(\tau_w(T_2\backslash a_2) = \sum_{i=1}^2 \min\{w(v_is1v_1), w(v_is1v_1)\} = 0\), contradicting (\(\alpha\)). Therefore \(y(v_is1v_1v_1)\) or \(y(v_is1v_1v_1)\) is a positive integer. So the above Claim is established.

From the above Claim and Lemma 3.2(iii), we conclude that \(\mathcal{D}(T, w)\) has an integral optimal solution.

**Lemma 5.13.** If \(T_2/S = G_5\), then \(\mathcal{D}(T, w)\) has an integral optimal solution.

**Proof.** Recall that \((b_2, a_2) = (v_2, v_6), s^* = v_5\), and \(Q = \{v_1, v_3, v_4\}\). Given an optimal solution \(y\) to \(\mathcal{D}(T, w)\), set \(\varphi(s_i) = \{u : z(us_i) > 0\} \text{ for } u \in V(T_2)\backslash a_2\) for each \(s_i \in S\). By Lemma 5.2(i) and (vi), we have

1. \(\varphi(s_i) \cap \varphi(s_j) = \emptyset\) whenever \(i \neq j\).

From (1) and Lemma 5.10(i), we see that

2. there exists at least one and at most three vertices \(s_i's\) in \(S\) with \(\varphi(s_i) \neq \emptyset\).

Lemma 5.2(i) allows us to assume that

3. if \(\varphi(s_i) \neq \emptyset\), then \(i \in \{1, 2, 3\}\).

By Lemma 5.10(ii), we obtain

4. \(w(e) = z(e) = 0\) for \(e \in \{v_1v_3, v_3v_4, v_4v_1\}\).

In the remainder of our proof, we reserve \(y\) for an optimal solution to \(\mathcal{D}(T, w)\) such that

5. \(y(C_2)\) is maximized; and

6. subject to (5), \((y(D_0), y(D_{q-1}), \ldots, y(D_3))\) is minimized lexicographically.

**Claim.** \(y(C)\) is integral for some \(C \in \mathcal{C}_2^g\).

To justify this, we consider three possible cases (see the structure of \(G_5\)), depending on the size of \(\varphi(s_i)\) for \(1 \leq i \leq 3\).

**Case 1.** \(|\varphi(s_i)| = 1\) for each \(1 \leq i \leq 3\).

In this case, by Lemma 5.10(i), (2) and (3), we may assume that \(\varphi(s_1) = \{v_1\}, \varphi(s_2) = \{v_3\},\) and \(\varphi(s_3) = \{v_4\}\). By (4), we obtain

7. \(C_2^g \subseteq \{v_2v_1s_1v_2, v_2v_3s_2v_2, v_2v_4s_3v_2\}\).

From Lemma 3.5(vii), we deduce that \(y(v_2v_1s_1v_2) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\}, y(v_2v_3s_2v_2) = \min\{w(v_2v_3), w(v_3s_2), w(s_2v_2)\},\) and \(y(v_2v_4s_3v_2) = \min\{w(v_2v_4), w(v_4s_3), w(s_3v_2)\}\). If \(y(v_2v_1s_1v_2), y(v_2v_3s_2v_2), y(v_2v_4s_3v_2)\) are all zero, then \(\tau_w(T_2\backslash a_2) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\} + \min\{w(v_2v_3), w(v_3s_2), w(s_2v_2)\} + \min\{w(v_2v_4), w(v_4s_3), w(s_3v_2)\} = 0\), contradicting (\(\alpha\)). Therefore, at least one of \(y(v_2v_1s_1v_2), y(v_2v_3s_2v_2), y(v_2v_4s_3v_2)\) is a positive integer.

**Case 2.** \(|\varphi(s_i)| = 1\) for exactly one \(i \in \{1, 2, 3\}\).

In this case, by Lemma 5.10(i), (2) and (3), we may assume that \(\varphi(s_1) = \{v_1\}, \varphi(s_2) = \{v_3, v_4\}\). By (4), we have

8. \(C_2^g \subseteq \{v_2v_1s_1v_2, v_2v_3s_2v_2, v_2v_4s_2v_2\}.

From Lemma 3.5(vii), we see that \(y(v_2v_1s_1v_2) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\}\). If \(y(v_2v_1s_1v_2) > 0\), we are done. So we assume that \(y(v_2v_1s_1v_2) = 0\). Since \(w(v_1s_1)w(s_1v_2) > 0\), we obtain \(w(v_2v_1) = \min\{w(v_2v_1), w(v_1s_1), w(s_1v_2)\} = 0\). By Lemma 5.2(iii), we have \(z(s_2v_2) = w(s_2v_2) > 0\).
Assume first that $s_2v_2$ is outside $C_0^y$. Then both $v_3s_2$ and $v_4s_2$ are outside $C_0^y$, and $s_2v_2$ is saturated by $y$ in $T_2$. Hence $y(v_2v_3s_2v_2) + y(v_2v_4s_2v_2) = w(s_2v_2) > 0$. Observe that both $y(v_2v_3s_2v_2)$ and $y(v_2v_4s_2v_2)$ are integral, for otherwise, since $0 < y(v_2v_3s_2v_2) < w(v_3s_2)$ for $i = 3, 4$, by Lemma 3.1(i) and (ii), we have $x(v_3s_2) = x(v_4s_2) = 0$, contradicting Lemma 5.9(iii). Hence both $y(v_2v_3s_2v_2)$ and $y(v_2v_4s_2v_2)$ are positive integers.

Assume next that $s_2v_2$ is contained in some cycle $C$ in $C_0^y$. From Lemma 3.5(vii), we see that $y(v_2v_3s_2v_2) = \min\{w(v_2v_3), w(v_3s_2)\}$ for $i = 3, 4$. If $y(v_2v_3s_2v_2) = 0$ for $i = 3, 4$, then $\tau_w(T_2\setminus a_2) = w(v_2v_1) + \sum_{i=3}^{4} \min\{w(v_2v_i), w(v_i s_2)\} = 0$, contradicting $(\alpha)$. Therefore $y(v_2v_3s_2v_2)$ or $y(v_2v_4s_2v_2)$ is a positive integer.

Case 3. $|\varphi(s_i)| \neq 1$ for any $i \in \{1, 2, 3\}$.

In this case, by Lemma 5.10(i), (2), and (3), we may assume that $\varphi(s_1) = \{v_1, v_3, v_4\}$ (see the structure of $G_5$). By (4), we obtain

$$C_0^y \subseteq \{v_2v_1s_1v_2, v_2v_3s_1v_2, v_2v_4s_1v_2\}.$$ By Lemma 5.2(iii), we have $z(s_1v_2) = w(s_1v_2) > 0$.

Assume first that $s_1v_2$ is outside $C_0^y$. Then $v_3s_1$ is outside $C_0^y$ for each $i \in \{1, 3, 4\}$, and $s_1v_2$ is saturated by $y$ in $T_2$. So $\sum_{i=1}^{3, 4} y(v_2v_3s_1v_2) = w(s_1v_2) > 0$. Observe that $y(v_2v_1s_1v_2)$ is integral for each $i \in \{1, 3, 4\}$, for otherwise, symmetry allows us to assume that $y(v_2v_1s_1v_2)$ is not integral. Then $y(v_2v_3s_1v_2) = y(v_2v_4s_1v_2)$ is not integral, say $y(v_2v_3s_1v_2) < w(v_3s_1)$ for $i = 1, 3, by Lemma 3.1(i) and (ii), we have $x(v_3s_1) = x(v_3s_1) = 0$, contradicting Lemma 5.9(iii). It follows that $y(v_2v_3s_1v_2)$ is a positive integer for each $i \in \{1, 3, 4\}$.

Assume next that $s_1v_2$ is contained in some cycle $C$ in $C_0^y$. From Lemma 3.5(vii), we deduce that $y(v_2v_1s_1v_2) = \min\{w(v_2v_1), w(v_1s_1)\}$ for $i \in \{1, 3, 4\}$. If $y(v_2v_1s_1v_2) = 0$ for each $i \in \{1, 3, 4\}$, then $\tau_w(T_2\setminus a_2) = \sum_{i=1}^{3, 4} \min\{w(v_2v_i), w(v_i s_1)\} = 0$, contradicting $(\alpha)$. Hence $y(v_2v_1s_1v_2)$ is a positive integer for some $i \in \{1, 3, 4\}$. This proves the Claim.

From the Claim and Lemma 3.2(iii), we conclude that $\mathbb{D}(T, w)$ has an integral optimal solution.

**Lemma 5.14.** If $T_2/S = G_6$, then $\mathbb{D}(T, w)$ has an integral optimal solution.

**Proof.** Recall that $(b_2, a_2) = (v_6, v_7)$, $s^* = v_5$, and $Q = \{v_1, v_2, v_3, v_4\}$. Given an optimal solution $y$ to $\mathbb{D}(T, w)$, set $\varphi(s_i) = \{u \in V(T_2\setminus a_2) \mid z(us_i) > 0 \}$ for $u \in V(T_2\setminus a_2)$ for each $s_i \in S$. By Lemma 5.2(i) and (vi), we have

(1) $\varphi(s_i) \cap \varphi(s_j) = \emptyset$ whenever $i \neq j$.

From (1) and Lemma 5.10(i), we see that

(2) there exists at least one and at most four vertices $s_i$’s in $S$ with $\varphi(s_i) \neq \emptyset$.

Lemma 5.2(i) allows us to assume that

(3) if $\varphi(s_i) \neq \emptyset$, then $1 \leq i \leq 4$.

By Lemma 5.10(ii), we obtain

(4) $w(e) = z(e) > 0$ for $e \in \{v_1v_3, v_3v_4, v_4v_1, v_1v_2, v_3v_2, v_4v_2\}$.

In the remainder of our proof, we reserve $y$ for an optimal solution to $\mathbb{D}(T, w)$ such that

(5) $y(C_2)$ is maximized; and

(6) subject to (5), $y(D_q), y(D_{q-1}), \ldots, y(D_3)$ is minimized lexicographically.

**Claim.** $y(C)$ is integral for some $C \in C_0^y$.

To justify this, we consider five possible cases (see the structure of $G_6$), depending on the size of $\varphi(s_i)$ for $1 \leq i \leq 4$. 55
Case 1. $|\varphi(s_i)| = 1$ for each $1 \leq i \leq 4$.

In this case, by Lemma 5.10(i), (2) and (3), we may assume that $\varphi(s_i) = \{v_i\}$ for each $1 \leq i \leq 4$. By (4), we obtain

(7) $C''_y \subseteq \{v_6 v_1 s_1 v_6, v_6 v_2 s_2 v_6, v_6 v_3 s_3 v_6, v_6 v_4 s_4 v_6\}$.

From Lemma 3.5(vii), we deduce that $y(v_6 v_1 s_1 v_6) = \min \{w(v_6 v_1), w(v_1 s_1), w(s_1 v_6)\}$ for each $1 \leq i \leq 4$. If $y(v_6 v_1 s_1 v_6) = 0$ for $1 \leq i \leq 4$, then $\tau_w(T_2 \setminus a_2) = \sum_{i=1}^{4} \min \{w(v_i v_1), w(v_i s_1), w(s_i v_6)\} = 0$, contradicting (a). Hence $y(v_6 v_1 s_1 v_6)$ is a positive integer for some $i \in \{1, 2, 3, 4\}$.

Case 2. $|\varphi(s_i)| = 1$ for exactly one $i \in \{1, 2, 3, 4\}$.

In this case, by Lemma 5.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1\}$, $\varphi(s_2) = \{v_2, v_3, v_4\}$. By (4), we have

(8) $C''_y \subseteq \{v_6 v_1 s_1 v_6, v_6 v_2 s_2 v_6, v_6 v_3 s_3 v_6, v_6 v_4 s_4 v_6\}$.

From Lemma 3.5(vii), we see that $y(v_6 v_1 s_1 v_6) = \min \{w(v_6 v_1), w(v_1 s_1), w(s_1 v_6)\}$. If $y(v_6 v_1 s_1 v_6) > 0$, we are done. So we assume that $y(v_6 v_1 s_1 v_6) > 0$. Since $w(v_1 s_1) w(s_1 v_6) > 0$, we obtain $w(v_6 v_1) = \min \{w(v_6 v_1), w(v_1 s_1), w(s_1 v_6)\} = 0$. By Lemma 5.2(iii), we have $z(s_2 v_6) = w(s_2 v_6) > 0$.

Assume first that $s_2 v_6$ is outside $C''_y$. Then $v_1 s_2$ is outside $C''_x$ for $i \in \{2, 3, 4\}$, and $s_2 v_6$ is saturated by $y$ in $T_2$. So $\sum_{i=2}^{4} y(v_6 v_1 s_2 v_6) = w(s_2 v_6) > 0$. Observe that $y(v_6 v_1 s_2 v_6)$ is integral for each $i \in \{2, 3, 4\}$, for otherwise, symmetry allows us to assume that $y(v_6 v_1 s_2 v_6)$ is not integral. Then one of $y(v_6 v_3 s_2 v_6)$ and $y(v_6 v_4 s_2 v_6)$ is not integral, say $y(v_6 v_3 s_2 v_6)$. Since $0 < y(v_6 v_3 s_2 v_6) < w(v_1 s_2)$ for $i = 2, 3$, by Lemma 3.1(i) and (ii), we have $x(v_2 s_2) = x(v_3 s_2) = 0$, contradicting Lemma 5.9(iii). It follows that $y(v_6 v_1 s_2 v_6)$ is a positive integer for each $i \in \{2, 3, 4\}$.

Assume next that $s_2 v_6$ is contained in some cycle $C \in C''_y$. By Lemma 3.5(vii), we obtain $y(v_6 v_1 s_2 v_6) = \min \{w(v_6 v_1), w(v_1 s_2)\}$ for $i \in \{2, 3, 4\}$. If $y(v_6 v_1 s_2 v_6) = 0$ for $i \in \{2, 3, 4\}$, then $\tau_w(T_2 \setminus a_2) = w(v_6 v_1) + \sum_{i=2}^{4} \min \{w(v_6 v_i), w(v_i s_2)\} = 0$ for $i = 1, 2, 3$. By Lemma 5.2(iii), we have $z(s_3 v_6) = w(s_3 v_6) > 0$.

Case 3. $|\varphi(s_i)| = 1$ for exactly two $i$’s in $\{1, 2, 3, 4\}$.

In this case, by Lemma 5.10(i), (2) and (3), we may assume that $\varphi(s_1) = \{v_1\}$ for $i = 1, 2$ and $\varphi(s_3) = \{v_3, v_4\}$. By (4), we obtain

(9) $C''_y \subseteq \{v_6 v_1 s_1 v_6, v_6 v_2 s_2 v_6, v_6 v_3 s_3 v_6, v_6 v_4 s_4 v_6\}$.

From Lemma 3.5(vii), we see that $y(v_6 v_1 s_1 v_6) = \min \{w(v_6 v_1), w(v_1 s_1), w(s_1 v_6)\}$ for $i = 1, 2$. If $y(v_6 v_1 s_1 v_6) > 0$, we are done. So we assume that $y(v_6 v_1 s_1 v_6) = 0$. Since $w(v_1 s_1) w(s_1 v_6) > 0$, we obtain $w(v_6 v_1) = \min \{w(v_6 v_1), w(v_1 s_1), w(s_1 v_6)\} = 0$ for $i = 1, 2$. By Lemma 5.2(iii), we have $z(s_3 v_6) = w(s_3 v_6) > 0$.

Assume first that $s_3 v_6$ is outside $C''_y$. Then $v_1 s_3$ is outside $C''_y$ for $i = 3, 4$, and $s_3 v_6$ is saturated by $y$ in $T_2$. So $y(v_6 v_3 s_3 v_6) + y(v_6 v_4 s_3 v_6) = w(s_3 v_6) > 0$. Observe that both $y(v_6 v_3 s_3 v_6)$ and $y(v_6 v_4 s_3 v_6)$ are integral, for otherwise, since $0 < y(v_6 v_3 s_3 v_6) < w(v_1 s_3)$ for $i = 3, 4$, by Lemma 3.1(i) and (ii), we have $x(v_3 s_3) = x(v_4 s_3) = 0$, contradicting Lemma 5.9(iii). It follows that $y(v_6 v_1 s_2 v_6)$ is a positive integer for $i = 3, 4$.

Assume next that $s_3 v_6$ is contained in some cycle $C \in C''_y$. By Lemma 3.5(vii), we obtain $y(v_6 v_1 s_3 v_6) = \min \{w(v_6 v_1), w(v_1 s_3)\}$ for $i = 3, 4$. If $y(v_6 v_1 s_3 v_6) = 0$ for $i = 3, 4$, then $\tau_w(T_2 \setminus a_2) = \sum_{i=1}^{2} w(v_6 v_i) + \sum_{i=3}^{4} \min \{w(v_6 v_i), w(v_i s_3)\} = 0$, contradicting (a). Hence $y(v_6 v_1 s_3 v_6)$ is a positive integer for $i = 3$ or 4.

Case 4. $1 < |\varphi(s_i)| < 4$ if $\varphi(s_i) \not= 0$, for $i \in \{1, 2, 3, 4\}$. 

56
In this case, by Lemma 5.10(i), (2) and (3), we may assume that \( \varphi(s_1) = \{v_1, v_2\} \) and \( \varphi(s_2) = \{v_3, v_4\} \). By (4), we obtain

\[
(10) \quad C_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_1v_6, v_6v_3s_2v_6, v_6v_4s_2v_6\}.
\]

By Lemma 5.2(iii), we have \( z(s_i, v_6) = w(s_i, v_6) > 0 \) for \( i = 1, 2 \).

Assume first that \( s_1v_6 \) is outside \( C_0^y \). Then both \( v_1s_1 \) and \( v_2s_1 \) are outside \( C_0^y \), and \( s_1v_6 \) is saturated by \( y \) in \( T_2 \). So \( y(v_6v_1s_1v_6) + y(v_6v_2s_1v_6) = w(s_1, v_6) > 0 \). Observe that both \( y(v_6v_1s_1v_6) \) and \( y(v_6v_2s_1v_6) \) are integral, for otherwise, since \( 0 < y(v_6v_1s_1v_6) < w(v_1s_1) \) for \( i = 1, 2 \), by Lemma 3.1(i) and (ii), we have \( x(v_1s_1) = x(v_2s_1) = 0 \), contradicting Lemma 5.9(iii).

It follows that \( y(v_6v_1s_1v_6) \) is a positive integer for \( i = 1, 2 \). Similarly, we can show that if \( s_2v_6 \) is outside \( C_0^y \), then \( y(v_6v_1s_2v_6) \) is a positive integer for \( i = 3, 4 \).

Assume next that \( s_1v_6 \) is contained in some cycle in \( C_0^y \) for \( i = 1, 2 \). By Lemma 3.5(vii), we have \( y(v_6v_1s_1v_6) = \min \{w(v_6v_1), w(v_1s_1)\} \) for \( i = 1, 2 \), and \( y(v_6v_1s_2v_6) = \min \{w(v_6v_1), w(v_1s_2)\} \) for \( i = 1, 2 \). If \( y(v_6v_1s_1v_6), y(v_6v_2s_1v_6), y(v_6v_3s_2v_6), y(v_6v_4s_2v_6) \), and \( y(v_6v_5s_2v_6) \), and \( y(v_6v_4s_2v_6) \) are all zero, then \( \tau_w(T_2, a_2) = \sum_{i=1}^{4} \min \{w(v_6v_1), w(v_1s_1)\} + \sum_{i=3}^{4} \min \{w(v_6v_1), w(v_1s_2)\} = 0 \), contradicting (\( \alpha \)). So at least one of \( y(v_6v_1s_1v_6), y(v_6v_2s_1v_6), y(v_6v_3s_2v_6), y(v_6v_4s_2v_6) \), and \( y(v_6v_4s_2v_6) \) is a positive integer.

Case 5. \( |\varphi(s_i)| > 2 \) if \( \varphi(s_i) \neq \emptyset \), for \( i = 1, 2, 3, 4 \).

In this case, by Lemma 5.10(i), (2) and (3), we may assume that \( \varphi(s_1) = \{v_1, v_2, v_3, v_4\} \). By (4), we obtain

\[
(11) \quad C_2^y \subseteq \{v_6v_1s_1v_6, v_6v_2s_1v_6, v_6v_3s_1v_6, v_6v_4s_1v_6\}.
\]

By Lemma 5.2(iii), we have \( z(s_1, v_6) = w(s_1, v_6) > 0 \).

Assume first that \( s_1v_6 \) is outside \( C_0^y \). Then \( \sum_{i=1}^{4} y(v_6v_1s_1v_6) = w(s_1, v_6) \). If \( y(v_6v_1s_1v_6) \) is a positive integer for some \( i \in \{1, 2, 3, 4\} \), we are done. So we assume the contrary. Thus at least two of \( y(v_6v_1s_1v_6), y(v_6v_2s_1v_6), y(v_6v_3s_1v_6) \), and \( y(v_6v_4s_1v_6) \) are not integral, say \( y(v_6v_1s_1v_6) \) and \( y(v_6v_2s_1v_6) \). Since \( 0 < y(v_6v_1s_1v_6) < w(v_1s_1) \) for \( i = 1, 2 \), by Lemma 3.1 (i) and (ii), we have \( x(v_1s_1) = x(v_2s_1) = 0 \), contradicting Lemma 5.9(iii).

Assume next that \( s_1v_6 \) is contained in some cycle of \( C_0^y \). By Lemma 3.5(vii), we have \( y(v_6v_1s_1v_6) = \min \{w(v_6v_1), w(v_1s_1)\} \) for \( 1 \leq i \leq 4 \). If \( y(v_6v_1s_1v_6) \) is zero for \( 1 \leq i \leq 4 \), then \( \tau_w(T_2, a_2) = \sum_{i=1}^{4} \min \{w(v_6v_1), w(v_1s_1)\} = 0 \), contradicting (\( \alpha \)). So \( y(v_6v_1s_1v_6) \) is a positive integer for some \( i \in \{1, 2, 3, 4\} \). This proves the Claim.

From the above Claim and Lemma 3.2(iii), we conclude that \( D(T, w) \) has an integral optimal solution. \( \square \)