Stochastic Homogenization

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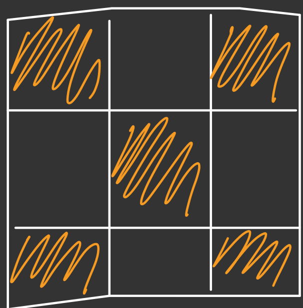
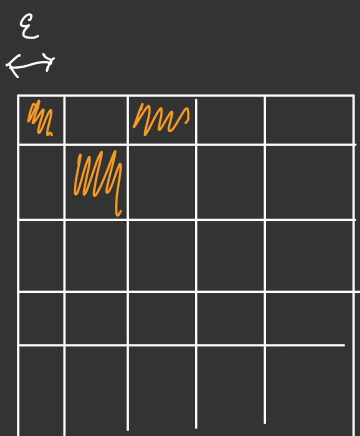
Stochastic Homogenization

$A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$

1-periodic

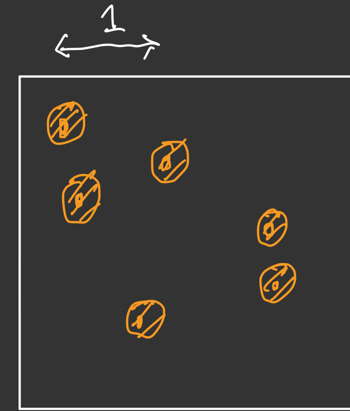


$$-\nabla\cdot a\nabla u = f$$



$$-\nabla\cdot a(x/\varepsilon)\nabla u^\varepsilon = f$$

what happens as $\varepsilon\to 0$?



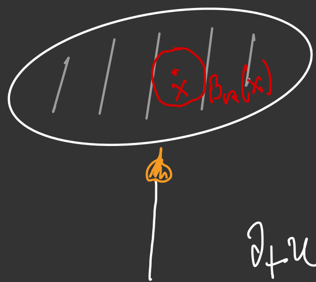
$$A : \mathbb{R}^d\times\Omega \to \mathbb{R}^{d\times d}$$

$$-\nabla\cdot A(x/\varepsilon, \omega)\nabla u^\varepsilon = f$$

$\exists$ a constant coefficient $\bar{A}$ such that a.s. $u^\varepsilon \to \bar{u}$ for

$$-\nabla\cdot \bar{A}\nabla\bar{u} = f$$

§ The Heat Equation



u = density of heat

f = source/sink = "flame"

$$\partial_t u = \Delta u + f$$

$$\partial_t \Big(\int_{B_r(x)} u \Big) = \int_{B_r(x)} f + \oint_{B_r(x)} \nabla u \cdot \nu$$

$$= \int_{B_R(x)} f + \int_{B_R(x)} \nabla \cdot (\nabla u)$$

$$= \int_{B_R} f + \int_{B_R} \Delta u$$

$$\Rightarrow \quad \partial_t u = \Delta u + f$$

As $t \to \infty$, $u \to$ equilibrium

$$\partial_t u \to 0$$

As $t \to \infty$, $u \to v$ for

$$\Delta v + f = 0.$$

$$0 = \partial_t \Big(\int_{B_R(x)} v \Big) = \int_{B_R(x)} f + \oint_{\partial B_R(x)} \nabla u \cdot \nu$$

$$= \int_{B_R(x)} f + \int_{B_R(x)} \Delta u$$

$$\Rightarrow \quad \Delta u + f = 0.$$

* assume homogeneity of material

{Diffusion processes

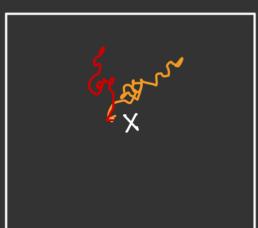
$\partial_t u = \Delta u$ on $\mathbb{R}^d \times (0,\infty)$

$u = u_0$ on $\mathbb{R}^d \times \{0\}$,

$$u(x,t) = \int_{\mathbb{R}^d} (2\pi t)^{-d/2} u_0(y) \exp\left(-\frac{|y-x|^2}{2t}\right) dy$$

A Brownian $(B_t^x)_{t\in[0,\infty)}$ starting from x satisfies

$$\mathbb{P}[B_t^x \in A] = \int_A (2\pi t)^{-d/2} \exp\left(-\frac{|y-x|^2}{2t}\right) dt$$



$$u(x,t) = \mathbb{E}\left[u_0(B_t^x)\right].$$

$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\partial_t u = \nabla \cdot A \nabla u$

For independent BM B_t^1, B_t^2,

$$dX_t = \begin{pmatrix} dX_t^1 \\ dX_t^2 \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_1}\, dB_t^1 \\ \sqrt{\lambda_2}\, dB_t^2 \end{pmatrix}$$

then

$$u(x,t) = \mathbb{E}_x\left[u_0(X_t)\right].$$

Assume $\lambda_1 >> 1 >> \lambda_2$

$$\frac{1}{2}\int_{\mathbb{R}^d} u(x,t)^2 + \int_0^t \int_{\mathbb{R}^d} \lambda_1 (\partial_1 u)^2 + \lambda_2 (\partial_2 u)^2 = \int_{\mathbb{R}^d} u_0^2$$

§ General divergence form equations

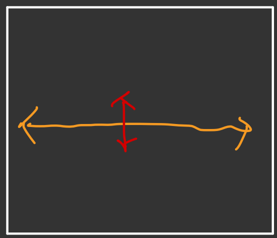
$A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$

$\partial_t u = \nabla\cdot A\nabla u + f.$

$$\partial_t\Big(\int_{B_r(x)} u\Big) = \int_{B_r(x)} f + \oint_{\partial B_r(x)} A(y)\nabla u(y)\cdot \nu$$

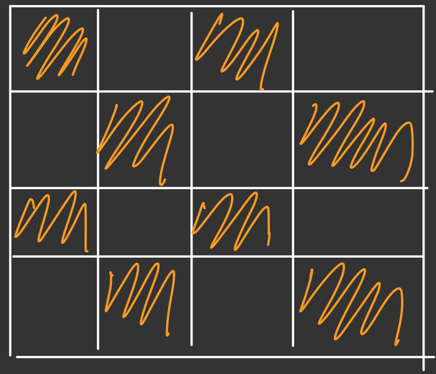
$$A = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \qquad = \int_{B_r(x)} f + \int_{B_r} \nabla\cdot A(y)\nabla u(y)$$

$\lambda_1 \gg 1 \gg \lambda_2$



$\Rightarrow \partial_t u = \nabla\cdot A\nabla u + f$

$-\nabla\cdot A\nabla v = f$ in equilibrium.



$A(y) = \lambda_1 \, Id$ on orange

$A(y) = \lambda_2 \, Id$ on black

$\lambda_1 \gg \lambda_2.$

$$\min \int \nabla u\cdot a\nabla u$$

$u-g \in H_0^1$

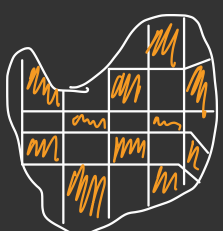
§ Periodic Homogenization

- $A \in \mathbb{R}^{d\times d}$ is uniformly elliptic if $\exists\ \lambda, \Lambda \in (0,\infty)$

$$|A\xi| \le \Lambda|\xi| \quad \text{and} \quad A\xi\cdot\xi \ge \lambda|\xi|^2 \ \forall \xi \qquad (*)$$

- $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, 1-periodic, $\exists\ \lambda, \Lambda$ such that $A(x)$ satisfies $(*)$ $\forall x \in \mathbb{R}^d$.

U



$$\boxed{\begin{cases} -\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon = f \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases}}$$

$$A(x+y) = A(x) \quad \forall y \in \mathbb{Z}^d$$

$$H^1_0(U) = \{u \in H^1(U) \mid \operatorname{Tr} u = g\}$$

$$= \overline{C^\infty(U)}^{H^1_0(U)}$$

$$\langle u, v\rangle_{H^1_0(U)} = \int_U \nabla u\cdot\nabla v. \qquad (*)$$

By Poincaré,

$$\|u\|_{L^2(U)} \le C\|\nabla u\|_{L^2(U)}$$

for all $u \in H^1_0(U)$

$$\|u\|_{H^1(U)} \le C\|\nabla u\|_{L^2}.$$

$$\|$$

$$\|u\|_{L^2} + \|\nabla u\|_{L^2}$$

$$\|u\|_{H^1_0(U)} = \|\nabla u\|_{L^2}.$$

§ Periodic Homogenization

$$\begin{cases} -\nabla \cdot A(x/\varepsilon) \nabla u^\varepsilon = f & \mathcal{U} \\ u = 0 & \partial \mathcal{U} \end{cases}$$

<u>Def</u> A weak a solution is a function $u \in H_0^1(\mathcal{U})$ such that

$$\int_{\mathcal{U}} A(x/\varepsilon) \nabla u \cdot \nabla v = \int_{\mathcal{U}} f \cdot v \quad \forall v \in H_0^1(\mathcal{U})$$

<u>Prop</u> $\forall \varepsilon \in (0,1)$, $\forall f \in H^{-1}(\mathcal{U}) = (H_0^1(\mathcal{U}))^*$ $\exists!$ weak solution u^ε.

Furthermore,

$$\|u^\varepsilon\|_{H_0^1} \leq \frac{2}{\lambda} \|f\|_{H^{-1}(\mathcal{U})}.$$

<u>Proof</u> By ellipticity, $\forall u, v \in H_0^1(\mathcal{U})$,

$$\left| \int A^\varepsilon \nabla u \cdot \nabla v \right| \leq C \|\nabla v\|_2 \cdot \|\nabla u\|_2 = c \|v\|_{H_0^1} \|u\|_{H_0^1}$$

$$\int_{\mathcal{U}} A^\varepsilon \nabla u \cdot \nabla u \geq \lambda \int_{\mathcal{U}} |\nabla u|^2 = \lambda \|u\|_{H_0^1}^2.$$

By Lax-Milgram, $\exists!$ u^ε such that

$$\int_{\mathcal{U}} A(x/\varepsilon) \nabla u^\varepsilon \cdot \nabla v = \langle f, v \rangle \quad \forall v \in H_0^1$$

"$f = f_1 + \nabla \cdot F_2$ for $f_1, f_2 \in L^2(\mathcal{U})$"

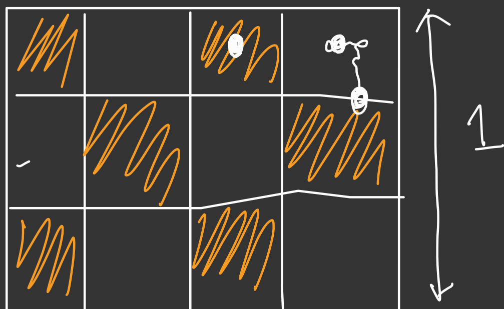
Take $v = u^\varepsilon$, by ellipticity,

$$\lambda \|u^\varepsilon\|_{H_0^1}^2 \leq \|f\|_{H^{-1}} \cdot \|u^\varepsilon\|_{H_0^1}.$$

$$\|u^\varepsilon\|_{H_0^1} \leq \frac{1}{\lambda} \|f\|_{H^{-1}}$$

§ An explicit example

$\bar{A} \neq \int_{\mathbb{T}^d} A(y)\,dy.$



$A \sim 0$ in orange

$A \sim 1$ in black

Then $\int_{\mathbb{T}^d} A(y)\,dy \approxeq 1/2.$

$A \to 0$ in orange regions

$u^\varepsilon \to \bar{u}$ $\qquad \partial_t \bar{u} = 0$

$$-\nabla \cdot a(x/\varepsilon) \nabla u^\varepsilon = f(x) \qquad (0,1)$$

$$u^\varepsilon = 0 \qquad \{0,1\}$$

Integrating,

$$a(x/\varepsilon) \nabla u^\varepsilon = c + \int_0^x f(y)\,dy$$

$$\nabla u^\varepsilon = \frac{-1}{a(x/\varepsilon)}\left[c + \int_0^x f(y)\,dy\right].$$

If f is 1-periodic, then

$$f(x/\varepsilon) \rightharpoonup \int_{\mathbb{T}^d} f(y)\,dy =: \langle f \rangle$$

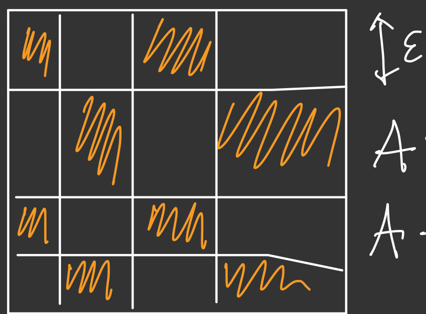
As $\varepsilon \to 0$,

$$\nabla u^\varepsilon \rightharpoonup -\langle a^{-1} \rangle \left[c + \int_0^x f(y)\,dy\right]$$

$$\nabla \bar{u} = -\langle a^{-1} \rangle \left[c + \int_0^x f(y)\,dy\right]$$

$$-\nabla \cdot \langle a^{-1} \rangle^{-1} \nabla \bar{u} = f$$

§ Periodic Homogenization



$A : \mathbb{R}^d \to \mathbb{R}^{d \times d}$

A - 1 periodic

uniformly elliptic

$\forall x \in \mathbb{R}^d, \ \forall y \in \mathbb{Z}^d$

$$A(x+z) = A(x)$$

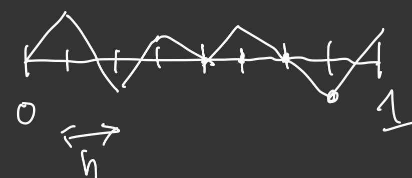
$\exists \ \lambda, \Lambda \in (0, \infty)$

$$|A\xi| \le \Lambda |\xi| \quad \& \quad A\xi \cdot \xi \ge \lambda |\xi|^2$$

$$-\nabla \cdot A(x/\varepsilon) \nabla u^\varepsilon = f \qquad \mathcal{U}$$

$$u^\varepsilon = 0 \qquad \partial \mathcal{U}$$

What happens in the $\varepsilon \to 0$ limit?



If $h >> \varepsilon$, $h = \varepsilon$,

numerical scheme approximate

$$-\nabla \langle A \rangle \nabla \tilde{u} = f, \quad \langle A \rangle = \int_{\mathbb{T}^d} A(y)\,dy$$

$-\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = f$

We postulate:

$$u^\varepsilon(x) = u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \dots$$
$$= \varepsilon^K u_K(x, x/\varepsilon)$$

Write x for slow variable, and '$y = x/\varepsilon$' for fast variable.

$$'\nabla = \nabla_x + \varepsilon^{-1}\nabla_y'$$

$$\nabla u_1(x,y) = \nabla_x u_1 + \varepsilon^{-1}\nabla_y u_1$$

$$\int_U A^\varepsilon \nabla u^\varepsilon \cdot \nabla \phi = \int_U A^\varepsilon(\varepsilon^{-1}\nabla_y u_0 + \nabla_x u_0 + \nabla_y u_1 + \varepsilon \nabla_x u_1 + O(\varepsilon)) \cdot \nabla \phi$$

$$-\tfrac{1}{2}\operatorname{tr}(A(x/\varepsilon)\nabla^2 u^\varepsilon) + \tfrac{1}{\varepsilon} b(x/\varepsilon)\cdot \nabla u^\varepsilon = f$$

*need u_2

$$u^\varepsilon = u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon)$$

$$-(\nabla_x + \varepsilon^{-1}\nabla_y)\cdot A(y)(\varepsilon^{-1}\nabla_y u_0 + \nabla_x u_0 + \nabla_y u_1 + \varepsilon \nabla_x u_1) = f$$

Order ε^{-2}:

$$-\nabla_y \cdot A(y)\nabla_y u_0(x,y) = 0 \quad \text{in } \mathbb{T}^d \quad \forall x \in U$$

So, $u_0(x,y) = u_0(x)$ is constant in y, $\forall x \in U$

$$u^\varepsilon(x) = u_0(x) + \varepsilon u_1(x,y)$$

$$-(\nabla_x + \varepsilon^{-1}\nabla_y)\cdot A(y)\left(\nabla_x u_0 + \nabla_y u_1 + \varepsilon \nabla_x u_1\right) = f$$

Order ε^{-1}:

$$-\nabla_y \cdot A(y)\left(\nabla_x u_0(x) + \nabla_y u_1(x,y)\right) = 0 \quad \text{in } \mathbb{T}^d \quad \forall x \in U$$

We postulate:

$$u_1(x,y) = \phi_i(y)\,\partial_i u_0(x)$$

the equation becomes

$$-\nabla_y \cdot A(y)\cdot\left(e_i + \nabla\phi_i(y)\right)\partial_j u_0(x) = 0.$$

$\forall i \in \{1,2,\dots,d\}$ we require

$$-\nabla_y \cdot A(y)\left(e_i + \nabla\phi_i\right) = 0 \quad \text{in } \mathbb{T}^d_i$$

* cell problem
* corrector equation
* ϕ_i homogenization correctors

Order 1:

$$-\nabla\cdot A(y)\left(e_i + \nabla\phi_i\right)\partial_i u_0 = f$$

Or, in ε-scaling,

$$-\nabla\cdot A(x/\varepsilon)\left(e_j + \nabla\phi_i(x/\varepsilon)\right)\partial_i u_0(x) = f(x)$$

$$u^\varepsilon = u_0(x) + \varepsilon \phi_i(x/\varepsilon)\,\partial_i u_0(x)$$

$$-\nabla\cdot A(y)(e_i + \nabla\phi_y) = 0 \quad \text{in } \mathbb{T}^d$$

$$-\nabla\cdot A(x/\varepsilon)\big(e_i + \nabla\phi_i(x/\varepsilon)\big)\,\partial_i u_0(x) = f(x) \quad \text{in } U$$

$$\varepsilon \to 0 \Big\downarrow$$

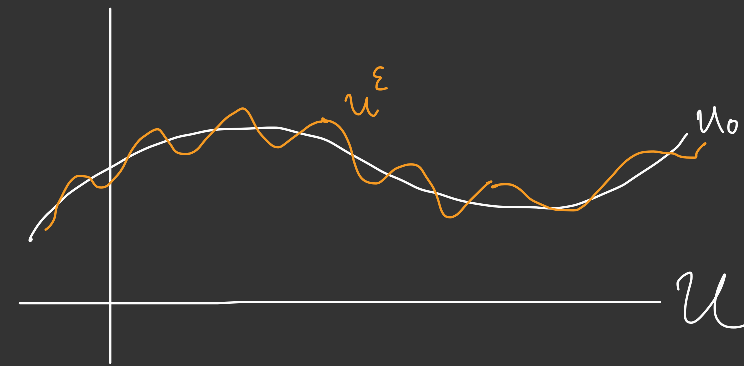
$$\int_{\mathbb{T}^d} A(y)\big(e_i + \nabla\phi_i(y)\big)\,dy = \langle A(e_i + \nabla\phi_i)\rangle.$$

We define $\bar{A}$ by the rule

$$\bar{A} e_i = \langle A(e_i + \nabla\phi_i)\rangle.$$

$$-\nabla\cdot \bar{A}\,\nabla u_0 = f \quad \text{in } U$$

$$u = 0 \quad \text{on } \partial U.$$



$$u^\varepsilon \not\to u \quad \text{in } H_0^1$$

$$u^\varepsilon - \varepsilon\phi_i(x/\varepsilon)\,\partial_i u_0 \to u_0 \quad \text{in } H_0^1$$

§ Correctors

$H_0^1(\mathbb{T}^d) = H^1$-functions with mean zero

By Poincaré,

$$\|u\|_{H_0^1(\mathbb{T}^d)} = \int_{\mathbb{T}^d} |\nabla u|^2 dx$$

$$\langle u, v \rangle_{H_0^1(\mathbb{T}^d)} = \int_{\mathbb{T}^d} \nabla u \cdot \nabla v$$

$\forall i \in \{1, 2, \dots, d\}$ $\exists!$ $\phi_i \in H_0^1(\mathbb{T}^d)$ with

$$-\nabla \cdot A(e_i + \nabla \phi_i) = 0 \quad \text{in } \mathbb{T}^d!$$

$u \in C^2(\mathbb{R}^d)$, $-\Delta u = 0$, $\lim_{|x|\to\infty} \frac{|u(x)|}{|x|^2} = 0$

Liouville $\Rightarrow$ $\exists\, c \in \mathbb{R}$, $\xi \in \mathbb{R}^d$ s.t.

$$u(x) = c + \xi \cdot x$$

For correctors,

$$-\nabla \cdot A(e_i + \nabla \phi_i) = -\nabla \cdot A \nabla(x_i + \phi_i) = 0$$

$\{x_i + \phi_i\}_{i=1}^d$ are 'A-harmonic'

If $u : \mathbb{R}^d \to \mathbb{R}$ solves

$-\nabla \cdot A \cdot \nabla u = 0$ and $\lim_{|x|\to\infty} \frac{|u(x)|}{|x|^2} = 0$,

then $\exists\, c \in \mathbb{R}$, $\xi \in \mathbb{R}^d$ so that

$$u(x) = c + \xi \cdot x + \phi_\xi(x)$$

where $\phi_\xi(x) = \xi_i \phi_i(x)$

§ Homogenized Coefficient

$$\bar{A} e_i = \langle A(e_i + \nabla\phi_i) \rangle$$

$$\bar{A}\xi = \langle A(\xi + \nabla\phi_\xi) \rangle$$

where for $\xi = (\xi_1, \ldots, \xi_d)$

$$\phi_\xi = \xi_i \phi_i.$$

$$u^\varepsilon = u_0 + \varepsilon \phi_i(x/\varepsilon)\partial_i u_0$$

$$-\nabla\cdot \bar{A}\nabla u_0 = f$$

* $\bar{A}$ is uniformly elliptic

$$|\bar{A}\xi| = \left| \int_{\mathbb{T}^d} A\,\xi_i (e_i + \nabla\phi_i) \right|$$

$$\leq \Lambda |\xi| \left(\sum_{i=1}^{d} \langle |e_i + \nabla\phi_i|^2 \rangle \right)^{1/2}.$$

$$\bar{A}\xi\cdot\xi = \langle A(\xi + \nabla\phi_\xi) \rangle \cdot \xi$$

$$= \langle A(\xi + \nabla\phi_\xi)\cdot \xi \rangle$$

$$= \langle A(\xi + \nabla\phi_\xi)\cdot(\nabla\phi_\xi + \xi) \rangle$$

$$\geq \lambda \langle |\xi + \nabla\phi_\xi|^2 \rangle$$

$$\geq \lambda |\langle \xi + \nabla\phi_\xi \rangle|^2$$

$$= \lambda |\xi|^2.$$

§ Adjoint

$$-\nabla \cdot A(e_i + \nabla \phi_i) = 0$$

$$-\nabla \cdot A^{+}(e_i + \nabla \phi_i^{+}) = 0$$

$$\bar{A} e_i = \langle A(\nabla \phi_i + e_i) \rangle$$

$$\tilde{A} e_i = \langle A^{+}(\nabla \phi_i^{+} + e_i) \rangle$$

$$\begin{aligned}
\bar{A}_{ji} &= \langle A(\nabla \phi_i + e_i) \rangle \cdot e_j \\
&= \langle A(\nabla \phi_i + e_i) \cdot (\nabla \phi_j^{+} + e_j) \rangle \\
&= \langle (\nabla \phi_i + e_i) \cdot A^{+} (\nabla \phi_j^{+} + e_j) \rangle \\
&= \tilde{A}_{ij}
\end{aligned}$$

So, we have that

$$\tilde{A} = \bar{A}^{+}.$$

If A is symmetric, then $\phi_i = \phi_i^{+}$ and

$$\bar{A} = \tilde{A} = \bar{A}^{+}.$$

So, $\bar{A} = \bar{A}^{+}$.

§ Helmholtz Decomposition

$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon = f$, $u^\varepsilon = 0$

$\|u^\varepsilon\|_{H_0^1} \le \frac{1}{\lambda}\|f\|$

$u^\varepsilon \rightharpoonup u$ in H_0^1

$A(x/\varepsilon)\nabla u^\varepsilon \rightharpoonup \bar{A}\nabla u$?

$A(x/\varepsilon) \rightharpoonup \langle A\rangle$

$\nabla u^\varepsilon \rightharpoonup \nabla u$

$A(x/\varepsilon)\nabla u^\varepsilon \not\rightharpoonup \langle A\rangle\nabla u$

$V\in L^2(U;\mathbb{R}^d)$

$\partial_i V_j = f$ if $-\int_U V_j\partial_i\phi = \int_U f\phi$ $\forall\phi\in C_c^\infty$

$L^2_{pot}(U) = \{V\in L^2 : V=\nabla w$ for some $w\in H_0^1(U)\}$

$L^2_{sol}(U) = \{V\in L^2 : \underbrace{\partial_i V_i}_{\nabla\cdot V} = 0\}$.

Prop) $L^2(U;\mathbb{R}^d) = L^2_{pot}(U)\oplus L^2_{sol}(U)$.

Proof) Let $\phi\in H_0^1$ solve $\Delta\phi = \nabla\cdot V$.

$(V-\nabla\phi)$ is div. free.

$V = \nabla\phi + (V-\nabla\phi)$

$\exists W\in L^2_{sol}, z\in H_0^1$ $\int_U \nabla z\cdot W = 0$.

§ Recap

$$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon = f$$

$$u^\varepsilon = u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2 + \cdots$$

$$u^\varepsilon = v(x) + \varepsilon u_1(x, x/\varepsilon).$$

$$\varepsilon u_1(x, x/\varepsilon) \cong \varepsilon \phi_i(x/\varepsilon)\,\partial_i v$$

$$-\nabla\cdot A(y)\underbrace{\left(e_i + \nabla\phi_i(y)\right)}_{\nabla(x_i+\phi_i)} = 0 \quad \text{in } \mathbb{T}^d$$

$\exists!\ \phi_i \in H^1_0(\mathbb{T}^d)$ satisfying this eq.

$$\bar{A}e_i = \langle A(e_i+\nabla\phi_i)\rangle = \int_{\mathbb{T}^d} A(e_i+\nabla\phi_i)\,dy.$$

$$-\nabla\cdot\bar{A}\nabla v = f.$$

$$-\nabla\cdot a(x/\varepsilon)\nabla u^\varepsilon = f \quad \text{in } (0,1)$$

$$\nabla u^\varepsilon = a(x/\varepsilon)^{-1}\left(c - \int_0^x f(y)\,dy\right)$$

$$\nabla u^\varepsilon \xrightarrow{\varepsilon\to 0} \langle a^{-1}\rangle\left(c - \int_0^x f(y)\,dy\right).$$

$$\nabla v = \langle a^{-1}\rangle\left(c - \int_0^x f(y)\,dy\right)$$

$$-\nabla\cdot\langle a^{-1}\rangle^{-1}\nabla v = 0.$$

$$\nabla u^\varepsilon \overset{\varepsilon\to 0}{\rightharpoonup} \nabla v \quad \text{in } L^2$$

$$\nabla u^\varepsilon \not\to \nabla v \quad \text{strongly in } L^2.$$

$$u^\varepsilon \overset{\varepsilon\to 0}{\rightharpoonup} v \quad \text{in } H^1_0$$

$$\nabla u^\varepsilon \cong \nabla v + \nabla\phi_i(x/\varepsilon)\,\partial_i v + \varepsilon\phi_i(x/\varepsilon)\,\nabla\partial_i v$$

$$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon = f \quad \text{in } \mathcal{U}$$

$$\|u^\varepsilon\|_{H_0^1} \lesssim C\|f\|_{L^2} \text{ unif. in } \varepsilon.$$

Along a subsequence,

$$u^\varepsilon \rightharpoonup v \quad \text{for } v\in H_0^1(\mathcal{U})$$

Need understand,

$$A(x/\varepsilon)\nabla u^\varepsilon \longrightarrow \ ?$$

Aim to show that $u^\varepsilon \rightharpoonup v$

for $\quad -\nabla\cdot\bar{A}\nabla v = f$ in $\mathcal{U}$.

$$L^2(\mathcal{U};\mathbb{R}^d) = L^2_{pot}(\mathcal{U}) \oplus L^2_{sol}(\mathcal{U})$$

$$L^2_{pot}(\mathcal{U}) = \{\nabla v : v\in H_0^1(\mathcal{U})\}$$

$$L^2_{sol}(\mathcal{U}) = \{V\in L^2(\mathcal{U}) : \partial_i V_i = 0\}.$$

If $W\in L^2_{pot}$ and $V\in L^2_{sol}$

$$\int_{\mathcal{U}} W\cdot V\,dx = 0.$$

$$W^\varepsilon \rightharpoonup W_0 \in L^2_{pot}$$

$$V^\varepsilon \rightharpoonup V_0 \in L^2_{sol}$$

$$\lim_{\varepsilon\to 0}\int W^\varepsilon\cdot V^\varepsilon = \int W_0\cdot V_0 = 0.$$

§ Div-Curl Lemma

Prop Let $\rho^\varepsilon \in L^2(U;\mathbb{R}^d)$, $v^\varepsilon \in L^2_{pot}(U)$

$$\rho^\varepsilon \xrightarrow{\varepsilon\to0} \rho_0, \quad v^\varepsilon \xrightarrow{\varepsilon\to0} v_0, \quad \nabla\cdot\rho^\varepsilon \xrightarrow[H^{-1}(U)]{\varepsilon\to0} \nabla\cdot\rho_0$$

Then, $\forall \varphi \in C_c^\infty(U)$,

$$\lim_{\varepsilon\to0}\int \rho^\varepsilon\cdot v^\varepsilon \varphi = \int \rho_0\cdot v_0\, \varphi.$$

Proof)

$$\int(\rho^\varepsilon-\rho_0)\cdot(v^\varepsilon-v_0)\varphi$$
$$= \int\rho^\varepsilon v^\varepsilon\varphi - \int\rho^\varepsilon v_0\varphi - \int v^\varepsilon\rho_0\varphi + \int v_0\rho_0\varphi$$

wlog

$$\rho^\varepsilon \to 0, \quad v^\varepsilon \to 0, \quad \nabla\cdot\rho^\varepsilon \xrightarrow[H^{-1}(U)]{} 0$$

$\int\rho^\varepsilon\cdot v^\varepsilon\varphi = \int\rho^\varepsilon\cdot\nabla u^\varepsilon\varphi$ for some $u^\varepsilon\in H_0^1$

$$= \int\rho^\varepsilon\cdot\nabla[u^\varepsilon\varphi] - \int\rho^\varepsilon\cdot\nabla\varphi\, u^\varepsilon.$$

$$\begin{array}{cc} \downarrow \varepsilon\to0 & \downarrow \varepsilon\to0 \\ 0 & 0 \end{array}$$

the final term vanishes by Sobolev,

$$u^\varepsilon \xrightarrow{\varepsilon\to0} 0 \text{ in } L^2$$

since $u^\varepsilon \xrightarrow{\varepsilon\to0} 0$ in H^1.

§ Perturbed Test Function Method

$$u^\varepsilon - v - \varepsilon \phi_i(x/\varepsilon)\, \partial_i v = w^\varepsilon$$

$$u^\varepsilon + \varepsilon \phi_i(x/\varepsilon)\, \partial_i u^\varepsilon.$$

Thm: $-\nabla \cdot A(x/\varepsilon) \nabla u^\varepsilon = f$ in U, $u^\varepsilon = 0$ ∂U

$-\nabla \cdot \bar{A} \nabla v = f$ in U, $v = 0$ ∂U

Then, as $\varepsilon \to 0$,

$$u^\varepsilon \rightharpoonup v \quad \text{in } H_0^1(U).$$

Proof) Along a subsequence,

$$u^\varepsilon \rightharpoonup v \quad \text{in } H_0^1(U).$$

wts that

$$\int_U \bar{A} \nabla v \cdot \nabla \psi = \int_U \psi f \quad \forall \psi \in C_c^\infty(U).$$

We have

$$\int_U A(x/\varepsilon) \nabla u^\varepsilon \cdot \nabla \psi = \int_U f \psi.$$

Let $\psi \in C_c^\infty(U)$.

Let $\psi^\varepsilon = \psi + \varepsilon \phi_i^+(x/\varepsilon)\, \partial_i \psi$ for

$$-\nabla \cdot A^+(\nabla \phi_i^+ + e_i) = 0.$$

Test with ψ^ε to obtain

$$\int_U A(x/\varepsilon) \nabla u^\varepsilon \cdot (e_i + \nabla \phi_i^+)\, \partial_i \psi$$

$$+ \int_U A(x/\varepsilon) \nabla u^\varepsilon \cdot \nabla(\partial_i \psi)\, \varepsilon \phi_i(x/\varepsilon)$$

$$= \int_U f \psi^\varepsilon.$$

$$\int_U \nabla u^\varepsilon \cdot A^+(x/\varepsilon)(e_i + \nabla\phi_i^+(x/\varepsilon))\, \partial_i \psi$$

$$+ \int_U A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla(\partial_i \psi)\, \varepsilon\phi_i^+(x/\varepsilon)$$

$$= \int_U f\psi^\varepsilon.$$

As $\varepsilon \to 0$, $\psi^\varepsilon \to \psi$ strongly in L^2.

$\varepsilon\phi_i(x/\varepsilon) \to 0$ strongly in L^2.

$$\lim_{\varepsilon\to 0}\left(\int_U \nabla u^\varepsilon \cdot A^+(x/\varepsilon)(e_i+\nabla\phi_i^+(x/\varepsilon))\partial_i\psi\right)$$

$$= \int_U f\psi.$$

Along a subsequence,

$$\nabla u^\varepsilon \xrightarrow{\varepsilon\to 0} \nabla v \qquad L^2_{pot}(U)$$

And, as $\varepsilon \to 0$,

$$A^+(x/\varepsilon)(e_i + \nabla\phi_i^+(x/\varepsilon))$$

$$\rightharpoonup \langle A^+(e_i + \nabla\phi_i)\rangle = \bar{A}^+ e_i$$

And,

$$-\nabla\cdot\left[A^+(x/\varepsilon)(e_i + \nabla\phi_i^+(x/\varepsilon))\right] = 0.$$

By div-curl lemma,

$$\int_U \nabla v \cdot \bar{A}^+ \nabla\psi = \int f\psi \quad \forall \psi \in C_c^\infty(U)$$

$$\int_U \bar{A}\nabla v \cdot \nabla\psi = \int_U f\psi \quad \forall \psi \in C_c^\infty(U)$$

§ The Flux

$$A(x/\varepsilon)\nabla u^\varepsilon \longrightarrow \bar{A}\nabla v \text{ in } L^2 ?$$

Test with $\xi\cdot x$ for $\xi\in\mathbb{R}^d$,

$$\int_U A(x/\varepsilon)\nabla u^\varepsilon\cdot\xi = \int_U f(\xi\cdot x).$$

For $\xi\in\mathbb{R}^d$, let $w_\xi^\varepsilon = \xi\cdot x + \varepsilon\phi_\xi^+(x/\varepsilon)$

for $\phi_\xi(y) = \xi_i\phi_i^+(y)$.

$$-\nabla\cdot A^+(\xi+\nabla\phi_\xi^+) = 0 \text{ in } \mathbb{T}^d.$$

Let $f\in C_c^\infty(U)$

Test with $f w_\xi^\varepsilon$.

Thm) $A(x/\varepsilon)\nabla u^\varepsilon \xrightarrow{\varepsilon\to 0} \bar{A}\nabla v$ in L^2.

Proof) Along a subsequence,

$$A(x/\varepsilon)\nabla u^\varepsilon \rightharpoonup F_0 \text{ in } L^2(U;\mathbb{R}^d)$$

$$\int_U F_0\cdot\nabla f = \int_U f f \quad \forall f\in C_c^\infty(U).$$

Test with $f w_\xi^\varepsilon$ to obtain

$$\int_U A(x/\varepsilon)\nabla u^\varepsilon\cdot[\xi+\nabla\phi_\xi^+]f$$
$$+\int_U A(x/\varepsilon)\nabla u^\varepsilon\cdot\nabla f\, w_\xi^\varepsilon = \int_U f f w_\xi^\varepsilon.$$

Test the w_ξ^ε equation with $f u^\varepsilon$,

$$\int_U A^+(x/\varepsilon)\nabla w_\xi^\varepsilon\cdot\nabla u^\varepsilon f + \int_U A^+(x/\varepsilon)\nabla w_\xi^\varepsilon\cdot\nabla f u^\varepsilon = 0.$$

So,

$$\int_{\mathcal{U}} A(x/\varepsilon)\nabla u^{\varepsilon}\cdot\nabla \phi\, w_{\xi}^{\varepsilon} - \int_{\mathcal{U}} A^{t}(x/\varepsilon)\nabla w_{\xi}^{\varepsilon}\cdot\nabla \phi\, u^{\varepsilon}$$

$$= \int_{\mathcal{U}} f\phi\, w_{\xi}^{\varepsilon}.$$

As $\varepsilon\to 0$, $w_{\xi}^{\varepsilon} \mapsto \xi\cdot x$ strongly in L^2.

Similarly, $u^{\varepsilon}\mapsto v$ in L^2 and

$$A^{t}\nabla w_{\xi}^{\varepsilon}(x/\varepsilon) \rightharpoonup \bar{A}^{t}\xi.$$

After passing $\varepsilon\to 0$, along a subsequence,

$$\int_{\mathcal{U}} F_0\cdot\nabla\phi\cdot(\xi\cdot x) - \int_{\mathcal{U}} \bar{A}^{t}\xi\cdot\nabla\phi\, v = \int_{\mathcal{U}} f\phi\,(\xi\cdot x).$$

$$\int_{\mathcal{U}} F_0\cdot\nabla\phi\,(\xi\cdot x) = \int_{\mathcal{U}} F_0\nabla(\phi(\xi\cdot x))$$

$$-\int_{\mathcal{U}} F_0\cdot\xi\phi$$

$$= \int_{\mathcal{U}} f\phi(\xi\cdot x) - \int_{\mathcal{U}} F_0\cdot\xi\cdot\phi.$$

$$-\int_{\mathcal{U}} \bar{A}^{t}\xi\,\nabla\phi\, v = \int_{\mathcal{U}} \bar{A}^{t}\xi\cdot\nabla v\,\phi.$$

Therefore,

$$\int_{\mathcal{U}} F_0\cdot\xi\,\phi = \int_{\mathcal{U}} \bar{A}\nabla v\cdot\xi\,\phi.$$

So, $F_0 = \bar{A}\nabla v.$

§ Strong convergence

$$w^\varepsilon = u^\varepsilon - v - \varepsilon \phi_i^\varepsilon(x/\varepsilon)\partial_i v.$$

$$-\nabla\cdot A(x/\varepsilon)\nabla w^\varepsilon$$

$$= \nabla\cdot\left(\left[A(x/\varepsilon)(e_i+\nabla\phi_i) - \bar{A}e_i\right]\partial_i v\right)$$

$$+ \nabla\cdot\left[A(x/\varepsilon)\,\varepsilon\phi_i(x/\varepsilon)\nabla[\partial_i v]\right].$$

$w^\varepsilon \to 0$ strongly in $H^1(\mathcal{U})$.

§ Recap

$$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon = f \quad \text{in } \mathcal{U}$$

$$u^\varepsilon = v(x) + \varepsilon\phi_i(x/\varepsilon)\partial_i v$$

$$-\nabla\cdot A(y)\underbrace{(\nabla\phi_i + e_i)}_{\nabla(\phi_i + x_i)} = 0 \quad \text{on } \mathbb{T}^d$$

$$\bar{A}e_i = \int_{\mathbb{T}^d} A(\nabla\phi_i + e_i)\,dy = \langle A(\nabla\phi_i+e_i)\rangle$$

$$-\nabla\cdot\bar{A}\nabla v = f \quad \text{in } \mathcal{U}$$

As $\varepsilon \to 0$

$$u^\varepsilon \rightharpoonup v \qquad H^1_0(\mathcal{U})$$

$$A(x/\varepsilon)\nabla u^\varepsilon \rightharpoonup \bar{A}\nabla v \quad \text{in } L^2(\mathcal{U};\mathbb{R}^d)$$

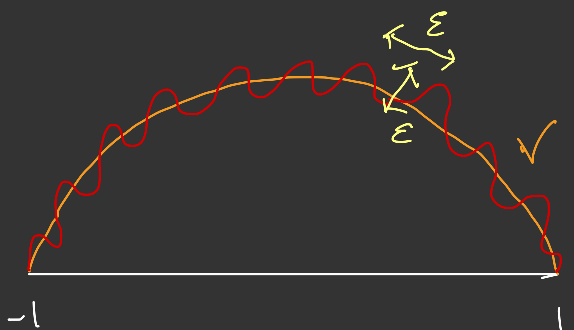
$$\sin(x/\varepsilon) \rightharpoonup 0 \quad \text{as} \quad \varepsilon \to 0$$

$$\sin^2(x/\varepsilon) \not\rightharpoonup 0 \quad \text{as} \quad \varepsilon \to 0$$

$$A(x/\varepsilon)\nabla u^\varepsilon \not\rightharpoonup \langle A \rangle \nabla v$$

$$A(x/\varepsilon) \rightharpoonup \langle A \rangle$$

$$\nabla u^\varepsilon \rightharpoonup \nabla v$$



$$u^\varepsilon = v + \varepsilon \phi_i(x/\varepsilon)\, \partial_i v$$

$$\nabla u^\varepsilon = \nabla v + \nabla \phi_i(x/\varepsilon)\, \partial_i v + \varepsilon \phi_i(x/\varepsilon)\, \nabla \partial_i v$$

§ The Homogenization Error

We define

$$w^\varepsilon = u^\varepsilon - v - \varepsilon \phi_i^\varepsilon\, \partial_i v$$

for $\phi_i^\varepsilon = \phi_i(x/\varepsilon)$

$$\nabla w^\varepsilon = \nabla u^\varepsilon - \nabla v - \nabla \phi_i^\varepsilon\, \partial_i v - \varepsilon \phi_i^\varepsilon\, \nabla \partial_i v$$

$$\begin{aligned} -\nabla \cdot A^\varepsilon \nabla w^\varepsilon &= f + \nabla \cdot A^\varepsilon \nabla v \\ &\quad + \nabla \cdot A^\varepsilon (\nabla \phi_i^\varepsilon\, \partial_i v) \\ &\quad + \nabla \cdot A^\varepsilon (\varepsilon \phi_i^\varepsilon\, \nabla(\partial_i v)). \\ &= \nabla \cdot (A^\varepsilon - \bar{A}) \nabla v \\ &\quad + \nabla \cdot A^\varepsilon (\nabla \phi_i^\varepsilon\, \partial_i v) \\ &\quad + \nabla \cdot A^\varepsilon (\varepsilon \phi_i^\varepsilon\, \nabla(\partial_i v)). \end{aligned}$$

$$-\nabla\cdot A^\varepsilon \nabla w^\varepsilon = \nabla\cdot (A^\varepsilon - \bar{A})\nabla v + \nabla\cdot A^\varepsilon (\nabla \phi_i^\varepsilon \partial_i v) + \nabla\cdot A^\varepsilon (\varepsilon \phi_i^\varepsilon \nabla(\partial_i v))$$

If we write $\nabla v = e_i \partial_i v$, we have

$$-\nabla\cdot A^\varepsilon \nabla w^\varepsilon = \nabla\cdot A^\varepsilon (\varepsilon \phi_i^\varepsilon \nabla(\partial_i v)) + \nabla\cdot \left([A^\varepsilon (\nabla \phi_i^\varepsilon + e_i) - \bar{A} e_i] \partial_i v \right)$$

For each i we write the flux

$$q_i(y) = A(y)(\nabla \phi_i(y) + e_i)$$

$$\langle q_i \rangle = \langle A(\nabla \phi_i + e_i) \rangle = \bar{A} e_i$$

So,

$$-\nabla\cdot A^\varepsilon \nabla w^\varepsilon = \nabla\cdot A^\varepsilon (\varepsilon \phi_i^\varepsilon \nabla(\partial_i v)) + \nabla\cdot (q_i^\varepsilon - \langle q_i \rangle)\partial_i v.$$

Need obtain strong convergence of

$$q_i^\varepsilon - \langle q \rangle \text{ to zero,}$$

Not true.

But, $(q_i - \langle q_i \rangle)$ is mean zero with

$$\nabla\cdot (q_i - \langle q_i \rangle) = 0.$$

§ De Rham Cohomology

$\mathbb{R}^3$

$f \in C^\infty$

$$f \longmapsto \nabla f = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$$

$$V = (V_1, V_2, V_3) = V_1 dx_1 + V_2 dx_2 + V_3 dx_3$$

$$dV = \left(\frac{\partial V_1}{\partial x_2} - \frac{\partial V_2}{\partial x_1}\right) dx_1 \wedge dx_2 = \nabla \times V$$
$$+ \left(\frac{\partial V_2}{\partial x_3} - \frac{\partial V_3}{\partial x_2}\right) dx_2 \wedge dx_3$$
$$+ \left(\frac{\partial V_3}{\partial x_1} - \frac{\partial V_1}{\partial x_3}\right) dx_1 \wedge dx_3$$

$$W = (W_1, W_2, W_3)$$

$$W = W_1\, dx_2 \wedge dx_3 + W_2\, dx_3 \wedge dx_1 + W_3\, dx_1 \wedge dx_2$$

$$dW = \left(\frac{\partial W_1}{\partial x_1} + \frac{\partial W_2}{\partial x_2} + \frac{\partial W_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3$$
$$= \nabla \cdot W$$

We have

$$\nabla \times (\nabla f) = 0.$$

$$\nabla \cdot (\nabla \times V) = 0,$$

$$H^n_{deRham} = \frac{\operatorname{Ker} d}{\operatorname{Im} d}.$$

Every divergence-free field is a curl,

If $\nabla \cdot W = 0$ then $\exists\, V$ such that

$$W = \nabla \times V.$$

§ Flux-Correction

$\widehat{dx_i \wedge dx_j} = dx_1 \wedge dx_2 \wedge \dots \wedge \hat{dx_i} \wedge dx_{i+1} \wedge \dots \wedge \hat{dx_j} \wedge \dots \wedge dx_d$

For each i, we construct a skew-symmetric matrix $\tau_i = (\tau_{ijk})$ satisfying

$$\nabla \cdot \tau_i = q_i - \langle q_i \rangle \quad (*)$$

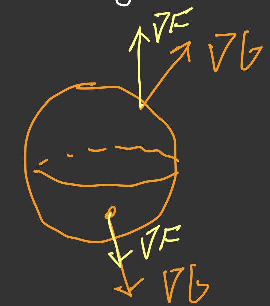
$$(\nabla \cdot \tau_i)_j = \partial_k \tau_{ijk}$$

Solve $(*)$ while minimizing

$$\sum_{j<k} \frac{1}{2} \int_{\mathbb{T}^d} |\nabla \tau_{ijk}|^2 \, dy =: E(\tau)$$

$$-\Delta \tau_{ijk} = \partial_j q_{ik} - \partial_k q_{ij}.$$

To optimize $F(x)$ subject to $\{G(x) = S\}$,



$\nabla F \parallel \nabla G$

$\nabla F = \lambda \nabla G.$

$$F(\tau) = \frac{1}{2} \int_{\mathbb{T}^d} |\nabla \cdot \tau|^2.$$

Take a variation in the jk^{th} coordinate by $v \in H_0^1(\mathbb{T}^d)$,

$$\int \nabla \tau_{ijk} \cdot \nabla v = \int q_{ij} \partial_k v - q_{ik} \partial_j v.$$

$$\int (\nabla \cdot \tau)_j \partial_k v - (\nabla \cdot \tau)_k \partial_j v$$
$$= \int q_{ij} \partial_k v - q_{ik} \partial_j v$$

§ Flux - corrector.

Let $q_i = (q_{ij}) = A(\nabla\phi_i + e_i)$

We define $\tau_i = (\tau_{ijk})$

by

$$-\Delta \tau_{ijk} = \partial_j q_{ik} - \partial_k q_{ij}.$$

$\exists!$ solutions $\tau_{ijk} \in H_0^1(\Pi^d)$.

We claim that

$$\nabla\cdot\tau_i = q_i - \langle q_i\rangle.$$

We show that

$$\Delta\left[(\nabla\cdot\tau)_j - q_{ij}\right] = 0.$$

$$\partial_k\partial_k(\partial_s\tau_{ijs} - q_{ij})$$

$$= \partial_s(\partial_k\partial_k\tau_{ijs} - \partial_k\partial_k q_{ij})$$

$$= \partial_s(\partial_s q_{ij} - \partial_j q_{is} - \partial_k\partial_k q_{ij})$$

$$= (\partial_s\partial_s q_{ij} - \partial_k\partial_k q_{ij}) - \partial_j\partial_s q_{is}$$

$$= \partial_j(\nabla\cdot q_{is})$$

$$= 0.$$

$$\boxed{\begin{aligned} &\text{So, } (\nabla\cdot\tau_i) - q_i = \text{constant} \\ &(\nabla\cdot\tau)_i - q_i = \langle(\nabla\cdot\tau)_i - q_i\rangle \\ &(\nabla\cdot\tau_i) = q_i - \langle q_i\rangle. \end{aligned}}$$

$\left[\int_{\Pi^d}((\nabla\cdot\tau)_j - q_{ij})\rho^\varepsilon\right]$ satisfies

$$\Delta_x\left[\int_{\Pi^d}((\nabla\cdot\tau)_j - q_{ij})(y)\rho^\varepsilon(y-x)\,dy\right] = 0$$

§ The Homogenization Error

$$-\nabla\cdot A^\varepsilon \nabla w^\varepsilon = \nabla\cdot A^\varepsilon\left(\varepsilon\phi_i^\varepsilon \nabla(\partial_i v)\right)$$
$$+\nabla\cdot\left[(q_i - \langle q_i\rangle)\partial_i v\right].$$
$$= \nabla\cdot A^\varepsilon\left(\varepsilon\phi_i^\varepsilon \nabla(\partial_i v)\right)$$
$$+\nabla\cdot\left[(\nabla\cdot\tau_i^\varepsilon)\partial_i v\right].$$

If $f\in \mathcal{C}_c^\infty(U)$, then

$$\int_U (\nabla\cdot\tau_i^\varepsilon)\partial_i v\cdot\nabla f = -\int_U \varepsilon\tau_i^\varepsilon \nabla\partial_i v\cdot\nabla f - \int_U \varepsilon\phi_i^\varepsilon : \tilde{\nabla} f\,\partial_i v$$
$$= \int_U \partial_i v\ \partial_k \tau_{ijk}^\varepsilon\,\partial_j f = -\int_U \varepsilon\tau_{ijk}^\varepsilon\,\partial_k\partial_i v\,\partial_j f$$
$$-\int_U \varepsilon\tau_{ijk}^\varepsilon\,\partial_i v\,\partial_j\partial_k f$$

$$\int_U (\nabla\cdot\tau_i^\varepsilon)\partial_i v\cdot\nabla f = -\int_U \varepsilon\tau_i^\varepsilon\nabla\partial_i v\cdot\nabla f$$

$$\nabla\cdot\left[(\nabla\cdot\tau_i^\varepsilon)\partial_i v\right] = -\nabla\cdot\left(\varepsilon\tau_i^\varepsilon\nabla\partial_i v\right),$$

Therefore,

$$-\nabla\cdot A^\varepsilon\nabla w^\varepsilon$$
$$= \nabla\cdot\left[A^\varepsilon\cdot\varepsilon\phi_i^\varepsilon - \varepsilon\tau_i^\varepsilon\right]\partial_i v$$

Thm) Assume $f \in C^{\alpha}(\mathcal{U})$.

$$\lim_{\varepsilon \to 0} \|w^{\varepsilon}\|_{H^1(\mathcal{U})} = 0.$$

Pf) We have

$$-\nabla \cdot A^{\varepsilon} \nabla w^{\varepsilon} = \nabla \cdot \left[(A^{\varepsilon} \varepsilon \phi_i^{\varepsilon} - \varepsilon \sigma_i^{\varepsilon}) \nabla(\partial_i v) \right]$$

And, we have

$$\|v\|_{C^{2,\alpha}(\mathcal{U})} \le C \|f\|_{C^{\alpha}(\mathcal{U})}.$$

We have

$$\int_{\mathcal{U}} |\nabla w^{\varepsilon}|^2 \le C \int_{\mathcal{U}} \left(|A^{\varepsilon}|^2 |\varepsilon \phi_i^{\varepsilon}|^2 + |\varepsilon \sigma_i^{\varepsilon}|^2 \right) |\nabla \partial_i v|^2$$

$$\int_{\mathcal{U}} |\nabla w^{\varepsilon}|^2 \le C \cdot \varepsilon^2 \|\nabla_v^2\|_{\infty} \cdot \int_{\mathcal{U}} |\phi_i^{\varepsilon}|^2 + |\sigma_i^{\varepsilon}|^2.$$

As $\varepsilon \to 0$,

$$\int_{\mathcal{U}} |\phi_i^{\varepsilon}|^2 + |\sigma_i^{\varepsilon}|^2 \to |\mathcal{U}| \cdot \int_{\mathbb{T}^d} |\phi_i|^2 + |\sigma_i|^2,$$

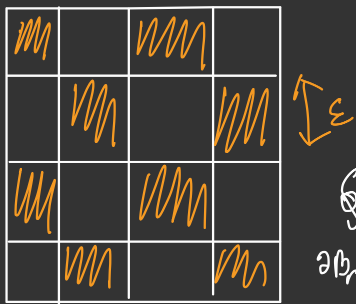
So,

$$\lim_{\varepsilon \to 0} \int_{\mathcal{U}} |\nabla w^{\varepsilon}|^2 = 0.$$

Hence,

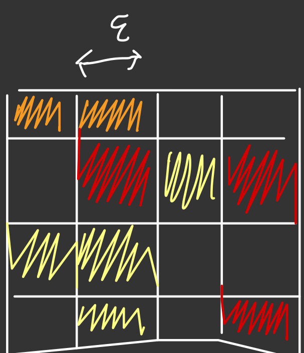
$$\lim_{\varepsilon \to 0} \|w^{\varepsilon}\|_{H^1(\mathcal{U})} = 0.$$

§ Stochastic Homogenization



$$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon = f$$

$$\int_{\partial B_r(x)} A(x/\varepsilon)\nabla u^\varepsilon\cdot\nu = \int_{B_r} f$$



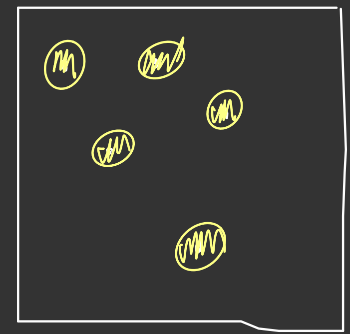
$(\Omega, \mathcal{F}, \mathbb{P})$

$A:\mathbb{R}^d\times\Omega\to\mathbb{R}^{d\times d}$

$$-\nabla\cdot A(x/\varepsilon,\omega)\nabla u^\varepsilon = f$$

Aim to obtain an almost sure characterization of u^ε, as $\varepsilon\to 0$

Point process

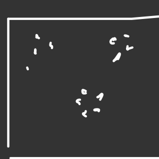


Random points x_i in $\mathbb{R}^d$,

$$A(x,\omega) = \alpha_1 \mathbb{1}_{\{\cup_i B_\delta(x_i)\}} + \alpha_2 \mathbb{1}_{\{\mathbb{R}^d\setminus \cup_i B_\delta(x_i)\}},$$

$\alpha_1 \ll \alpha_2$.

$A \sim \sum_i W(x-x_i)$.



§ Poisson Point Process

We consider the space $\mathcal{N}$ of elements

$$\omega = \sum_{i \in \mathcal{J}} \delta_{x_i}, \quad \text{for a countable index } \mathcal{J},\ x_i \in \mathbb{R}^d$$

and for every bounded Borel set B,

$$\omega(B) = \#\{x_i \in B : i \in \mathcal{J}\} < \infty.$$

To construct the Poisson point process of intensity $\lambda \in (0,\infty)$, we equip $\mathcal{N}$ with the smallest σ-algebra $\mathcal{F}$ such that, $\forall B \subseteq \mathbb{R}^d$,

$$\omega \longmapsto \omega(B) \text{ is measurable,}$$

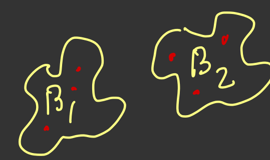
and the measure $\mathbb{P}_\lambda$ satisfying

i.) $\mathbb{E}_\lambda[\omega(B)] = \lambda |B|, \ \forall B \in \mathcal{B}(\mathbb{R}^d)$

ii.) If $B_1, \dots, B_N$ are disjoint Borel sets, then the variables

$$\omega \longmapsto \omega(B_k),\ k \in \{1, \dots, N\}$$

are independent



iii.) $\forall A \in \mathcal{F}$, $\mathbb{P}_\lambda[A] = \mathbb{P}_\lambda[A+y]$ $\forall y \in \mathbb{R}^d$

where $A+y = \{\omega(\cdot + y) : \omega \in A\}$.

§ Stationarity and Ergodicity

Let $(\Omega, \mathcal{F}, \mathbb{P}_\lambda)$ be a Poisson point process.

By stationarity, the space Ω comes equipped with a measure-preserving transformation group $\{T_x\}_{x \in \mathbb{R}^d}$,

$$T_x \omega(\cdot) = \omega(\cdot - x).$$

$$A(x, \omega) = \alpha_1 \mathbb{1}\{\cup_i B_\delta(x_i)\} + \alpha_2 \mathbb{1}\{\cup_i B_\delta(x_i)^c\}$$

$$A(x+y, \omega) = A(y, T_x \omega).$$

In this way, whenever $|x-y| > 2\delta$,

$A(x, \cdot), A(y, \cdot)$ are independent.

We will assume that the group $\{T_x\}_{x \in \mathbb{R}^d}$ is ergodic:

$\mathbb{P}_\lambda[A \Delta T_x A] = 0 \quad \forall x \in \mathbb{R}^d$ if and only if $\mathbb{P}_\lambda[A] = 0, 1$

$\Rightarrow$ σ-algebra of T_x-invariant sets is trivial

$A < \mathrm{Id}$ | $A \geq 2\,\mathrm{Id}$

§ The Ergodic Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{T_x\}_{x \in \mathbb{R}^d}$ be an ergodic, measure-preserving transformation group.

<u>Thm</u> For every $f \in \mathcal{L}^1(\Omega)$, we have almost surely that

$$\lim_{R \to \infty} ⨍_{B_R} f(T_x \omega)\, dx = \mathbb{E}[f].$$

Cor) Let $f \in \mathcal{L}^p(\Omega)$.

Then, a.s. as $\varepsilon \to 0$,

$$f(T_{x/\varepsilon}\omega) \rightharpoonup \mathbb{E}f \text{ in } \mathcal{L}^p_{loc}(\mathbb{R}^d)$$

$$-\nabla \cdot A(x/\varepsilon, \omega) \nabla u^\varepsilon = f$$

$$\Big\downarrow \varepsilon \to 0, \text{ a.s.}$$

$$-\nabla \cdot \bar{A} \nabla v = f$$

$\bar{A}$ = deterministic constant coefficient

$\forall \varphi \in C_c^\infty$, $f \in \mathcal{L}^2 \Rightarrow \varphi \in \mathcal{L}^2$ with compact support

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} f(T_{x/\varepsilon}\omega)\varphi\, dx = \mathbb{E}[f] \int_{\mathbb{R}^d} \varphi$$

§ Stochastic Homogenization

$(\Omega, \mathcal{F}, \mathbb{P})$ – prob. space

$\{T_x\}_{x\in\mathbb{R}^d}$ – ergodic, measure-preserving

$a(x,\omega): \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$, uniformly elliptic

Almost surely,

$\forall x, y$ $\quad a(x+y,\omega) = a(y, T_x\omega)$.

$$-\nabla\cdot A(x/\varepsilon, \omega)\nabla u^\varepsilon = f \text{ in } U, \; u^\varepsilon = 0 \; \partial U.$$

$$u^\varepsilon(x,\omega) = u_0(x, x/\varepsilon, \omega) + \varepsilon u_1(x, x/\varepsilon, \omega) + \varepsilon^2 \dots$$

As in the periodic case:

$$u^\varepsilon(x,\omega) = v(x) + \varepsilon\phi_i(x/\varepsilon, \omega)\partial_i v.$$

Here, ϕ_i solves

$$-\nabla\cdot a(y,\omega)(\nabla\phi_i + e_i) = 0 \quad \text{on } \mathbb{R}^d.$$

And v solves:

$$-\nabla\cdot \bar{a}\nabla v = f \text{ in } U, \; v=0 \text{ on } \partial U,$$

for $\bar{a}$ defined by

$$\bar{a}e_i = \mathbb{E}\left[A(\omega)(\nabla\phi_i(0,\omega) + e_i)\right]$$

where $A(\omega) = a(0,\omega)$.

$\forall x, \quad a(x,\omega) = a(0, T_x\omega) = A(T_x\omega)$.

$\nabla\phi_i(x,\omega) = \Phi_i(T_x\omega)$ for

$\Phi_i : \Omega \to \mathbb{R}^d$.

§ The Correctors

$$-\nabla \cdot a(y,\omega)(\nabla\phi_i + e_i) = 0 \quad \text{on } \mathbb{R}^d$$

why not $\phi_i(x) = -x_i$?

$u^\varepsilon = v + \varepsilon\phi_i(x/\varepsilon)\partial_i v.$

If $u^\varepsilon \to v$, need

$$\lim_{\varepsilon\to 0} \varepsilon\phi_i(x/\varepsilon) \simeq \lim_{|x|\to\infty} \frac{1}{|x|}|\phi_i(x)| = 0.$$

Need sublinearity:

$$\lim_{R\to\infty} \frac{1}{R}\left(\fint_{B_R}|\phi_i|^2\right)^{1/2} = 0.$$

§ Test functions

$f \in L^\infty(\Omega)$, $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\varphi_f(\omega) = \int_{\mathbb{R}^d} f(T_x\omega)\varphi(x)\,dx,$$

$\mathcal{D}(\Omega) =$ space of all such functions

For each $i \in \{1,\dots,d\}$ we define

$$\mathcal{D}(D_i) = \left\{ f \in L^2(\Omega) : \lim_{h\to 0} \frac{f(T_{he_i}\omega) - f(\omega)}{h} \text{ exists strongly in } L^2(\Omega) \right\}.$$

and $D_i : \mathcal{D}(D_i) \to L^2(\Omega)$ by

$$D_i f = \lim_{h\to 0} \frac{f(T_{he_i}\omega) - f(\omega)}{h}.$$

$$\mathcal{D}(\Omega) \subseteq \mathcal{D}(D_i) \quad \forall i \in \{1,\dots,d\}.$$

§ $\mathcal{H}^1(\Omega)$

$$D_i f = \lim_{h \to 0} \frac{f(T_{he_i}\omega) - f(\omega)}{h} \in \mathcal{L}^2(\Omega)$$

densely defined on $\mathcal{L}^2(\Omega)$, domain $\mathcal{D}(D_i)$.

$$\mathcal{H}^1(\Omega) = \bigcap_{i=1}^{d} \mathcal{D}(D_i).$$

If $f \in \mathcal{H}^1(\Omega)$ we can write $Df = (D_1 f, \dots, D_d f)$. We can try to solve

$$-D \cdot A(Df + e_i) = 0 \quad \text{in } \mathcal{H}^1.$$

Fails.

$$\mathcal{L}^2_{pot}(\Omega) = \overline{\{Df : f \in \mathcal{H}^1(\Omega)\}}^{\mathcal{L}^2(\Omega;\mathbb{R}^d)},$$

is the space of mean zero, curl free fields,

We show that $\exists!\ \Phi_i \in \mathcal{L}^2_{pot}$ such that

$$\mathbb{E}[A(\Phi_i + e_i) \cdot f] = 0$$

$$\forall f \in \mathcal{L}^2_{pot}.$$

§ Stochastic Homogenization

$(\Omega, \mathcal{F}, \mathbb{P}) \sim \omega \in \Omega$ is an environment

$\{T_x : \Omega \to \Omega\}_{x \in \mathbb{R}^d}$

$$T_x \circ T_y = T_{x+y} \quad \forall x, y \in \mathbb{R}^d$$

$$\mathbb{P}[T_x(A)] = \mathbb{P}[A] \quad \forall A \in \mathcal{F}.$$

The group $\{T_x\}_{x \in \mathbb{R}^d}$ is ergodic

$\mathbb{P}[A \,\Delta\, T_x[A]] = 0 \quad \forall x \in \mathbb{R}^d$ iff $\mathbb{P}[A] = 0, 1$

$A \in \mathcal{L}^\infty(\Omega; \mathbb{R}^{d \times d})$ uniformly elliptic

$a : \mathbb{R}^d \times \Omega \to \mathbb{R}^{d \times d}$

$a(x, \omega) = A(T_x \omega)$. / stationarity

Stationarity: $\forall x_1, \dots, x_N, y \in \mathbb{R}^d$

$(a(x_1, \cdot), \dots, a(x_N, \cdot)) \in \mathcal{L}^\infty(\Omega, (\mathbb{R}^{d \times d})^N)$

$\sim (a(x_1 + y, \cdot), \dots, a(x_N + y, \cdot))$

in law.

If a satisfies this condition,

$a : \mathbb{R}^d \times \Omega \to \mathbb{R}^{d \times d}$,

then $\exists$ a measure-preserving group $\{T_x\}_{x \in \mathbb{R}^d}$ such that

$$a(x + y, \omega) = a(y, T_x \omega).$$

Or, for $A(\omega) = a(0, \omega)$,

$$a(x, \omega) = A(T_x \omega).$$

§ Homogenization Corrector

$$-\nabla \cdot a(x/\varepsilon, \omega) \nabla u^\varepsilon = f \text{ in } U, \quad u^\varepsilon = 0 \ \partial U$$

$$u^\varepsilon(x,\omega) = v(x) + \varepsilon \phi_i(x/\varepsilon, \omega)\, \partial_i v(x)$$

$$-\nabla \cdot a(y,\omega) (\nabla \phi_i(y,\omega) + e_i) = 0 \text{ in } \mathbb{R}^d$$

$$\bar{a} e_i = \mathbb{E}[A(\nabla \phi_i + e_i)].$$

$$-\nabla \bar{a} \nabla v = f \text{ in } U, \quad v = 0 \ \partial U.$$

The validity of expansion requires

$$\lim_{\varepsilon \to 0} \varepsilon \phi_i(x/\varepsilon) \simeq \lim_{|x| \to \infty} \frac{1}{|x|} \phi(x) = 0.$$

rules out $\phi_i(x) = -x_i$.

$$D_i : \mathcal{D}(D_i) \to L^2(\Omega)$$

$$D_i f(\omega) = \lim_{h \to 0} \frac{f(T_{he_i}\omega) - f(\omega)}{h} \in L^2(\Omega)$$

$$\mathcal{D}(D_i) = \left\{ f \in L^2(\Omega) : \text{limit exits strongly in } L^2(\Omega) \right\}$$

$$H^1(\Omega) = \bigcap_{i=1}^{d} \mathcal{D}(D_i).$$

$$D\phi = (D_1\phi, \dots, D_d\phi).$$

$$a(x,\omega) = A(T_x\omega),$$

Solve

$$-D \cdot A(D\phi + e_i) = 0 \text{ in } H^1(\Omega)$$

- loss of compactness
- no Poincaré

§ Corrector

$f \in \mathcal{H}^1(\Omega)$.

$$\mathbb{E}[D_i f] = \mathbb{E}\left[\lim_h \frac{f(T_{he_i}\omega) - f(\omega)}{h}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\mathbb{E}\left[f(T_{he_i}\omega) - f(\omega)\right]$$

$$= 0.$$

$$\mathcal{D}(\Omega) = \left\{ \int_{\mathbb{R}^d} f(T_x\omega)\phi(x)\,dx : \begin{matrix} f \in \mathcal{L}^\infty(\Omega) \\ \phi \in C^\infty_c(\Omega) \end{matrix} \right\}$$

$\mathcal{D}(\Omega) \subseteq \mathcal{H}^1(\Omega)$.

Similarly, as distributions,

$$D_i D_j f = D_j D_i f \quad \forall f \in \mathcal{H}^1(\Omega).$$

$$\mathcal{L}^2_{pot}(\Omega) = \overline{\{Df : f \in \mathcal{H}^1(\Omega)\}}^{\mathcal{L}^2(\Omega;\mathbb{R}^d)}.$$

$V \in \mathcal{L}^2_{pot}(\Omega)$

$\mathbb{E}[V] = 0$

$D_i V_j = D_j V_i \quad \forall i,j \in \{1,\dots,d\}$.

Prop) For each $i \in \{1,\dots,d\}$
$\exists!\ \Phi_i \in \mathcal{L}^2_{pot}(\Omega)$ such that

$$\mathbb{E}[A(\Phi_i + e_i)\cdot f] = 0 \quad \forall f \in \mathcal{L}^2_{pot}(\Omega)$$

Pf.) Lax-Milgram
$\mathcal{L}^2_{pot}(\Omega)$ a Hilbert Space

"$\Phi_i = D\phi_i$ so that $-D\cdot A(D\phi_i + e_i) = 0$"
ϕ_i itself need not exist

§Physical Corrector

<u>Prop</u> Almost surely $\exists!\ \phi_i \in H^1_{loc}(\mathbb{R}^d)$ satisfying

"$\phi_i(0,\omega) = 0\ \forall \omega$"

"$\phi_i(x,\omega) = \phi_i(0, \tau_x\omega) = 0\ \forall x$"

i.) $\fint_{B_1} \phi_i = 0$

ii.) $\nabla\phi_i(x,\omega) = \Phi_i(T_x\omega)$

iii.) For every $f \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} a(x,\omega)(\nabla\phi_i(x,\omega)+e_i)\cdot\nabla f = 0.$$

Pf.) Existence by integration.

Uniqueness ... $\mathbb{E}[|\Phi_i|^2] < \infty$

$$\mathbb{E}\left[\int_{B_R} |\Phi_i(T_x\omega)|^2 dx\right] = |B_R|\,\mathbb{E}[|\Phi_i|^2] < \infty.$$

a.s. $\Phi_i(\cdot,\omega) \in L^2_{loc}(\mathbb{R}^d;\mathbb{R}^d)$

curl-free

$A \in \mathcal{F}$,

$$\mathbb{E}\left[\int_{\mathbb{R}^d} a(x,\omega)(\nabla\phi_i(x,\omega)+e_i)\cdot\nabla f\, dx\, \mathbb{1}_A(\omega)\right]$$

$$= \mathbb{E}\left[\int_{\mathbb{R}^d} A(T_x\omega)(\Phi_i(T_x\omega)+e_i)\nabla f(x)\mathbb{1}_A(\omega)\right]$$

$$= \mathbb{E}\left[A(\omega)(\Phi_i(\omega)+e_i)\int_{\mathbb{R}^d} \mathbb{1}_A(T_{-x}\omega)\nabla f(x)\,dx\right]$$

$$= -\mathbb{E}\left[A(\omega)(\Phi_i(\omega)+e_i)\underbrace{\int_{\mathbb{R}^d} \mathbb{1}_A(T_x\omega)\nabla f(-x)\,dx}_{\text{gradient in } \mathcal{D}(\mathcal{L})}\right]$$

$= 0.$

Because A was arbitrary,

$\int a(x,\omega)(\nabla\phi_i(x,\omega)+e_i)\nabla f\,dx = 0$ a.s. | use separability

§ Sublinearity

$$\nabla\phi_i(x,\omega) = \Phi_i(T_x\omega) \qquad \mathbb{E}[\Phi_i] = 0.$$

$$\phi_i(x) - \phi_i(0) = \underbrace{\int_0^1 \Phi_i(T_{sx})\cdot x\,ds}_{\text{cancellations due to } \mathbb{E}[\Phi_i]=0 \text{ + ergodicity}}.$$

$$\lim_{|x|\to\infty} \frac{|\phi_i(x)-\phi_i(0)|}{|x|} = \frac{1}{|x|}\left|\int_0^1 \Phi_i(T_{sx})\cdot x\,ds\right|$$

$= 0$ by ergodic theorem

__Prop__ For each $i \in \{1, \dots, d\}$, almost surely,

$$\lim_{R\to\infty} \frac{1}{R}\left(⨍_{B_R} |\phi_i|^2\right)^{1/2} = 0.$$

not true if $\phi_i(x) = -x_i$.

$$\lim_{R\to\infty} \frac{1}{R}\left(⨍_{B_R} |\phi_i|^2\right)^{1/2} = \lim_{\varepsilon\to 0}\left(⨍_{B_1} |\varepsilon\phi_i^\varepsilon|^2\right)^{1/2}$$

for $\phi_i^\varepsilon(x) = \phi_i(x/\varepsilon)$.

$$\langle\phi_i^\varepsilon\rangle = ⨍_{B_1}\phi_i^\varepsilon$$

We will show

$$\lim_{\varepsilon\to 0}\left(⨍_{B_1} |\varepsilon\phi_i^\varepsilon - \varepsilon\langle\phi_i^\varepsilon\rangle|^2\right)^{1/2} = 0.$$

$\tilde\phi_i^\varepsilon = \phi_i^\varepsilon - \langle\phi_i^\varepsilon\rangle$, $\lim_{\varepsilon\to 0}\left(⨍_{B_1}|\varepsilon\tilde\phi_i^\varepsilon|^2\right)^{1/2}$.

$$\nabla(\varepsilon\tilde\phi_i^\varepsilon) = \nabla\phi_i(x/\varepsilon,\omega) = \Phi_i(T_{x/\varepsilon}\omega).$$

By Poincaré,

$$\left(⨍_{B_1}|\varepsilon\tilde\phi_i^\varepsilon|^2\right)^{1/2} \le C\left(⨍_{B_1}|\Phi_i(T_{x/\varepsilon}\omega)|^2\,dx\right)^{1/2}$$

$$\lim_{\varepsilon\to 0}\left(⨍_{B_1}|\nabla(\varepsilon\tilde\phi_i)|^2\right)^{1/2} = \lim_{\varepsilon\to 0}\left(⨍_{B_1}|\Phi_i(T_{x/\varepsilon}\omega)|^2\right)^{1/2}.$$

§ Sublinearity

$$\tilde{\phi}_i^{\varepsilon} = \phi_i^{\varepsilon} - \langle \phi_i^{\varepsilon} \rangle$$

$$\left(\fint_{B_1} |\varepsilon \tilde{\phi}_i^{\varepsilon}|^2 \right)^{1/2} \le C \left(\fint_{B_1} |\Phi_i(T_{x/\varepsilon}\omega)|^2 \right)^{1/2}$$

$$\left(\fint_{B_1} |\nabla(\varepsilon \phi_i^{\varepsilon})|^2 \right)^{1/2} = \left(\fint_{B_1} |\Phi_i(T_{x/\varepsilon}\omega)|^2 \right)^{1/2}.$$

By the ergodic theorem, a.s. as $\varepsilon \to 0$,

$$\Phi_i(T_{x/\varepsilon}\omega) \longrightarrow \mathbb{E}[\Phi_i] = 0 \quad \text{in } L^2(B_1)$$

$$\sup_{\varepsilon \in (0,1)} \left(\fint_{B_1} |\Phi_i(T_{x/\varepsilon}\omega)|^2 \right)^{1/2} < \infty \quad \text{a.s.}$$

Or, since a.s. $\lim_{\varepsilon \to 0} \left(\fint_{B_1} |\Phi_i(T_{x/\varepsilon})|^2 \right)^{1/2} = \mathbb{E}[|\Phi_i|^2]^{1/2}$.

So, $(\varepsilon \tilde{\phi}_i^{\varepsilon})_{\varepsilon \in (0,1)}$ is uniformly bounded in $H^1(B_1)$.

$$\fint_{B_1} \varepsilon \tilde{\phi}_i^{\varepsilon} = 0 \quad \forall \varepsilon \in (0,1)$$

$$\nabla(\varepsilon \tilde{\phi}_i^{\varepsilon}) \longrightarrow 0 \quad \text{in } L^2.$$

By Sobolev embedding, along a subsequence,

$$\varepsilon \tilde{\phi}_i^{\varepsilon} \to \text{constant} = 0.$$

strongly in $L^2(B_1)$.

The full sequence

$$\varepsilon \tilde{\phi}_i^{\varepsilon} \to 0 \quad \text{strongly in } L^2(B_1).$$

$$\lim_{R \to \infty} \frac{1}{R} \left(\fint_{B_R} |\phi_i - \langle \phi_i \rangle_R|^2 \right)^{1/2} = 0.$$

§ The homogenized coefficient

$|A\xi| \le \Lambda|\xi|$, $A\xi\cdot\xi \ge \lambda|\xi|^2$

$\bar{a}e_i = \mathbb{E}[A(\Phi_i + e_i)]$ "$\Phi_i = \nabla\phi_i$"

$|\bar{a}\xi| = |\xi_i \mathbb{E}[A(\Phi_i + e_i)]|$

$\le \Lambda|\xi_i|\mathbb{E}[|\Phi_i + e_i|]$

$\le \Lambda|\xi|\left(\sum_{i=1}^{d}\mathbb{E}[|\Phi_i + e_i|^2]\right)^{1/2}$

$\bar{a}\xi\cdot\xi = \mathbb{E}[A(\Phi_\xi + \xi)\cdot\xi]$ $\Phi_\xi = \xi_i\Phi_i$

$= \mathbb{E}[A(\Phi_\xi + \xi)(\Phi_\xi + \xi)]$

$\ge \lambda\mathbb{E}[|\Phi_\xi + \xi|^2] \ge \lambda|\mathbb{E}[\Phi_\xi + \xi]|^2$

$= \lambda|\xi|^2$

$-\nabla\cdot A(\Phi_i + e_i) = 0$

$-\nabla\cdot A^{\dagger}(\Phi_i^{\dagger} + e_i) = 0$

$\bar{a} = \mathbb{E}[A(\Phi_i + e_i)]$

$\tilde{a} = \mathbb{E}[A^{\dagger}(\Phi_i^{\dagger} + e_i)]$

$\tilde{a} = \bar{a}^{\dagger}$

If A is symmetric, then $\bar{a}$ is symmetric.

$-\nabla\cdot\bar{a}\nabla v = f$ U, $v = 0$ ∂U

§ Stochastic Homogenization

$(\Omega, \mathcal{F}, \mathbb{P})$

$\{T_x : \Omega \to \Omega\}_{x \in \mathbb{R}^d}$ ergodic & measure preserving

$A : \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$, uniformly elliptic

$A(x+y, \omega) = A(y, T_x \omega)$,

$-\nabla \cdot a^\varepsilon \nabla u^\varepsilon = f$ in U, $u^\varepsilon = 0$ on ∂U

$a^\varepsilon(x, \omega) = A(T_{x/\varepsilon}\omega) = A(x/\varepsilon, \omega)$.

$$-\nabla \cdot a(y,\omega) \underbrace{(\nabla \phi_i + e_i)}_{\nabla(\phi_i + x_i)} = 0$$

$d\tilde{X}_t = \sigma(\tilde{X}_t)\, dB_t + \nabla \cdot A(\tilde{X}_t)\, dt$

$X^\varepsilon = \varepsilon \tilde{X}_{t/\varepsilon^2}$

a.s., $X^\varepsilon \xrightarrow{\text{law}} \bar{\sigma} B_t$

$u^\varepsilon = v + \varepsilon \phi_i^\varepsilon \partial_i v$

for $\phi_i^\varepsilon(x) = \phi_i(x/\varepsilon, \omega)$,

if $u^\varepsilon \to v$ as $\varepsilon \to 0$

need

$$\lim_{\varepsilon \to 0} \varepsilon \phi_i(x/\varepsilon, \omega) = 0.$$

Thm Almost surely, for each $i \in \{1, \dots, d\}$, $\exists\, \phi_i \in H^1_{loc}(\mathbb{R}^d)$ satisfying

$$-\nabla \cdot a(y,\omega)(\nabla \phi_i + e_i) = 0,$$

and

$$\lim_{R \to 0} \frac{1}{R} \left(\fint_{B_R} |\phi_i|^2 \right)^{1/2} = 0.$$

§ The Perturbed Test Function Method

$\Phi_i \in \mathcal{L}^2_{pot}(\mathcal{U})$ $\quad -D\cdot A(\Phi_i + e_i) = 0$

$\bar{a} \in \mathbb{R}^{d\times d}$

$\bar{a} e_i = \mathbb{E}\left[A(\Phi_i + e_i)\right]$

$\bar{a}$ is uniformly elliptic

$\Phi_i^+ \in \mathcal{L}^2_{pot}$ $\quad -D\cdot A^+(\Phi_i^+ + e_i) = 0$

$\tilde{a} = \mathbb{E}\left[A^+(\Phi_i^+ + e_i)\right]$

$\tilde{a} = \bar{a}^+$.

$u^\varepsilon \cong v + \varepsilon \phi_i^\varepsilon \partial_i v$

$-\nabla\cdot\bar{a}\nabla v = f$ in U, $v = 0$ on ∂U

Thm Let $f \in \mathcal{L}^2(U)$. Then, almost surely as $\varepsilon \to 0$,

$$u^\varepsilon \rightharpoonup v \quad \text{in } H_0^1(U).$$

Pf Since a.s.,

$$\sup_{\varepsilon\in(0,1)} \|u^\varepsilon\|_{H_0^1(U)} < \infty,$$

along a subsequence

$$u^\varepsilon \rightharpoonup \tilde{v} \quad \text{for } \tilde{v} \in H_0^1(U).$$

We'll show that $\tilde{v} = v$.

Need show, $\forall \psi \in C_c^\infty(U)$,

$$\int_U \bar{a}\nabla\tilde{v}\cdot\nabla\psi = \int_U f\psi.$$

Let $\psi \in C_c^\infty(U)$.

$$\psi^\varepsilon = \psi + \varepsilon \phi_i^+(x/\varepsilon, \omega)\,\partial_i \psi.$$

Test u^ε-equation with ψ^ε,

$$\int_U a^\varepsilon \nabla u^\varepsilon \cdot \left(\nabla\phi_i^+(x/\varepsilon) + e_i\right)\cdot \partial_i \psi$$

$$+ \int_U \left(\varepsilon\phi_i^+(x/\varepsilon)\right) a^\varepsilon \nabla u^\varepsilon \cdot \nabla(\partial_i\psi)$$

$$= \int_U f \cdot \psi^\varepsilon.$$

For the first term, we have

$$\int_U \underbrace{\nabla u^\varepsilon}_{\text{curl free}} \cdot \underbrace{(a^\varepsilon)^+\left(\nabla\phi_i^+(x/\varepsilon) + e_i\right)}_{\text{div-free}} \cdot \partial_i \psi.$$

$\nabla u^\varepsilon \rightharpoonup \nabla v$ By ergodic thm $\xrightarrow{\varepsilon\to 0} \bar{a}^+ e_i$

As $\varepsilon \to 0$, by div-curl,

$$\int_U a^\varepsilon \nabla u^\varepsilon \cdot (\nabla\phi_i^+ + e_i)\,\partial_i\psi$$

$$\to \int \nabla\tilde{v} \cdot \bar{a}^+ \nabla\psi = \int \bar{a}\nabla\tilde{v}\cdot\nabla\psi.$$

For the second term,

$$\left|\int_U \left(\varepsilon\phi_i^+(x/\varepsilon)\right) a^\varepsilon \nabla u^\varepsilon\, \nabla(\partial_i\psi)\right|$$

$$\lesssim \|\nabla u^\varepsilon\|_{L^2}\left(\int_U \left|\varepsilon\phi_i^+(x/\varepsilon)\right|^2\right)^{1/2}.$$

A.S. as $\varepsilon\to 0$, the RHS vanishes.

Similarly, a.s., as $\varepsilon \to 0$,

$$\psi^\varepsilon = \left(\psi + \varepsilon\phi_i^+(x/\varepsilon)\,\partial_i\psi\right) \to \psi \quad \text{strongly in } L^2.$$

$$\int \bar{a}\nabla\tilde{v}\cdot\nabla\psi = \int \psi f. \quad / \quad v = \tilde{v}.$$

§ Strong Convergence

$$\nabla u^\varepsilon \simeq \nabla v + \nabla \phi_i^\varepsilon \partial_i v + \varepsilon \phi_i^\varepsilon \nabla(\partial_i v)$$

$$w^\varepsilon = u^\varepsilon - v - \varepsilon \phi_i^\varepsilon \partial_i v$$

Almost as $\varepsilon \to 0$,

$w^\varepsilon \to 0$ strongly in $H^1(\Omega)$.

We saw that

$$-\nabla \cdot a^\varepsilon \nabla w^\varepsilon = \nabla \cdot \left[q_i^\varepsilon \cdot \partial_i v + a^\varepsilon \cdot \varepsilon \phi_i^\varepsilon \nabla(\partial_i v) \right]$$

$$q_i^\varepsilon = a^\varepsilon (\nabla \phi_i^\varepsilon + e_i) - \bar{a} e_i$$

q_i^ε ~ mean zero, divergence free

$\exists$ skew-symmetric matrix τ_i defined by

$$\nabla \cdot \tau_i = q_i \qquad (\nabla \cdot \tau_i)_j = \partial_k \tau_{ijk}$$

with

$$-\Delta \tau_{ijk} = \partial_k q_{ij} - \partial_j q_{ik}.$$

After introducing τ_i,

$$-\nabla \cdot a^\varepsilon \nabla w^\varepsilon = \nabla \cdot \left[(a^\varepsilon \varepsilon \phi_i^\varepsilon - \varepsilon \tau_i^\varepsilon) \nabla(\partial_i v) \right]$$

§ The Flux Corrector

$$Q_i = A(\Phi_i + e_i) - \bar{a} e_i \in \mathcal{L}^2(\Omega; \mathbb{R}^d)$$

" $-\Delta \tau_{ijk} = \partial_k q_{ij} - \partial_j q_{ik}$ ".

Need find $\Sigma_{ijk} \in \mathcal{L}^2_{pot}(\Omega)$,

for " $\Sigma_{ijk} = \nabla \tau_{ijk}$ " such that

$$\mathbb{E}[\Sigma_{ijk} t] = \mathbb{E}[Q_{ik} t_j - Q_{ij} t_k]$$

for every $t \in \mathcal{L}^2_{pot}(\Omega)$.

Prop) $\forall i,j,k \in \{1,\dots,d\}$ $\exists !$ $\Sigma_{ijk} \in \mathcal{L}^2_{pot}(\Omega)$ such that

$$\mathbb{E}[\Sigma_{ijk} \cdot t] = \mathbb{E}[Q_{ik} t_j - Q_{ij} t_k] \quad \forall t \in \mathcal{L}^2_{pot}$$

Prop) For each $i \in \{1,\dots,d\}$ let

$\tau_i = (\tau_{ijk}) \in H^1_{loc}(\mathbb{R}^d, \mathbb{R}^{d \times d})$

be the unique function with

$$\fint_{B_1} \tau_{ijk} = 0 \quad \& \quad \nabla \tau_{ijk}(x,\omega) = \Sigma_{ijk}(\tau_x \omega).$$

Then, τ_i is skew-symmetric and

$$\nabla \cdot \tau_i = q_i \quad \text{on } \mathbb{R}^d,$$

(almost surely)

Proof) Skew-symmetry follows by uniqueness since

$$\Sigma_{ijk} = -\Sigma_{ikj}.$$

In the periodic case, we proved

$$\Delta[(\nabla\cdot T_i)_j - q_{ij}] = 0$$

On the probability space,

$$\partial_s\partial_s[\partial_k T_{ijk} - q_{ij}]$$

$$= \partial_s\partial_s[(\Sigma_{ijk})_k - q_{ij}].$$

$$= 0.$$

The components are distributional solutions of

$$\Delta[(\Sigma_{ijk})_k - q_{ij}] = 0.$$

If $f \in L^2(\Omega)$ is a distributional solution of

$$\Delta f = 0.$$

For each $\varepsilon > 0$ let

$$f^\varepsilon = \int_{\mathbb{R}^d} f(T_x\omega)\,\rho^\varepsilon(x)\,dx$$

Then, $\Delta f^\varepsilon = 0$ on Ω.

$F^\varepsilon(x,\omega) = f^\varepsilon(T_x\omega)$.

Almost surely,

$$\Delta F^\varepsilon(\cdot,\omega) = 0 \quad \text{on } \mathbb{R}^d.$$

So, $F^\varepsilon(\cdot,\omega) = c(\omega) \in \mathbb{R}$ on $\mathbb{R}^d$.

By ergodic thm, a.s.,

$$c(\omega) = \lim_{R\to\infty} \fint F^\varepsilon(x,\omega)\,dx = \mathbb{E}[f^\varepsilon]$$

a.s. on Ω

$$f^\varepsilon = \mathbb{E}[f^\varepsilon]$$

$$f = \mathbb{E}[f].$$

§ Stochastic Homogenization

$$* \lim_{\varepsilon \to 0} \left(\int_{B_R} |\varepsilon \sigma_i^\varepsilon|^2 \right)^{1/2} = 0.$$

Thm Let $f \in C^\alpha(U)$.

Then, a.s. as $\varepsilon \to 0$,

$$w^\varepsilon = u^\varepsilon - v - \varepsilon \phi_i^\varepsilon \partial_i v \to 0 \text{ in } H^1(U).$$

Pf.)

$$-\nabla \cdot a^\varepsilon \nabla w^\varepsilon = \nabla \cdot (a^\varepsilon \varepsilon \phi_i^\varepsilon - \sigma_i^\varepsilon) \nabla (\partial_i v).$$

$$\|v\|_{C^{2,\alpha}} \le C \|f\|_\alpha.$$

$\eta_\rho = 1 \quad d(x, \partial U) \ge \rho$

$\eta_\rho = 0 \quad d(x, \partial U) < \rho/2$

$|\nabla \eta_\rho| \le \frac{C}{\rho}$

$$w^{\varepsilon,\rho} = u^\varepsilon - v - \varepsilon \phi_i^\varepsilon (\eta_\rho \partial_i v).$$

$$-\nabla \cdot a^\varepsilon \nabla w^{\varepsilon,\rho} = \nabla \cdot [(1-\eta_\rho)(a^\varepsilon - \bar{a}) \partial_i v]$$

$$+ \nabla \cdot [a^\varepsilon \varepsilon \phi_i^\varepsilon - \sigma_i^\varepsilon] \nabla (\eta_\rho \partial_i v)$$

$$\int_U |\nabla w^{\varepsilon,\rho}|^2 \lesssim \|f\|_\alpha \int_U (1-\eta_\rho)^2$$

$$+ \frac{\|f\|_\alpha}{\rho} \int_U |\varepsilon \phi_i^\varepsilon|^2 + |\varepsilon \sigma_i^\varepsilon|^2$$

$$\limsup_{\varepsilon\to 0} \int |\nabla w^{\varepsilon,\rho}|^2 \lesssim \rho \|f\|_{\alpha}.$$

$$\nabla w^{\varepsilon} = \nabla w^{\varepsilon,\rho} + \nabla\left[\varepsilon\phi_i^{\varepsilon}(1-\eta_\rho)\partial_i v\right].$$

$$\limsup_{\varepsilon\to 0} \int |\nabla(\varepsilon\phi_i^{\varepsilon}(1-\eta_\rho)\partial_i v)|^2$$

$$\leq c\|f\|_{\alpha}\cdot\rho\cdot \mathbb{E}\left[|\Phi_i|^2\right]. \quad \text{ergodic thm.}$$

$$\limsup_{\varepsilon\to 0} \int_U |\nabla w^{\varepsilon}|^2 \lesssim \rho \|f\|_{\alpha}.$$

$$\nabla w^{\varepsilon} \to 0 \quad \text{a.s. as } \varepsilon\to 0$$

strongly in L^2.

$$w^{\varepsilon} = u^{\varepsilon} - v - \underbrace{\varepsilon\phi_i^{\varepsilon}\partial_i v}_{\substack{\text{sublinearity} \\ \to 0 \text{ strongly in } L^2}}$$

By perturbed test function and Sobolev,

$$u^{\varepsilon} \to v \quad \text{strongly in } L^2.$$

So, a.s.,

$$w^{\varepsilon} \to 0 \quad \text{strongly in } L^2.$$

Hence, a.s., as $\varepsilon\to 0$,

$$w^{\varepsilon} \to 0 \quad \text{strongly in } H^1(U).$$

$$\|u^{\varepsilon} - v\| \leq f(\varepsilon)?$$

§ Equations in non-divergence form

$A : \mathbb{R}^d \times \Omega \to \mathbb{R}^{d \times d}$, stat, ergodic, uniformly elliptic

$\sigma : \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$, $\sigma\sigma^t = 2A$

$b : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, drift

$$dX_t = \sigma(X_t, \omega)\, dB_t + b(X_t, \omega)\, dt$$

Can we characterize the rescalings

$$X_t^\varepsilon = \varepsilon X_{t/\varepsilon^2},$$

a.s. as $\varepsilon \to 0$ in law.

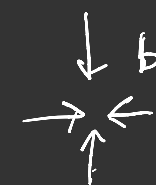
$$dX_t^\varepsilon = \sigma\left(X^\varepsilon/\varepsilon, \omega\right) dB_t + \tfrac{1}{\varepsilon} b\left(X^\varepsilon/\varepsilon, \omega\right) dt$$

$$dX = dB_t + dt$$

$$X_t = B_t + t$$

$$\varepsilon X_{t/\varepsilon^2} = \tilde{B}_t + t/\varepsilon$$

$$\tilde{X}_t^\varepsilon = \varepsilon X_{t/\varepsilon^2} - t/\varepsilon = \tilde{B}_t.$$



$$dX = -X_t\, dt + dB_t.$$

$$\varepsilon X_{t/\varepsilon^2} \xrightarrow{\varepsilon \to 0} 0.$$

$$\begin{cases} \partial_t u^\varepsilon = \operatorname{tr}\left(A(x/\varepsilon, \omega)\, \nabla^2 u^\varepsilon\right) + \frac{1}{\varepsilon} b(x/\varepsilon, \omega) \cdot \nabla u^\varepsilon \\ u = f \end{cases}$$

Feynman-Kac $\Rightarrow$ $u^\varepsilon(x,t) = \mathbb{E}_x\left[f(X_t^\varepsilon)\right]$

§ Ergodicity & Periodic Diffusions

$$\begin{cases} \partial_t u = \frac{1}{2}\operatorname{tr}(A\nabla_x^2 u) + b\cdot\nabla u & \text{in } \mathbb{T}^d_x(0,\infty) \\ u = f & \text{in } \mathbb{T}^d_x\{0\}. \end{cases}$$

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt$$

σ, b are periodic, $\sigma\sigma^T = A$.

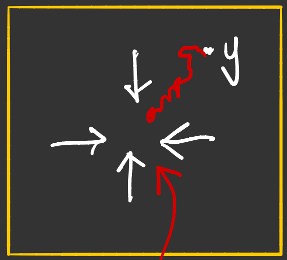
$$u(x,t) = P_t f(x) = \mathbb{E}_x[f(X_t)].$$

Def) A measure π on $\mathbb{T}^d$ is invariant if $\forall f \in \mathcal{L}^\infty(\mathbb{T}^d)$, $t \geq 0$,

$$\int_{\mathbb{T}^d} f\,d\pi = \int_{\mathbb{T}^d} P_t f\,d\pi.$$

$$\Updownarrow$$

$$\pi = P_t^* \pi \quad \forall\, t \geq 0.$$



$\mathbb{T}^d$

π assigns more mass here

$$\int_{\mathbb{T}^d} f(y)\,d\pi = \int_{\mathbb{T}^d} \mathbb{E}_y[f(X_t)]\,d\pi$$

Thm $\exists!$ invariant probability measure π on $\mathbb{T}^d$ so that

$$\lim_{t\to\infty}\left(\sup_{x\in\mathbb{T}^d}\left|P_t f(x) - \int_{\mathbb{T}^d} f\,d\pi\right|\right) \leq C\|f\|_\infty e^{-ct}$$

Pf.) Bensoussan, Lions, Papanicolaou, Chapter III.

§ The Adjoint Equation

$$\mathcal{L}u = \frac{1}{2}\operatorname{tr}(A\nabla^2 u) + b\cdot\nabla u$$

$$\mathcal{L}^* m = \frac{1}{2}(a_{ij}m)_{x_i x_j} - \nabla\cdot(mb).$$

$$\begin{cases} \partial_t u = \mathcal{L}u \\ u = f \end{cases} \quad \Big| \quad \mathcal{L}^* m = 0.$$

$$\partial_t\left(\int_{\mathbb{T}^d} u(x,t)\, m(x)\, dx\right) = \int_{\mathbb{T}^d} \mathcal{L}u\cdot m$$

$$= \int_{\mathbb{T}^d} u\, \mathcal{L}^* m = 0.$$

$\forall t \geq 0,\quad \int_{\mathbb{T}^d} P_t f(x)\, m(x)\, dx = \int_{\mathbb{T}^d} f(x)\, m(x)\, dx$

So, "$d\pi = m(x)\,dx$"

The Fredholm Alternative:

We can solve the equation

$$\mathcal{L}u = f \quad \text{in } \mathbb{T}^d$$

if and only if

$$f \perp \operatorname{Ker}(\mathcal{L}^*)$$

And, $\dim(\operatorname{Ker}\mathcal{L}) = \dim(\operatorname{Ker}\mathcal{L}^*)$.

($\forall\, \mathcal{L}^* m = 0$, we have $\int_{\mathbb{T}^d} f m\, dx = 0$)

Here, $\dim(\operatorname{Ker}\mathcal{L}) = \dim(\text{constants}) = 1$

by max principle

$\exists$ nonzero solution $\mathcal{L}^* m = 0$.

Necessarily, $m(x)\,dx = d\pi$.

$m > 0$ a.e., and $\exists!$ with $\int_{\mathbb{T}^d} m(x)\,dx = 1$

§ Asymptotic Expansion

$$\frac{1}{2}\mathrm{tr}(A(x/\varepsilon)\nabla^2 u^\varepsilon) + \frac{1}{\varepsilon} b(x/\varepsilon)\cdot\nabla u^\varepsilon = f.$$

$$u^\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2(x, x/\varepsilon) + \sim$$

"$y = x/\varepsilon$" $\quad \nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y.$

$\mathcal{O}(\varepsilon^{-2})$: $\quad \frac{1}{2}\mathrm{tr}(A\nabla_y^2 u_0) + b\nabla_y u_0 = 0$

$$u_0(x,y) = v(x)$$

$\mathcal{O}(\varepsilon^{-1})$: $\quad \frac{1}{2}\mathrm{tr}(A\nabla_y^2 u_1) + b\cdot(\nabla_x v + \nabla_y u_1) = 0$

$$u_1 = \phi_i \partial_i v$$

$$\frac{1}{2}\mathrm{tr}(A\nabla^2\phi_i) + b\cdot\nabla\phi_i = -b_i$$

$\mathcal{O}(1)$:

$$\frac{1}{2}\mathrm{tr}(A\nabla_y^2 w) + b\cdot\nabla_y w = f - \mathrm{tr}(\tilde{A}(y)\nabla^2 v)$$

for some $\tilde{A} : \mathbb{T}^d \to \mathbb{R}^{d\times d}$.

By Fredholm, for $\mathcal{L}^* m = 0$,

$\forall i$ we need $\quad \int_{\mathbb{T}^d} b_i(y)\, m(y)\, dy = 0$

That is, $b \perp m$.

To solve for w, we need

$f - \mathrm{tr}(\tilde{A}(y)\nabla^2 v) \perp m.$

$\bar{A} = \int_{\mathbb{T}^d} \tilde{A}(y) m(y) dy$, $\quad \mathrm{tr}(\bar{A}\nabla^2 v) = f.$

$\Rightarrow \bar{A}$ is u.e. $\Rightarrow$ homogenization

§ Ballistic Behavior

$\mathcal{L}^* m = 0$.

$$\bar{b} = \int_{\mathbb{T}^d} b(y) m(y)\, dy$$

$$\partial_t u = \frac{1}{2} \operatorname{tr}(A \nabla^2 u) + b \cdot \nabla u = \mathcal{L} u$$

$$u = f$$

$$\partial_t \tilde{u} = \frac{1}{2} \operatorname{tr}(A \nabla^2 \tilde{u}) + (b - \bar{b}) \cdot \nabla \tilde{u}$$

$$\tilde{u} = f$$

Then, basically,

$$\tilde{u}(x,t) = u(x - \bar{b}t, t).$$

$$(b - \bar{b}) \perp m.$$

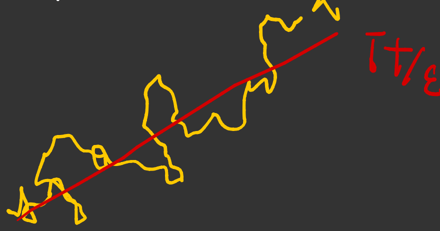
$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt$$

$$X_t^\varepsilon = \varepsilon X_{t/\varepsilon^2}$$

$$dX_t^\varepsilon = \sigma\left(X^\varepsilon_{t/\varepsilon}\right) dB_t + \frac{1}{\varepsilon} b\left(X^\varepsilon_{t/\varepsilon}\right) dt$$

$\exists\ \bar{\sigma} \in \mathbb{R}^{d \times d}$ such that, as $\varepsilon \to 0$,

$$\left(X_t^\varepsilon - \frac{t\bar{b}}{\varepsilon}\right) \xrightarrow{\text{law}} \bar{\sigma} B_t.$$



§ Stochastic Case

Corrector equation

$$A D^2\phi_i + B(D\phi_i + e_i) = 0 \text{ on } \Omega.$$

Solvability is not clear

- no invariant measure in general on Ω
- no compactness $\Longleftrightarrow$ no Fredholm

$$dX_t = \sigma(X_t,\omega)\,dB_t + b(X_t,\omega)\,dt$$

As $\varepsilon \to 0$, $\varepsilon X_{t/\varepsilon^2} \xrightarrow[\text{a.s}]{\text{law}}$?

- Divergence-form (Kozlov, Papanicolaou & Varadhan)

$$-\nabla\cdot A\nabla u = 0$$

$$\parallel$$

$$-\mathrm{tr}(A\nabla^2 u) - (\nabla\cdot A)\cdot\nabla u = 0.$$

$$\mathcal{L}^* m = -\nabla\cdot A\nabla m = 0 \text{ on } \Omega$$

So, $m = 1$.

$$\mathbb{E}[(\nabla\cdot A)\cdot m] = \text{"} \int_{\mathbb{T}^d} (\nabla\cdot A)\, m\, dy \text{"}$$

$$= \mathbb{E}[(\nabla\cdot A)]$$

$$= 0.$$

- $B = 0$ (Papanicolaou & Varadhan)
  - ABP-estimate
  - approximation of random environment by periodic environments

$dX_t = \tau(X_t, \omega) dB_t$

- martingale
- homogenization reduces to an ergodic thm using characteristic function / exponential martingale

- $B = \nabla U$, gradient stat. field

$\mathcal{L}v = \Delta v + \nabla U \cdot \nabla v = 0$

$\mathcal{L}v = e^{-U} \nabla \cdot (e^{U} \nabla v)$.

$\mathcal{L}^* m = \nabla \cdot (e^{U} \nabla (e^{-U} m))$

up to normalization,

$m = \frac{e^{U}}{\mathbb{E}[e^{U}]}$, Gibb's measure

$\boxed{\text{Olla}}$

$\mathbb{E}[b m] = \frac{1}{\mathbb{E}[e^{U}]} \mathbb{E}[\nabla U e^{U}]$

$= \frac{1}{\mathbb{E}[e^{U}]} \mathbb{E}[\nabla e^{U}] = 0$

- Divergence-free

$$\partial_t u = \Delta u + b \cdot \nabla u$$

$$\mathcal{L}^* m = \Delta m - \nabla \cdot (mb)$$

$m = 1$ if $\nabla \cdot b = 0$, $\mathbb{E}[b] = 0$

motivated by the flux-convection,

$$\mathbb{E}[b] = \mathbb{E}[\nabla \cdot \tau] = 0.$$

§ Sznitman & Zeitouni

Bricmont & Kupiainen

$d \geq 3$, A, b are small perturbations of Δ, satisfying a finite range of dependence and restricted isotropy condition in law.

The $\exists\ \bar{\tau}$ such that, a.s.,

$$X_t^\varepsilon \longrightarrow \bar{\tau} B_t \text{ in law.}$$

§ Quantitative Theory of Homogenization

- Avellaneda & Lin (periodic)
- Armstrong & Smart
- Armstrong, Kuusi, Mourrat
- Gloria, Neukamm, Otto
- Gloria, Otto

Quantify the convergence

$$u^\varepsilon \to v.$$

Quantify the sublinearity of correctors

$$w^\varepsilon = u^\varepsilon - v - \varepsilon \phi_i^\varepsilon \partial_i v$$

$$-\nabla \cdot a^\varepsilon \nabla w^\varepsilon = \nabla \cdot [(\varepsilon \phi_i^\varepsilon a^\varepsilon - \varepsilon \sigma_i^\varepsilon) \nabla (\partial_i v)].$$

A rate of homogenization amounts to obtaining a rate for the convergence

$$\varepsilon \phi_i^\varepsilon, \ \varepsilon \sigma_i^\varepsilon \to 0 \quad \text{in } L^2.$$