Toward a quantitative stochastic homogenization

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The coefficient field: the map $a\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ satisfies

$$\xi \cdot a\xi \geq \lambda|\xi|^2 \;\text{ and }\; \xi \cdot a^{-1}\xi \geq \Lambda^{-1}|\xi|^2,$$

and, for a measure-preserving transformation group $\{\tau_x\colon \Omega \to \Omega\}_{x\in\mathbb{R}^d}$,

$$a(x+y,\omega) = a(y,\tau_x\omega) \;\text{ for all } x, y \in \mathbb{R}^d \text{ and } \omega \in \Omega.$$

Furthermore, the transformation group is ergodic:

$$f(\omega) = f(\tau_x\omega) \in L^\infty(\Omega) \;\text{ for all } x \in \mathbb{R}^d \text{ if and only if } f \text{ is constant.}$$

The equation: for a fixed environment $\omega \in \Omega$, and $U \subseteq \mathbb{R}^d$,

$$-\nabla \cdot a^\varepsilon \nabla u_\varepsilon = f \;\text{ in }\; U \;\text{ with }\; u_\varepsilon = 0 \;\text{ on }\; \partial U.$$

Qualitative homogenization: there exists $\overline{a} \in \mathbb{R}^{d\times d}$ such that, $\mathbb{P}$-a.s.,

$$u_\varepsilon \rightharpoonup \overline{u} \;\text{ weakly in } H^1(U) \;\text{ and }\; u_\varepsilon \to \overline{u} \;\text{ strongly in } L^2(U),$$

for $\overline{u}$ solving

$$-\nabla \cdot \overline{a}\nabla\overline{u} = f \;\text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

I. Qualitative stochastic homogenization

The equation: for a symmetric diffusion matrix a, the solution

$$-\nabla \cdot a^{\varepsilon} \nabla u_{\varepsilon} = f \;\text{ in } U \text{ with } u_{\varepsilon} = 0 \text{ on } \partial U,$$

satisfies

$$u_{\varepsilon} = \operatorname{argmin}_{v \in H_0^1(U)} \int_U \big(\nabla v \cdot a^{\varepsilon} \nabla v - v f \big).$$

The approximate gradient: by the two-scale expansion,

$$\nabla u_{\varepsilon} \simeq \nabla \overline{u} + \partial_i \overline{u} \nabla \phi_i^{\varepsilon} = (e_i + \nabla \phi_i^{\varepsilon}) \partial_i \overline{u},$$

and, therefore,

$$\int_U \nabla v \cdot a^{\varepsilon} \nabla v \simeq \sum_{ij} \int_U \big((e_i + \nabla \phi_i^{\varepsilon}) \cdot a^{\varepsilon} (e_j + \nabla \phi_j^{\varepsilon}) \big) \partial_i \overline{u} \partial_j \overline{u}.$$

The effective energy: we have that

$$\frac{1}{|U|} \int_U (e_i + \nabla \phi_i^{\varepsilon}) \cdot a^{\varepsilon} (e_j + \nabla \phi_j^{\varepsilon}) = \fint_{U/\varepsilon} (e_i + \nabla \phi_i) \cdot a (e_j + \nabla \phi_j).$$

Polarization: we will analyze on large-scales, for $\xi \in \mathbb{R}^d$,

$$\mu(U, \xi) = \frac{1}{2} \fint_U (\xi + \nabla \phi_{\xi}) \cdot a (\xi + \nabla \phi_{\xi}),$$

for $\phi_{\xi} = \xi_i \phi_i$ satisfying $-\nabla \cdot a(\xi + \nabla \phi_{\xi}) = 0$.

The effective coefficient: by the subadditive ergodic theorem, for $\xi, \xi' \in \mathbb{R}$,

$$\frac{1}{2}\xi' \cdot \overline{a}\xi = \lim_{m\to\infty}\left(\frac{1}{4}\left(\mu(\xi'+\xi,\square_m) - \mu(\xi'-\xi,\square_m)\right)\right) \text{ for } \square_m = \left[-\frac{3^m}{2},\frac{3^m}{2}\right].$$

The finite-volume correctors: for the minimizers $v_{m,\xi}$, for $\ell_\xi(x) = x\cdot\xi$,

$$\mu(\xi,\square_m) = \inf_{v\in\ell_\xi + H^1_0(\square_m)} \frac{1}{2}\fint_{\square_m} \nabla v\cdot a\nabla v,$$

the finite-volume correctors $\phi_{m,\xi} = v_{m.\xi} - \ell_\xi$ satisfy

$$-\nabla\cdot a(\xi+\nabla\phi_{m,\xi}) = 0 \text{ in } \square_m \text{ with } \phi_{m,\xi} = 0 \text{ on } \partial\square_M.$$

The two-scale expansion: if $3^{m-1} \le \varepsilon^{-1} < 3^m$,

$$u_\varepsilon \simeq \tilde{u}_{m,\varepsilon} = \overline{u} + \varepsilon\phi^\varepsilon_{m,i}\partial_i\overline{u}.$$

Sublinearity of correctors: for the finite-volume flux correctors $\sigma_{m,ijk}$

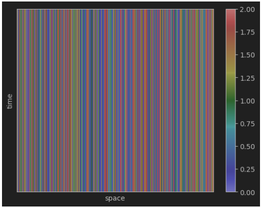
$$\limsup_{m\to\infty} 3^{-m}\left(\|\phi_{m,i}\|_{\underline{L}^2(\square_m)} + \|\sigma_{m,ijk}\|_{\underline{L}^2(\square_m)}\right)$$

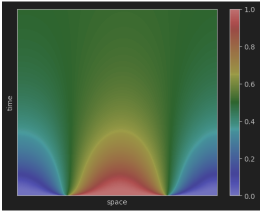
$$\le \limsup_{m\to\infty} c|\xi|^2\left(3^{-m} + \sum_{k=0}^m 3^{k-m}\left(|\overline{\sum}_{z\in 3^n\mathbb{Z}^d\cap\square_m} a(z+\square_n) - \overline{a}|\right)^{\frac{1}{2}}\right).$$

I. Qualitative stochastic homogenization

The equation: for $\varepsilon = {}^1\!/_{1000}$ in $\mathbb{T}^1 \times (0,\infty)$,

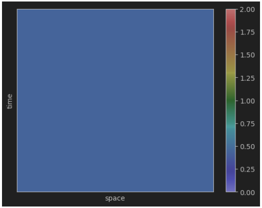
$$\partial_t u_\varepsilon = ({}^1\!/_2)\nabla \cdot a^\varepsilon \nabla u_\varepsilon \;\text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1\!/_4,{}^3\!/_4\}}.$$

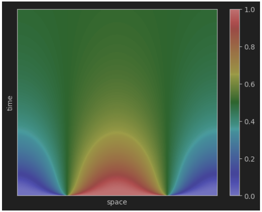




A macroscopic limit: choosing $\overline{a} = \langle a^{-1}\rangle^{-1} \simeq 0.43588$,

$$\partial_t \overline{u} = ({}^1\!/_2)\nabla \cdot \overline{a} \nabla \overline{u} \;\text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1\!/_4,{}^3\!/_4\}}.$$





II. The dual quantity

The energy: we have that

$$\mu(\square_m, \xi) = \inf_{v \in \ell_\xi + H_0^1(\square_m)} \frac{1}{2}\left(\fint_{\square_m} \nabla v \cdot a \nabla v \right) = \frac{1}{2}\left(\xi \cdot a(\square_m)\xi\right).$$

The convex dual: the convex dual of the map $\xi \to (1/2)(\xi \cdot a(\square_m)\xi)$,

$$\mu^*(\square_m, \xi) = \sup_{p \in \mathbb{R}^d} \left(\xi \cdot p - \mu(\square_m, \xi)\right) = \frac{1}{2}\left(\xi \cdot a(\square_m)^{-1}\xi\right).$$

The Fenchel–Young inequality: for all $p, q \in \mathbb{R}^d$,

$$\mu(\square_m, p) + \mu^*(\square_m, q) \geq p \cdot q.$$

with equality if $q \in \partial f(p)$.

II. The dual quantity

The energy: we have that

$$\mu(\square_m,\xi)=\inf_{v\in\ell_\xi+H^1_0(\square_m)}\frac{1}{2}\Big(\fint_{\square_m}\nabla v\cdot a\nabla v\Big)=\frac{1}{2}\big(\xi\cdot a(\square_m)\xi\big).$$

The approximating convex dual: we define

$$\mu_*(\square_m,q)=\sup_{w\in H^1(\square_m)}\fint_{\square_m}\big(-\frac{1}{2}(\nabla w\cdot a\nabla w)+q\cdot\nabla w\big),$$

where the minimizer $w_{m,q}$ satisfies

$$\fint_{\square_m}\nabla v\cdot a\nabla w_{m,q}=\fint_{\square_m}q\cdot\nabla v\ \text{ for all } v\in H^1(\square_m),$$

which implies formally that

$$\frac{1}{|\square_m|}\int_{\partial\square_m}(a\nabla w_{m,q}\cdot\nu)v+\fint_{\square_m}\big(-\nabla\cdot a\nabla w_{m,q}\big)v=\frac{1}{|\square_m|}\fint_{\square_m}v(q\cdot\nu).$$

The equation: the minimizer $w_{m,q}$ satisfies

$$-\nabla\cdot a\nabla w_{m,q}=0\ \text{ in } \square_m \text{ with } a\nabla w_{m,q}\cdot\nu=q\cdot\nu \text{ on } \partial\square_M.$$

The energy: we have that

$$\mu(\square_m,\xi)=\inf_{v\in\ell_\xi+H_0^1(U)}\frac{1}{2}\Big(\fint_{\square_m}\nabla v\cdot a\nabla v\Big)=\frac{1}{2}\big(\xi\cdot a(\square_m)\xi\big).$$

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where the minimizer $w_{m,q}$ satisfies

$$-\nabla\cdot a\nabla w_{m,q}=0\;\text{ in }\square_m\text{ with }a\nabla w_{m,q}\cdot\nu=q\cdot\nu\text{ on }\partial\square_M.$$

The dual matrix: there exists $a_*(\square_m)^{-1}\in\mathbb{R}^{d\times d}$ such that

$$\begin{aligned}\mu_*(\square_m,q)&=\sup_{w\in H^1(U)}\fint_{\square_m}\big(-\frac{1}{2}\big(\nabla w\cdot a\nabla w\big)+q\cdot\nabla w\big)\\&=\frac{1}{2}\big(q\cdot a_*(\square_m)^{-1}q\big).\end{aligned}$$

II. The dual quantity

The energy: we have that

$$\mu(\square_m, \xi) = \inf_{v \in \ell_\xi + H_0^1(U)} \frac{1}{2}\Big(\fint_{\square_m} \nabla v \cdot a \nabla v \Big) = \frac{1}{2}\big(\xi \cdot a(\square_m)\xi\big).$$

The approximating convex dual: we define

$$\mu_*(\square_m, q) = \sup_{w \in H^1(U)} \fint_{\square_m} \big(-\frac{1}{2}\big(\nabla w \cdot a \nabla w\big) + q \cdot \nabla w\big) = \frac{1}{2}\big(q \cdot a_*(\square_m)^{-1} q\big).$$

where

$$\mu_*(\square_m, q) \geq \sup_{p \in \mathbb{R}^d} \Big(q \cdot p - p \cdot \big(\fint_{\square_m} a\big) p\Big) = \frac{1}{2}\big(q \cdot \big(\fint_{\square_m} a\big)^{-1} q\big) \geq \frac{\Lambda^{-1}}{2}|q|^2.$$

The approximate Fenchel-Young inequality: for the minimizer $v_{m,p}$ of μ,

$$\begin{aligned}
&\mu(\square_m, p) + \mu_*(\square_m, q) = \frac{1}{2}\Big(p \cdot a(\square_m)p + q \cdot a_*^{-1}(\square_m) q\Big) \\
&\geq \frac{1}{2}\Big(\fint_{\square_m} \nabla v_{m,p} \cdot a \nabla v_{m,p}\Big) + \fint_{\square_m} \big(-\frac{1}{2}\big(\nabla v_{m,p} \cdot a \nabla v_{m,p}\big) + q \cdot \nabla v_{m,p}\big) \\
&= q \cdot \fint_{\square_m} \nabla v_{m,p} = q \cdot p.
\end{aligned}$$

Monotonicity: by choosing $q = a_*(\square_m)p$,

$$\frac{1}{2}\big(p \cdot (a(\square_m) - a_*(\square_m))p \geq p \cdot a_*(\square_m)p \geq \lambda |p|^2.$$

Monotonicity: since for every $p \in \mathbb{R}^d$,

$$\frac{1}{2}\left(p\cdot(a(\square_m)-a_*(\square_m)\right)p \geq p\cdot a_*(\square_m)p \geq \lambda|p|^2, \quad \text{we have} \quad a(\square_m) \geq a_*(\square_m).$$

Ellipticity upper bound: for every $p \in \mathbb{R}^d$, choosing ℓ_p in $\mu(\xi,p)$,

$$\frac{1}{2}\left(p\cdot a(\square_m)p\right) \leq \frac{1}{2}\fint_{\square_m} p\cdot ap = \frac{1}{2}\Big(p\cdot\big(\fint_{\square_m} a\big)p\Big) \leq \frac{\Lambda}{2}|p|^2.$$

Ellipticity lower bound: for every $w \in H^1(\square_m)$,

$$\fint_{\square_m}\Big(q\cdot\nabla w - \frac{1}{2}\left(\nabla w\cdot a^{-1}\nabla w\right)\Big) \leq \fint_{\square_m}\sup_{p\in\mathbb{R}^d}\Big(q\cdot p - \frac{1}{2}\left(p\cdot a^{-1}p\right)\Big)$$

$$= \fint_{\square_m}\frac{1}{2}q\cdot a^{-1}q = \frac{1}{2}\Big(q\cdot\big(\fint_{\square_m} a^{-1}\big)q\Big) \leq \frac{\lambda^{-1}}{2}|q|^2,$$

and, after taking the supremum over $w \in H^1(\square_m)$,

$$\frac{1}{2}\left(q\cdot a_*^{-1}(\square_m)q\right) \leq \frac{\lambda^{-1}}{2}|q|^2.$$

The conclusion: we have that

$$\lambda\cdot\mathrm{Id} \leq \big(\int_{\square_m} a^{-1}\big)^{-1} \leq a_*(\square_m) \leq a(\square_m) \leq \fint_{\square_m} a \leq \Lambda\cdot\mathrm{Id}.$$

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$$\mu(\square_m, \xi) = \inf_{v \in \ell_\xi + H_0^1(\square_m)} \frac{1}{2}\Big(\fint_{\square_m} \nabla v \cdot a \nabla v \Big) = \frac{1}{2}\big(\xi \cdot a(\square_m)\xi\big).$$

The approximating convex dual: we define

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The approximate Fenchel-Young inequality: for every $p, q \in \mathbb{R}^d$,

$$\mu(\square_m, p) + \mu_*(\square_m, q) \geq q \cdot p.$$

The "master" quantity: for every $p, q \in \mathbb{R}^d$,

$$\begin{aligned} J(\square_m, p, q) &= \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q \\ &= \frac{1}{2}\big(p \cdot a(\square_m) p\big) + \frac{1}{2}\big(q \cdot a_*(\square_m)^{-1} q\big) - p \cdot q \\ &= \frac{1}{2}\big(p \cdot (a(\square_m) - a_*(\square_m))p\big) + \frac{1}{2}\big((q - a_*(\square_m))p \cdot a_*(U)^{-1}(q - a_*(\square_m))p\big) \end{aligned}$$

The homogenization error: if $q = \overline{a}p$,

$$\begin{aligned} &J(\square_m, p, \overline{a}p) \\ &= \frac{1}{2}\big(p \cdot (a(\square_m) - a_*(\square_m))p\big) + \frac{1}{2}\big((\overline{a} - a_*(\square_m))p \cdot a_*(U)^{-1}(\overline{a} - a_*(\square_m))p\big)\big). \end{aligned}$$

The energy: we have that

$$\mu(\square_m, \xi) = \inf_{v \in \ell_\xi + H_0^1(U)} \frac{1}{2}\Big(\fint_{\square_m} \nabla v \cdot a \nabla v \Big) = \frac{1}{2}\big(\xi \cdot a(\square_m)\xi\big).$$

The approximating convex dual: we define

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The "master" quantity: for every $p, q \in \mathbb{R}^d$,

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The homogenization error: if $q = \overline{a}p$,

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Sublinearity of correctors: for the finite-volume correctors,

$$\begin{aligned} &\limsup_{m \to \infty} 3^{-m} \Big(\|\phi_{m,i}\|_{\underline{L}^2(\square_m)} + \|\sigma_{m,ijk}\|_{\underline{L}^2(\square_m)} \Big) \\ &\leq \limsup_{m \to \infty} c\Big(3^{-m} + \sum_{k=0}^{m} 3^{k-m} \Big(\big| \overline{\sum}_{z \in 3^n \mathbb{Z}^d \cap \square_m} a(z + \square_n) - \overline{a} \big| \Big)^{\frac{1}{2}} \Big). \end{aligned}$$

III. Properties of the "master" quantity

The "master" quantity: for every $p, q \in \mathbb{R}^d$,

$$J(\square_m, p, q) = \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q,$$

The variational form: for $\xi \in \mathbb{R}^d$, for

$$\mathcal{H}(\square_m) = \{u \in H^1(\square_m) \colon -\nabla \cdot a\nabla u = 0\},$$

and for the minimizer v of $\mu(\square_m, p)$, for every $w \in \mathcal{H}(\square_m)$,

$$\begin{aligned}
&\mu(\square_m, p) + \fint_{\square_m} (\nabla w \cdot q - \frac{1}{2}\nabla w \cdot a\nabla w) - p \cdot q \\
&= \fint_{\square_m} \frac{1}{2}\nabla v \cdot a\nabla v - \frac{1}{2}\nabla w \cdot a\nabla w + (\nabla w - \nabla v) \cdot q \\
&= \fint_{\square_m} -\frac{1}{2}(\nabla w - \nabla v) \cdot a(\nabla w - \nabla v) - \nabla v \cdot a(\nabla w - \nabla v) + q \cdot (\nabla w - \nabla v) \\
&= \fint_{\square_m} -\frac{1}{2}(\nabla w - \nabla v) \cdot a(\nabla w - \nabla v) - p \cdot a(\nabla w - \nabla v) + q \cdot (\nabla w - \nabla v),
\end{aligned}$$

where the equality uses that $v - \ell_p \in H^1_0(\square_m)$ and $(w - v) \in \mathcal{H}(\square_m)$.

The conclusion: after taking the supremum over $w \in \mathcal{H}(U)$,

$$J(\square_m, p, q) = \sup_{v \in \mathcal{H}(\square_m)} \fint_{\square_m} -\frac{1}{2}\nabla v \cdot a\nabla v - p \cdot a\nabla v + q \cdot \nabla v.$$

The "master" quantity: for every $p, q \in \mathbb{R}^d$,

$$J(\square_m, p, q) = \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q$$
$$= \sup_{v \in \mathcal{H}(\square_m)} \fint_{\square_m} -\frac{1}{2}\nabla v \cdot a\nabla v - p \cdot a\nabla v + q \cdot \nabla v,$$

where, for the minimizers $v_{m,p.q}$,

$$(p, q) \to v_{m,p,q} \text{ is linear.}$$

The first variation: taking a variation in $w \in \mathcal{H}(\square_m)$,

$$\fint_{\square_m} q \cdot \nabla w - \fint_{\square_m} p \cdot a\nabla w = \fint_{\square_m} \nabla w \cdot a\nabla v_{m,p,q}.$$

Quadratic response: choosing $w - v_{m,p,q}$ in the above,

$$J(\square_m, p, q) - \left(-\fint_{\square_m} \nabla w \cdot a\nabla w - \fint_{\square_m} p \cdot a\nabla w + \fint_{\square_m} q \cdot \nabla w \right)$$
$$= \frac{1}{2}\fint_{\square_m} (\nabla v_{m,p.q} - \nabla w) \cdot a(\nabla v_{m,p.q} - \nabla w).$$

An energy identity: if we take $w = 0$ in the above,

$$J(\square_m, p, q) = \frac{1}{2}\fint_{\square_m} \nabla v_{m,p.q} \cdot a\nabla v_{m,p.q}.$$

III. Properties of the "master" quantity

The "master" quantity: for every $p, q \in \mathbb{R}^d$,

$$J(\square_m, p, q) = \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q$$
$$= \sup_{v \in \mathcal{H}(\square_m)} \fint_{\square_m} -\frac{1}{2}\nabla v \cdot a\nabla v - p \cdot a\nabla v + q \cdot \nabla v.$$

Polarization identity: using the first variation, polarization, and the energy,

$$q' \cdot \fint_{\square_m} \nabla v_{m,p,q} - p' \cdot \fint_{\square_m} a\nabla v_{m,p.q} = \fint_{\square_m} \nabla v_{m,p,q} \cdot a\nabla v_{m,p',q'}$$
$$= \frac{1}{4}\left(\fint_{\square_m} \nabla v_{m,p+p',q+q'} \cdot a\nabla v_{m,p+p',q+q'} - \nabla v_{m,p-p',q-q'} \cdot a\nabla v_{m,p-p',q-q'}\right)$$
$$= \frac{1}{2}\left(J(\square_m, p+p', q+q') + J(\square_m, p-p', q-q')\right)$$
$$= p' \cdot a(\square_m)p + q' \cdot a_*^{-1}(\square_m)q - p \cdot q' - q \cdot p'$$
$$= q' \cdot (a_*^{-1}(\square_m)q - p) - p' \cdot (q - a(\square_m)p).$$

Spatial averages of the gradient and flux: we therefore have that

$$\fint_{\square_m} \nabla v_{m,p,q} = \left(a_*^{-1}(\square_m)q - p\right) \text{ and } \fint_{\square_m} a\nabla v_{m,p.q} = \left(q - a(\square_m)p\right).$$

The "master" quantity: for every $p, q \in \mathbb{R}^d$,

$$J(\square_m, p, q) = \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q$$
$$= \sup_{v \in \mathcal{H}(\square_m)} \fint_{\square_m} -\frac{1}{2}\nabla v \cdot a\nabla v - p \cdot a\nabla v + q \cdot \nabla v.$$

Subadditivty: the variational form proves that, if $U = \coprod_{i=1}^n U_i$,

$$J(U, p, q) = \sup_{v \in \mathcal{H}(U_i)} \fint_{\square_m} -\frac{1}{2}\nabla v \cdot a\nabla v - p \cdot a\nabla v + q \cdot \nabla v$$
$$\leq \sum_{i=1}^N \sup_{v \in \mathcal{H}(U)} \frac{|U_i|}{|U|} \fint_{U_i} -\frac{1}{2}\nabla v \cdot a\nabla v - p \cdot a\nabla v + q \cdot \nabla v$$
$$\leq \sum_{i=1}^N \sup_{v \in \mathcal{H}(U_i)} \frac{|U_i|}{|U|} \int_{U_i} -\frac{1}{2}\nabla v \cdot a\nabla v - p \cdot a\nabla v + q \cdot \nabla v$$
$$= \sum_{i=1}^N \frac{|U_i|}{|U|} J(U_i, p, q).$$

Subadditivity of the course-grained coefficients: we have that, if $n \leq m$,

$$a(\square_m) \leq \overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} a(z + \square_n) \ \text{ and } \ a_*(\square_m) \geq \left(\overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} a_*(z + \square_n)^{-1}\right)^{-1}.$$

The "master" quantity: for every $p, q \in \mathbb{R}^d$,

$$\begin{aligned} J(\square_m, p, q) &= \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q \\ &= \sup_{v \in \mathcal{H}(\square_m)} \fint_{\square_m} -\frac{1}{2}\nabla v \cdot a \nabla v - p \cdot a\nabla v + q \cdot \nabla v. \\ &= \frac{1}{2}\big(p \cdot (a(\square_m) - a_*(\square_m)p)\big) + \frac{1}{2}\big((q - a_*(\square_m)p) \cdot a_*(U)^{-1}(q - a_*(\square_m)p)\big). \end{aligned}$$

Controlling $a - a_*$: for every symmetric matrix b and $p \in \mathbb{R}^d$,

$$\begin{aligned} &J(\square_m, p, Bp) \\ &= \frac{1}{2}\big(p \cdot (a(\square_m) - a_*(\square_m))p)\big) + \frac{1}{2}\big((B - a_*(\square_m))p \cdot a_*(U)^{-1}(B - a_*(\square_m)p)\big). \end{aligned}$$

The uniform ellipticity of a_* and $a \geq a_*$ prove that

$$|a(\square_m) - a_*(\square_m)| + |B - a_*(\square_m)|^2 \leq c \sup_{|p|=1} J(\square_m, p, Bp) \leq c \sum_{i=1}^{d} J(\square_m, e_i, Be_i),$$

and, therefore,

$$|a(\square_m) - a_*(\square_m)| + |B - a_*(\square_m)|^2 + |B - a(\square_m)|^2 \leq c \sum_{i=1}^{d} J(\square_m, e_i, Be_i).$$

An inequality of means: if $\{B_i\}_{i=1}^n$ and B symmetric, positive definite matrices,

$$\overline{\sum}_{i=1}^n B_i = \Big(\overline{\sum}_{i=1}^n B_i^{-1}\Big)^{-1} + \overline{\sum}_{i=1}^n (B - B_i) B_i^{-1} (B - B_i)$$
$$- \Big(\Big(\overline{\sum}_{i=1}^n B_i^{-1}\Big)^{-1} - B\Big) \Big(\overline{\sum}_{i=1}^n B_i^{-1}\Big) \Big(\Big(\overline{\sum}_{i=1}^n B_i^{-1}\Big)^{-1} - B\Big),$$

and, therefore,

$$\overline{\sum}_{i=1}^n B_i \leq \Big(\overline{\sum}_{i=1}^n B_i^{-1}\Big)^{-1} + \overline{\sum}_{i=1}^n (B - B_i) B_i^{-1} (B - B_i).$$

The additivity defect: applying this to a and a_* for $n \leq m$,

$$\begin{aligned}
& a(\square_m) \leq a_*(\square_m) + c\Sigma_{i=1}^d J(\square_m, e_i, Be_i) \\
& \leq \overline{\Sigma}_{z\in 3^n\mathbb{Z}^d\cap\square_m} a_*(z+\square_n) + c\Sigma_{i=1}^d J(\square_m, e_i, Be_i)\mathrm{Id} \\
& \leq \Big(\overline{\Sigma}_{z\in 3^n\mathbb{Z}^d\cap\square_m} a_*^{-1}(z+\square_n)\Big)^{-1} + c\Sigma_{i=1}^d J(\square_m, e_i, Be_i)\mathrm{Id} \\
& \quad + \overline{\Sigma}_{z\in 3^n\mathbb{Z}^d\cap\square_m} (B - a_*(z+\square_n)) a_*^{-1}(z+\square_n)(B - a_*(z+\square_n) \\
& \leq \Big(\overline{\Sigma}_{z\in 3^n\mathbb{Z}^d\cap\square_m} a_*^{-1}(z+\square_n)\Big)^{-1} + c\sum_{i=1}^d J(\square_m, e_i, Be_i)\mathrm{Id} \\
& \leq a_*(\square_m) + c\Sigma_{i=1}^d J(\square_m, e_i, Be_i) \leq a(\square_m) + c\Sigma_{i=1}^d J(\square_m, e_i, Be_i)\mathrm{Id}.
\end{aligned}$$

III. Properties of the “master” quantity

The “master” quantity: for every $p, q \in \mathbb{R}^d$,

$$J(\square_m, p, q) = \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q$$
$$= \sup_{v \in \mathcal{H}(\square_m)} \fint_{\square_m} -\frac{1}{2}\nabla v \cdot a\nabla v - p \cdot a\nabla v + q \cdot \nabla v.$$

The additivity defect: we have that

$$\overline{\sum}_{z\in 3^n\mathbb{Z}^d\cap\square_m} a_*(z+\square_n) - \left(\overline{\sum}_{z\in 3^n\mathbb{Z}^d\cap\square_m} a_*^{-1}(z+\square_n)\right)^{-1} \leq c\sum_{i=1}^d J(\square_m, e_i, Be_i)\mathrm{Id}.$$

Subadditivity: if $\ell \leq n \leq m$,

$$\overline{\sum}_{z\in 3^n\mathbb{Z}^d\cap\square_m} J(z+\square_n, p, q) \leq \overline{\sum}_{z\in 3^\ell\mathbb{Z}^d\cap\square_m} J(z+\square_\ell, p, q)$$
$$= \mathbb{E}[J(\square_\ell, p, q)] + \overline{\sum}_{z\in 3^\ell\mathbb{Z}^d\cap\square_m}\left(J(z+\square_\ell, p, q) - \mathbb{E}[J(\square_\ell, p, q)\right)$$

while if $n < \ell \leq m$,

$$\overline{\sum}_{z\in 3^n\mathbb{Z}^d\cap\square_m} J(z+\square_n, p, q) \leq \overline{\sum}_{z\in 3^\ell\mathbb{Z}^d\cap\square_m}\overline{\sum}_{z'\in 3^n\mathbb{Z}^d\cap(z+\square_m)} J(z'+\square_n, p, q)$$
$$= \mathbb{E}[J(\square_n, p, q)] + \overline{\sum}_{z\in 3^\ell\mathbb{Z}^d\cap\square_m}\overline{\sum}_{z'\in 3^n\mathbb{Z}^d\cap(z+\square_m)}\left(J(z'+\square_n, p, q) - \mathbb{E}[J(\square_n, p, q)]\right).$$

The "master" quantity: for every $p, q \in \mathbb{R}^d$,

$$\begin{aligned} J(\square_m, p, q) &= \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q \\ &= \sup_{v \in \mathcal{H}(\square_m)} \fint_{\square_m} -\frac{1}{2}\nabla v \cdot a\nabla v - p \cdot a\nabla v + q \cdot \nabla v \\ &= \frac{1}{2}\big(p \cdot (a(\square_m) - a_*(\square_m)p)\big) + \frac{1}{2}\big((q - a_*(\square_m)p) \cdot a_*(U)^{-1}(q - a_*(\square_m)p)\big). \end{aligned}$$

A course-grained identity: for every $w \in \mathcal{H}(U)$,

$$\begin{aligned} e_i \cdot \fint_{\square_m} (a_*(\square_m) - a) \cdot \nabla w &= \fint_{\square_m} \nabla w \cdot a\nabla v_{m,e_i,a_*(\square_m)e_i} \\ &\le \left(\fint_{\square_m} \nabla v_{m,e_i,a_*(\square_m)e_i} \cdot a\nabla v_{m,e_i,a_*(\square_m)e_i}\right)^{\frac{1}{2}} \left(\fint_{\square_m} \nabla w \cdot a\nabla w\right)^{\frac{1}{2}} \\ &\le (2J(\square_m, e_i, a_*(\square_m)e_i))^{\frac{1}{2}} \left(\fint_{\square_m} \nabla w \cdot a\nabla w\right)^{\frac{1}{2}} \\ &= (e_i \cdot (a(U) - a_*(U))e_i)^{\frac{1}{2}} \left(\fint_{\square_m} \nabla w \cdot a\nabla w\right)^{\frac{1}{2}}. \end{aligned}$$

The conclusion: we have, for every $p \in \mathbb{R}^d$ and $w \in \mathcal{H}(\square_m)$,

$$p \cdot \fint_{\square_m} (a_*(\square_m) - a) \cdot \nabla w \le (p \cdot (a(U) - a_*(U))p)^{\frac{1}{2}} \left(\fint_{\square_m} \nabla w \cdot a\nabla w\right)^{\frac{1}{2}}.$$

III. Properties of the "master" quantity

The "master" quantity: for every $p, q \in \mathbb{R}^d$,

$$J(\square_m, p, q) = \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q$$
$$= \sup_{v \in \mathcal{H}(\square_m)} \fint_{\square_m} -\frac{1}{2}\nabla v \cdot a\nabla v - p \cdot a\nabla v + q \cdot \nabla v.$$

A course-grained energy inequality: for every $u \in H^1(\square_m)$, for $p = \fint_{\square_m} \nabla u$,

$$\frac{1}{2}(p \cdot a_*(\square_m)p) = \sup_{q \in \mathbb{R}^d} \left(p \cdot q - \frac{1}{2}(q \cdot a_*(U)^{-1} q\right)$$
$$= \sup_{q \in \mathbb{R}^d} \sup_{v \in H^1(U)} \fint_{\square_m} \left(q \cdot (\nabla u - \nabla v) + \frac{1}{2}\nabla v \cdot a\nabla v\right)$$
$$\leq \frac{1}{2}\fint_{\square_m} \nabla u \cdot a\nabla u,$$

which implies that

$$\frac{1}{2}\left(\fint_{\square_m} \nabla u\right) \cdot a_*(\square_m)\left(\fint_{\square_m} \nabla u\right) \leq \frac{1}{2}\fint_{\square_m} \nabla u \cdot a\nabla u.$$

Another course-grained energy inequality: we have similarly that

$$\frac{1}{2}\left(\fint_{\square_m} \nabla u\right) \cdot a^{-1}(\square_m)\left(\fint_{\square_m} \nabla u\right) \leq \frac{1}{2}\fint_{\square_m} \nabla u \cdot a^{-1}\nabla u.$$

IV. Summarizing the properties of J

Ordering of the course-grained matrices: we have that

$$\lambda \cdot \mathrm{Id} \leq \Big(\fint_{\square_m} a^{-1} \Big)^{-1} \leq a_*(\square_m) \leq a(\square_m) \leq \fint_{\square_m} a \leq \Lambda \cdot \mathrm{Id}.$$

The variational form: for every $p, q \in \mathbb{R}^d$,

$$\begin{aligned} J(\square_m, p, q) &= \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q \\ &= \sup_{v \in \mathcal{H}(\square_m)} \fint_{\square_m} -\frac{1}{2} \nabla v \cdot a \nabla v - p \cdot a \nabla v + q \cdot \nabla v \\ &= \frac{1}{2} \big(p \cdot (a(\square_m) - a_*(\square_m) p) \big) + \frac{1}{2} \big((q - a_*(\square_m) p) \cdot a_*(U)^{-1} (q - a_*(\square_m) p) \big). \end{aligned}$$

Locality and subadditivity: $J(U, p, q)$ is $\mathcal{F}(U)$-measurable and, if $U = \coprod U_i$,

$$J(U, p, q) \leq \sum_{i=1}^{n} \frac{|U_i|}{|U|} J(U_i, p, q).$$

The minimzer: we have that

$$v_{m,p,q} \text{ is the difference of the minimizers of } \mu \text{ and } \mu_*,$$

and the map

$$(p, q) \to v_{m,p,q} \text{ is linear.}$$

IV. Summarizing the properties of J

The energy: for the minimizer $v_{m,p,q}$,

$$J(\square_m, p, q) = \frac{1}{2} \fint_{\square_m} \nabla v_{m,p.q} \cdot a \nabla v_{m,p.q}.$$

Spatial averages of the gradient and flux: we therefore have that

$$\fint_{\square_m} \nabla v_{m,p,q} = \left(a_*^{-1}(\square_m) q - p\right) \text{ and } \fint_U a \nabla v_{m,p.q} = \left(q - a(\square_m) p\right).$$

The first variation: for every $w \in \mathcal{H}(\square_m)$,

$$\fint_{\square_m} q \cdot \nabla w - \fint_{\square_m} p \cdot a \nabla w = \fint_{\square_m} \nabla w \cdot a \nabla v_{m,p,q}.$$

Quadratic response: we have that

$$\begin{aligned} &J(\square_m, p, q) - \left(- \fint_{\square_m} \nabla w \cdot a \nabla w - \fint_{\square_m} p \cdot a \nabla w + \fint_{\square_m} q \cdot \nabla w \right) \\ &= \frac{1}{2} \fint_{\square_m} (\nabla v_{m,p.q} - \nabla w) \cdot a (\nabla v_{m,p.q} - \nabla w). \end{aligned}$$

IV. Summarizing the properties of J

The energy: for the minimizer $v_{m,p,q}$,

$$J(\square_m, p, q) = \frac{1}{2} \fint_{\square_m} \nabla v_{m,p.q} \cdot a \nabla v_{m,p.q}.$$

Spatial averages of the gradient and flux: we therefore have that

$$\fint_{\square_m} \nabla v_{m,p,q} = \left(a_*^{-1}(\square_m) q - p\right) \text{ and } \fint_U a \nabla v_{m,p.q} = \left(q - a(\square_m) p\right).$$

The first variation: for every $w \in \mathcal{H}(\square_m)$,

$$\fint_{\square_m} q \cdot \nabla w - \fint_{\square_m} p \cdot a \nabla w = \fint_{\square_m} \nabla w \cdot a \nabla v_{m,p,q}.$$

Quadratic response: we have that

$$J(\square_m, p, q) - \left(- \fint_{\square_m} \nabla w \cdot a \nabla w - \fint_{\square_m} p \cdot a \nabla w + \fint_{\square_m} q \cdot \nabla w \right)$$
$$= \frac{1}{2} \fint_{\square_m} (\nabla v_{m,p.q} - \nabla w) \cdot a (\nabla v_{m,p.q} - \nabla w).$$

IV. Summarizing the properties of J

Controlling $a - a_*$ using the convex duality defect: we have that

$$|a(\square_m) - a_*(\square_m)| + |B - a_*(\square_m)|^2 + |B - a(\square_m)|^2 \leq c \sum_{i=1}^{d} J(\square_m, e_i, Be_i).$$

The additivity defect: we have that

$$\overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} a_*(z + \square_n) - \left(\overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} a_*^{-1}(z + \square_n)\right)^{-1} \leq c \sum_{i=1}^{d} J(\square_m, e_i, Be_i)\mathrm{Id}.$$

A course-grained flux inequality: we have, for every $p \in \mathbb{R}^d$ and $w \in \mathcal{H}(\square_m)$,

$$p \cdot \fint_{\square_m} (a_*(\square_m) - a) \cdot \nabla w \leq (p \cdot (a(U) - a_*(U))p)^{\frac{1}{2}} \left(\fint_{\square_m} \nabla w \cdot a \nabla w\right)^{\frac{1}{2}}.$$

Course-grained energy inequalities: for every $w \in H^1(U)$,

$$\frac{1}{2}\left(\fint_{\square_m} \nabla u\right) \cdot a_*(\square_m)\left(\fint_{\square_m} \nabla u\right) \leq \frac{1}{2} \fint_{\square_m} \nabla u \cdot a \nabla u,$$

and

$$\frac{1}{2}\left(\fint_{\square_m} \nabla u\right) \cdot a^{-1}(\square_m)\left(\fint_{\square_m} \nabla u\right) \leq \frac{1}{2} \fint_{\square_m} \nabla u \cdot a^{-1} \nabla u.$$

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