Quantitative Periodic and Qualitative Stochastic Homogenization

Benjamin Fehrman

LSU

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I. The homogenization error

The equation: for a 1-periodic $a\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}$,

$$-\nabla \cdot a^{\varepsilon}\nabla u_\varepsilon = f \ \text{ in } \ U \ \text{ with } \ u_\varepsilon = 0 \ \text{ on } \ \partial U.$$

The homogenization correctors: for $\phi_i \in H^1(\mathbb{T}^d)$,

$$-\nabla \cdot a(\nabla\phi_i + e_i) = 0 \ \text{ in } \ \mathbb{T}^d \ \text{ with } \ \langle \phi_i \rangle_{\mathbb{T}^d} = 0.$$

The homogenized coefficient: we define $\overline{a} \in \mathbb{R}^{d\times d}$ by

$$\overline{a}e_i = \langle a(\nabla\phi_i + e_i)\rangle_{\mathbb{T}^d}.$$

The homogenized equation: for $\overline{u} \in H^1(U)$,

$$-\nabla \cdot \overline{a}\nabla\overline{u} = f \ \text{ in } \ U \ \text{ with } \ \overline{u} = 0 \ \text{ on } \ \partial U.$$

The two-scale expansion: we have that

$$\tilde{u}_\varepsilon = \overline{u} + \varepsilon\phi_i^\varepsilon\partial_i\overline{u},$$

Qualitative homogenization: as $\varepsilon \to 0$,

$$u_\varepsilon \rightharpoonup \overline{u} \ \text{ weakly in } H^1(U) \ \text{ and } \ u_\varepsilon \to \overline{u} \ \text{ strongly in } L^2(U).$$

The homogenization error: comparing u_ε to the two-scale expansion,

$$w_\varepsilon = u_\varepsilon - \overline{u} - \varepsilon\phi_i^\varepsilon \partial_i \overline{u}.$$

The flux: we have that

$$\nabla w_\varepsilon = \nabla u_\varepsilon - \nabla \overline{u} - \nabla \phi_i^\varepsilon \partial_i \overline{u} + \varepsilon \phi_i^\varepsilon \nabla \partial_i \overline{u},$$

and, therefore,

$$\begin{aligned} a^\varepsilon \nabla w_\varepsilon &= a^\varepsilon \nabla u_\varepsilon - a^\varepsilon \nabla \overline{u} - (a^\varepsilon \nabla \phi_i^\varepsilon)\partial_i \overline{u} + \varepsilon \phi_i^\varepsilon (a^\varepsilon \nabla \partial_i \overline{u}) \\ &= a^\varepsilon \nabla u_\varepsilon - a^\varepsilon (e_i + \nabla \phi_i^\varepsilon)\partial_i \overline{u} + \varepsilon \phi_i^\varepsilon (a^\varepsilon \nabla \partial_i \overline{u}). \end{aligned}$$

The equation: we have that

$$\begin{aligned} &- \nabla \cdot a^\varepsilon \nabla w_\varepsilon \\ &= -\nabla \cdot a^\varepsilon \nabla u_\varepsilon + \big(\nabla \cdot a^\varepsilon (e_i + \nabla \phi_i)\big)\partial_i \overline{u} + a^\varepsilon (e_i + \nabla \phi_i) \cdot \nabla \partial_i \overline{u} + \nabla \cdot (\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla \partial_i \overline{u}) \\ &= f + a^\varepsilon (e_i + \nabla \phi_i) \cdot \nabla \partial_i \overline{u} + \nabla \cdot (\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla \partial_i \overline{u}) \\ &= -\nabla \cdot \overline{a} \nabla \overline{u} + a^\varepsilon (e_i + \nabla \phi_i) \cdot \nabla \partial_i \overline{u} + \nabla \cdot (\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla \partial_i \overline{u}) \\ &= -\overline{a} e_i \cdot \nabla \partial_i \overline{u} + a^\varepsilon (e_i + \nabla \phi_i) \cdot \nabla \partial_i \overline{u} + \nabla \cdot (\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla \partial_i \overline{u}) \\ &= \big(a^\varepsilon (e_i + \nabla \phi_i) - \overline{a} e_i\big) \cdot \nabla \partial_i \overline{u} + \nabla \cdot (\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla \partial_i \overline{u}). \end{aligned}$$

I. The homogenization error

The homogenization error: we have that

$$w_\varepsilon = u_\varepsilon - \overline{u} - \varepsilon\phi_i^\varepsilon \partial_i \overline{u}.$$

The equation: for the corrected flux $q_i = a(e_i + \nabla\phi_i) - \overline{a}e_i$,

$$\begin{aligned} -\nabla \cdot a^\varepsilon \nabla w_\varepsilon &= \left(a^\varepsilon(e_i + \nabla\phi_i) - \overline{a}e_i\right) \cdot \nabla\partial_i\overline{u} + \nabla \cdot (\varepsilon\phi_i^\varepsilon a^\varepsilon \nabla\partial_i\overline{u}) \\ &= q_i^\varepsilon \cdot \nabla\partial_i\overline{u} + \nabla \cdot (\varepsilon\phi_i^\varepsilon a^\varepsilon \nabla\partial_i\overline{u}). \end{aligned}$$

The stream matrix: since

$$\nabla \cdot q_i = 0 \;\text{ and }\; \langle q_i \rangle_{\mathbb{T}^d} = 0,$$

there exists a skew-symmetric "stream matrix" $\sigma_i = (\sigma_{ikj}) \in H^1(\mathbb{T}^d)^{d\times d}$ such that

$$\nabla \cdot \sigma_i = q_i \;\text{ for }\; (\nabla \cdot \sigma_i)_k = \partial_j \sigma_{ikj}.$$

The equation: if such matrices σ_i exist, then as distributions

$$q_i^\varepsilon \cdot \nabla\partial_i\overline{u} = (\nabla \cdot \sigma_i) \cdot \nabla\partial_i\overline{u} = \nabla \cdot \left(\sigma_i^t \nabla\partial_i\overline{u}\right) - \mathrm{tr}(\sigma_i^t \nabla^2 \partial_i\overline{u}) = -\nabla \cdot (\sigma_i \nabla\partial_i\overline{u}),$$

and

$$-\nabla \cdot a^\varepsilon \nabla w_\varepsilon = \nabla \cdot \left((\varepsilon\phi_i^\varepsilon a^\varepsilon - \varepsilon\sigma_i^\varepsilon)\nabla\partial_i\overline{u}\right).$$

I. The homogenization error

The stream matrix: for the corrected flux $q_i = a(e_i + \nabla\phi_i) - \overline{a}e_i$ satisfying

$$\nabla \cdot q_i = 0 \text{ with } \langle q_i \rangle_{\mathbb{T}^d} = 0,$$

minimize, over the set of 1-periodic $d \times d$ skew-symmetric matrices,

$$\int_{\mathbb{T}^d} |\nabla \sigma|^2 \text{ subject to } \nabla \cdot \sigma = q_i.$$

The equation: by Lax–Milgram there exist unique $\sigma_{ijk} \in H^1(\mathbb{T}^d)$ with

$$-\Delta \sigma_{ijk} = \partial_j q_{ij} - \partial_k q_{ik} \text{ and } \langle \sigma_{ijk} \rangle_{\mathbb{T}^d} = 0.$$

If $\sigma_i = (\sigma_{ijk})$ then

$$\begin{aligned}
\Delta\big((\nabla \cdot \sigma_i - q_i)_j\big) &= \Delta\big(\partial_r \sigma_{ijr}\big) - \Delta q_{ij} \\
&= \partial_r\big(\Delta \sigma_{ijr}\big) - \Delta q_{ij} \\
&= \partial_r\big(\partial_r q_{ij} - \partial_j q_{ir}\big) - \Delta q_{ij} \\
&= \Delta q_{ij} - \partial_j\big(\nabla \cdot q_i\big) - \Delta q_{ij} \\
&= 0.
\end{aligned}$$

We therefore conclude that

$$\nabla \cdot \sigma_i - q_i = \langle \nabla \cdot \sigma_i - q_i \rangle_{\mathbb{T}^d} = 0 \text{ and, therefore, } \nabla \cdot \sigma_i = q_i.$$

I. The homogenization error

The equation: for the homogenization error

$$w_\varepsilon = u_\varepsilon - \overline{u} - \varepsilon\phi_i^\varepsilon \partial_i \overline{u},$$

for the flux

$$q_i = a(e_i + \nabla\phi_i) - \overline{a}e_i,$$

and for the stream matrix $\sigma_i = (\sigma_{ijk})$ satisfying

$$-\Delta\sigma_{ijk} = \partial_j q_{ij} - \partial_k q_{ik} \;\text{ and }\; \langle\sigma_{ijk}\rangle_{\mathbb{T}^d} = 0,$$

we have that $\nabla\cdot\sigma_i = q_i$ and that

$$\begin{aligned}-\nabla\cdot a^\varepsilon\nabla w_\varepsilon &= q_i^\varepsilon\cdot\nabla\partial_i\overline{u} + \nabla\cdot(\varepsilon\phi_i^\varepsilon a^\varepsilon\nabla\partial_i\overline{u}) \\ &= \nabla\cdot\big((\varepsilon\phi_i^\varepsilon a^\varepsilon - \varepsilon\sigma_i^\varepsilon)\nabla\partial_i\overline{u}\big).\end{aligned}$$

The energy estimate: formally this suggests that, for c depending on Λ and U,

$$\begin{aligned}\lambda^{\frac{1}{2}}\left(\int_U |\nabla w_\varepsilon|^2\right)^{\frac{1}{2}} &\leq c\varepsilon\,\|\overline{u}\|_{W^{2,\infty}}\left(\int_U \left(|\phi_i^\varepsilon|^2 + |\sigma_i^\varepsilon|^2\right)\right)^{\frac{1}{2}} \\ &\leq c\varepsilon\,\|\overline{u}\|_{W^{2,\infty}}\left(\int_{\mathbb{T}^d}\left(|\phi_i|^2 + |\sigma_i|^2\right)\right)^{\frac{1}{2}}.\end{aligned}$$

The boundary layer: however, $w_\varepsilon \notin H_0^1(U)$!

II. Quantitative periodic homogenization

The equation: for a 1-periodic $a\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}$,

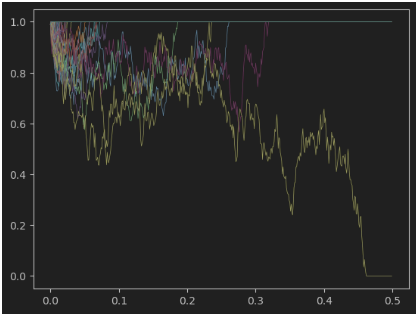
$$-\nabla \cdot a^{\varepsilon} \nabla u_\varepsilon = f \;\text{ in }\; U \;\text{ with }\; u_\varepsilon = 0 \;\text{ on }\; \partial U.$$

The Feynman–Kac formula: for the diffusion process, for $\sigma\sigma^t = a$,

$$\mathrm{d}X_t = \sqrt{2}\sigma(X_t)\,\mathrm{d}B_t + (\nabla \cdot a^t)(X_t)\,\mathrm{d}t,$$

we have that, for the exit time τ_U from U,

$$u_\varepsilon = E_{x/\varepsilon}\left[\int_0^{\tau_U} f\left(\varepsilon X_{t/\varepsilon^2}\right)\mathrm{d}s\right].$$



The boundary layer: for $0 < r << 1$,

$$U_r = \{x \in U\colon\ \mathrm{d}(x, \partial U) \geq r\}.$$

Let $\eta_r \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ satisfy

$$\eta_r(x) = 1 \ \text{ in } U_r \text{ and } \ \eta_r(x) = 0 \ \text{ in } \ \mathbb{R}^d \setminus U^\circ_{r/2} \ \text{ and } \ |\nabla \eta_r| \leq {}^c/_r \ \text{ on } \mathbb{R}^d.$$

The two-scale expansion: we define $w_{\varepsilon,r} \in H^1_0(U)$ as

$$w_{\varepsilon,r} = u_\varepsilon - \overline{u} - \varepsilon \eta_r \phi_i^\varepsilon \partial_i \overline{u}.$$

The flux: similarly to before,

$$a^\varepsilon \nabla w_{\varepsilon,r} = a^\varepsilon \nabla u_\varepsilon - a^\varepsilon \nabla \overline{u} - (a^\varepsilon \nabla \phi_i^\varepsilon)\eta_r \partial_i \overline{u} - \varepsilon \phi_i^\varepsilon a^\varepsilon \nabla(\eta_r \partial_i \overline{u}).$$

The equation: nearly as before,

$$\begin{aligned}
&-\nabla \cdot a^\varepsilon \nabla w_{\varepsilon,r} \\
&= -\nabla \cdot \overline{a} \nabla \overline{u} + \nabla \cdot (1-\eta_r) a^\varepsilon \nabla \overline{u} + \nabla \cdot \big(a^\varepsilon(e_i + \nabla \phi_i^\varepsilon)\eta_r \partial_i \overline{u}\big) + \nabla \cdot \big(\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla(\eta_r \partial_i \overline{u})\big) \\
&= \nabla \cdot \big((1-\eta_r)(a^\varepsilon - \overline{a})\nabla \overline{u}\big) + \nabla \cdot \big((a^\varepsilon(e_i + \nabla \phi_i^\varepsilon) - \overline{a} e_i)\eta_r \partial_i \overline{u}\big) + \nabla \cdot \big(\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla(\eta_r \partial_i \overline{u})\big) \\
&= \nabla \cdot \big((1-\eta_r)(a^\varepsilon - \overline{a})\nabla \overline{u}\big) + q_i^\varepsilon \cdot \nabla(\eta_r \partial_i \overline{u}) + \nabla \cdot \big(\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla(\eta_r \partial_i \overline{u})\big) \\
&= \nabla \cdot \big((1-\eta_r)(a^\varepsilon - \overline{a})\nabla \overline{u}\big) + \nabla \cdot \big((\varepsilon \phi_i^\varepsilon a^\varepsilon - \varepsilon \sigma_i^\varepsilon)\nabla(\eta_r \partial_i \overline{u})\big).
\end{aligned}$$

The two-scale expansion: for $w_{\varepsilon,r} \in H_0^1(U)$ defined by

$$w_{\varepsilon,r} = u_\varepsilon - \overline{u} - \varepsilon\eta_r\phi_i^\varepsilon\partial_i\overline{u},$$

we have that

$$-\nabla\cdot a^\varepsilon\nabla w_{\varepsilon,r} = \nabla\cdot\Big((1-\eta_r)(a^\varepsilon-\overline{a})\nabla\overline{u}\Big) + \nabla\cdot\Big((\varepsilon\phi_i^\varepsilon a^\varepsilon - \varepsilon\sigma_i^\varepsilon)\nabla(\eta_r\partial_i\overline{u})\Big).$$

The energy estimate: for $c\in(0,\infty)$ depending on λ, Λ, and U,

$$\begin{aligned}\int_U |\nabla w_{\varepsilon,r}|^2 \leq cr\left\|\nabla^2\overline{u}\right\|^2_{L^\infty(U\setminus U_r)} \\ &+ c\varepsilon^2 r^{-2}\left\|\nabla\overline{u}\right\|^2_{L^\infty(U_r\setminus U_{r/2})}\left\|(\phi_i^\varepsilon,\sigma_i^\varepsilon)\right\|^2_{L^2(U_r\setminus U_{r/2})} \\ &+ c\varepsilon^2\left\|\nabla^2\overline{u}\right\|^2_{L^\infty(U_{r/2})}\left\|(\phi_i^\varepsilon,\sigma_i^\varepsilon)\right\|^2_{L^2(U_{r/2})}.\end{aligned}$$

Due to the periodicity, for $r\geq\varepsilon$,

$$\left\|(\phi_i^\varepsilon,\sigma_i^\varepsilon)\right\|^2_{L^2(U_r\setminus U_{r/2})} \leq c|U_r\setminus U_{r/2}|\left\|(\phi_i,\sigma_i)\right\|^2_{L^2(\mathbb{T}^d)} \leq cr\left\|(\phi_i,\sigma_i)\right\|^2_{L^2(\mathbb{T}^d)}.$$

and, after choosing $r=\varepsilon$, for $c\in(0,\infty)$ depending on λ, Λ, and U,

$$\begin{aligned}\int_U |\nabla w_{\varepsilon,r}|^2 \leq c\varepsilon\Big(\left\|\nabla^2\overline{u}\right\|^2_{L^\infty(U)} + \left\|\nabla\overline{u}\right\|^2_{L^\infty(U)}\left\|(\phi_i,\sigma_i)\right\|^2_{L^2(\mathbb{T}^d)}\Big) \\ &+ c\varepsilon^2\left\|\nabla^2\overline{u}\right\|^2_{L^\infty(U)}\left\|(\phi_i,\sigma_i)\right\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

The energy estimate: for $w_{\varepsilon,r} = u_\varepsilon - \overline{u} - \varepsilon\eta_r\phi_i^\varepsilon\partial_i\overline{u}$,

$$\int_U |\nabla w_{\varepsilon,r}|^2 \leq c\varepsilon\left(\left\|\nabla^2\overline{u}\right\|^2_{L^\infty(U)} + \|\nabla\overline{u}\|^2_{L^\infty(U)} \|(\phi_i,\sigma_i)\|^2_{L^2(\mathbb{T}^d)} \right) + c\varepsilon^2 \left\|\nabla^2\overline{u}\right\|^2_{L^\infty(U)} \|(\phi_i,\sigma_i)\|^2_{L^2(\mathbb{T}^d)}.$$

The full two-scale expansion: for $w_\varepsilon = u_\varepsilon - \overline{u} - \varepsilon\phi_i^\varepsilon\partial_i\overline{u}$, for $r = \varepsilon$,

$$\begin{aligned}
&\|\nabla w_\varepsilon - \nabla w_{\varepsilon,r}\|^2_{L^2(U)} \\
&= \|\nabla(\varepsilon(1-\eta_r)\phi_i^\varepsilon\partial_i\overline{u}\|^2_{L^2(U)} \\
&\leq c\varepsilon\left(\|\nabla\overline{u}\|^2_{L^\infty(U)} \|\phi_i\|^2_{L^2(U)} + \|\nabla\overline{u}\|^2_{L^\infty(U)} \|\nabla\phi_i\|^2_{L^2(\mathbb{T}^d)} + \varepsilon \left\|\nabla^2\overline{u}\right\|^2_{L^\infty(U)} \|\phi_i\|^2_{L^2(\mathbb{T}^d)} \right).
\end{aligned}$$

Quantitative periodic homogenization

There exists $c \in (0,\infty)$ depending on λ, Λ, U, and

$$\max_{i\in\{1,\ldots,d\}} \|(\phi_i,\sigma_i)\|_{L^2(\mathbb{T}^d)} \quad \text{and} \quad \max_{i\in\{1,\ldots,d\}} \|\nabla\phi_i\|_{L^2(\mathbb{T}^d)},$$

such that, for the two-scale expansion $w_\varepsilon = u_\varepsilon - \overline{u} - \varepsilon\phi_i^\varepsilon\partial_i\overline{u}$,

$$\|\nabla w_\varepsilon\|_{L^2(U)} \leq c\varepsilon^{\frac{1}{2}} \left(\|\nabla\overline{u}\|_{L^\infty(U)} + \left\|\nabla^2\overline{u}\right\|_{L^\infty(U)} \right).$$

III. A random environment

Locally finite point configurations: the set

$$\Omega = \{\omega = \sum_{i\in I} \delta_{x_i} \colon \{x_i\}_{i\in I} \subseteq \mathbb{R}^d \text{ is locally finite}\}.$$

with sigma algebra $\mathcal{F}$ generated by the maps

$$\omega \to \omega(B) = \#\{x_i \in B\} \text{ for all Borel } B \subseteq \mathbb{R}^d.$$

Poisson point process: there exists a unique $\mathbb{P}_\lambda$ on $(\Omega, \mathcal{F})$ such that

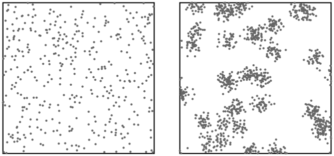
$$\mathbb{E}_\lambda[\omega(B)] = \lambda|B|,$$

such that, if $B_1, B_2, \ldots, B_N \subseteq \mathbb{R}^d$ are disjoint,

$$\{\omega \to \omega(B_i)\}_{i\in\{1,\ldots,N\}} \text{ are independent},$$

and such that, for every $y \in \mathbb{R}^d$,

$$\mathbb{P}_\lambda(A) = \mathbb{P}_\lambda(A+y) \text{ for } A + y = \{\omega(\cdot - y) \colon \omega \in A\}.$$



III. A random environment

A random diffusion matrix: for the Poisson point process $(\Omega, \mathcal{F}, \mathbb{P}_\lambda)$,

$$a\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$$

is defined by

$$a(\cdot,\omega) = ({}^{19}/_{10})\mathbf{1}_{\cup_{i\in I} B_1(x_i)} + ({}^{1}/_{10})\mathbf{1}_{(\cup_{i\in I} B_1(x_i))^c}.$$

A diffusion in random environment: for $\sigma\sigma^t = 2a$,

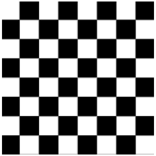
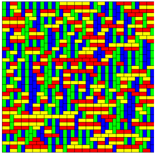
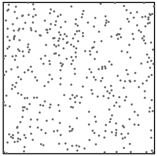
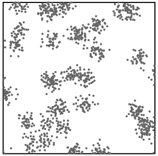
$$\mathrm{d}X_t^\omega = \sigma(X_t^\omega,\omega)\,\mathrm{d}B_t + (\nabla\cdot a^t)(X_t^\omega,\omega)\,\mathrm{d}t.$$

Stationarity: the measure-preserving group $\tau_x\omega(\cdot) = \omega(\cdot - x)$,

$$a(y,\tau_x\omega) = a(y+x,\omega)\ \text{ for all } x,y\in\mathbb{R}^d \text{ and } \omega\in\Omega.$$

Ergodicity: for a measurable $f\colon \Omega\to\mathbb{R}$,

$$f \text{ is } \tau_x\text{-invariant } \forall x\in\mathbb{R}^d \text{ if and only if } f \text{ is constant.}$$

IV. Stochastic homogenization

The probability space: the space of environments $(\Omega, \mathcal{F}, \mathbb{P})$.

The coefficient field: a measurable, uniformly elliptic $a\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$.

Stationarity: for a measure-preserving transformation group $\{\tau_x\}_{x\in\mathbb{R}^d}$,

$$a(y, \tau_x\omega) = a(x+y, \omega) \text{ for all } x, y \in \mathbb{R}^d \text{ and } \omega \in \Omega.$$

Ergodicity: a measurable function $f\colon \Omega \to \mathbb{R}$ satisfies

$$f(\omega) = f(\tau_x\omega) \;\; \mathbb{P}\text{-a.s. } \forall x \in \mathbb{R}^d \text{ if and only if } f \text{ is constant.}$$

The equation: for $\varepsilon \in (0,1)$, for $\omega \in \Omega$ fixed and for $a^\varepsilon(x,\omega) = a(x/\varepsilon, \omega)$,

$$-\nabla \cdot a^\varepsilon \nabla u_\varepsilon = f \;\text{ in U }\; u = 0 \;\text{ on }\; \partial U.$$

Quenched qualitative homogenization: there exists $\overline{a} \in \mathbb{R}^{d\times d}$ such that,

$$u_\varepsilon \to \overline{u} \;\text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

for the solution

$$-\nabla \cdot \overline{a}\nabla\overline{u} = f \;\text{ in } U \text{ with } \;\overline{u} = 0 \;\text{ on } \partial U.$$

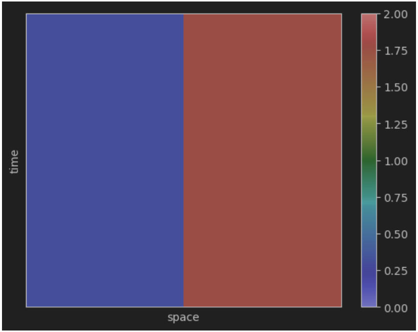
The coefficient field: in $d = 1$,

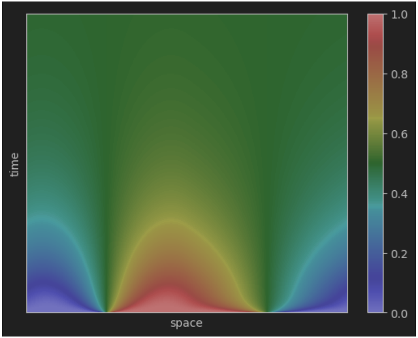
$$a(x,\omega) = 1 + (9/10)\sin(2\pi\mathcal{N}^{\varepsilon}(x,\omega)),$$

for $\mathcal{N} = \sum_{k\in\mathbb{Z}} \mathrm{Unif}_k(0,1)\mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = 1/2$ in $\mathbb{T}^1 \times (0,\infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2}\nabla\cdot a^{\varepsilon}\nabla u_\varepsilon \;\text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4,3/4\}}.$$





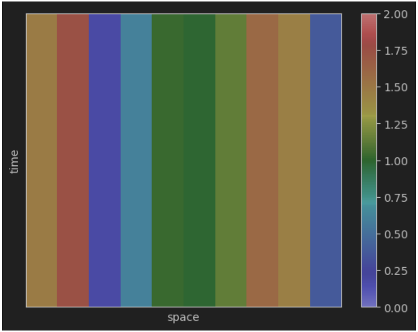
The coefficient field: in $d = 1$,

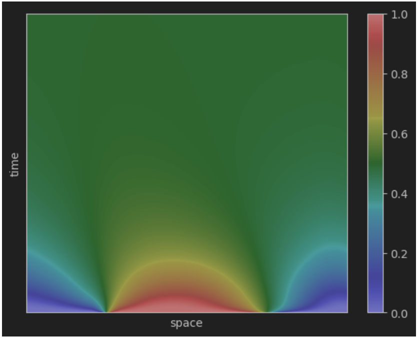
$$a(x,\omega) = 1 + (9/10)\sin(2\pi\mathcal{N}^{\varepsilon}(x,\omega)),$$

for $\mathcal{N} = \sum_{k\in\mathbb{Z}} \mathrm{Unif}_k(0,1)\mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = 1/10$ in $\mathbb{T}^1 \times (0,\infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2}\nabla\cdot a^{\varepsilon}\nabla u_\varepsilon \ \text{ in } \mathbb{T}^1\times(0,\infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4,3/4\}}.$$





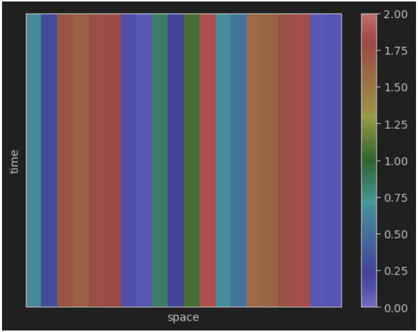
The coefficient field: in $d = 1$,

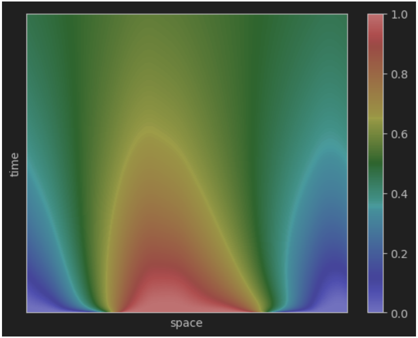
$$a(x, \omega) = 1 + (9/10)\sin(2\pi \mathcal{N}^{\varepsilon}(x, \omega)),$$

for $\mathcal{N} = \sum_{k \in \mathbb{Z}} \mathrm{Unif}_k(0, 1)\mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = 1/20$ in $\mathbb{T}^1 \times (0, \infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2}\nabla \cdot a^{\varepsilon}\nabla u_\varepsilon \;\text{ in } \mathbb{T}^1 \times (0, \infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4, 3/4\}}.$$





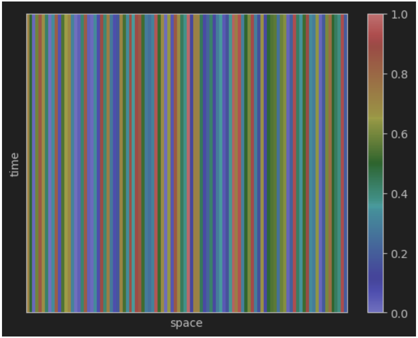
The coefficient field: in $d = 1$,

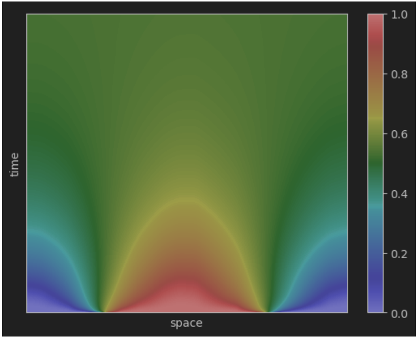
$$a(x, \omega) = 1 + ({}^{9}\!/_{10}) \sin(2\pi \mathcal{N}^{\varepsilon}(x, \omega)),$$

for $\mathcal{N} = \sum_{k \in \mathbb{Z}} \mathrm{Unif}_k(0,1) \mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = {}^{1}\!/_{100}$ in $\mathbb{T}^1 \times (0, \infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2} \nabla \cdot a^{\varepsilon} \nabla u_\varepsilon \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^{1}\!/_{4}, {}^{3}\!/_{4}\}}.$$





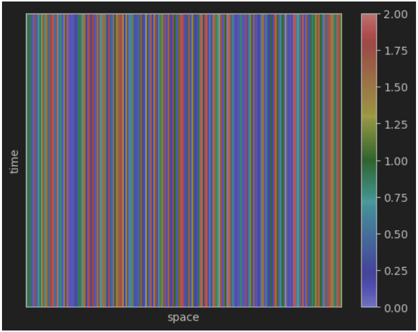
The coefficient field: in $d = 1$,

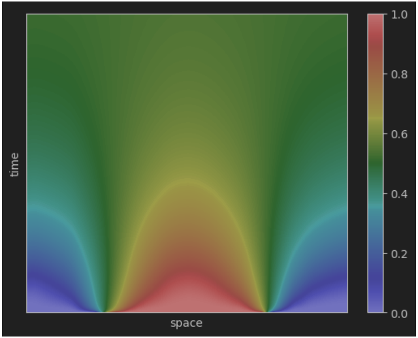
$$a(x,\omega) = 1 + (9/10)\sin(2\pi\mathcal{N}^{\varepsilon}(x,\omega)),$$

for $\mathcal{N} = \sum_{k\in\mathbb{Z}} \mathrm{Unif}_k(0,1)\mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = 1/200$ in $\mathbb{T}^1 \times (0,\infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2}\nabla\cdot a^{\varepsilon}\nabla u_\varepsilon \ \text{ in } \mathbb{T}^1\times(0,\infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4,3/4\}}.$$





The coefficient field: in $d = 1$,

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The equation: for $\varepsilon = 1/1000$ in $\mathbb{T}^1 \times (0,\infty)$,

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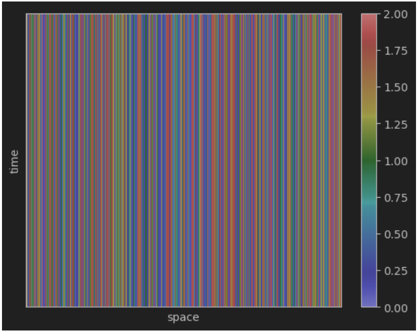

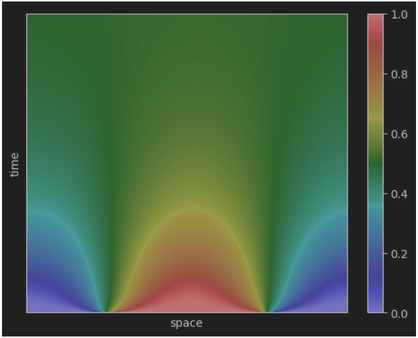

The coefficient field: in $d = 1$,

$$a(x,\omega) = 1 + ({}^{9}/_{10})\sin(2\pi\mathcal{N}^{\varepsilon}(x,\omega)),$$

for $\mathcal{N} = \sum_{k\in\mathbb{Z}} \mathrm{Unif}_k(0,1)\mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = {}^{1}/_{1000}$ in $\mathbb{T}^1 \times (0,\infty)$,

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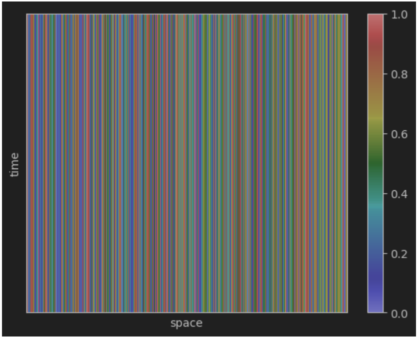
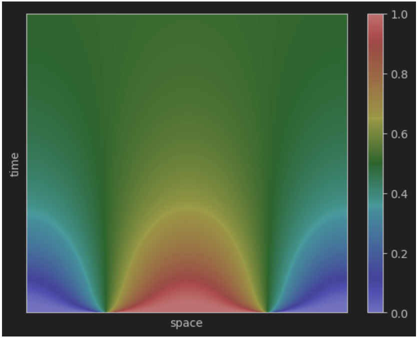




The coefficient field: in $d = 1$,

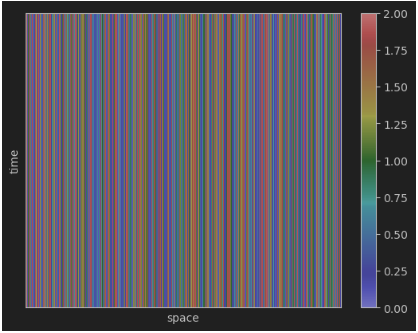
$$a(x) = 1 + (9/10)\sin(2\pi * \mathcal{N}(x,\omega)),$$

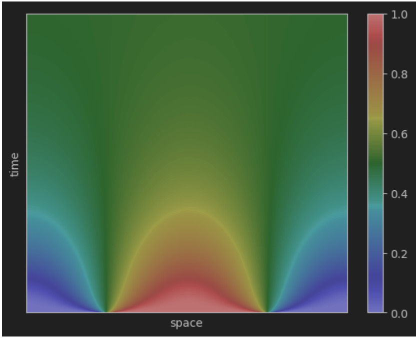
for

$$N(x,\omega) = \sum_{k\in\mathbb{Z}} \mathbf{1}_{[k,k+1)} \mathrm{Unif}_k(0,1).$$

The equation: for $\varepsilon = 1/1000$ in $\mathbb{T}^1 \times (0,\infty)$,

$$\partial_t u_\varepsilon = \nabla \cdot a^\varepsilon \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u(x,0) = \mathbf{1}_{\{(1/4,3/4\}}.$$

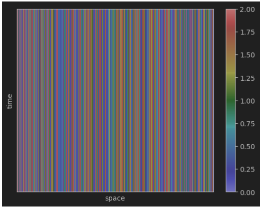


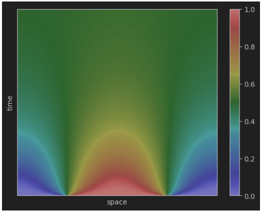


IV. Stochastic homogenization

The equation: for $\varepsilon = {}^1/{}_{1000}$ in $\mathbb{T}^1 \times (0,\infty)$,

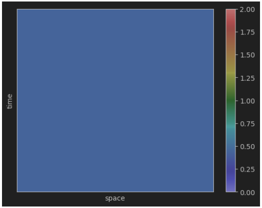
$$\partial_t u_\varepsilon = ({}^1/{}_2)\nabla \cdot a^\varepsilon \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1/{}_4, {}^3/{}_4)\}}.$$

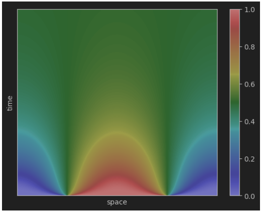




A macroscopic limit: choosing $\overline{a} = \langle a^{-1} \rangle^{-1} \simeq 0.43588$,

$$\partial_t \overline{u} = ({}^1/{}_2)\nabla \cdot \tilde{a} \nabla \overline{u} \ \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1/{}_4, {}^3/{}_4)\}}.$$





IV. Stochastic homogenization

The equation: for $\omega \in \Omega$ fixed,

$$-\nabla \cdot a^{\varepsilon} \nabla u_{\varepsilon} = f \;\text{ in } U \quad u = 0 \;\text{ on }\; \partial U.$$

The asymptotic expansion: we postulate the approximation

$$\tilde{u}_{\varepsilon}(x) = \overline{u}(x) + \varepsilon \phi_i({}^{x}/_{\varepsilon}) \partial_i \overline{u}(x),$$

for the homogenization correctors

$$-\nabla \cdot a(e_i + \nabla \phi_i) = 0 \;\text{ in }\; \mathbb{R}^d \;\text{ with }\; \fint_{B_1} \phi_i = 0.$$

The equation: we have that

$$-\nabla \cdot a^{\varepsilon} \nabla \tilde{u}_{\varepsilon} = -a^{\varepsilon}(e_i + \nabla \phi_i^{\varepsilon}) \cdot \nabla_x \partial_i \overline{u} + \mathcal{O}_{H^{-1}}(\varepsilon).$$

The ergodic theorem: for $F \in L^2(\Omega)$ and $f(x, \omega) = F(\tau_x \omega)$,

$$f^{\varepsilon} = f({}^{x}/_{\varepsilon}, \omega) \rightharpoonup \langle F \rangle = \mathbb{E}[F] \;\text{ weakly in } L^2(U) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

for every smooth, bounded $U \subseteq \mathbb{R}^d$ [Becker; 1981].

The homogenized coefficient: if $\nabla \phi_i(x, \omega) = \Phi_i(\tau_x \omega)$ for $\Phi_i \in L^2(\Omega)^d$,

$$a^{\varepsilon}(e_i + \nabla \phi_i^{\varepsilon}) \rightharpoonup \overline{a} e_i \;\text{ and }\; -\nabla \cdot \overline{a} \nabla \overline{u} = f.$$

The corrector equation: on the whole space,

$$-\nabla \cdot a(e_i + \nabla\phi_i) = 0 \text{ in } \mathbb{R}^d \text{ with } \fint_{B_1} \phi_i = 0.$$

Martingale decomposition: for the diffusion process in environment ω,

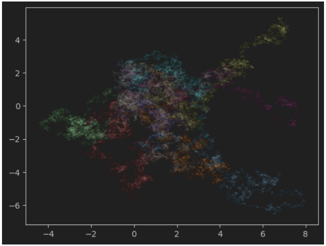
$$\mathrm{d}X_t^\omega = \sigma(X_t^\omega, \omega)\,\mathrm{d}B_t + (\nabla \cdot a^t)(X_t, \omega)\,\mathrm{d}t,$$

for $\sigma\sigma^t = 2a$, the process

$$M_t^\omega = X_t^\omega + \phi(X_t^\omega, \omega) \text{ is a martingale for } \phi = (\phi_1, \phi_2, \ldots, \phi_d).$$

The diffusive scaling limit: We then have that

$$\varepsilon X^\omega_{t/\varepsilon^2} = \varepsilon M^\omega_{t/\varepsilon^2} + \varepsilon\phi(X^\omega_{t/\varepsilon^2}).$$



The corrector equation: on the whole space,

$$-\nabla \cdot a(e_i + \nabla \phi_i) = 0 \;\text{ in }\; \mathbb{R}^d \;\text{ with }\; \fint_{B_1} \phi_i = 0.$$

The horizontal derivatives: for $f \in L^2(\Omega)$,

$$D_i f(\omega) = \lim_{h \to 0} \frac{f(\tau_{he_i}\omega) - f(\omega)}{h} \;\text{ strongly in }\; L^2(\Omega),$$

where $\mathbb{E}[D_i f] = 0$.

A class of test functions: for $f \in L^2(\Omega)$ and $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$(\psi * f)(\omega) = \int_{\mathbb{R}^d} \psi(x) f(\tau_x\omega)\,\mathrm{d}x \;\text{ for which }\; D_i(\psi * f)(\omega) = -\int_{\mathbb{R}^d} (\partial_i \psi)(x) f(\tau_x \omega)\,\mathrm{d}x.$$

The generalized Sobolev space: the space $\mathcal{H}^1(\Omega)$ is the space

$$\mathcal{H}^1(\Omega) = L^2(\Omega) \cap \left(\cap_{i=1}^d \mathcal{D}(D_i) \right),$$

equipped with the Hilbert space inner product

$$\langle f, g \rangle_{\mathcal{H}^1} = \mathbb{E}[f, g] + \mathbb{E}[Df \cdot Dg].$$

IV. Stochastic homogenization

The approximate corrector equation: on Ω for $A(\omega) = a(0,\omega)$, for $\alpha \in (0,1)$,

$$\alpha\Phi_i^\alpha - D \cdot A(e_i + D\Phi_i^\alpha) = 0 \text{ in } \mathcal{H}^1(\Omega),$$

in the sense that

$$\mathbb{E}[\alpha\Phi_i^\alpha F] + \mathbb{E}[DF \cdot A(e_i + D\Phi_i^\alpha)] = 0 \text{ for all } F \in \mathcal{H}^1(\Omega).$$

Existence and energy: there exists a unique solution $\Phi_i^\alpha \mathcal{H}^1(\Omega)$, and Φ_i^α satisfies

$$\alpha\mathbb{E}[(\Phi_i^\alpha)^2] + \mathbb{E}[\Phi_i^\alpha \cdot A\Phi_i^\alpha] \leq c\mathbb{E}[|Ae_i|^2],$$

by the Lax–Milgram theorem.

The potential space: the space of gradient-like vector fields,

$$L^2_{\text{pot}}(\Omega) = \overline{\{DF \colon F \in \mathcal{H}^1(\Omega)\}}^{L^2(\Omega)\text{-strongly}},$$

where, for every $\Psi \in L^2_{\text{pot}}(\Omega)$,

$$D_i\Psi_j = D_j\Psi_i \text{ as distributions on } \Omega.$$

The corrector equation: there exists a unique $D\Phi_i \in L^2_{\text{pot}}(\Omega)$ satisfying

$$\mathbb{E}[D\Psi \cdot aD\Phi] = -\mathbb{E}[D\Psi \cdot Ae_i] \text{ for all } D\Psi \in L^2_{\text{pot}}(\Omega).$$

Warning!: $D\Phi_i$ is generally not the gradient of a measurable $\Phi_i \colon \Omega \to \mathbb{R}$!

The corrector equation: there exists a unique $D\Phi_i \in L^2_{\mathrm{pot}}(\Omega)$ satisfying

$$\mathbb{E}[D\Psi \cdot aD\Phi] = -\mathbb{E}[D\Psi \cdot Ae_i] \ \text{ for all } D\Psi \in L^2_{\mathrm{pot}}(\Omega).$$

The gradient in the physical space: $\mathbb{P}$-a.s. we have that

$$\nabla\phi_i(x,\omega) = D\Phi_i(\tau_x\omega) \ \text{ is in } L^2_{\mathrm{loc}}(\mathbb{R}^d),$$

since $\mathbb{E}[\int_{B_R} |\nabla\phi_i|^2] = |B_R|\mathbb{E}[|D\Phi_i|^2]$.

The corrector in the physical space: there $\mathbb{P}$-a.s. exists a unique

$$\phi_i \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \ \text{ with } \ \nabla\phi_i(x,\omega) = D\Phi_i(\tau_x\omega) \ \text{ and } \ \fint_{B_1} \phi_i = 0.$$

The equation: let $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^d)$, let $\tilde{\psi}(x) = \psi(-x)$, and let $B \subseteq \Omega$,

$$\begin{aligned}\mathbb{E}[D(\tilde{\psi} * \mathbf{1}_B) \cdot A(e_i + D\Phi_i)] &= \mathbb{E}[\int_{\mathbb{R}^d} \mathbf{1}_B(\tau_x\omega)\nabla\psi(-x) \cdot A(\omega)(e_i + \nabla\phi_i(\omega)\,\mathrm{d}x] \\ &= \mathbb{E}[\mathbf{1}_B(\omega) \int_{\mathbb{R}^d} \nabla\psi(-x) \cdot A(\tau_{-x}\omega)(e_i + \nabla\phi_i(0, \tau_{-x}\omega))\,\mathrm{d}x] \\ &= \mathbb{E}[\mathbf{1}_B(\omega) \int_{\mathbb{R}^d} \nabla\psi(x) \cdot a(x,\omega)(e_i + \nabla\phi_i(x,\omega))\,\mathrm{d}x],\end{aligned}$$

and, $\mathbb{P}$-a.s. for every $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^d)$, we have $\int_{\mathbb{R}^d} \nabla\psi \cdot a(e_i + \nabla\phi_i)\,\mathrm{d}x = 0$.

IV. Stochastic homogenization

The equation: for a smooth and bounded domain $U \subseteq \mathbb{R}^d$, for a fixed $\omega \in \Omega$,

$$-\nabla \cdot a^{\varepsilon} \nabla u_\varepsilon = f \text{ in } U \text{ with } u_\varepsilon = 0 \text{ on } \partial U.$$

The homogenization correctors: for $\mathbb{P}$-a.e. environment $\omega \in \Omega$,

$$-\nabla \cdot a(\nabla \phi_i + e_i) = f \text{ with } \phi_i \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \nabla \phi_i \text{ stationary.}$$

The homogenized coefficient: we define $\overline{a} \in \mathbb{R}^{d \times d}$ by

$$\overline{a} e_i = \langle A(e_i + \nabla \phi_i) \rangle,$$

which is uniformly elliptic and symmetric (if a is symmetric).

The macroscopic limit: the function $\overline{u} \in H^1(U)$ solves

$$-\nabla \cdot \overline{a} \nabla \overline{u} = f \text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

The two-scale expansion: we postulate that

$$u_\varepsilon \simeq \tilde{u}_\varepsilon = \overline{u} + \varepsilon \phi_i^{\varepsilon} \partial_i \overline{u}.$$

Sublinearity: we require that

$$\lim_{\varepsilon \to 0} \|\varepsilon \phi_i^{\varepsilon}\|_{L^2(U)} = \lim_{r \to \infty} \frac{1}{r} \left(\int_{rU} |\phi_i|^2 \right)^{\frac{1}{2}} = 0.$$

Sublinearity: we aim to prove $\mathbb{P}$-a.s. that, for every $R \in (0,\infty)$,

$$\lim_{\varepsilon\to 0} \|\varepsilon\phi_i^\varepsilon\|_{L^2(B_R)} = \lim_{r\to\infty} \frac{1}{r}\left(\int_{B_{rR}} |\phi_i|^2\right)^{\frac{1}{2}} = 0.$$

Subtracting the mean: we may assume that $\langle \phi_i^\varepsilon\rangle_{B_R} = 0$, as otherwise

$$\tilde{\phi}_i^\varepsilon = \phi_i^\varepsilon - \langle\phi_i^\varepsilon\rangle_{B_R} \text{ is an equivalent sequence of correctors.}$$

The Poincaré inequality and ergodic theorem: we have $\mathbb{P}$-a.s. that

$$\limsup_{\varepsilon\to 0} \|\varepsilon\phi_i^\varepsilon\|_{L^2(B_R)} \leq cR \limsup_{\varepsilon\to 0} \|\nabla\phi_i^\varepsilon\|_{L^2(B_R)} = cR|B_R|^{\frac{1}{2}} \cdot \mathbb{E}[|\nabla\phi_i|^2]^{\frac{1}{2}} < \infty.$$

Compactness: $\mathbb{P}$-a.s., by the Sobolev embedding theorem, for some $\tilde{\phi}_i \in H^1(B_r)$,

$$\phi_i^\varepsilon \rightharpoonup \tilde{\phi}_i \text{ weakly in } H^1(B_R) \text{ and } \phi_i^\varepsilon \to \tilde{\phi}_i \text{ strongly in } L^2(B_R).$$

The strong limit: since $\mathbb{P}$-a.s. by the ergodic theorem $\nabla\phi_i^\varepsilon \rightharpoonup \mathbb{E}[\nabla\phi_i^\varepsilon] = 0$,

$$\nabla\tilde{\phi}_i = 0 \text{ and } \langle\tilde{\phi}_i\rangle_{B_R} = \lim_{\varepsilon\to 0}\langle\phi_i^\varepsilon\rangle_{B_R} = 0 \text{ proves that } \tilde{\phi}_i = 0,$$

and, therefore, that

$$\lim_{\varepsilon\to 0} \|\varepsilon\phi_i^\varepsilon\|_{L^2(B_R)} = 0.$$

IV. Stochastic homogenization

The equation: for a smooth and bounded domain $U \subseteq \mathbb{R}^d$, for a fixed $\omega \in \Omega$,

$$-\nabla \cdot a^{\varepsilon} \nabla u_{\varepsilon} = f \text{ in } U \text{ with } u_{\varepsilon} = 0 \text{ on } \partial U.$$

The energy estimate: we have that, for c depending on λ and U,

$$\|u_{\varepsilon}\|^2_{H^1_0(U)} \leq c \|f\|^2 .$$

The perturbed test function: for $\psi \in \mathrm{C}^{\infty}_c(U)$,

$$\psi_{\varepsilon} = \psi + \varepsilon \phi_i^{\varepsilon,t} \partial_i \psi,$$

for the transpose correctors $-\nabla \cdot a^t(e_i + \nabla \phi_i^t) = 0$.

The limit: we have that

$$\begin{aligned}\int_U \nabla \psi_{\varepsilon} \cdot a^{\varepsilon} \nabla u_{\varepsilon} &= \int_U f \psi_{\varepsilon} \\ &= \int_U a^{\varepsilon,t}(e_i + \nabla \phi_i^{\varepsilon,t}) \partial_i \psi \cdot \nabla u_{\varepsilon} + \int_U \varepsilon \phi_i^{\varepsilon,t} \nabla \partial_i \psi \cdot a^{\varepsilon} \nabla u_{\varepsilon} \\ &= -\int_U u_{\varepsilon} \left(a^{\varepsilon,t}(e_i + \nabla \phi_i^{\varepsilon,t}) \cdot \nabla \partial_i \psi \right) + \mathcal{O}(\left\| \varepsilon \phi^{\varepsilon,t} \right\|_{L^2(U)}),\end{aligned}$$

where the ergodic theorem and sublinearity prove that, if $u_{\varepsilon} \rightharpoonup \overline{u}$ in $H^1_0(U)$,

$$-\nabla \cdot \overline{a} \nabla \overline{u} = 0 \text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

A first qualitative homogenization statement

Let $U \subseteq \mathbb{R}^d$ be a smooth and bounded domain, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $a \colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ be a stationary and ergodic coefficient field, let $f \in L^2(U)$, and for every $\varepsilon \in (0,1)$ let $u_\varepsilon \in H_0^1(U)$ be the unique Lax–Milgram solution of the equation

$$-\nabla \cdot a^\varepsilon \nabla u_\varepsilon = f \ \text{ in } U \text{ with } u_\varepsilon = 0 \text{ on } \partial U.$$

There $\mathbb{P}$-a.s. exist sublinear correctors $\phi_i \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ with stationary gradients solving

$$-\nabla \cdot a(e_i + \nabla\phi_i) = 0 \ \text{ in } \ \mathbb{R}^d.$$

For the homogenized coefficient $\overline{a} \in \mathbb{R}^{d\times d}$ defined by

$$\overline{a} e_i = \langle a(e_i + \nabla\phi_i)\rangle,$$

we have $\mathbb{P}$-a.s. that, as $\varepsilon \to 0$,

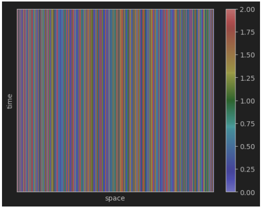
$$u_\varepsilon \rightharpoonup \overline{u} \ \text{ weakly in } H_0^1(U) \text{ and } \ u_\varepsilon \to \overline{u} \ \text{ strongly in } L^2(U),$$

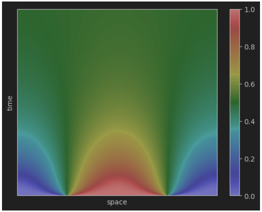
for $\overline{u} \in H_0^1(U)$ satisfying

$$-\nabla \cdot \overline{a} \nabla \overline{u} = f \ \text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

The equation: for $\varepsilon = {}^1\!/_{500}$ in $\mathbb{T}^1 \times (0,\infty)$,

$$\partial_t u_\varepsilon = ({}^1\!/_2)\nabla \cdot a^\varepsilon \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1\!/_4, {}^3\!/_4\}}.$$





A macroscopic limit: choosing $\overline{a} = \langle a^{-1} \rangle^{-1} \simeq 0.43588$,

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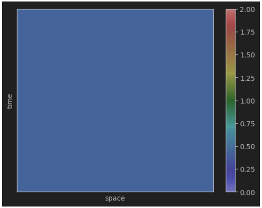

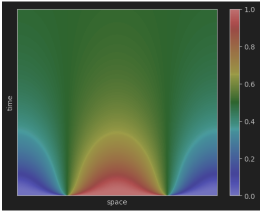

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