An algebraic rate of homogenization

Benjamin Fehrman

LSU

10 July 2026

I. Stochastic homogenization

The coefficient field: the map $a\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ satisfies

$$\xi \cdot a\xi \geq \lambda|\xi|^2 \;\text{ and }\; \xi \cdot a^{-1}\xi \geq \Lambda^{-1}|\xi|^2,$$

and, for a measure-preserving transformation group $\{\tau_x\colon \Omega \to \Omega\}_{x\in\mathbb{R}^d}$,

$$a(x+y,\omega) = a(y,\tau_x\omega) \;\text{ for all } x, y \in \mathbb{R}^d \text{ and } \omega \in \Omega.$$

Furthermore, the transformation group is ergodic:

$$f(\omega) = f(\tau_x\omega) \in L^\infty(\Omega) \;\text{ for all } x \in \mathbb{R}^d \text{ if and only if } f \text{ is constant.}$$

The equation: for a fixed environment $\omega \in \Omega$, and $U \subseteq \mathbb{R}^d$,

$$-\nabla \cdot a^\varepsilon \nabla u_\varepsilon = f \;\text{ in }\; U \;\text{ with }\; u_\varepsilon = 0 \;\text{ on }\; \partial U.$$

Qualitative homogenization: there exists $\overline{a} \in \mathbb{R}^{d\times d}$ such that, $\mathbb{P}$-a.s.,

$$u_\varepsilon \rightharpoonup \overline{u} \;\text{ weakly in }\; H^1(U) \;\text{ and }\; u_\varepsilon \to \overline{u} \;\text{ strongly in }\; L^2(U),$$

for $\overline{u}$ solving

$$-\nabla \cdot \overline{a}\nabla\overline{u} = f \;\text{ in } U \text{ with } \overline{u} = 0 \text{ on } \partial U.$$

I. Stochastic homogenization

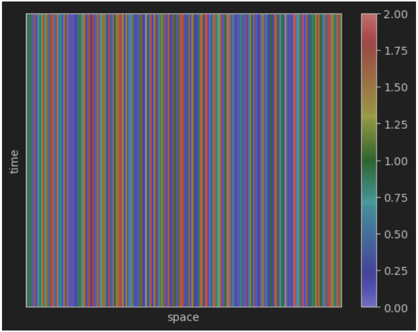
The coefficient field: in $d = 1$,

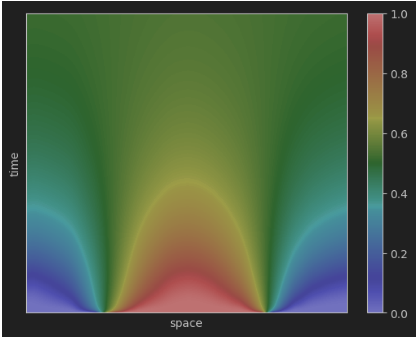
$$a(x,\omega) = 1 + ({}^9\!/\!{}_{10})\sin(2\pi\mathcal{N}^\varepsilon(x,\omega)),$$

for $\mathcal{N} = \sum_{k\in\mathbb{Z}} \mathrm{Unif}_k(0,1)\mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = {}^1\!/\!{}_{200}$ in $\mathbb{T}^1 \times (0,\infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2}\nabla\cdot a^\varepsilon\nabla u_\varepsilon \text{ in } \mathbb{T}^1\times(0,\infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4,3/4\}}.$$





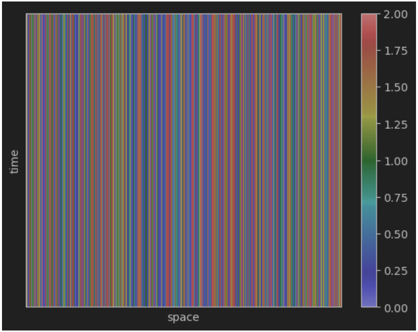
The coefficient field: in $d = 1$,

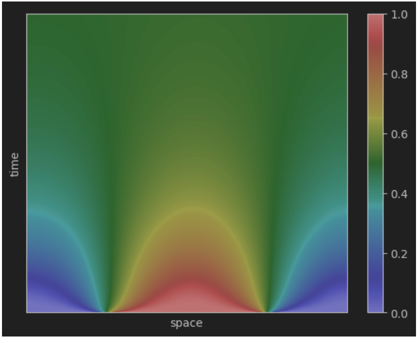
$$a(x,\omega) = 1 + (9/10)\sin(2\pi\mathcal{N}^{\varepsilon}(x,\omega)),$$

for $\mathcal{N} = \sum_{k\in\mathbb{Z}} \mathrm{Unif}_k(0,1)\mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = 1/1000$ in $\mathbb{T}^1 \times (0,\infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2}\nabla\cdot a^{\varepsilon}\nabla u_\varepsilon \text{ in } \mathbb{T}^1\times(0,\infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4,3/4\}}.$$





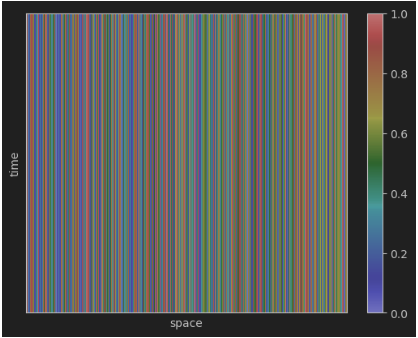
The coefficient field: in $d = 1$,

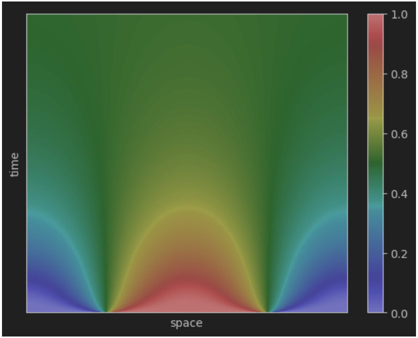
$$a(x, \omega) = 1 + (9/10) \sin(2\pi \mathcal{N}^{\varepsilon}(x, \omega)),$$

for $\mathcal{N} = \sum_{k \in \mathbb{Z}} \mathrm{Unif}_k(0, 1) \mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = 1/1000$ in $\mathbb{T}^1 \times (0, \infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2} \nabla \cdot a^{\varepsilon} \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0, \infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4, 3/4\}}.$$

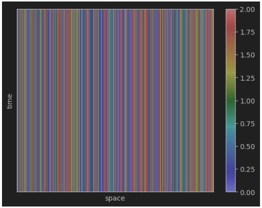


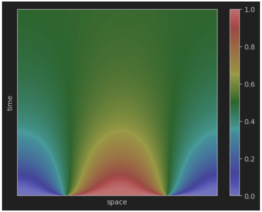


I. Stochastic homogenization

The equation: for $\varepsilon = {}^1/_{1000}$ in $\mathbb{T}^1 \times (0,\infty)$,

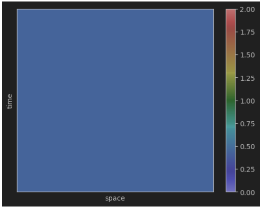
$$\partial_t u_\varepsilon = ({}^1/_2)\nabla \cdot a^\varepsilon \nabla u_\varepsilon \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1/_4,{}^3/_4\}}.$$

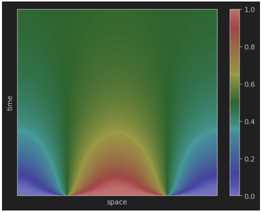




A macroscopic limit: choosing $\overline{a} = \langle a^{-1}\rangle^{-1} \simeq 0.43588$,

$$\partial_t \overline{u} = ({}^1/_2)\nabla \cdot \overline{a}\nabla \overline{u} \text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1/_4,{}^3/_4\}}.$$





II. The subadditive quantities

The finite-volume corrector: for $p \in \mathbb{R}^d$, $\ell_p = x \cdot p$, and for $\square_m = [-{}^{3^m}/{}_2, {}^{3^m}/{}_2]^d$,

$$\mu(\square_m, p) = \inf_{v \in \ell_p + H_0^1(\square_m)} \frac{1}{2} \fint_{\square_m} \nabla v \cdot a \nabla v,$$

where for the minimizer $v_{m,p}$ and $\phi_{m,p} = v_{m,p} - \ell_p$,

$$-\nabla \cdot a(p + \nabla v_{m,p}) = 0 \text{ in } \square_m \text{ with } \phi_{m,p} = 0 \text{ on } \partial \square_m.$$

The "dual" quantity: we have that

$$\mu_*(\square_m, q) = \sup_{w \in H^1(U)} \fint_{\square_m} -\nabla w \cdot a \nabla + q \cdot w,$$

where the minimzer $w_{m,q}$ satisfies

$$-\nabla \cdot a \nabla w_{m,q} = 0 \text{ in } \square_m \text{ with } a \nabla w_{m,q} \cdot \nu = q \cdot \nu \text{ on } \partial \square_m.$$

The coarse-grained matrices: we have that

$$\mu(\square_m, p) = \frac{1}{2}\big(p \cdot a(\square_m) p\big) \text{ and } \mu_*(\square_m, q) = \frac{1}{2}\big(q \cdot a_*(U)^{-1} q\big),$$

where $\overline{a} = \lim_{m \to \infty} a_*(\square_m)$.

II. The subadditive quantities

The master quantity: for every $p, q \in \mathbb{R}^d$ and $m \in \mathbb{N}_0$,

$$
\begin{aligned}
J(\square_m, p, q) &= \mu(\square_m, p) + \mu_*(\square_m, q) - p \cdot q \\
&= \frac{1}{2}\left(p \cdot a(\square_m) p\right) + \frac{1}{2}\left(q \cdot a_*(\square_m)^{-1} q\right) - p \cdot q \\
&= \frac{1}{2}\left(p \cdot (a(\square_m) - a_*(\square_m)) p\right) + \frac{1}{2}\left((q - a_*(\square_m)) p \cdot a_*(U)^{-1} (q - a_*(\square_m)) p\right).
\end{aligned}
$$

Locality and subadditivity: $J(U, p, q)$ is $\mathcal{F}(U)$-measurable and, if $U = \coprod U_i$,

$$
J(U, p, q) \leq \sum_{i=1}^{n} \frac{|U_i|}{|U|} J(U_i, p, q).
$$

Controlling $a - a_*$ using the convex duality defect: we have that

$$
|a(\square_m) - a_*(\square_m)| + |B - a_*(\square_m)|^2 + |B - a(\square_m)|^2 \leq c \sum_{i=1}^{d} J(\square_m, e_i, B e_i).
$$

Sublinearity of correctors: for the finite-volume correctors,

$$
\begin{aligned}
&\limsup_{m \to \infty} 3^{-m} \left(\|\phi_{m,i}\|_{\underline{L}^2(\square_m)} + \|\sigma_{m,ijk}\|_{\underline{L}^2(\square_m)} \right) \\
&\leq \limsup_{m \to \infty} c \left(3^{-m} + \sum_{k=0}^{m} 3^{k-m} \left(|\overline{\sum}_{z \in 3^n \mathbb{Z}^d \cap \square_m} a(z + \square_n) - \overline{a}| \right)^{\frac{1}{2}} \right).
\end{aligned}
$$

III. A first quantitative rate

Subadditivity: if $n \leq m$ by $\mathbb{Z}^d$-stationarity,

$$J(\square_m, p, q) \leq \overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} J(z + \square_n, p, q)$$
$$= \mathbb{E}[J(\square_n, p, q)] + \overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} \big(J(z + \square_n, p, q) - \mathbb{E}[J(\square_n, p, q)]\big).$$

The mixing condition: a finite range of dependence, for $U, V \subseteq \mathbb{R}^d$,

$$\mathrm{d}(U, V) \geq 1 \ \text{ implies that } \mathcal{F}_U \text{ and } F_V \text{ are independent,}$$

for

$$\mathcal{F}_U = \sigma\Big(\omega \to \int_{\mathbb{R}^d} a(x,\omega)\psi(x) \colon \psi \in \mathrm{C}_c^\infty(U)\Big).$$

Other choices: spectral grap, logarithmic Sobolev-inequality, and more...

The stochastic error: we will first analyze

$$\overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} \big(J(z + \square_n, p, q) - \mathbb{E}[J(\square_n, p, q)]\big).$$

III. A first quantitative rate

Azuma's inequality: let $(M_n)_{n\in\mathbb{N}_0}$ be a martingale with $M_0 = 0$ and

$$|M_n - M_{n-1}| \leq c \;\text{ for every } n \in \mathbb{N}.$$

Then,

$$\mathbb{P}\big(M_n > \lambda\big) \leq \exp\Big(-\frac{\lambda^2}{2nc^2}\Big).$$

In particular,

$$\mathbb{P}\big({}^{M_n}\!/_{n} > \lambda\big) \leq \exp\Big(-\frac{n\lambda^2}{2c^2}\Big)$$

Proof assuming independent increments: for $\tilde{M}_n = {}^{M_n}\!/\!/_{c}$,

$$\begin{aligned}\mathbb{P}\big(\tilde{M}_n > \lambda\big) &= \mathbb{P}\Big(\exp(s\tilde{M}_n) > \exp(s\lambda)\Big)\\ &\leq e^{-s\lambda}\prod_{i=1}^{n}\mathbb{E}[\exp(s(\tilde{M}_n - \tilde{M}_{n-1}))]\\ &\leq e^{-s\lambda}\mathbb{E}[\exp(s(\tilde{M}_n - \tilde{M}_{n-1}))]^n\\ &\leq e^{-s\lambda + s^2 n/2}.\end{aligned}$$

Choosing $s = {}^{\lambda}\!/_{n}$,

$$\mathbb{P}\big(\tilde{M}_n > \lambda\big) \leq \exp\Big(-\frac{\lambda^2}{2n}\Big) \text{ and, so, } \mathbb{P}\big(\tilde{M}_n > \lambda\big) \leq \exp\Big(-\frac{\lambda^2}{2nc^2}\Big).$$

Azuma's inequality: let $(M_n)_{n\in\mathbb{N}_0}$ be a martingale with $M_0=0$ and

$$|M_n - M_{n-1}| \leq c \;\text{ for every } n\in\mathbb{N}.$$

Then,

$$\mathbb{P}\big(M_n>\lambda\big)\leq \exp\Big(-\frac{\lambda^2}{2nc^2}\Big).$$

In particular,

$$\mathbb{P}\big({}^{M_n}\!/_{n}>\lambda\big)\leq \exp\Big(-\frac{n\lambda^2}{2c^2}\Big)$$

. **The stochastic error**: for the error term,

$$\overline{\sum}_{z\in 3^n\mathbb{Z}^d\cap\square_m}\big(J(z+\square_n,p,q)-\mathbb{E}[J(\square_n,p,q)]\big).$$

using that $|J(z+\square_n,p,q)-\mathbb{E}[J(\square_n,p,q)]\big)|\leq c(|p|^2+|q|^2)(1+\Lambda)=\overline{c}_{p,q}$,

$$\mathbb{P}\Big[\Big(\overline{\sum}_{z\in 3^n\mathbb{Z}^d\cap\square_m}\big(J(z+\square_n,p,q)-\mathbb{E}[J(\square_n,p,q)]\big)\Big)>3^{-\frac{d}{2}(m-n)}\lambda\Big]$$
$$\leq \exp\Big(-\frac{\lambda^2}{2\overline{c}_{p,q}^2}\Big).$$

III. A first quantitative rate

Subadditivity: if $n \le m$ by $\mathbb{Z}^d$-stationarity,

$$J(\square_m, p, q) \le \overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} J(z + \square_n, p, q)$$
$$= \mathbb{E}[J(\square_n, p, q)] + \overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} \big(J(z + \square_n, p, q) - \mathbb{E}[J(\square_n, p, q)] \big).$$

The stochastic error: we have that

$$\mathbb{P}\Big[\Big(\overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} \big(J(z + \square_n, p, q) - \mathbb{E}[J(\square_n, p, q)] \big) \Big) > 3^{-\frac{d}{2}(m-n)} \lambda \Big]$$
$$\le \exp\Big(-\frac{\lambda^2}{2\overline{c}_{p,q}^2} \Big),$$

where, since $3^{-m} = \varepsilon$,

$$3^{-\frac{d}{2}(m-n)} = \varepsilon^{\frac{d(m-n)}{2m}}.$$

The "deterministic" error: it remains to bound

$$\mathbb{E}[J(\square_n, p, q)].$$

III. A first quantitative rate

The averaged coarse-grained matrices: for every $m \in \mathbb{N}$,

$$\overline{a}(\square_m) = \mathbb{E}[a(\square_m)] \text{ and } \overline{a}_*(\square_m) = \mathbb{E}[a_*(\square_m)]^{-1}.$$

Monotonicity: due to the subbadditivity of μ and μ_*,

$$\overline{a}(\square_{m+1}) \leq \overline{a}(\square_m) \text{ and } \overline{a}_*(\square_m) \leq \overline{a}_*(\square_{m+1}).$$

The homogenized matrix: we define $\overline{a} \in \mathbb{R}^{d\times d}$ by

$$\overline{a} = \lim_{m\to\infty} a_*(\square_m).$$

Algebraic rate of homogenization: for constants depending on α, λ, and Λ,

$$|a(\square_m) - \overline{a}| + |a_*(\square_m) - \overline{a}| \leq c3^{-m\alpha} \simeq c\varepsilon^{\alpha}.$$

Expectation of the master quantity: for every $p, q \in \mathbb{R}^d$ and $m \in \mathbb{N}_0$,

$$\begin{aligned}
&\mathbb{E}[J(\square_m, p, q)] = \mathbb{E}[\mu(\square_m, p)] + \mathbb{E}[\mu_*(\square_m, q)] - p\cdot q \\
&= \frac{1}{2}\big(p\cdot \overline{a}(\square_m)p\big) + \frac{1}{2}\big(q\cdot \overline{a}_*(\square_m)^{-1}q\big) - p\cdot q \\
&= \frac{1}{2}\big(p\cdot(\overline{a}(\square_m) - \overline{a}_*(\square_m))p\big) + \frac{1}{2}\big((q - \overline{a}_*(\square_m))p\cdot \overline{a}_*(\square_m)^{-1}(q - \overline{a}_*(\square_m))p\big).
\end{aligned}$$

III. A first quantitative rate

Algebraic rate of homogenization: for constants depending on α, λ, and Λ,

$$|a(\square_m) - \overline{a}| + |a_*(\square_m) - \overline{a}| \leq c3^{-m\alpha} \simeq c\varepsilon^{\alpha}.$$

Expectation of the master quantity: for every $p, q \in \mathbb{R}^d$ and $m \in \mathbb{N}_0$,

$$\begin{aligned}
&\mathbb{E}[J(\square_m, p, q)] = \mathbb{E}[\mu(\square_m, p)] + \mathbb{E}[\mu_*(\square_m, q)] - p \cdot q \\
&= \frac{1}{2}\big(p \cdot (\overline{a}(\square_m) - \overline{a}_*(\square_m))p\big) + \frac{1}{2}\big((q - \overline{a}_*(\square_m))p \cdot \overline{a}_*(\square_m)^{-1}(q - \overline{a}_*(\square_m))p\big),
\end{aligned}$$

where, if $q = \overline{a}_*(\square_m)p$,

$$\begin{aligned}
\mathbb{E}[J(\square_m, p, \overline{a}_*(\square_m)p)] &= \frac{1}{2}\big(p \cdot (\overline{a}(\square_m) - \overline{a}_*(\square_m))p\big) \\
&= \frac{1}{2}\big(p \cdot (\overline{a}(\square_m) - \overline{a})p\big) + \frac{1}{2}\big(p \cdot (\overline{a} - \overline{a}_*(\square_m))p\big).
\end{aligned}$$

Bounding $\overline{a} - \overline{a}_*(\square_m)$ with J from above: we have that

$$\begin{aligned}
&|\overline{a}(\square_m) - \overline{a}| + |\overline{a} - \overline{a}_*(\square_m)| \\
&\leq c \max_{|p|=1} \left(\frac{1}{2}\big(p \cdot (\overline{a}(\square_m) - \overline{a})p\big) + \frac{1}{2}\big(p \cdot (\overline{a} - \overline{a}_*(\square_m))p\big)\right) \\
&\leq c \sup_{|p|\leq 1} \mathbb{E}[J(\square_m, p, \overline{a}_*(\square_m)p)] \\
&\leq c \sup_{|p|\leq 1} \mathbb{E}[J(\square_m, p, \overline{a}p)].
\end{aligned}$$

III. A first quantitative rate

Algebraic rate of homogenization: for constants depending on α, λ, and Λ,

$$|a(\square_m) - \overline{a}| + |a_*(\square_m) - \overline{a}| \leq c3^{-m\alpha} \simeq c\varepsilon^{\alpha}.$$

Expectation of the master quantity: for every $q = \overline{a}p \in \mathbb{R}^d$,

$$\begin{aligned}
&\mathbb{E}[J(\square_m, p, \overline{a}p)] = \mathbb{E}[\mu(\square_m, \overline{a}p)] + \mathbb{E}[\mu_*(\square_m, \overline{a}p)] - p \cdot \overline{a}p \\
&= \frac{1}{2}\big(p \cdot (\overline{a}(\square_m) - \overline{a}_*(\square_m))p\big) + \frac{1}{2}\big((\overline{a} - \overline{a}_*(\square_m))p \cdot \overline{a}_*(\square_m)^{-1}(\overline{a} - \overline{a}_*(\square_m))p\big).
\end{aligned}$$

Bounding $\overline{a} - \overline{a}_*(\square_m)$ with J from below: we have that

$$\begin{aligned}
&\sup_{|p|\leq 1} \mathbb{E}[J(\square_m, p, \overline{a}p)] \\
&\leq \sup_{|p|\leq 1} \left(\frac{1}{2}\big(p \cdot (\overline{a}(\square_m) - \overline{a}_*(\square_m))p\big) + \frac{1}{2}\big((\overline{a} - \overline{a}_*(\square_m))p \cdot \overline{a}_*(\square_m)^{-1}(\overline{a} - \overline{a}_*(\square_m))p\big)\right) \\
&\leq c\big(|\overline{a}(\square_m) - \overline{a}_*(\square_m)| + |\overline{a} - \overline{a}_*(\square_m)|\big) \\
&\leq c\big(|\overline{a}(\square_m) - \overline{a}| + |\overline{a} - \overline{a}_*(\square_m)|\big).
\end{aligned}$$

Conclusion: for constants depending on Λ,

$$c\big(|\overline{a}(\square_m) - \overline{a}| + |\overline{a} - \overline{a}_*(\square_m)|\big) \leq \sup_{|p|\leq 1} \mathbb{E}[J(\square_m, p, \overline{a}p)] \leq C\big(|\overline{a}(\square_m) - \overline{a}| + |\overline{a} - \overline{a}_*(\square_m)|\big).$$

Algebraic rate of homogenization: for constants depending on α, λ, and Λ,

$$|a(\square_m) - \overline{a}| + |a_*(\square_m) - \overline{a}| \leq c3^{-m\alpha} \simeq c\varepsilon^{\alpha}.$$

The additivity defect: we define the quantity

$$\tau_m = |\overline{a}(\square_m) - \overline{a}(\square_{m-1})| + |\overline{a}_*(\square_m) - \overline{a}_*(\square_m)|,$$

where, by the monotonicity of the matrices and

$$\mathbb{E}[J(\square_m, p, \overline{a}_*(\square_m)p)] = \frac{1}{2}\big(p \cdot (\overline{a}(\square_m) - \overline{a}_*(\square_m))p\big),$$

we have that

$$\begin{aligned}
\tau_m &\leq c\sum_{i=1}^{d} e_i \cdot \big(\overline{a}(\square_{m-1}) - \overline{a}(\square_{m-1}) + \overline{a}_*(\square_m) - \overline{a}^*(\square_{m-1})\big) \cdot e_i \\
&\leq c\sum_{i=1}^{d} \mathbb{E}[J(\square_{m-1}, e_i, \overline{a}_*(\square_{m-1})e_i)] - \mathbb{E}[J(\square_m, e_i, \overline{a}_*(\square_m)e_i)] \\
&\leq c\tau_{m-1}.
\end{aligned}$$

III. A first quantitative rate

Algebraic rate of homogenization: for constants depending on α, λ, and Λ,

$$|a(\square_m) - \overline{a}| + |a_*(\square_m) - \overline{a}| \leq c3^{-m\alpha} \simeq c\varepsilon^{\alpha}.$$

Control of J: we will show that

$$\mathbb{E}[J(\square_m, e, \overline{a}_*(\square_m)e)] \leq c\sum_{n=0}^{m} 3^{n-m}(\tau_n + \mathrm{var}(\overline{a}_*^{-1}(\square_n)) + c3^{-2m}.$$

for

$$\tau_m = |\overline{a}(\square_m) - \overline{a}(\square_{m-1})| + |\overline{a}_*(\square_m) - \overline{a}_*(\square_m)|.$$

The energy identity: using $J(\square_m, p.q) = \frac{1}{2}\fint_{\square_m} \nabla v_{m,p,q} \cdot a\nabla v_{m,p.q}$,

$$\begin{aligned} J(\square_{m-1}, p, q) \leq c3^{-2m} &\min_{k\in\mathbb{R}} \|v_{m,p,q} - k\|^2_{\underline{L}^2(\square_m)} \\ &+ \sum_{z\in 3^{m-1}\mathbb{Z}^d\cap\square_m} 2(J(z+\square_{m-1}, p, q) - J(\square_m, p, q)). \end{aligned}$$

The multiscale Poincaré inequality: we have that

$$\begin{aligned} 3^{-m}\min_{k\in\mathbb{R}} \|v_{m,p,q} - k\|_{\underline{L}^2(\square_m)} \leq c3^{-m} &+ c\sum_{n=0}^{m} 3^{n-m}\left(\overline{\sum_y}(J(y+\square_n, p, q) - J(\square_m, p, q)\right)^{\frac{1}{2}} \\ &+ c\sum_{n=0}^{m} 3^{n-m}\left(\overline{\sum_y}|a_*^{-1}(y+\square_n)q - p|^2\right)^{\frac{1}{2}}. \end{aligned}$$

III. A first quantitative rate

Algebraic rate of homogenization: for constants depending on α, λ, and Λ,

$$|a(\square_m) - \overline{a}| + |a_*(\square_m) - \overline{a}| \leq c3^{-m\alpha} \simeq c\varepsilon^{\alpha}.$$

Variance of the course-grained matrices: we will show that, for symmetric B,

$$\begin{aligned}&\operatorname{var}[a(\square_m)] + \operatorname{var}(a_*^{-1}(\square_m)] \\ &\leq c\min\Big\{\sup_{|p|=1}\mathbb{E}[J(\square_n, e, Be)]^2, \sum_{k=n+1}^{M}\tau_k\Big\} + c(1+|B|^4)3^{-d(m-n)},\end{aligned}$$

using that, by the triangle inequality,

$$\operatorname{var}[a(\square_m)] \leq 2\operatorname{var}\Big[\overline{\sum}_z a(z+\square_n)\Big] + 2\mathbb{E}\Big[\big|a(\square_m) - \overline{\sum}_z a(z+\square_n)\big|^2\Big].$$

Azuma's inequality: integrating Azuma's inequality,

$$\operatorname{var}\Big[\overline{\sum}_z a(z+\square_n)\Big] \leq c3^{-d(m-n)}.$$

The conclusion: we have that

$$\mathbb{E}\Big[\big|a(\square_m) - \overline{\sum}_z a(z+\square_n)\big|^2\Big] \leq c\min\Big\{\sup_{|p|=1}\mathbb{E}[J(\square_n, e, Be)]^2, \sum_{k=n+1}^{M}\tau_k\Big\}.$$

III. A first quantitative rate

Algebraic rate of homogenization: for constants depending on α, λ, and Λ,

$$|a(\square_m) - \overline{a}| + |a_*(\square_m) - \overline{a}| \leq c3^{-m\alpha} \simeq c\varepsilon^{\alpha}.$$

The iteration: we have that

$$\begin{aligned}
&\sup_{|p|\leq 1} \mathbb{E}[J(\square_m, e, \overline{a}_*(\square_m)] \\
&\leq c\sum_{j=n}^{m}(\tau_j + \mathrm{var}[a_*^{-1}(\square_j)]) + c3^{-(m-n)} \\
&\leq c\min\Big\{\sum_{j=k}^{m}\tau_j, \sup_{|p|\leq 1}\mathbb{E}[J(\square_k, e, \overline{a}_*(\square_k)e]^2\Big\} \\
&\quad + c\big(3^{-(m-n)} + 3^{-d(n-k)}\big) + c\sum_{j=n}^{m}3^{j-m}\tau_j.
\end{aligned}$$

A logarithmic rate: we have that

$$\mathbb{E}[\sup_{|p|\leq 1}\mathbb{E}[J(\square_n, p, \overline{a}_*(\square_n)p)] \leq \frac{c\log(n)}{n} + \exp\Big(-\frac{(1-\beta)n}{c}\Big).$$

III. A first quantitative rate

Subadditivity: if $n \leq m$ by $\mathbb{Z}^d$-stationarity,

$$J(\square_m, p, q) \leq \overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} J(z + \square_n, p, q)$$
$$= \mathbb{E}[J(\square_n, p, q)] + \overline{\sum}_{z \in 3^n\mathbb{Z}^d \cap \square_m} \big(J(z + \square_n, p, q) - \mathbb{E}[J(\square_n, p, q)]\big).$$

Equivalence of J and the averaged matrices: for constants depending on Λ,

$$c\big(|\overline{a}(\square_m) - \overline{a}| + |\overline{a} - \overline{a}_*(\square_m)|\big) \leq \sup_{|p| \leq 1} \mathbb{E}[J(\square_m, p, \overline{a}p)] \leq C\big(|\overline{a}(\square_m) - \overline{a}| + |\overline{a} - \overline{a}_*(\square_m)|\big).$$

An algebraic rate for the coarse-grained matrices

There exists $\alpha \in (0, 1)$ and $c \in (0, \infty)$ depending on d, λ, and Λ such that, for a constant $\overline{c}$ satisfying, for all $\beta \geq 1$,

$$\mathbb{P}(\overline{c} > \beta) \leq \exp\big(-\frac{\beta^2}{c}\big),$$

we have that

$$|a(\square_m) - a_*(\square_m)| + |a(\square_m) - \overline{a}|^2 + |a_*(\square_m) - \overline{a}|^2 \leq c3^{-n\alpha} + \overline{c}3^{-\frac{d}{2}(m-n)}.$$

Algebraic rate: equilibrating in n yields a rate of order

$$3^{-m\alpha} = e^{\alpha}.$$

An algebraic rate of stochastic homogenization

There exists a constant $c \in (0,\infty)$ and $\alpha \in (0,1)$ depending on d, U, λ, and Λ such that, for a constant $\overline{c} \in (0,\infty)$ depending on d, λ, and Λ satisfying, for all $\beta \geq 1$,

$$\mathbb{P}(\overline{c} \geq \beta) \leq e^{-\frac{\beta^2}{c}},$$

such that, for the solutions u_ε satisfying

$$-\nabla \cdot a^\varepsilon u_\varepsilon = f \;\text{ in } U \text{ with } u_\varepsilon = 0 \text{ on } \partial U,$$

and for $\overline{u}$ satisfying

$$-\nabla \cdot \overline{a}\nabla\overline{u} = f \;\text{ in } U \text{ with } u_\varepsilon = 0 \text{ on } \partial U,$$

we have that, for every $0 < r << 1$,

$$\begin{aligned} &\|\nabla u_\varepsilon - \nabla u\|_{H^{-1}(U)} + \|a^\varepsilon \nabla u^\varepsilon - \overline{a}\nabla\overline{u}\|_{H^{-1}(U)} \\ &\leq c\,\|\nabla\overline{u}\|_{L^2(U\setminus U_r)} + r^{-1}\overline{c}\varepsilon^\alpha \,\|\nabla\overline{u}\|_{L^\infty(U_r)}. \end{aligned}$$

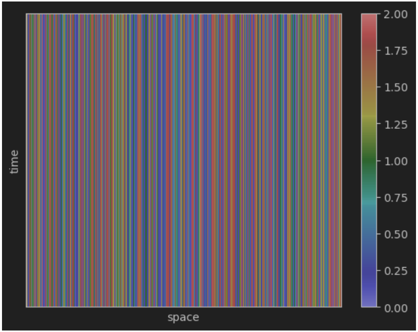
The coefficient field: in $d = 1$,

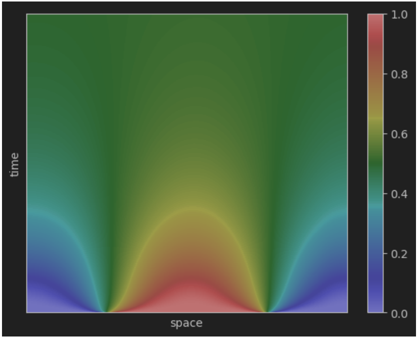
$$a(x,\omega) = 1 + (9/10)\sin(2\pi\mathcal{N}^{\varepsilon}(x,\omega)),$$

for $\mathcal{N} = \sum_{k\in\mathbb{Z}} \mathrm{Unif}_k(0,1)\mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = 1/1000$ in $\mathbb{T}^1 \times (0,\infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2}\nabla\cdot a^{\varepsilon}\nabla u_\varepsilon \ \text{ in } \mathbb{T}^1\times(0,\infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4,3/4\}}.$$





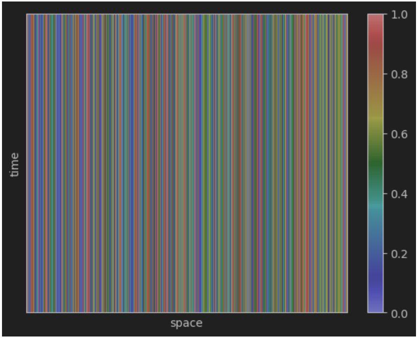
The coefficient field: in $d = 1$,

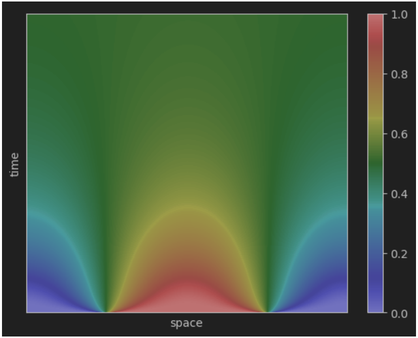
$$a(x, \omega) = 1 + (9/10) \sin(2\pi \mathcal{N}^{\varepsilon}(x, \omega)),$$

for $\mathcal{N} = \sum_{k \in \mathbb{Z}} \mathrm{Unif}_k(0,1) \mathbf{1}_{[k,k+1)}$.

The equation: for $\varepsilon = 1/1000$ in $\mathbb{T}^1 \times (0, \infty)$,

$$\partial_t u_\varepsilon = \frac{1}{2} \nabla \cdot a^{\varepsilon} \nabla u_\varepsilon \ \text{ in } \mathbb{T}^1 \times (0, \infty) \text{ with } u_0 = \mathbf{1}_{\{(1/4, 3/4\}}.$$

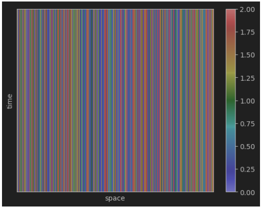


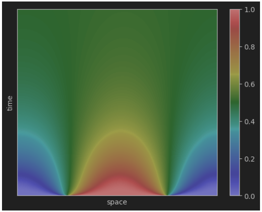


I. Stochastic homogenization

The equation: for $\varepsilon = {}^1\!/_{1000}$ in $\mathbb{T}^1 \times (0,\infty)$,

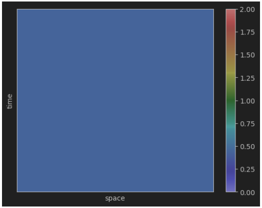
$$\partial_t u_\varepsilon = ({}^1\!/_2)\nabla \cdot a^\varepsilon \nabla u_\varepsilon \;\text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1\!/_4, {}^3\!/_4\}}.$$

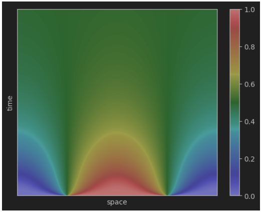




A macroscopic limit: choosing $\overline{a} = \langle a^{-1}\rangle^{-1} \simeq 0.43588$,

$$\partial_t \overline{u} = ({}^1\!/_2)\nabla \cdot \overline{a} \nabla \overline{u} \;\text{ in } \mathbb{T}^1 \times (0,\infty) \text{ with } u_0 = \mathbf{1}_{\{({}^1\!/_4, {}^3\!/_4\}}.$$





V. References

S. Armstrong and T. Kuusi
Elliptic homogenization from qualitative to quantitative
arXiv:2210.06488, 2024.

S. Armstrong and T. Kuusi and J.-C. Mourrat
Quantitative stochastic homogenization and large-scale regularity
Grundlehren der Mathematischen WissenschaftenSpringer, 352, 2019.

S. Armstrong and C. Smart
Quantitative stochastic homogenization of convex integral functionals
Ann. Sci. Éc. Norm. Supér., 49(2):423–481, 2016.