The variational approach to homogenization

Benjamin Fehrman

LSU

8 July 2026

I. The stochastic homogenization error

The coefficient field: the map $a\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ satisfies

$$\xi \cdot a\xi \geq \lambda|\xi|^2 \;\text{ and }\; \xi \cdot a^{-1}\xi \geq \Lambda^{-1}|\xi|^2,$$

and, for a measure-preserving transformation group $\{\tau_x\colon \Omega \to \Omega\}_{x\in\mathbb{R}^d}$,

$$a(x+y,\omega) = a(y,\tau_x\omega) \;\text{ for all } x,y\in\mathbb{R}^d \text{ and } \omega\in\Omega.$$

Furthermore, the transformation group is ergodic:

$$f(\omega) = f(\tau_x\omega) \in L^\infty(\Omega) \;\text{ for all } x\in\mathbb{R}^d \text{ if and only if } f \text{ is constant.}$$

The equation: for a fixed environment $\omega\in\Omega$, and $U\subseteq\mathbb{R}^d$,

$$-\nabla\cdot a^\varepsilon\nabla u_\varepsilon = f \;\text{ in }\; U \;\text{ with }\; u_\varepsilon = 0 \;\text{ on }\; \partial U.$$

Qualitative homogenization: there exists $\overline{a}\in\mathbb{R}^{d\times d}$ such that, $\mathbb{P}$-a.s.,

$$u_\varepsilon \rightharpoonup \overline{u} \;\text{ weakly in } H^1(U) \;\text{ and }\; u_\varepsilon \to \overline{u} \;\text{ strongly in } L^2(U),$$

for $\overline{u}$ solving

$$-\nabla\cdot\overline{a}\nabla\overline{u} = f \;\text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

I. The stochastic homogenization error

The homogenization corrector: there $\mathbb{P}$-a.s. exist $\phi_i \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying

$$-\nabla \cdot a(\nabla \phi_i + e_i) = 0 \ \text{ in} \in \mathbb{R}^d \ \text{ with } \nabla \phi_i \text{ stationary.}$$

That is, there exists $D\Phi_i \in L^2_{\mathrm{pot}}(\Omega)$ such that

$$\nabla \phi_i(x, \omega) = D\Phi_i(\tau_x \omega).$$

Sublinearity: for $\mathbb{P}$-a.e. $\omega \in \Omega$, for every $R \in (0, \infty)$,

$$\lim_{\varepsilon \to 0} \|\varepsilon \phi_i^\varepsilon\|_{L^2(B_R)} = \lim_{r \to \infty} \frac{1}{r} \left(\fint_{B_{rR}} |\phi_i|^2 \right) = 0.$$

The homogenized coefficient: as in the periodic case,

$$\overline{a} e_i = \langle a(e_i + \nabla \phi_i) \rangle,$$

is uniformly elliptic (with the same constants) and symmetric (if a is).

The two-scale expansion: we define

$$u_\varepsilon \simeq \tilde{u}_\varepsilon = \overline{u} + \varepsilon \phi_i^\varepsilon \partial_i \overline{u},$$

for $\overline{u}$ solving

$$-\nabla \cdot \overline{a} \nabla \overline{u} = f \ \text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

I. The stochastic homogenization error

The two-scale expansion: we define

$$u_\varepsilon \simeq \tilde{u}_\varepsilon = \overline{u} + \varepsilon \phi_i^\varepsilon \partial_i \overline{u}.$$

The stochastic homogenization error: we have

$$w_\varepsilon = u_\varepsilon - \overline{u} - \varepsilon \phi_i^\varepsilon \partial_i \overline{u}.$$

The equation: as in the periodic case,

$$\begin{aligned} -\nabla \cdot a^\varepsilon \nabla w_\varepsilon &= \big(a^\varepsilon (e_i + \nabla \phi_i) - \overline{a} e_i\big) \cdot \nabla \partial_i \overline{u} + \nabla \cdot (\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla \partial_i \overline{u}) \\ &= q_i^\varepsilon \cdot \nabla \partial_i \overline{u} + \nabla \cdot (\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla \partial_i \overline{u}), \end{aligned}$$

for the flux $q_i^\varepsilon = a^\varepsilon (e_i + \nabla \phi_i) - \overline{a} e_i$.

The flux correction: in the periodic case,

$$-\Delta \sigma_{ijk} = \partial_j q_{ik} - \partial_k q_{ij} \ \text{ and } \ \langle \sigma_{ijk} \rangle_{\mathbb{T}^d} = 0,$$

which, when lifted to Ω, for $D_i \Psi = \lim_{h \to 0} h^{-1}(\Psi(\tau_{he_i}\omega) - \Psi(\omega))$,

$$-D \cdot D \Sigma_{ijk} = D_j Q_{ik} - D_k Q_{ij} \ \text{ in } L^2_{\mathrm{pot}}(\Omega),$$

for $q_i(x,\omega) = Q_i(\tau_x \omega)$, and in the weak form

$$\mathbb{E}\big[D\Psi \cdot D\Sigma_{ijk}\big] = \mathbb{E}\big[Q_{ik} D_j \Psi - Q_{ij} D_k \Psi\big] \ \text{ for all } D\Psi \in L^2_{\mathrm{pot}}(\Omega).$$

I. The stochastic homogenization error

The flux correction: there exists a unique $D\Sigma_{ijk} \in L^2_{\mathrm{pot}}(\Omega)$ satisfying

$$\mathbb{E}\big[D\Psi \cdot D\Sigma_{ijk}\big] = \mathbb{E}\big[Q_{ik}D_j\Psi - Q_{ij}D_k\Psi\big] \text{ for all } D\Psi \in L^2_{\mathrm{pot}}(\Omega).$$

As with the correctors, there $\mathbb{P}$-a.s. exist $\sigma_{ijk} \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying

$$-\Delta\sigma_{ijk} = \partial_j q_{ik} - \partial_k q_{ij} \text{ in } \mathbb{R}^d \text{ with } \nabla\sigma_{ijk} \text{ stationary.}$$

Subinearity: since the gradient is stationary with $\mathbb{E}[\nabla\sigma_{ijk}] = 0$,

$$\lim_{\varepsilon\to 0}\left\|\varepsilon\sigma^{\varepsilon}_{ijk}\right\|_{L^2(B_R)} = \lim_{r\to\infty}\frac{1}{r}\left(\fint_{B_{rR}}|\sigma_{ijk}|^2\right) = 0.$$

The equation: $\mathbb{P}$-a.s. for the skew-symmetric matrix $\sigma_i = (\sigma_{ijk})$, using $\nabla\cdot q_i = 0$,

$$-\Delta\big((\nabla\cdot\sigma_i)_j - q_{ij}\big) = 0 \text{ in } \mathbb{R}^d,$$

which implies $(\nabla\cdot\sigma_i)_j - q_{ij}$ is constant, and by the ergodic theorem

$$\lim_{r\to\infty}\fint_{B_r}\big((\nabla\cdot\sigma_i)_j - q_{ij}\big) = \mathbb{E}\big[(\nabla\cdot\sigma_i)_j - q_{ij}\big] = 0.$$

Therefore,

$$\nabla\cdot\sigma_i = q_i \text{ for } (\nabla\cdot\sigma_i)_j = \partial_k\sigma_{ijk}.$$

I. The stochastic homogenization error

The stochastic homogenization error: we have

$$w_\varepsilon = u_\varepsilon - \overline{u} - \varepsilon \phi_i^\varepsilon \partial_i \overline{u}.$$

The equation: as in the periodic case, using the skew-symmetry of σ_i,

$$\begin{aligned} -\nabla \cdot a^\varepsilon \nabla w_\varepsilon &= q_i^\varepsilon \cdot \nabla \partial_i \overline{u} + \nabla \cdot (\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla \partial_i \overline{u}) \\ &= (\nabla \cdot \sigma_i^\varepsilon) \cdot \nabla \partial_i \overline{u} + \nabla \cdot (\varepsilon \phi_i^\varepsilon a^\varepsilon \nabla \partial_i \overline{u}) \\ &= \nabla \cdot \big((\varepsilon \phi_i^\varepsilon a^\varepsilon - \varepsilon \sigma_i^\varepsilon) \nabla \partial_i \overline{u} \big). \end{aligned}$$

The boundary layer: for $0 < r << 1$ let

$$U_r = \{ x \in U \colon \mathrm{d}(x, \partial U) \geq r \}$$

and for $\eta_r = 1$ on U_r, $\eta_r = 0$ on $\mathbb{R}^d \setminus U_{r/2}^\circ$, and $|\nabla \eta_r| \leq {}^c/_r$,

$$w_{\varepsilon,r} = u_\varepsilon - \overline{u} - \varepsilon \phi_i^\varepsilon \eta_r \partial_i \overline{u} \in H_0^1(U).$$

The equation: we have that

$$-\nabla \cdot a^\varepsilon \nabla w_{\varepsilon,r} = \nabla \cdot \big((1-\eta_r)(a^\varepsilon - \overline{a}) \nabla \overline{u} \big) + \nabla \cdot \big((\varepsilon \phi_i^\varepsilon a^\varepsilon - \varepsilon \sigma_i^\varepsilon) \nabla (\eta_r \partial_i \overline{u}) \big).$$

I. The stochastic homogenization error

The boundary expansion: since $w_{\varepsilon,r} = u_\varepsilon - \overline{u} - \varepsilon\eta_r\phi_i^\varepsilon\partial_i\overline{u} \in H_0^1(U)$,

$$-\nabla\cdot a^\varepsilon\nabla w_{\varepsilon,r} = \nabla\cdot\Big((1-\eta_r)(a^\varepsilon-\overline{a})\nabla\overline{u}\Big) + \nabla\cdot\Big((\varepsilon\phi_i^\varepsilon a^\varepsilon - \varepsilon\sigma_i^\varepsilon)\nabla(\eta_r\partial_i\overline{u})\Big).$$

The energy estimate: for $c\in(0,\infty)$ depending on λ, Λ, and U,

$$\int_U |\nabla w_{\varepsilon,r}|^2 \leq cr\left\|\nabla^2\overline{u}\right\|^2_{L^\infty(U\setminus U_r)} + cr^{-2}\left\|\nabla\overline{u}\right\|^2_{L^\infty(U_r\setminus U_{r/2})}\left\|\varepsilon(\phi_i^\varepsilon,\sigma_i^\varepsilon)\right\|^2_{L^2(U_r\setminus U_{r/2})}$$
$$+ c\left\|\nabla^2\overline{u}\right\|^2_{L^\infty(U_{r/2})}\left\|\varepsilon(\phi_i^\varepsilon,\sigma_i^\varepsilon)\right\|^2_{L^2(U_{r/2})}.$$

The full two-scale expansion: for $w_\varepsilon = u_\varepsilon - \overline{u} - \varepsilon\phi_i^\varepsilon\partial_i\overline{u}$, for $r=\varepsilon$,

$$\begin{aligned}
&\left\|\nabla w_\varepsilon - \nabla w_{\varepsilon,r}\right\|^2_{L^2(U)} = \left\|\nabla(\varepsilon(1-\eta_r)\phi_i^\varepsilon\partial_i\overline{u}\right\|^2_{L^2(U)}\\
&\leq c\Big(\varepsilon^{-1}\left\|\nabla\overline{u}\right\|^2_{L^\infty(U\setminus U_r)}\left\|\varepsilon\phi_i^\varepsilon\right\|^2_{L^2(U\setminus U_r)} + \left\|\nabla\overline{u}\right\|^2_{L^\infty(U\setminus U_r)}\left\|\nabla\phi_i^\varepsilon\right\|^2_{L^2(U\setminus U_r)}\Big)\\
&\quad + c\varepsilon\left\|\nabla^2\overline{u}\right\|^2_{L^\infty(U\setminus U_r)}\left\|\phi_i^\varepsilon\right\|^2_{L^2(U\setminus U_r)}.
\end{aligned}$$

Qualitative homogenization: if $\overline{u}\in W^{2,\infty}(U)$,

$$\limsup_{\varepsilon\to 0}\int_U |\nabla w_\varepsilon|^2 = 0.$$

I. The stochastic homogenization error

Qualitative stochastic homogenization

Let $U \subseteq \mathbb{R}^d$ be a smooth, bounded domain, let $f \in \mathrm{C}^\alpha(\overline{U})$ for some $\alpha \in (0,1)$, and let $a\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ be a uniformly elliptic, stationary, and ergodic diffusion matrix. There $\mathbb{P}$-a.s. exist homogenization correctors $\phi_i \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying

$$-\nabla \cdot a(e_i + \nabla\phi_i) = 0 \text{ in } \mathbb{R}^d \text{ with } \nabla\phi_i \text{ stationary},$$

that define the homogenized matrix

$$\overline{a}e_i = \langle a(e_i + \nabla\phi_i)\rangle,$$

and the macroscopic limit $\overline{u}$ solving

$$-\nabla \cdot \overline{a}\nabla\overline{u} = f \text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

The stochastic homogenization error

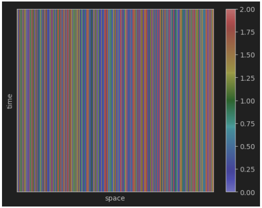
$$w_\varepsilon = u_\varepsilon - \overline{u} - \varepsilon\phi_i^\varepsilon\partial_i\overline{u},$$

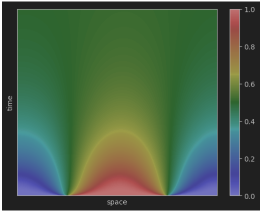
satisfies

$$\lim_{\varepsilon\to 0}\int_U |\nabla w_\varepsilon|^2 = 0 \text{ and, so, } \nabla u_\varepsilon \simeq \nabla\overline{u} + \partial_i\overline{u}\nabla\phi_i^\varepsilon.$$

The equation: for $\varepsilon = {}^{1}/{}_{1000}$ in $\mathbb{T}^1 \times (0,\infty)$,

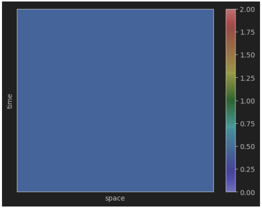
$$\partial_t u_\varepsilon = ({}^{1}/{}_{2})\nabla \cdot a^{\varepsilon}\nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^{1}/{}_{4},{}^{3}/{}_{4})\}}.$$

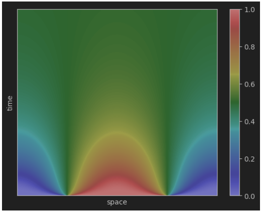




A macroscopic limit: choosing $\overline{a} = \langle a^{-1}\rangle^{-1} \simeq 0.43588$,

$$\partial_t \overline{u} = ({}^{1}/{}_{2})\nabla \cdot \overline{a}\nabla \overline{u} \ \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^{1}/{}_{4},{}^{3}/{}_{4})\}}.$$





II. The variational approach

The equation: for a symmetric diffusion matrix a, the solution

$$-\nabla \cdot a^{\varepsilon}\nabla u_\varepsilon = f \;\text{ in } U \text{ with } u_\varepsilon = 0 \text{ on } \partial U,$$

satisfies

$$u_\varepsilon = \operatorname{argmin}_{v\in H_0^1(U)} \int_U \big(\nabla v \cdot a^{\varepsilon}\nabla v - vf\big).$$

The approximate gradient: by the two-scale expansion,

$$\nabla u_\varepsilon \simeq \nabla \overline{u} + \partial_i \overline{u} \nabla \phi_i^{\varepsilon} = (e_i + \nabla \phi_i^{\varepsilon})\partial_i \overline{u},$$

and, therefore,

$$\int_U \nabla v \cdot a^{\varepsilon}\nabla v \simeq \sum\nolimits_{ij} \int_U \big((e_i + \nabla\phi_i^{\varepsilon}) \cdot a^{\varepsilon}(e_j + \nabla \phi_j^{\varepsilon})\big)\partial_i\overline{u}\partial_j\overline{u}.$$

The effective energy: we have that

$$\frac{1}{|U|}\int_U (e_i + \nabla\phi_i^{\varepsilon}) \cdot a^{\varepsilon}(e_j + \nabla\phi_j^{\varepsilon}) = \fint_{U/\varepsilon} (e_i + \nabla\phi_i)\cdot a(e_j + \nabla\phi_j).$$

Polarization: we will analyze on large-scales, for $\xi \in \mathbb{R}^d$,

$$\mu(U,\xi) = \frac{1}{2}\fint_U (\xi + \nabla\phi_\xi)\cdot a(\xi + \nabla\phi_\xi),$$

for $\phi_\xi = \xi_i\phi_i$ satisfying $-\nabla \cdot a(\xi + \nabla\phi_\xi) = 0$.

II. The variational approach

The energy: for every $V \subseteq \mathbb{R}^d$, for $\xi \in \mathbb{R}^d$, and for $\ell_\xi(x) = x \cdot \xi$,

$$\mu(V, \xi) = \inf_{v \in \ell_\xi + H_0^1(U)} \frac{1}{2} \fint_V \nabla v \cdot a \nabla v.$$

The Euler–Lagrange equation: the minimizer $v_{\xi,V}$ satisfies

$$-\nabla \cdot a \nabla v_{\xi,V} = 0 \text{ in } V \text{ with } v_{\xi,V} = \ell_\xi \text{ on } \partial V.$$

The finite-volume corrector: if $\phi_{\xi,V} = v_{\xi,V} - \ell_\xi$,

$$-\nabla \cdot a(\nabla \phi_{\xi,V} + \xi) = 0 \text{ in } V \text{ with } \phi_{\xi,V} = 0 \text{ on } \partial V.$$

Approximate correctors: for $\alpha \in (0,1)$,

$$\alpha \phi_{i,\alpha} - \nabla \cdot a(e_i + \nabla \phi_{i,\alpha}) = 0 \text{ in } \mathbb{R}^d,$$

satisfy that

$$\phi_{i,\alpha}(x) = \int_0^\infty e^{-\alpha t} \rho_i(x,t) \, \mathrm{d}t,$$

for

$$\partial_t \rho_i = \nabla \cdot a \nabla \rho_i \text{ in } \mathbb{R}^d \times (0,\infty) \text{ with } \rho_i(\cdot, 0) = \nabla \cdot (a e_i).$$

The energy: for every $V \subseteq \mathbb{R}^d$, for $\xi \in \mathbb{R}^d$, and for $\ell_\xi(x) = x \cdot \xi$,

$$\mu(V,\xi) = \inf_{v \in \ell_\xi + H_0^1(U)} \frac{1}{2} \fint_V \nabla v \cdot a \nabla v = \frac{1}{2} \fint_V (\xi + \nabla \phi_{\xi,V}) \cdot a(\xi + \nabla \phi_{\xi,V}),$$

for $-\nabla \cdot a(\xi + \nabla \phi_{\xi,V}) = 0$ in V with $\phi_{\xi,V} = 0$ on ∂V.

The coarse-grained matrix: since the map $\xi \to (\xi + \nabla \phi_{\xi,V})$ is linear,

the map $\xi \to \mu(V,\xi)$ is quadratic in $\xi \in \mathbb{R}^d$.

There exists a symmetric $a(V) \in \mathbb{R}^{d \times d}$ such that

$$\mu(V,\xi) = \frac{1}{2}\big(\xi \cdot a(V)\xi\big) \text{ for every } \xi \in \mathbb{R}^d.$$

Subadditivity: if $V = \coprod_{i=1}^N V_i$ then

$$|V|\mu(V,\xi) \leq \sum_{i=1}^N |V_i|\mu(V_i,\xi) \text{ and, so, } \mu(V,\xi) \leq \sum_{i=1}^N \frac{|V_i|}{|V|}\mu(V_i,\xi).$$

Upper bound: for every $V \subseteq \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$, choosing $v = \ell_\xi$,

$$\mu(V,\xi) \leq \frac{1}{2}\Big(\xi \cdot \big(\fint_V a\big)\xi\Big) \leq \Lambda|\xi|^2.$$

A subadditive quantity: for the set $\mathcal{V}$ of bounded, open subsets of $\mathbb{R}^d$,

$$s\colon \Omega \times \mathcal{V} \to \mathbb{R},$$

is subadditive if $\mathbb{P}$-a.s. whenever $V = \coprod_{i=1}^{n} V_i$, we have $s(\omega, V) \leq \sum_{i=1}^{N} s(\omega, V_i)$.

Kingman subadditive ergodic theorem [Akcoglu, Krengel; 1981]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with an ergodic, $\mathbb{Z}^d$-stationary probability measure $\mathbb{P}$: for an ergodic measure-preserving transformation group $\{\tau_z : \Omega \to \Omega\}_{z \in \mathbb{Z}^d}$, for every $f \in L^\infty(\Omega)$,

$$f(\omega) = f(\tau_z \omega) \ \text{ in } L^\infty(\Omega) \text{ for every } z \in \mathbb{Z}^d.$$

If $s\colon \Omega \times \mathcal{V} \to \mathbb{R}$ is a subadditive quantity that is $\mathbb{Z}^d$ stationary:

$$s(\omega, V + z) = s(\tau_z \omega, V) \ \text{ for every } V \in \mathcal{V} \text{ and } z \in \mathbb{Z}^d,$$

and bounded: for some $\Lambda \in (0, \infty)$, we have $s(V) \leq \Lambda$ $\mathbb{P}$-a.s. for every $V \in \mathcal{V}$.

Then, there exists $\overline{u} \in \mathbb{R}$ such that, $\mathbb{P}$-a.s. for every $V \in \mathcal{V}$,

$$\lim_{t \to \infty} s(\omega, tV) = \overline{\mu}.$$

The energy: for every $V \subseteq \mathbb{R}^d$, for $\xi \in \mathbb{R}^d$, and for $\ell_\xi(x) = x \cdot \xi$,

$$\mu(V,\xi) = \inf_{v \in \ell_\xi + H_0^1(U)} \frac{1}{2} \fint_V \nabla v \cdot a \nabla v = \frac{1}{2} \fint_V (\xi + \nabla \phi_{\xi,V}) \cdot a(\xi + \nabla \phi_{\xi,V}).$$

The coarse-grained matrix: there exists a symmetric $a(V) \in \mathbb{R}^{d\times d}$ such that

$$\mu(V,\xi) = \frac{1}{2}\big(\xi \cdot a(V)\xi\big) \text{ for every } \xi \in \mathbb{R}^d.$$

Subadditivity: if $V = \coprod_{i=1}^N V_i$ then

$$|V|\mu(V,\xi) \leq \sum\nolimits_{i=1}^N |V_i|\mu(V_i,\xi) \text{ and, so, } \mu(V,\xi) \leq \sum\nolimits_{i=1}^N \frac{|V_i|}{|V|}\mu(V_i,\xi).$$

Upper and lower bounds: for every $V \in \mathcal{V}$ and $\xi \in \mathbb{R}^d$, choosing $v = \ell_\xi$,

$$\frac{\lambda}{2}|\xi|^2 \leq \frac{\lambda}{2}\fint_V |\xi + \phi_{\xi,V}|^2 \leq \mu(V,\xi) \leq \frac{1}{2}\big(\xi \cdot \big(\fint_V a\big)\xi\big) \leq \frac{\Lambda}{2}|\xi|^2.$$

The subadditive ergodic theorem: for every $\xi \in \mathbb{R}^d$ and $V \in \mathcal{V}$,

$$\lim_{t\to\infty} \mu(tV,\xi) = \overline{\mu}_\xi.$$

The homogenized matrix: for a symmetric, uniformly elliptic $\overline{a} \in \mathbb{R}^{d\times d}$,

$$\lim_{t\to\infty} a(tV) = \overline{a}.$$

A useful decomposition: for every $n \in \mathbb{N}$, up to a set of Lebesgue measure zero,

$$\square_n = \left[-\frac{3^n}{2}, \frac{3^n}{2}\right] \text{ so } \mathbb{R}^d = \coprod\nolimits_{z \in 3^n \mathbb{Z}^d} \left(z + \square_n\right).$$

The finite-volume correctors: for every $n \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$, for the minimizer $v_{n,\xi}$,

$$-\nabla \cdot a \nabla v_{n,\xi} = 0 \text{ in } \square_n \text{ with } v_{n,\xi} = \ell_\xi \text{ on } \partial \square_n.$$

we define the finite volume corrector

$$\phi_{n,\xi} = v_{n,\xi} - \ell_\xi \text{ satisfies } -\nabla \cdot a(\xi + \nabla \phi_{n,\xi}) = 0 \text{ in } \square_n \text{ with } \phi_{n,\xi} = 0 \text{ on } \partial \square_n.$$

The finite volume flux: since we have that

$$\int_{\square_n} a(\nabla \phi_{n,\xi} + \xi) = \int_{\square_n} a \nabla v_{n,\xi} = a(\square_n)\xi.$$

the finite-volume flux is

$$q_{n,\xi} = a(\nabla \phi_{n,\xi} + \xi) - a(\square_n)\xi,$$

which satisfies, $\mathbb{P}$-a.s.,

$$\nabla \cdot q_{n,\xi} = 0 \text{ and } \fint_{\square_n} q_{n,\xi} = 0.$$

A useful decomposition: for every $n \in \mathbb{N}$, up to a set of Lebesgue measure zero,

$$\square_n = \left[-\frac{3^n}{2}, \frac{3^n}{2}\right] \text{ so } \mathbb{R}^d = \coprod_{z \in 3^n \mathbb{Z}^d} (z + \square_n).$$

The finite-volume correctors: for every $n \in \mathbb{N}$,

$$-\nabla \cdot a(e_i + \nabla \phi_{n,i}) = 0 \text{ in } \square_n \text{ with } \phi_{n,i} = 0 \text{ on } \partial \square_n.$$

The finite volume flux: for every $n \in \mathbb{N}$,

$$q_{n,i} = a(e_i + \nabla \phi_{n,i}) - a(\square_n) e_i,$$

which satisfies, $\mathbb{P}$-a.s.,

$$\nabla \cdot q_{n,i} = 0 \text{ and } \fint_{\square_n} q_{n,i} = 0.$$

The finite volume flux-corrector: for 3^n-periodic skew-symmetric matrices $\sigma_{i,n}$,

$$\nabla \cdot \sigma_{n,i} = q_{n,i} \text{ in } \mathbb{R}^d,$$

with $\sigma_{n,i} = (\sigma_{n,ijk})$ for

$$-\Delta \sigma_{n,ijk} = \partial_j q_{n,ik} - \partial_k q_{n,ij} \text{ in } \mathbb{R}^d \text{ with } \fint_{\square_n} \sigma_{n,ijk} = 0.$$

II. The variational approach

The equation: we have

$$-\nabla \cdot a^{\varepsilon} \nabla u_{\varepsilon} = f \ \text{ in } U \text{ with } u_{\varepsilon} = 0 \text{ on } \partial U.$$

The two-scale expansion: if $U \subseteq \square_0$ and $3^{m-1} \leq \varepsilon^{-1} < 3^m$ then $(U/\varepsilon) \subseteq \square_m$,

$$u_{\varepsilon} \simeq \tilde{u}_{\varepsilon,m} = u_{\varepsilon} - \overline{u} - \varepsilon \partial_i \phi^{\varepsilon}_{m,i} \partial_i \overline{u}.$$

for the finite-volume correctors

$$-\nabla \cdot a(e_i + \nabla \phi_{m,i}) = 0 \ \text{ in } \square_m \text{ with } \phi_{m,i} = 0 \text{ on } \partial \square_m.$$

The finite volume flux: for every $n \in \mathbb{N}$,

$$q_{n,i} = a(e_i + \nabla \phi_{n,i}) - a(\square_n) e_i,$$

which satisfies, $\mathbb{P}$-a.s.,

$$\nabla \cdot q_{n,i} = 0 \ \text{ and } \ \fint_{\square_n} q_{n,i} = 0.$$

The finite volume flux-corrector: for 3^n-periodic skew-symmetric matrices $\sigma_{i,n}$,

$$\nabla \cdot \sigma_{n,i} = q_{n,i} \ \text{ in } \mathbb{R}^d,$$

with $\sigma_{n,i} = (\sigma_{n,ijk})$ for

$$-\Delta \sigma_{n,ijk} = \partial_j q_{n,ik} - \partial_k q_{n,ij} \ \text{ in } \mathbb{R}^d \text{ with } \fint_{\square_n} \sigma_{n,ijk} = 0.$$

II. The variational approach

The equation: we have

$$-\nabla \cdot a^{\varepsilon}\nabla u_{\varepsilon} = f \ \text{ in } U \text{ with } u_{\varepsilon} = 0 \text{ on } \partial U.$$

The two-scale expansion: the homogenization error, for $U \subseteq \square_0$ and $3^{m-1} \leq \varepsilon^{-1} < 3^m$,

$$w_{\varepsilon} = u_{\varepsilon} - \overline{u} - \varepsilon\phi^{\varepsilon}_{m,i}\partial_i\overline{u},$$

satisfies

$$\begin{aligned}-\nabla \cdot a^{\varepsilon}\nabla w_{\varepsilon} &= q^{\varepsilon}_{m,i} \cdot \nabla\partial_i\overline{u} + \nabla \cdot (a(\square_m) - \overline{a})\nabla\overline{u} + \nabla \cdot (\varepsilon\phi^{\varepsilon}_{m,i}\nabla\partial_i\overline{u}) \\ &= \nabla \cdot (a(\square_m) - \overline{a})\nabla\overline{u} + \nabla \cdot \left(\varepsilon\phi^{\varepsilon}_{m,i} - \varepsilon\sigma^{\varepsilon}_{m,i}\right)\nabla\partial_i\overline{u}.\end{aligned}$$

Sublinearity: since $3^{m-1} \leq \varepsilon^{-1} < 3^m$,

$$\begin{aligned}\left\|\varepsilon\phi^{\varepsilon}_{m,i}\right\|_{L^2(U)} + \left\|\varepsilon\sigma^{\varepsilon}_{m,i}\right\|_{L^2(U)} &= \varepsilon|U|^{\frac{1}{2}}\left(\fint_{U/\varepsilon} |\phi_{m,i}|^2 + |\sigma_{m,i}|^2\right)^{\frac{1}{2}} \\ &\leq c3^{-m}\left(\fint_{\square_m} |\phi_{m,i}|^2 + |\sigma_{m,i}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

It suffices to prove that

$$\lim_{m\to\infty} 3^{-m}\left(\|\phi_{m,i}\|_{\underline{L}^2(U)} + \|\sigma_{m,i}\|_{\underline{L}^2(U)} \right) = 0.$$

III. The multiscale Poincaré inequality

The Poincaré inequality: for every $U \subseteq \mathbb{R}^d$,

$$\|u - \langle u \rangle_U\|_{L^2(U)} \leq c\,\mathrm{diam}(U)\,\|\nabla u\|_{L^2(U)}\,.$$

Scale-corrected norms: we define the following volume-normalized function spaces,

$$\|u\|_{\underline{L}^2(U)} = \Big(\fint_U u^2\Big)^{\frac{1}{2}},$$
$$\|u\|_{\underline{H}^1(U)} = \Big(|U|^{-1}\fint_U u^2\Big)^{\frac{1}{2}} + \Big(\fint_U |\nabla u|^2\Big)^{\frac{1}{2}},$$
$$\|u\|_{\underline{H}^{-1}(U)} = \sup\Big\{\fint_U uv\colon\ \|v\|_{\underline{H}^1(U)} \leq 1\Big\}.$$

The scaling factor: we have that

$$\|v^\varepsilon\|_{\underline{L}^2(\varepsilon U)} = \|v\|_{\underline{L}^2(U)} \quad \text{and} \quad \|v^\varepsilon\|_{\underline{H}^1(\varepsilon U)} = \varepsilon^{-1}\,\|v\|_{\underline{H}^1(U)}\,.$$

The scaling $v \to \varepsilon v^\varepsilon$ preserves the $\underline{H}^1$-norm.

III. The multiscale Poincaré inequality

A useful characterization of the L^2-norm: we aim to show that

$$\|u - \langle u \rangle_U\|_{\underline{L}^2(U)} \leq c \|\nabla u\|_{\underline{H}^{-1}(U)}.$$

The Neumann solution: let w solve

$$-\Delta w = (u - \langle u \rangle_U) \text{ in } U \text{ with } \nabla w \cdot \nu = 0 \text{ on } \partial U.$$

Maximal regularity: we have that

$$\|\nabla w\|_{\underline{H}^1(U)} \leq c \|u - \langle u \rangle_U\|_{\underline{L}^2(U)},$$

using the $\nabla w \to \varepsilon (\nabla w)^\varepsilon$ scaling.

The estimate: we have that

$$\begin{aligned} \|u - \langle u \rangle_U\|^2_{\underline{L}^2(U)} &= \fint_U \nabla w \cdot \nabla u \\ &\leq \|\nabla w\|_{\underline{H}^1(U)} \|\nabla u\|_{\underline{H}^{-1}(U)} \\ &\leq c \|u - \langle u \rangle_U\|_{\underline{L}^2(U)} \|\nabla u\|_{\underline{H}^{-1}(U)}. \end{aligned}$$

For $c \in (0, \infty)$ independent of U,

$$\|u - \langle u \rangle_U\|^2_{\underline{L}^2(U)} \leq c \|\nabla u\|_{\underline{H}^{-1}(U)}.$$

The Lebesgue decomposition: for $m \in \mathbb{N}$ and $g \in L^1(\square_m)$,

$$g = \langle g \rangle_{\square_m} + \sum_{k=-\infty}^{m} \sum_{z \in 3^k \mathbb{Z}^d \cap \square_m} \sum_{z' \in 3^{k-1}\mathbb{Z}^d \cap (z+\square_k)} \left(\langle g \rangle_{z'+\square_{k-1}} - \langle g \rangle_{z+\square_k} \right) \mathbf{1}_{z'+\square_{k-1}},$$

and, therefore,

$$\fint_{\square_m} fg = \langle f \rangle_{\square_m} \langle g \rangle_{\square_m} + \sum\nolimits_k \overline{\sum}\nolimits_{z,z'} \left(\langle g \rangle_{z'+\square_{k-1}} - \langle g \rangle_{z+\square_k} \right) \langle f \rangle_{z'+\square_{k-1}}.$$

Hölder's inequality shows that

$$\begin{aligned}
&|\fint_{\square_m} fg| \leq \langle f \rangle_{\square_m} \langle g \rangle_{\square_m} + \sum\nolimits_k \overline{\sum}\nolimits_{z,z'} |\langle g \rangle_{z'+\square_{k-1}} - \langle g \rangle_{z+\square_k})||\langle f \rangle_{z'+\square_{k-1}}| \\
&\leq \langle f \rangle_{\square_m} \langle g \rangle_{\square_m} + \sum\nolimits_k \overline{\sum}\nolimits_z 3^d \fint_{z+\square_k} |g - \langle g \rangle_{z+\square_k}| \overline{\sum}\nolimits_{z'} |\langle f \rangle_{z'+\square_{k-1}}| \\
&\leq \langle f \rangle_{\square_m} \langle g \rangle_{\square_m} + 3^d \sum\nolimits_k \left(\overline{\sum}\nolimits_z \left(\fint_{z+\square_k} |g - \langle g \rangle_{z+\square_k}| \right)^2 \right)^{\frac{1}{2}} \left(\overline{\sum}\nolimits_{z'} |\langle f \rangle_{z'+\square_{k-1}}|^2 \right)^{\frac{1}{2}} \\
&\leq \langle f \rangle_{\square_m} \langle g \rangle_{\square_m} + 3^{d+1} \sup_{k \leq m} 3^{-k} \left(\overline{\sum}\nolimits_z \left(\fint_{z+\square_k} |g - \langle g \rangle_{z+\square_k}| \right)^2 \right)^{\frac{1}{2}} \\
&\qquad \times \sum_k 3^k \left(\overline{\sum}\nolimits_{z'} |\langle f \rangle_{z'+\square_{k-1}}|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The Poincaré inequality: since we have that, if $\|g\|_{\underline{H}^1(\square_m)} \leq 1$,

$$\sup_{k\leq m} 3^{-k}\Big(\overline{\sum}_z \big(⨍_{z+\square_k} |g-\langle g\rangle_{z+\square_k}|\big)^2\Big)^{\frac{1}{2}} \leq c\sup_{k\leq m}\Big(\overline{\sum}_z ⨍_{z+\square_k}|\nabla g|^2\Big)^{\frac{1}{2}}$$
$$\leq c\,\|\nabla g\|_{\underline{L}^2(\square_m)}$$
$$\leq c.$$

The conclusion: since $\langle g\rangle_{\square_m} \leq 3^m$ if $\|g\|_{\underline{H}^1(\square_m)} \leq 1$,

$$\|f\|_{\underline{H}^{-1}(\square_m)} = \sup_{\|g\|_{\underline{H}^1(\square_m)}} |⨍_{\square_m} fg| \leq \sum_{k=-\infty}^{m} 3^k\Big(\overline{\sum}_{z\in 3^k\mathbb{Z}^d\cap\square_m} |\langle f\rangle_{z+\square_k}|^2\Big)^{\frac{1}{2}}.$$

and, using that $\|u-\langle u\rangle_U\|^2_{\underline{L}^2(U)} \leq c\,\|\nabla u\|_{\underline{H}^{-1}(U)}$,

$$\|u-\langle u\rangle_{\square_m}\|^2_{\underline{L}^2(\square_m)} \leq c\sum_{k=-\infty}^{m} 3^k\Big(\overline{\sum}_{z\in 3^k\mathbb{Z}^d\cap\square_m} |\langle \nabla u\rangle_{z+\square_k}|^2\Big)^{\frac{1}{2}}.$$

In particular, for $c\in(0,\infty)$ independent of m,

$$\|u-\langle u\rangle_{\square_m}\|^2_{\underline{L}^2(\square_m)} \leq c\,\|\nabla u\|_{\underline{L}^2(\square_m)} + c\sum_{k=0}^{m} 3^k\Big(\overline{\sum}_{z\in 3^k\mathbb{Z}^d\cap\square_m} |\langle \nabla u\rangle_{z+\square_k}|^2\Big)^{\frac{1}{2}}.$$

The finite-volume correctors: for $v_{n,\xi}$ minimizing $\mu(\square_n, \xi)$,

$$-\nabla \cdot a\nabla v_{n,\xi} = 0 \;\text{ in } \square_n \text{ with } v_{n,\xi} = \ell_\xi \text{ on } \partial\square_n,$$

we have that $\phi_{n,\xi} = v_{n,\xi} - \ell_\xi$ and, since $\phi_{n,\xi} = 0$ on $\partial\square_n$,

$$\langle \phi_{n,\xi} \rangle_{\square_n} = 0 \text{ implies } \langle v_{n,\xi} \rangle_{\square_n} = \xi.$$

Large-scale averages: we have that, for every $n \leq m \in \mathbb{N}_0$,

$$\begin{aligned}
\left|\langle \nabla \phi_{m,\xi} \rangle_{\square_m}\right|^2 &= \left|\langle \nabla v_{m,\xi} - \xi \rangle_{\square_m}\right|^2 \\
&= \left|\langle \nabla v_{m,\xi} \rangle_{\square_m} - \langle \nabla v_{n,\xi} \rangle_{\square_m}\right|^2 \\
&= \left| \fint_{\square_m} \left(\nabla v_{m,\xi} - \nabla v_{n,\xi}\right)\right|^2 \\
&\leq \fint_{\square_m} \left|\nabla v_{m,\xi} - \nabla v_{n,\xi}\right|^2.
\end{aligned}$$

Large-scale averages: for the finite-volume corrector, for every $n \leq m \in \mathbb{N}_0$,

$$\left|\langle \nabla \phi_{m,\xi} \rangle_{\square_m}\right|^2 \leq \fint_{\square_m} \left|\nabla v_{m,\xi} - \nabla v_{n,\xi}\right|^2.$$

Asymptotic flatness: for every $n \leq m \in \mathbb{N}_0$,

$$\begin{aligned}
\frac{\lambda}{2} \fint_{\square_m} \left|\nabla v_{m,\xi} - \nabla v_{n,\xi}\right|^2 &\leq \frac{1}{2} \fint_{\square_m} (\nabla v_{m,\xi} - \nabla v_{n,\xi}) \cdot a(\nabla v_{m,\xi} - \nabla v_{n,\xi}) \\
&= \frac{1}{2} \fint_{\square_m} \nabla v_{n,\xi} \cdot a \nabla v_{n,\xi} - \frac{1}{2} \fint_{\square_m} \nabla v_{n,\xi} \cdot a \nabla v_{m,\xi} \\
&= \frac{1}{2} \fint_{\square_m} \nabla v_{n,\xi} \cdot a \nabla v_{n,\xi} - \frac{1}{2} \fint_{\square_m} \nabla v_{m,\xi} \cdot a \nabla v_{m,\xi} \\
&= \overline{\sum}_{z \in 3^n \mathbb{Z}^d \cap \square_M} \left(\mu(z + \square_n, \xi) - \mu(\square_m, \xi)\right) \\
&= \overline{\sum}_{z \in 3^n \mathbb{Z}^d \cap \square_M} \left(\xi \cdot a(z + \square_n)\xi - \xi \cdot a(\square_m)\xi\right) \\
&= \overline{\sum}_{z \in 3^n \mathbb{Z}^d \cap \square_M} \left(\xi \cdot (a(z + \square_n) - a(\square_m))\xi\right).
\end{aligned}$$

Large-scale averages: for the finite-volume corrector, for every $n \le m \in \mathbb{N}_0$,

$$\left|\langle \nabla \phi_{m,\xi} \rangle_{\square_m}\right|^2 \le \fint_{\square_m} \left|\nabla v_{m,\xi} - \nabla v_{n,\xi}\right|^2.$$

Asymptotic flatness: for every $n \le m \in \mathbb{N}_0$,

$$\frac{\lambda}{2} \fint_{\square_m} \left|\nabla v_{m,\xi} - \nabla v_{n,\xi}\right|^2 = \overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_M} \big(\xi \cdot (a(z+\square_n) - a(\square_m))\xi\big).$$

The multiscale Poincaré inequality: we have that

$$\begin{aligned}
&\left\|\phi_{m,\xi} - \langle \phi_{m,\xi} \rangle_{\square_m}\right\|_{\underline{L}^2(\square_m)} \\
&\le c\bigg(\left\|\nabla \phi_{m,\xi}\right\|_{\underline{L}^2} + \Sigma_{k=0}^m 3^k \Big(\overline{\sum}_{z \in 3^k\mathbb{Z}^d \cap \square_m} \left|\langle \nabla \phi_{m,\xi} \rangle_{z+\square_k}\right|^2\Big)^{\frac{1}{2}}\bigg) \\
&\le c\bigg(|\xi|^2 + \sum_{k=0}^{m-1} 3^k \Big(\overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} \big(\xi \cdot (a(z+\square_n) - a(\square_m))\xi\big)\Big)^{\frac{1}{2}}\bigg) \\
&\le c|\xi|^2\bigg(1 + \sum_{k=0}^{m-1} 3^k \Big(\Big|a(\square_m) - \overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} a(z+\square_n)\Big|\Big)^{\frac{1}{2}}\bigg) \\
&\le c|\xi|^2\bigg(1 + \sum_{k=0}^{m} 3^k \Big(\Big|\overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} a(z+\square_n) - \overline{a}\Big|\Big)^{\frac{1}{2}}\bigg).
\end{aligned}$$

Large-scale averages: for the finite-volume corrector, for every $n \le m \in \mathbb{N}_0$,

$$\left|\langle \nabla\phi_{m,\xi}\rangle_{\square_m}\right|^2 \le \fint_{\square_m} \left|\nabla v_{m,\xi} - \nabla v_{n,\xi}\right|^2.$$

Asymptotic flatness: for every $n \le m \in \mathbb{N}_0$,

$$\frac{\lambda}{2}\fint_{\square_m} \left|\nabla v_{m,\xi} - \nabla v_{n,\xi}\right|^2 = \overline{\sum}_{z\in 3^n\mathbb{Z}^d\cap\square_M} \big(\xi\cdot(a(z+\square_n) - a(\square_m))\xi\big).$$

The multiscale Poincaré inequality: we have that

$$\begin{aligned} &\|\phi_{m,\xi} - \langle\phi_{m,\xi}\rangle_{\square_m}\|_{\underline{L}^2(\square_m)} \\ &\le c|\xi|^2\left(1+\sum_{k=0}^m 3^k\Big(|\overline{\sum}_{z\in 3^n\mathbb{Z}^d\cap\square_m} a(z+\square_n) - \overline{a}|\Big)^{\frac{1}{2}}\right). \end{aligned}$$

Sublinearity: by the subadditive ergodic theorem and dominated convergence,

$$\begin{aligned} &\limsup_{m\to\infty} 3^{-m}\Big(\,\|\phi_{m,\xi} - \langle\phi_{m,\xi}\rangle_{\square_m}\|_{\underline{L}^2(\square_m)}\,\Big) \\ &\le \limsup_{m\to\infty} c|\xi|^2\left(3^{-m}+\sum_{k=0}^m 3^{k-m}\Big(|\overline{\sum}_{z\in 3^n\mathbb{Z}^d\cap\square_m} a(z+\square_n) - \overline{a}|\Big)^{\frac{1}{2}}\right) \\ &= 0. \end{aligned}$$

IV. Qualitative stochastic homogenization

The finite volume flux: for every $n \in \mathbb{N}$,

$$q_{m,i} = a(e_i + \nabla \phi_{m,i}) - a(\square_m)e_i.$$

The finite volume flux-corrector: for 3^m-periodic skew-symmetric matrices $\sigma_{i,m}$ with

$$-\Delta \sigma_{m,ijk} = \partial_j q_{m,ik} - \partial_k q_{m,ij} \text{ in } \mathbb{R}^d \text{ with } \fint_{\square_n} \sigma_{m,ijk} = 0.$$

The L^2-norm: for the 3^m-periodic function w satisfying

$$-\Delta w = \sigma_{m,ijk} \text{ in } \mathbb{R}^d \text{ with } \langle w \rangle_{\square_m} = 0,$$

we have using the maximal regularity estimate that

$$\begin{aligned}
\|\sigma_{m,ijk}\|^2_{\underline{L}^2(\square_m)} \leq \fint_{\square_m} \nabla w \cdot \nabla \sigma_{m,ijk} &= \fint_{\square_m} q_{m,ij}\partial_k w - q_{m,ik}\partial_j w \\
&\leq \|q_{m,i}\|_{\underline{H}^{-1}(\square_m)} \|\nabla w\|_{\underline{H}^1(\square_m)} \\
&\leq \|q_{m,i}\|_{\underline{H}^{-1}(\square_m)} \|\sigma_{m,ijk}\|_{\underline{L}^2(\square_m)},
\end{aligned}$$

and, therefore, $\|\sigma_{m,ijk}\|_{\underline{L}^2(\square_m)} \leq \|q_{m,i}\|_{\underline{H}^{-1}(\square_m)}$.

Sublinearity: applying the multiscale Poincaré inequality to $q_{m,i}$.

$$\limsup_{m\to\infty} 3^{-m}\left(\|\sigma_{m,\xi} - \langle \sigma_{m,\xi}\rangle_{\square_m}\|_{\underline{L}^2(\square_m)} \right) = 0.$$

Qualitative stochastic homogenization

Let $U \subseteq \square_0$ be a smooth, bounded domain, let $f \in \mathrm{C}^\alpha(\overline{U})$ for some $\alpha \in (0,1)$, and let $a\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ be a symmetric, uniformly elliptic, stationary, and ergodic diffusion matrix. Then, there exists a deterministic, uniformly elliptic $\overline{a} \in \mathbb{R}^{d\times d}$ defined $\mathbb{P}$-a.s. by

$$\frac{1}{2}\big(\xi \cdot \overline{a}\xi\big) = \lim_{m\to\infty} \mu(\square_m, \xi),$$

such that, $\mathbb{P}$-a.s., the local two scale expansion, for $3^{m-1} \leq \varepsilon^{-1} < 3^m$,

$$w_\varepsilon = u_\varepsilon - \varepsilon\phi^\varepsilon_{m,i}\partial_i\overline{u},$$

defined by the finite-volume correctors $\phi_{m,i}$ and macroscopic limit

$$-\nabla \cdot \overline{a}\nabla\overline{u} = f \;\text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U,$$

satisfies

$$\lim_{\varepsilon\to 0}\int_U |\nabla w_\varepsilon|^2 = 0 \;\text{ and, so, }\; \nabla u_\varepsilon \simeq \nabla\overline{u} + \partial_i\overline{u}\nabla\phi^\varepsilon_i.$$

Polarization: for every $\xi, \xi' \in \mathbb{R}^d$,

$$\xi \cdot \overline{a}\xi' = \frac{1}{4}\Big((\xi+\xi')\cdot\overline{a}(\xi+\xi') - (\xi-\xi')\cdot\overline{a}(\xi-\xi')\Big).$$

V. References

M. A. Akcoglu and U. Krengel
Ergodic theorems for superadditive processes
J. Reine Angew. Math., 323:53–67, 1981.

S. Armstrong and T. Kuusi and J.-C. Mourrat
Quantitative stochastic homogenization and large-scale regularity
Grundlehren der Mathematischen WissenschaftenSpringer, 352, 2019.

E. De Giorgi and S. Spagnolo
Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine
Boll. Un. Mat. Ital., 8(4):391-411, 1973.

G. Dal Maso and L. Modica
Nonlinear stochastic homogenization
Ann. Mat. Pura Appl., 144(4):347-389, 1986.

G. Dal Maso and L. Modica
Nonlinear stochastic homogenization and ergodic theory
J. Reine Angew. Math., 368:28–42, 1986.