Qualitative Periodic Homogenization

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I. Elliptic equations in divergence form

Divergence form elliptic equations: for a smooth and bounded domain $U \subseteq \mathbb{R}^d$,

$$-\nabla \cdot a\nabla u = f \;\text{ in } U \text{ with } \; u = 0 \;\text{ on } \partial U.$$

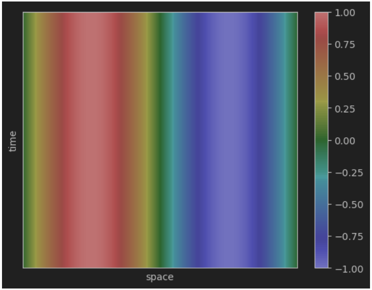
The coefficient field: $a \in L^\infty(U)^{d\times d}$ satisfying, for $\lambda \leq \Lambda \in (0,\infty)$,

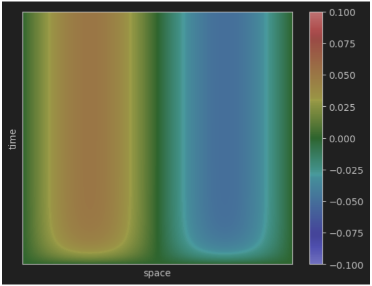
$$\xi \cdot a\xi \geq \lambda|\xi|^2 \;\text{ and }\; \xi \cdot a^{-1}\xi \geq \Lambda^{-1}|\xi|^2.$$

The flux: for every sub-domain $D \subseteq U^\circ$,

$$\int_D \big(-\nabla \cdot a\nabla u\big) = -\int_{\partial D} \nu_{\text{out}} \cdot a\nabla u = \int_D f.$$

Example: we have for $\partial_t u = \frac{1}{2}\Delta u + \sin(2\pi x)$ on $(0,1)\times(0,\infty)$ with $u = 0$ on ∂_p,





I. Elliptic equations in divergence form

Divergence form elliptic equations: for a smooth and bounded domain $U \subseteq \mathbb{R}^d$,

$$-\nabla \cdot a\nabla u = f \text{ in } U \text{ with } u = 0 \text{ on } \partial U.$$

The L^2-Poincaré inequality: for some $c = c(U) \in (0, \infty)$,

$$\int_U v^2 \leq c \int_U |\nabla v|^2 \text{ for every } v \in H_0^1(U).$$

The Energy estimates: using the Poincaré inequality and Hölder's inequality,

$$\lambda \int_U |\nabla u|^2 \leq \int_U \nabla u \cdot a\nabla u = \int_U uf \leq c\Big(\int_U |\nabla u|^2\Big)^{\frac{1}{2}}\Big(\int_U f^2\Big)^{\frac{1}{2}},$$

and, for every $u, v \in H_0^1(U)$,

$$|\int_U \nabla u \cdot a\nabla v| \leq \Lambda \int_U |\nabla u||\nabla v| \leq \Lambda\Big(\int_U |\nabla u|^2\Big)^{\frac{1}{2}}\Big(\int_U |\nabla v|^2\Big)^{\frac{1}{2}}.$$

Lax–Milgram theorem: there exists a unique $u \in H_0^1(U)$ satisfying

$$\int_U \nabla v \cdot a\nabla u = \int_U fv \text{ for every } v \in H_0^1(U).$$

Divergence form elliptic equations: for a smooth and bounded domain $U \subseteq \mathbb{R}^d$,

$$-\nabla \cdot a\nabla u = f \text{ in } U \text{ with } u = 0 \text{ on } \partial U.$$

Energy minimization: u solves the above for a symmetric a if and only if

$$u = \operatorname*{argmin}_{v \in H_0^1(U)} \left(\int_U (\nabla v \cdot a\nabla v - fv) \right).$$

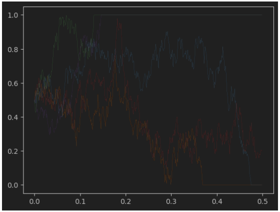
The Feynman–Kac formula: for symmetric a, for $\sigma\sigma^t = \sqrt{2}a$, and for

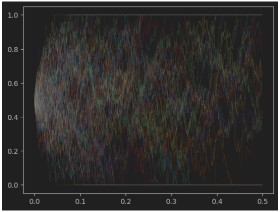
$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + (\nabla \cdot a^t)(X_t)\,\mathrm{d}t,$$

we have that

$$u(x) = \mathbb{E}_x\left[g(X_{\tau_U}) + \int_0^{\tau_U} f(X_s)\,\mathrm{d}s\right],$$

if $u = g$ on ∂U, where $\tau_U = \inf\{s \in [0,\infty) \colon X_s \notin U\}$.





II. Periodic homogenization

The coefficient field: a uniformly elliptic, not necessarily symmetric, 1-periodic

$$a\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}.$$

The microscopic scale: for every $\varepsilon \in (0,1)$,

$$a^\varepsilon(x) = a(x/\varepsilon).$$

The equation: for $\varepsilon \in (0,1)$, a smooth bounded $U \subseteq \mathbb{R}^d$, and $f \in L^2(U)$,

$$-\nabla \cdot a^\varepsilon \nabla u_\varepsilon = f \;\text{ in } U \text{ with } u = 0 \text{ on } \partial U.$$

Qualitative homogenization: identify $\overline{a} \in \mathbb{R}^{d\times d}$ such that, as $\varepsilon \to 0$,

$$u_\varepsilon \to \overline{u},$$

for $\overline{u}$ solving

$$-\nabla \cdot \overline{a}\nabla\overline{u} = f \;\text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

The diffusion process: for $\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + (\nabla\cdot a^t)(X_t)\,\mathrm{d}t$,

$$\varepsilon X_{t/\varepsilon^2} \rightharpoonup \overline{\sigma} B_t \;\text{ in law},$$

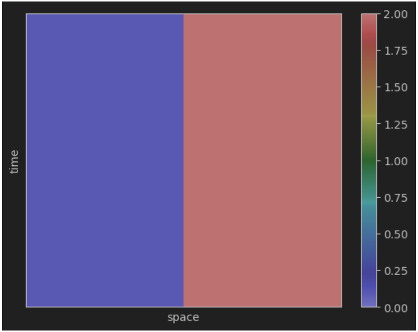
for $\sigma\sigma^t = 2a$ and $\overline{\sigma\sigma}^2 = 2\overline{a}$.

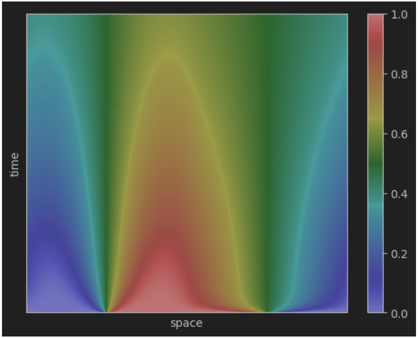
The coefficient field: in $d = 1$,

$$a(x) = \frac{1}{10} + \frac{19}{10} \sum_{k \in \mathbb{Z}} \mathbf{1}_{2k+1}(x)$$

The equation: for $\varepsilon = {}^1\!/_2$ in $\mathbb{T}^1 \times (0, \infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2} \nabla \cdot a^\varepsilon \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0, \infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4, 3/4\}}.$$



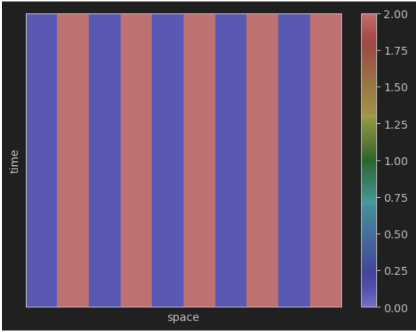


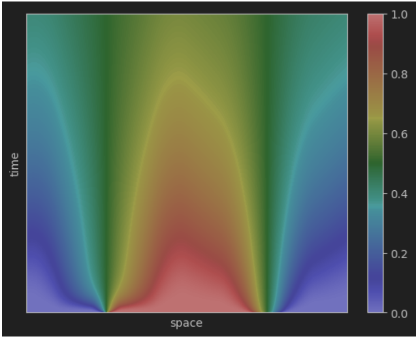
The coefficient field: in $d = 1$,

$$a(x) = \frac{1}{10} + \frac{19}{10} \sum_{k \in \mathbb{Z}} \mathbf{1}_{2k+1}(x)$$

The equation: for $\varepsilon = {}^{1}/_{10}$ in $\mathbb{T}^1 \times (0, \infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2} \nabla \cdot a^\varepsilon \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0, \infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4, 3/4\}}.$$





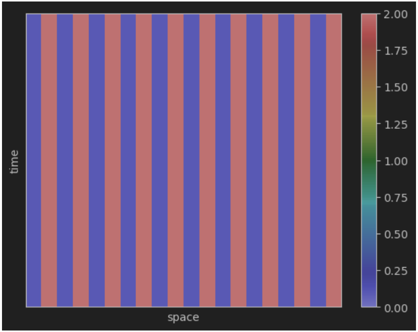
II. Periodic homogenization

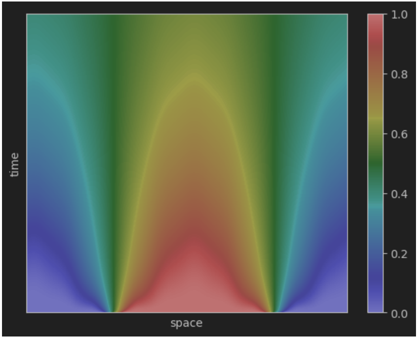
The coefficient field: in $d = 1$,

$$a(x) = \frac{1}{10} + \frac{19}{10}\sum_{k\in\mathbb{Z}} \mathbf{1}_{2k+1}(x)$$

The equation: for $\varepsilon = {}^1\!/_{50}$ in $\mathbb{T}^1 \times (0,\infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2}\nabla\cdot a^\varepsilon \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4,3/4\}}.$$



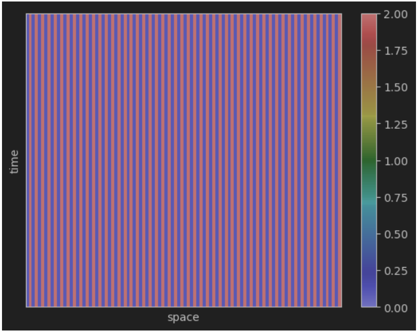


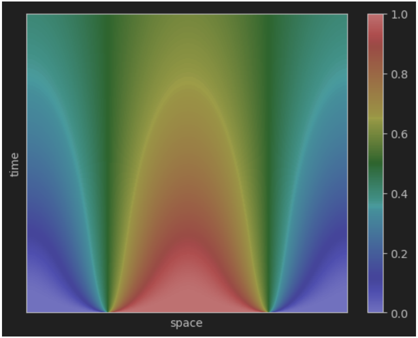
The coefficient field: in $d = 1$,

$$a(x) = \frac{1}{10} + \frac{19}{10} \sum_{k \in \mathbb{Z}} \mathbf{1}_{2k+1}(x)$$

The equation: for $\varepsilon = {}^1\!/_{100}$ in $\mathbb{T}^1 \times (0, \infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2} \nabla \cdot a^\varepsilon \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0, \infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4, 3/4\}}.$$



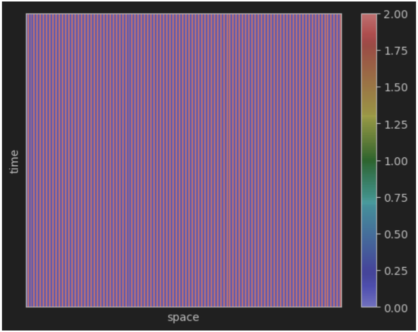


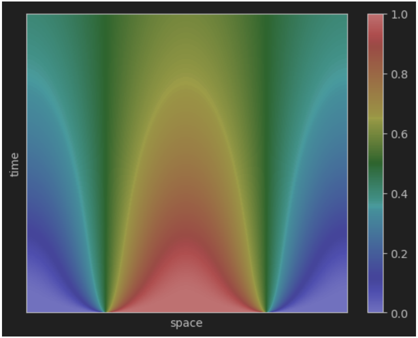
The coefficient field: in $d = 1$,

$$a(x) = \frac{1}{10} + \frac{19}{10} \sum_{k \in \mathbb{Z}} \mathbf{1}_{2k+1}(x)$$

The equation: for $\varepsilon = 1/200$ in $\mathbb{T}^1 \times (0, \infty)$,

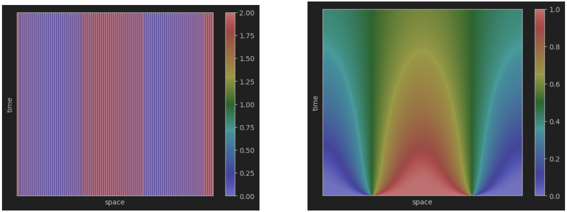
$$\partial_t u_\varepsilon = \frac{1}{2} \nabla \cdot a^\varepsilon \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0, \infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4, 3/4\}}.$$





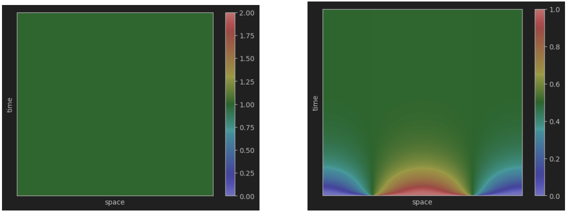
The equation: for $\varepsilon = {}^{1}/{}_{500}$ in $\mathbb{T}^1 \times (0,\infty)$,

$$\partial_t u_\varepsilon = ({}^{1}/{}_{2})\nabla \cdot a^{\varepsilon} \nabla u_\varepsilon \;\text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^{1}/{}_{4},{}^{3}/{}_{4})\}}.$$



A macroscopic limit: choosing $\tilde{a} = \langle a \rangle_{\mathbb{T}^1} = 1$,

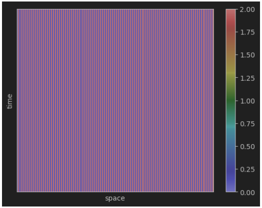
$$\partial_t \overline{u} = ({}^{1}/{}_{2})\nabla \cdot \tilde{a} \nabla \overline{u} \;\text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^{1}/{}_{4},{}^{3}/{}_{4})\}}.$$

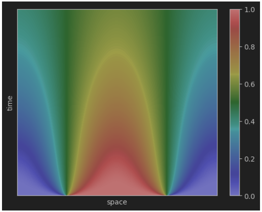


II. Periodic homogenization

The equation: for $\varepsilon = {}^1/_{500}$ in $\mathbb{T}^1 \times (0,\infty)$,

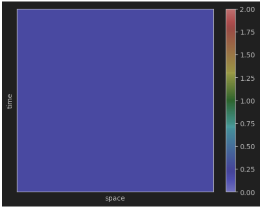
$$\partial_t u_\varepsilon = ({}^1/_2)\nabla \cdot a^\varepsilon \nabla u_\varepsilon \;\text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1/_4, {}^3/_4\}}.$$

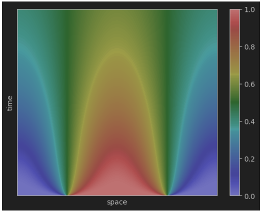




A macroscopic limit: choosing $\overline{a} = 0.19$,

$$\partial_t \overline{u} = ({}^1/_2)\nabla \cdot \overline{a}\nabla \overline{u} \;\text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1/_4, {}^3/_4\}}.$$





II. The asymptotic expansion

The asymptotic expansion: we postulate the approximation

$$\tilde{u}_\varepsilon(x) = u_0(x, {}^x/_\varepsilon) + \varepsilon u_1(x, {}^x/_\varepsilon) + \varepsilon^2 u_2(x, {}^x/_\varepsilon) + \ldots .$$

The approximate flux: for the slow variable x and the fast variable $y = {}^x/_\varepsilon$,

$$a^\varepsilon \nabla \tilde{u}_\varepsilon(x) = \varepsilon^{-1} a^\varepsilon \nabla_y u_0 + a^\varepsilon(\nabla_x u_0 + \nabla_y u_1) + \mathcal{O}_{L^2}(\varepsilon). \tag{1}$$

The equation: evaluated at a point $(x, {}^x/_\varepsilon)$,

$$\begin{aligned} -\nabla \cdot a^\varepsilon \nabla \tilde{u}_\varepsilon = & -\varepsilon^{-2} \nabla_y \cdot a^\varepsilon \nabla_y u_0 \\ & - \varepsilon^{-1} \nabla_y \cdot a^\varepsilon(\nabla_x u_0 + \nabla_y u_1) \\ & - \nabla_x \cdot a^\varepsilon(\nabla_x u_0 + \nabla_y u_1) \\ & + \mathcal{O}_{H^{-1}}(\varepsilon). \end{aligned}$$

The macroscopic limit: imposing periodicity in the y-variable, for every $x \in U$,

$$-\nabla_y \cdot a(y) \nabla_y u_0(x, y) = 0 \ \text{ in } \ \mathbb{T}^d.$$

Energy estimates show that $\nabla_y u_0(x, y) = 0$ and, therefore, that

$$u_0(x, y) = u_0(x).$$

II. The asymptotic expansion

The asymptotic expansion: we postulate the approximation

$$\tilde{u}_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, x/\varepsilon).$$

The equation: since $u_0(x,y) = u_0(x)$,

$$\begin{aligned} -\nabla \cdot a^\varepsilon \nabla \tilde{u}_\varepsilon = & -\varepsilon^{-1} \nabla_y \cdot a^\varepsilon (\nabla_x u_0 + \nabla_y u_1) \\ & - \nabla_x \cdot a^\varepsilon (\nabla_x u_0 + \nabla_y u_1) \\ & + \mathcal{O}_{H^{-1}}(\varepsilon). \end{aligned}$$

Scale separation: we postulate that, summing over $i \in \{1, \ldots, d\}$,

$$u_1(x,y) = \phi_i(y) \partial_i u_0(x),$$

for which, since $\nabla_x u_0(x) = \partial_i u_0 \cdot e_i$,

$$-\nabla_y \cdot a^\varepsilon (\nabla_x u_0 + \nabla_y u_1) = -\big(\nabla_y \cdot a^\varepsilon (e_i + \nabla_y \phi_i)\big) \partial_i u_0.$$

The homogenization correctors: for $i \in \{1, \ldots, d\}$,

$$-\nabla \cdot a(y)(e_i + \nabla \phi_i(y)) = 0 \text{ in } \mathbb{T}^d \text{ with } \fint_{\mathbb{T}^d} \phi_i = 0.$$

The asymptotic expansion: we postulate the approximation

$$\tilde{u}_\varepsilon(x) = u_0(x) + \varepsilon\phi_i(x/\varepsilon)\partial_i u_0(x),$$

for the homogenization correctors

$$-\nabla\cdot a(y)(e_i + \nabla\phi_i(y)) = 0 \ \text{ in } \ \mathbb{T}^d \ \text{ with } \ \fint_{\mathbb{T}^d}\phi_i = 0.$$

The equation: we have that

$$\begin{aligned} -\nabla\cdot a^\varepsilon\nabla\tilde{u}_\varepsilon &= -\nabla_x\cdot a^\varepsilon(e_i + \nabla\phi_i^\varepsilon)\partial_i u_0 + \mathcal{O}_{H^{-1}}(\varepsilon) \\ &= -a^\varepsilon(e_i + \nabla\phi_i^\varepsilon)\cdot\nabla_x\partial_i u_0 + \mathcal{O}_{H^{-1}}(\varepsilon). \end{aligned}$$

"Periodic ergodic theorem": for $f \in L^2(\mathbb{T}^d)$ and $U \subseteq \mathbb{R}^d$ smooth and bounded,

$$f^\varepsilon \rightharpoonup \fint_{\mathbb{T}^d} f \ \text{ weakly in } L^2(U).$$

That is, for every $g \in L^2(U)$,

$$\lim_{\varepsilon\to 0}\int_U f(x/\varepsilon)g(x)\,\mathrm{d}x = \int_U \langle f\rangle_{\mathbb{T}^d}\, g = \langle f\rangle_{\mathbb{T}^d}\int_U g.$$

II. The asymptotic expansion

The asymptotic expansion: we postulate the approximation

$$\tilde{u}_\varepsilon(x) = u_0(x) + \varepsilon\phi_i(x/\varepsilon)\partial_i u_0(x),$$

for the homogenization correctors

$$-\nabla \cdot a(y)(e_i + \nabla\phi_i(y)) = 0 \ \text{ in } \ \mathbb{T}^d \ \text{ with } \ \fint_{\mathbb{T}^d} \phi_i = 0.$$

The equation: we have that

$$-\nabla \cdot a^\varepsilon \nabla \tilde{u}_\varepsilon = -a^\varepsilon(e_i + \nabla\phi_i^\varepsilon) \cdot \nabla_x \partial_i u_0 + \mathcal{O}_{H^{-1}}(\varepsilon).$$

The homogenized coefficient: as $\varepsilon \to 0$, weakly in $L^2(U)$,

$$a^\varepsilon(e_i + \nabla\phi_i^\varepsilon) \rightharpoonup \langle a(e_i + \nabla\phi_i)\rangle_{\mathbb{T}^d} = \overline{a}e_i.$$

The coefficient $\overline{a}$ is the matrix whose rows are $\langle a(e_i + \nabla\phi_i)\rangle_{\mathbb{T}^d}$.

The homogenized equation: the macroscopic equation

$$-\nabla \cdot \overline{a}\nabla\overline{u} = f \ \text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

II. The asymptotic expansion

The asymptotic expansion: we postulate the approximation

$$\tilde{u}_\varepsilon(x) = \overline{u}(x) + \varepsilon\phi_i({}^{x}/_{\varepsilon})\partial_i\overline{u}(x),$$

for the homogenization correctors

$$-\nabla\cdot a(y)(e_i + \nabla\phi_i(y)) = 0 \;\text{ in }\; \mathbb{T}^d \;\text{ with }\; \fint_{\mathbb{T}^d}\phi_i = 0,$$

and for the macroscopic limit, for $\overline{a}e_i = \langle a(e_i + \nabla\phi_i)\rangle_{\mathbb{T}^d}$,

$$-\nabla\cdot\overline{a}\nabla\overline{u} = f \;\text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

The equation: we have that

$$-\nabla\cdot a^\varepsilon\nabla\tilde{u}_\varepsilon = -a^\varepsilon(e_i + \nabla\phi_i^\varepsilon)\cdot\nabla_x\partial_i\overline{u} + \mathcal{O}_{H^{-1}}(\varepsilon).$$

The $\varepsilon \to 0$ limit: as $\varepsilon \to 0$, by the periodic ergodic theorem,

$$-a^\varepsilon(e_i + \nabla\phi_i^\varepsilon)\cdot\nabla_x\partial_i\overline{u} \to -\overline{a}e_i\cdot\nabla_x\partial_i\overline{u} = -\nabla\cdot\overline{a}\nabla\overline{u} = f.$$

Formally in H^{-1}, for small ε,

$$-\nabla\cdot a_\varepsilon\nabla\tilde{u}_\varepsilon \simeq f = -\nabla\cdot a^\varepsilon\nabla u^\varepsilon \;\text{ with }\; \tilde{u}_\varepsilon = -\varepsilon\phi_i^\varepsilon\partial_i\overline{u} \simeq 0 \;\text{ on }\; \partial U.$$

II. The asymptotic expansion

The asymptotic expansion: we postulate the approximation

$$\tilde{u}_\varepsilon(x) = \overline{u}(x) + \varepsilon\phi_i(x/\varepsilon)\partial_i\overline{u}(x).$$

Periodic L^2-Poincaré inequality: for every $\psi \in H^1(\mathbb{T}^d)$,

$$\|\psi - \langle\psi\rangle_{\mathbb{T}^d}\|_2 \leq c\,\|\nabla\psi\|_2 \;\text{ for }\; \|\psi\|_2 = \left(\int_{\mathbb{T}^d}\psi^2\right)^{\frac{1}{2}}.$$

The homogenization correctors: there exist unique Lax–Milgram solutions $\phi_i \in H^1(\mathbb{T}^d)$ of the equation

$$-\nabla\cdot a(e_i + \nabla\phi_i) = 0 \;\text{ in }\; \mathbb{T}^d \;\text{ with }\; \langle\phi_i\rangle_{\mathbb{T}^d} = 0.$$

"Boundedness" of the correctors: we have the energy estimate

$$\|\phi_i\|_2^2 + \|\nabla\phi_i\|_2^2 \leq \frac{c}{\lambda}\,\|ae_i\|_2^2\,.$$

The diffusion process: for $\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + (\nabla\cdot a^t)(X_t)\,\mathrm{d}t$,

$$M_t = X_t + \phi(X_t) \;\text{ is a martingale,}$$

for $\phi = (\phi_1, \phi_2, \ldots, \phi_d)$, for $\sigma\sigma^t = 2a$, and

$$\varepsilon X_{t/\varepsilon^2} = \varepsilon M_{t/\varepsilon^2} + \varepsilon\phi(X_{t/\varepsilon^2}).$$

The asymptotic expansion: we postulate the approximation

$$\tilde{u}_\varepsilon(x) = \overline{u}(x) + \varepsilon\phi_i(x/\varepsilon)\partial_i\overline{u}(x).$$

The homogenized coefficient: the matrix $\overline{a} \in \mathbb{R}^{d\times d}$,

$$\overline{a}e_i = \langle a(e_i + \nabla\phi_i)\rangle_{\mathbb{T}^d}.$$

Lower bound: for $\phi_\xi = \xi_i\phi_i$ solving $-\nabla\cdot a(\xi+\nabla\phi_\xi) = 0$,

$$\begin{aligned}\xi\cdot\overline{a}\xi = \langle\xi\cdot\overline{a}\xi\rangle_{\mathbb{T}^d} &= \langle\xi\cdot a(\xi+\nabla\phi_\xi)\rangle_{\mathbb{T}^d} = \langle(\xi+\nabla\phi_\xi)\cdot\overline{a}(\xi+\nabla\phi_\xi)\rangle_{\mathbb{T}^d} \\ &\geq \lambda\langle|\xi+\nabla\phi_\xi|^2\rangle_{\mathbb{T}^d} = \lambda\big(|\xi|^2 + \langle|\nabla\phi_\xi|^2\rangle_{\mathbb{T}^d}\big) \geq \lambda|\xi|^2.\end{aligned}$$

Upper bound: for $q_\xi = a(\xi+\nabla\phi_\xi) - \overline{a}\xi$,

$$\begin{aligned}\Lambda^{-1}\big(\overline{a}\xi\cdot\overline{a}\xi\big) &\leq \Lambda^{-1}\langle(\overline{a}\xi+q_\xi)\cdot(\overline{a}\xi+q_\xi)\rangle_{\mathbb{T}^d} = \Lambda^{-1}\langle a(\xi+\nabla\phi_\xi)\cdot a(\xi+\nabla\phi_\xi)\rangle_{\mathbb{T}^d} \\ &\leq \langle a(\xi+\nabla\phi_\xi)\cdot a^{-1}a(\xi+\nabla\phi_\xi)\rangle_{\mathbb{T}^d} = \langle(\xi+\nabla\phi_\xi)\cdot a(\xi+\nabla\phi_\xi)\rangle_{\mathbb{T}^d} = \xi\cdot\overline{a}\xi.\end{aligned}$$

Since the lower bound shows that $\overline{a}$ is invertible,

$$\Lambda^{-1}\big(\overline{a}\xi\cdot\overline{a}\xi\big) \leq (\overline{a}\xi)\cdot\overline{a}^{-1}(\overline{a}\xi) \text{ implies } \Lambda^{-1}|\tilde{\xi}|^2 \leq \tilde{\xi}\cdot\overline{a}^{-1}\tilde{\xi} \text{ for all } \tilde{\xi}\in\mathbb{R}^d.$$

II. The asymptotic expansion

The asymptotic expansion: we postulate the approximation

$$\tilde{u}_\varepsilon(x) = \overline{u}(x) + \varepsilon\phi_i(x/\varepsilon)\partial_i\overline{u}(x).$$

The homogenized coefficient: the matrix $\overline{a} \in \mathbb{R}^{d\times d}$,

$$\overline{a}e_i = \langle a(e_i + \nabla\phi_i)\rangle_{\mathbb{T}^d}.$$

Another upper bound: if $|a\xi| \leq \Lambda|\xi|$ then

$$|\overline{a}\xi| = |\langle a(\xi + \nabla\phi_\xi)\rangle_{\mathbb{T}^d}| \leq |\Lambda|\big(|\xi| + \langle|\nabla\phi_\xi|\rangle_{\mathbb{T}^d}\big) \leq c\Lambda|\xi|,$$

for c depending on $\max_{i\in\{1,2,\ldots,d\}}\langle|\nabla\phi_i|\rangle_{\mathbb{T}^d}$.

Transpose: for the transpose correctors ϕ_i^t solving

$$-\nabla\cdot a^t(e_i + \nabla\phi_i^t) = 0 \text{ in } \mathbb{T}^d \text{ with } \langle\phi_i^t\rangle_{\mathbb{T}^d} = 0,$$

we have that

$$\begin{aligned}\overline{a}_{ij} = e_i\cdot\overline{a}e_j &= \langle e_i\cdot a(e_j+\nabla\phi_j)\rangle_{\mathbb{T}^d} = \langle(e_i + \nabla\phi_i^t)\cdot a^t(e_j+\nabla\phi_j)\rangle_{\mathbb{T}^d} \\ &= \langle((e_j+\nabla\phi_j)\cdot a(e_i+\nabla\phi_i^t)\rangle_{\mathbb{T}^d} = \langle e_j\cdot a(e_i+\nabla\phi_i^t)\rangle_{\mathbb{T}^d} = e_j\cdot\overline{a^t}e_i = \overline{a^t}_{ji}.\end{aligned}$$

So, $\overline{a}^t = \overline{a^t}$ and $\overline{a}$ is symmetric if a is symmetric.

The asymptotic expansion: we postulate the approximation

$$\tilde{u}_\varepsilon(x) = \overline{u}(x) + \varepsilon\phi_i({}^x\!/_\varepsilon)\partial_i\overline{u}(x).$$

The homogenized coefficient: the matrix $\overline{a} \in \mathbb{R}^{d\times d}$,

$$\overline{a}e_i = \langle a(e_i + \nabla\phi_i)\rangle_{\mathbb{T}^d}.$$

A one-dimensional example: in one-dimension $\overline{a} \in \mathbb{R}$ and the corrector ϕ satisfy

$$-\partial_x\big(a(1+\partial_x\phi)\big) = 0 \ \text{ in } \mathbb{T}^1,$$

with $\langle\phi\rangle_{\mathbb{T}^1} = 0$ and with $\overline{a} = \langle a(1+\nabla\phi)\rangle_{\mathbb{T}^1}$.

Therefore,

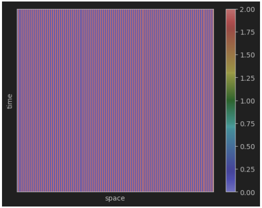
$$\partial_x\phi = \frac{c}{a} - 1 \ \text{ where } \langle\nabla\phi\rangle_{\mathbb{T}^1} = 0 \text{ implies } \ c = \langle a^{-1}\rangle_{\mathbb{T}^1}^{-1},$$

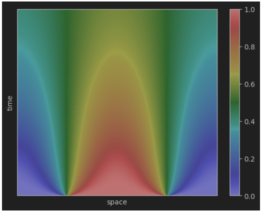
and

$$\overline{a} = \langle a(1+\nabla\phi)\rangle_{\mathbb{T}^1} = \langle\langle a^{-1}\rangle_{\mathbb{T}^1}^{-1}\rangle_{\mathbb{T}^1} = \langle a^{-1}\rangle_{\mathbb{T}^1}^{-1}.$$

The equation: for $\varepsilon = {}^1\!/_{500}$ in $\mathbb{T}^1 \times (0,\infty)$,

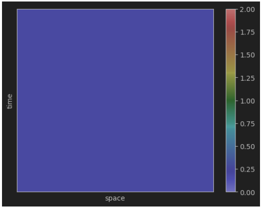
$$\partial_t u_\varepsilon = ({}^1\!/_2)\nabla \cdot a^\varepsilon \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1\!/_4, {}^3\!/_4\}}.$$

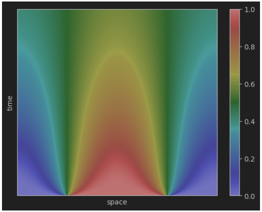




A macroscopic limit: choosing $\overline{a} = \langle a^{-1}\rangle_{\mathbb{T}^1}^{-1} = 0.19$,

$$\partial_t \overline{u} = ({}^1\!/_2)\nabla \cdot \overline{a}\nabla \overline{u} \ \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1\!/_4, {}^3\!/_4\}}.$$





The asymptotic expansion: we postulate the approximation

$$\tilde{u}_\varepsilon(x) = \overline{u}(x) + \varepsilon\phi_i(x/\varepsilon)\partial_i\overline{u}(x).$$

The homogenized equation: for the homogenized matrix $\overline{a} \in \mathbb{R}^{d\times d}$,

$$\overline{a}e_i = \langle a(e_i + \nabla\phi_i)\rangle_{\mathbb{T}^d},$$

we have that $\overline{u}$ solves

$$-\nabla\cdot\overline{a}\nabla\overline{u} = f \;\text{ in }\; U \;\text{ with }\; \overline{u} = 0 \;\text{ on }\; \partial U.$$

Maximal regularity: we have that

$$\left\|\nabla^2\overline{u}\right\|_{L^2(U)} \leq c\left\|f\right\|_{L^2(U)} \;\text{ and, therefore, }\; \left\|u\right\|_{H^2(U)} \leq c\left\|f\right\|_{L^2(U)}.$$

Schauder estimates: for $\alpha \in (0,1)$, if $U \subseteq \mathbb{R}^d$ is a $\mathrm{C}^{2,\alpha}$-domain,

$$f \in \mathrm{C}^\alpha(\overline{U}) \;\text{ implies that }\; u \in \mathrm{C}^{2,\alpha}(\overline{U}).$$

In particular, $u \in W^{2,\infty}(\overline{U})$.

III. The perturbed test function method

The equation: for a one-periodic a,

$$-\nabla \cdot a^{\varepsilon} \nabla u_{\varepsilon} = f \;\text{ in }\; U \;\text{ with }\; u = 0 \;\text{ on }\; \partial U.$$

Compactness: the energy estimate, for $c \in (0, \infty)$ independent of ε,

$$\|u_{\varepsilon}\|^2_{L^2(U)} + \|\nabla u_{\varepsilon}\|^2_{L^2(U)} \leq c \|f\|^2_{L^2(U)}.$$

implies that

$$\{u_{\varepsilon}\}_{\varepsilon \in (0,1)} \;\text{ is weakly compact in }\; H^1(U),$$

and that, by the Sobolev inequality,

$$\{u_{\varepsilon}\}_{\varepsilon \in (0,1)} \;\text{ is strongly compact in }\; L^2(U).$$

A subsequential limit: along a subsequence, for some $u \in H^1_0(U)$,

$$\nabla u_{\varepsilon} \rightharpoonup \nabla u \;\text{ weakly in }\; L^2(U)^d \;\text{ and }\; u_{\varepsilon} \to u \;\text{ strongly in }\; L^2(U).$$

Characterizing the limit: if $u_{\varepsilon} \simeq \tilde{u}_{\varepsilon} = \overline{u} + \varepsilon \phi^{\varepsilon}_i \partial_i \overline{u}$, then $u = \overline{u}$ and

$$\lim_{\varepsilon \to 0} \int_U \nabla v \cdot a^{\varepsilon} \nabla u_{\varepsilon} = \int_U f v = \int_U \nabla v \cdot \overline{a} \nabla u \;\text{ for all }\; v \in H^1_0(U).$$

III. The perturbed test function method

The equation: for a one-periodic a,

$$-\nabla \cdot a^{\varepsilon} \nabla u_{\varepsilon} = f \ \text{ in } \ U \ \text{ with } \ u = 0 \ \text{ on } \partial U.$$

The perturbed test function: for every $v \in \mathrm{C}_c^{\infty}(U)$,

$$v_{\varepsilon} = v + \varepsilon \phi_i^{t,\varepsilon} \partial_i v,$$

for the transpose correctors solving $-\nabla \cdot a^t(e_i + \nabla \phi_i^t) = 0$ in $\mathbb{T}^d$.

The equation: for every $v \in H_0^1(U)$,

$$\begin{aligned}\int_U f v_{\varepsilon} &= \int_U \nabla v_{\varepsilon} \cdot a^{\varepsilon} \nabla u_{\varepsilon} \\ &= \int_U (e_i + \nabla \phi_i^{t,\varepsilon}) \partial_i v \cdot a^{\varepsilon} \nabla u_{\varepsilon} + \varepsilon \int_U \phi_i^{t,\varepsilon} \nabla \partial_i v \cdot a^{\varepsilon} \nabla u_{\varepsilon} \\ &= \int_U \nabla u_{\varepsilon} \cdot a^{t,\varepsilon} (e_i + \nabla \phi_i^{t,\varepsilon}) \partial_i v + \mathcal{O}_v(\varepsilon).\end{aligned}$$

The div-curl lemma: after integrating by parts,

$$\int_U \nabla v_{\varepsilon} \cdot a^{\varepsilon} \nabla u_{\varepsilon} = -\int_U u_{\varepsilon} \, a^{t,\varepsilon} (e_i + \nabla \phi_i^{t,\varepsilon}) \cdot \nabla \partial_i v + \mathcal{O}_v(\varepsilon).$$

III. The perturbed test function method

The equation: for a one-periodic a,

$$-\nabla\cdot a^{\varepsilon}\nabla u_{\varepsilon}=f\ \text{ in }\ U\ \text{ with }\ u=0\ \text{ on }\partial U.$$

A subsequential limit: along a subsequence, for some $u\in H^1_0(U)$,

$$\nabla u_{\varepsilon}\rightharpoonup\nabla u\ \text{ weakly in }L^2(U)^d\ \text{ and }\ u_{\varepsilon}\to u\ \text{ strongly in }L^2(U).$$

The equation: for every $v\in H^1_0(U)$, for $v_{\varepsilon}=v+\varepsilon\phi_i^{t,\varepsilon}\partial_i v$,

$$\int_U\nabla v_{\varepsilon}\cdot a^{\varepsilon}\nabla u_{\varepsilon}=-\int_U u_{\varepsilon}\,a^{t,\varepsilon}(e_i+\nabla\phi_i^{t,\varepsilon})\cdot\nabla\partial_i v+\mathcal{O}_v(\varepsilon).$$

The $\varepsilon\to 0$ limit: since, as $\varepsilon\to 0$,

$$a^{t,\varepsilon}(e_i+\nabla\phi_i^{t,\varepsilon})\rightharpoonup\langle a^t(e_i+\nabla\phi_i^t)\rangle_{\mathbb{T}^d}=\overline{a}^te_i\ \text{ weakly in }L^2(U)^d,$$

and $v_{\varepsilon}\to v$ strongly in $L^2(U)$, along a subsequence,

$$\begin{aligned}\int_U fv=\lim_{\varepsilon\to0}\int_U fv_{\varepsilon}&=\lim_{\varepsilon\to0}\Big(-\int_U u_{\varepsilon}\,a^{t,\varepsilon}(e_i+\nabla\phi_i^{t,\varepsilon})\cdot\nabla\partial_i v+\mathcal{O}_v(\varepsilon)\Big)\\&=-\int_U u\,\overline{a}^te_i\cdot\nabla\partial_i v=\int_U\nabla v\cdot\overline{a}\nabla u.\end{aligned}$$

III. The perturbed test function method

The equation: for a one-periodic a,

$$-\nabla \cdot a^{\varepsilon} \nabla u_{\varepsilon} = f \;\text{ in }\; U \;\text{ with }\; u = 0 \;\text{ on }\; \partial U.$$

A subsequential limit: along a subsequence, for some $u \in H_0^1(U)$,

$$\nabla u_{\varepsilon} \rightharpoonup \nabla u \;\text{ weakly in }\; L^2(U)^d \;\text{ and }\; u_{\varepsilon} \to u \;\text{ strongly in }\; L^2(U).$$

The equation: for every $v \in H_0^1(U)$, for $v_{\varepsilon} = v + \varepsilon \phi_i^{t,\varepsilon} \partial_i v$,

$$\int_U \nabla v_{\varepsilon} \cdot a^{\varepsilon} \nabla u_{\varepsilon} = -\int_U u_{\varepsilon}\, a^{t,\varepsilon}(e_i + \nabla \phi_i^{t,\varepsilon}) \cdot \nabla \partial_i v + \mathcal{O}_v(\varepsilon).$$

The $\varepsilon \to 0$ limit: since, as $\varepsilon \to 0$,

$$a^{t,\varepsilon}(e_i + \nabla \phi_i^{t,\varepsilon}) \rightharpoonup \langle a^t (e_i + \nabla \phi_i^t) \rangle_{\mathbb{T}^d} = \overline{a}^t e_i \;\text{ weakly in }\; L^2(U)^d,$$

and $v_{\varepsilon} \to v$ strongly in $L^2(U)$, along a subsequence,

$$\begin{aligned} \int_U f v = \lim_{\varepsilon \to 0} \int_U f v_{\varepsilon} &= \lim_{\varepsilon \to 0} \Big(-\int_U u_{\varepsilon}\, a^{t,\varepsilon}(e_i + \nabla \phi_i^{t,\varepsilon}) \cdot \nabla \partial_i v + \mathcal{O}_v(\varepsilon) \Big) \\ &= -\int_U u\, \overline{a}^t e_i \cdot \nabla \partial_i v = \int_U \nabla v \cdot \overline{a} \nabla u. \end{aligned}$$

Qualitative Periodic Homogenization

Let $U \subseteq \mathbb{R}^d$ be a bounded domain, let $a\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be one-periodic and uniformly elliptic, let $f \in L^2(U)$, for every $\varepsilon \in (0,1)$ let $u_\varepsilon \in H^1_0(U)$ be the unique Lax–Milgram solution of the equation

$$-\nabla \cdot a^\varepsilon \nabla u_\varepsilon = f \;\text{ in }\; U \;\text{ with }\; u = 0 \;\text{ on }\; \partial U,$$

and let $\overline{u} \in H^1_0(U)$ be the unique Lax–Milgram solution of the equation

$$-\nabla \cdot \overline{a} \nabla \overline{u} = f \;\text{ in }\; U \;\text{ with }\; \overline{u} = 0 \;\text{ on }\; \partial U.$$

Then, as $\varepsilon \to 0$,

$$u_\varepsilon \rightharpoonup \overline{u} \;\text{ weakly in }\; H^1_0(U) \;\text{ and }\; u_\varepsilon \to \overline{u} \;\text{ strongly in }\; L^2(U).$$

The diffusion process: for $\overline{\sigma\sigma}^t = \sqrt{2}\overline{a}$,

$$\varepsilon X_{t/_{\varepsilon^2}} \rightharpoonup \overline{\sigma} B_t \;\text{ in law.}$$

Convergence of the gradient: based on $u_\varepsilon \simeq \tilde{u}_\varepsilon = \overline{u} + \varepsilon \phi_i^\varepsilon \partial_i \overline{u}$,

$$\nabla u_\varepsilon \simeq \nabla \tilde{u}_\varepsilon = \nabla \overline{u} + \partial_i \overline{u} \nabla \phi_i^\varepsilon + \mathcal{O}_{L^2}(\varepsilon).$$

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