Diffusions in random environment and stochastic homogenization

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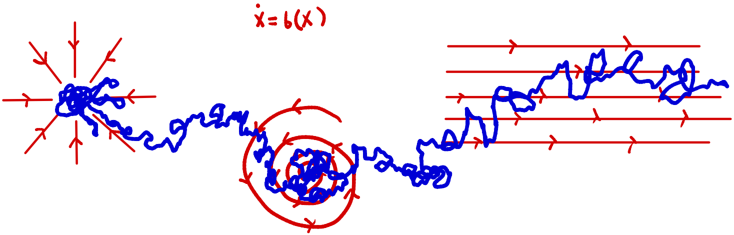
I. Diffusion processes and PDEs

In these talks, we will consider the longtime behavior of a diffusion process

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t \ \text{ for } \ t \in (0, \infty).$$

- σ quantifies the diffusion
  — thermal fluctuations / microscopic collisions driving a Brownian particle
- b quantifies the drift
  — mean macroscopic motion / wind or current in a fluid flow



If X_t solves

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t \ \text{ for } \ t \in (0,\infty),$$

the central limit scaling $X^\varepsilon = \varepsilon X_{t/\varepsilon^2}$ solves

$$\mathrm{d}X^\varepsilon_t = \sigma({}^{X^\varepsilon_t}/_\varepsilon)\,\mathrm{d}W^\varepsilon_t + \varepsilon^{-1} b({}^{X^\varepsilon_t}/_\varepsilon)\,\mathrm{d}t.$$

What happens (in law) as $\varepsilon \to 0$?

- (*diffusive*) If $\sigma = I$ and $b = 0$ then, in law for every $\varepsilon \in (0,1)$,

$$X^\varepsilon_t = B_t.$$

- (*ballistic*) If $\sigma = I$ and $b = \overline{b} \in \mathbb{R}^d \setminus \{0\}$ then, in law for every $\varepsilon \in (0,1)$,

$$X^\varepsilon_t = B_t + \varepsilon^{-1} t\overline{b} \ \text{ and almost surely } \ |X^\varepsilon_t| \to \infty \ \text{ as } \ \varepsilon \to 0.$$

- (*degenerate / trapped*) For the Ornstein-Uhlenbeck process

$$\mathrm{d}X_t = \mathrm{d}B_t - X_t\,\mathrm{d}t \ \text{ and } \ \mathrm{d}X^\varepsilon_t = \mathrm{d}W^\varepsilon_t - \varepsilon^{-2} X^\varepsilon_t\,\mathrm{d}t,$$

  and

$$|X^\varepsilon_t| \to 0 \ \text{ almost surely as } \varepsilon \to 0.$$

I. Diffusion processes and PDEs

We are interested in the behavior of X_t in *law*.

If X_t solves

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t \;\text{ for }\; t \in (0, \infty),$$

how can we characterize

$$\mathbb{P}[X_t \in A] \;\text{ for every measurable }\; A \subseteq \mathbb{R}^d?$$

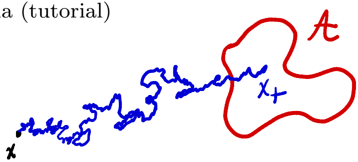
The *Feynman-Kac Formula*: if ρ solves the equation

$$\partial_t \rho = \operatorname{tr}(a\nabla^2\rho) + b \cdot \nabla\rho \;\text{ in }\; \mathbb{R}^d \times (0, \infty) \;\text{ with }\; \rho(\cdot, 0) = \rho_0,$$

for the covariance matrix $a = \frac{1}{2}\sigma\sigma^2$, then we have the formula

$$\rho(x, t) = \mathbb{E}_x\left[\rho_0(X_t)\right].$$

- the heat equation and Brownian motion
- the solution is the average of the initial data with respect to the diffusion
  — regularizing / smoothing properties of parabolic equations
- proof using Itô's formula (tutorial)



I. Diffusion processes and PDEs

In the central limit scaling $X_t^\varepsilon = \varepsilon X_{t/\varepsilon^2}$,

$$\mathrm{d}X_t^\varepsilon = \sigma(X_t^\varepsilon/\varepsilon)\,\mathrm{d}W_t^\varepsilon + \varepsilon^{-1} b(X_t^\varepsilon/\varepsilon)\,\mathrm{d}t,$$

and the solution ρ^ε of the equation

$$\partial_t \rho^\varepsilon = \mathrm{tr}(a(x/\varepsilon)\nabla^2\rho^\varepsilon) + \varepsilon^{-1} b(x/\varepsilon)\cdot\nabla\rho^\varepsilon \ \text{ in } \ \mathbb{R}^d \times (0,\infty) \ \text{ with } \ \rho(\cdot,0) = \rho_0,$$

for $a = \frac{1}{2}\sigma\sigma^2$ satisfies

$$\rho(x,t) = \mathbb{E}_x\left[\rho_0(X_t^\varepsilon)\right] = \mathbb{E}_{\frac{x}{\varepsilon}}\left[\rho_0(\varepsilon X_{t/\varepsilon^2})\right].$$

- (*diffusive*) If $\sigma = I$ and $b = 0$ then, for every $\varepsilon \in (0,1)$,

$$\rho^\varepsilon = \overline{\rho} \ \text{ for } \ \partial_t\overline{\rho} = \frac{1}{2}\Delta\overline{\rho}.$$

- (*ballistic*) If $\sigma = I$ and $b = \overline{b} \in \mathbb{R}^d \setminus \{0\}$ then

$$\left(\lim_{\varepsilon\to 0} \rho^\varepsilon(x,t)\right) = \left(\lim_{s\to\infty} \rho_0(x + s\overline{b})\right).$$

- (*degenerate / trapped*) In the case of the Ornstein-Uhlenbeck process,

$$\partial_t\rho^\varepsilon = \frac{1}{2}\Delta\rho^\varepsilon - \varepsilon^{-2} x\cdot\nabla\rho^\varepsilon,$$

  and $(\lim_{\varepsilon\to 0}\rho^\varepsilon(x,t)) = \rho_0(0)$.

I. Diffusion processes and PDEs

Characterizing the limiting behavior, as $\varepsilon \to 0$, of the solution

$$\mathrm{d}X_t^\varepsilon = \sigma(X_t^\varepsilon/\varepsilon)\,\mathrm{d}W_t^\varepsilon + \varepsilon^{-1} b(X_t^\varepsilon/\varepsilon)\,\mathrm{d}t,$$

in law is equivalent to characterizing the limiting behavior, as $\varepsilon \to 0$, of the solutions

$$\partial_t \rho^\varepsilon = \mathrm{tr}(a(x/\varepsilon)\nabla^2 \rho^\varepsilon) + \varepsilon^{-1} b(x/\varepsilon) \cdot \nabla \rho^\varepsilon,$$

for arbitrary smooth initial data.

- The Feynan-Kac formula:

$$\rho^\varepsilon(x,t) = \mathbb{E}_x\left[\rho_0(X_t^\varepsilon)\right].$$

- As $\varepsilon \to 0$, we have $X^\varepsilon \to \overline{X}$ in law, for $\overline{X}$ solving

$$\mathrm{d}\overline{X}_t = \overline{\sigma}\,\mathrm{d}B_t \ \text{ for some } \ \overline{\sigma} \in \mathbb{R}^{d\times d},$$

  if and only if we have $\rho^\varepsilon \to \overline{\rho}$, for $\overline{\rho}$ solving

$$\partial_t \overline{\rho} = \mathrm{tr}(\overline{a}\nabla^2 \overline{\rho}) \ \text{ for } \ \overline{a} = \frac{1}{2}\overline{\sigma}\overline{\sigma}^t.$$

- The divergence-form case / a reversible diffusion:

$$-\nabla \cdot (a(x/\varepsilon)\nabla \rho^\varepsilon) = -\,\mathrm{tr}(a(x/\varepsilon)\nabla^2 \rho^\varepsilon) - \varepsilon^{-1}(\nabla \cdot a^t(x/\varepsilon)) \cdot \nabla \rho^\varepsilon.$$

II. Ergodic properties of diffusions on the torus

We will restrict (for now) to periodic coefficient fields.

- For 1-periodic coefficients σ and b, we have the diffusion X on $\mathbb{R}^d$:

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t \;\text{ in }\; \mathbb{R}^d.$$

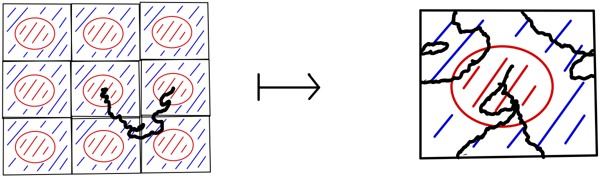
  Lift this to a diffusion $\overline{X}$ on the torus $\mathbb{T}^d$:

$$\mathrm{d}\overline{X}_t = \sigma(\overline{X}_t)\,\mathrm{d}B_t + b(\overline{X}_t)\,\mathrm{d}t \;\text{ in }\; \mathbb{T}^d.$$

- For $\rho_0 \in \mathrm{C}^\infty(\mathbb{T}^d)$, the function

$$\rho(x,t) = (\overline{P}_t\rho_0)(x) = \mathbb{E}_x[\rho_0(\overline{X}_t)] \;\text{ solves }\; \partial_t\rho = \mathrm{tr}(a\nabla^2\rho) + b\cdot\nabla\rho \;\text{ in }\; \mathbb{T}^d.$$

- What are the averaging / ergodic properties of the semigroup $\overline{P}_t$?



II. Ergodic properties of diffusions on the torus

The invariant measure [Section 3.2, Asym. Anal. for Per. Struct.]

Assume that σ and b are sufficiently regular, and assume that $a = \frac{1}{2}\sigma\sigma^t$ is uniformly elliptic: there exist $\lambda \leq \Lambda \in (0, \infty)$ such that

$$\lambda\,|\xi|^2 \leq \langle a(x)\xi, \xi\rangle \leq \Lambda\,|\xi|^2 \;\text{ for every }\; x \in \mathbb{T}^d \;\text{ and }\; \xi \in \mathbb{R}^d.$$

Then there exists a unique probability measure π on $\mathbb{T}^d$ and constants $c, \rho \in (0, \infty)$ such that, for every $f \in L^\infty(\mathbb{T}^d)$,

$$\sup_{x \in \mathbb{T}^d} \left| \mathbb{E}_x\left[f(\overline{X}_t)\right] - \int_{\mathbb{T}^d} f(y)\pi(\,\mathrm{d}y)\right| \leq c\,\|f\|_{L^\infty(\mathbb{T}^d)}\exp(-\rho t).$$

- *uniform ellipticity* yields exponential convergence to the invariant distribution
- The semigroup $\overline{P}_t$ on functions defines an adjoint semigroup on $\overline{P}_t^*$ on measures:

$$\int_{\mathbb{T}^d} f(y)(\overline{P}_t^*\mu)(\,\mathrm{d}y) := \int_{\mathbb{T}^d} \overline{P}_t f(y)\mu(\,\mathrm{d}y) = \int_{\mathbb{T}^d} \mathbb{E}_y[f(\overline{X}_t)]\mu(\,\mathrm{d}y).$$

- *Invariance*: we have that $(\overline{P}_t^*\pi) = \pi$ for every $t \in [0, \infty)$, since

$$\int_{\mathbb{T}^d} \overline{P}_t f(y)\pi(\,\mathrm{d}y) = \int_{\mathbb{T}^d} f(y)\pi(\,\mathrm{d}y) \;\text{ for every }\; t \in [0, \infty),$$

- *Uniqueness / absolute continuity with respect to Lebesgue measure* (tutorial)

II. Ergodic properties of diffusions on the torus

For 1-periodic coefficients,

$$\mathrm{d}\overline{X}_t = \sigma(\overline{X}_t)\,\mathrm{d}B_t + b(\overline{X}_t)\,\mathrm{d}t \ \text{ in } \ \mathbb{T}^d.$$

- We have the unique, mutually, absolutely continuous invariant measure $(\overline{P}^*_t\pi) = \pi$:

$$\int_{\mathbb{T}^d} E_y\left[F(\overline{X}_t)\right]\pi(\,\mathrm{d}y) = \int_{\mathbb{T}^d} \overline{P}_t f(y)\pi(\,\mathrm{d}y) = \int_{\mathbb{T}^d} f(y)\pi(y)\,\mathrm{d}y.$$

- By absolute continuity, the invariant measure π has a positive density m in $L^1(\mathbb{T}^d)$:

$$\mathrm{d}\pi = m(y)\,\mathrm{d}y.$$

- By Feynman-Kac if $\partial_t\rho = \mathrm{tr}(a\nabla^2\rho) + b\cdot\nabla\rho$ in $\mathbb{T}^d$ then

$$\int_{\mathbb{T}^d} \rho_0(y)m(y)\,\mathrm{d}y = \int_{\mathbb{T}^d} \rho(y,t)m(y)\,\mathrm{d}y \ \text{ for every } \ t\in[0,\infty).$$

- For the differential operator

$$\mathcal{L}g = \mathrm{tr}(a\nabla^2 g) + b\cdot\nabla g \ \text{ and its adjoint } \ \mathcal{L}^*g = (a_{ij}g)_{x_ix_j} - \nabla\cdot(gb),$$

  we have that

$$0 = \partial_t\left(\int_{\mathbb{T}^d}\rho(y,t)m(y)\,\mathrm{d}y\right) = \int_{\mathbb{T}^d}\left(\mathcal{L}\rho(y,t)\right)m(y)\,\mathrm{d}y = \int_{\mathbb{T}^d}\rho(x,t)\mathcal{L}^*m(y)\,\mathrm{d}y.$$

- The density solves the adjoint equation $\mathcal{L}^*m = 0$.

II. Ergodic properties of diffusions on the torus

The Fredholm alternative [Section 3.3, Asym. Anal. for Per. Struct.]

Let σ and b be sufficiently regular, and let a be uniformly elliptic. Consider the equations

$$\operatorname{tr}(a\nabla^2\rho) + b \cdot \nabla\rho = \mathcal{L}\rho = 0 \;\text{ in }\; \mathbb{T}^d, \tag{1}$$

and

$$(a_{ij}z)_{x_i x_j} - \nabla \cdot (zb) = \mathcal{L}^* z = 0 \;\text{ in }\; \mathbb{T}^d. \tag{2}$$

Then up to a multiplicative constant there exists a unique solution of (1) and (2) (namely, $\rho = 1$ and $z = m$, the density invariant measure). Furthermore, for $\phi, \psi \in L^\infty(\mathbb{T}^d)$ satisfying

$$\int_{\mathbb{T}^d} \phi(y)m(y)\,\mathrm{d}y = 0 \;\text{ and }\; \int_{\mathbb{T}^d} \psi(y)\,\mathrm{d}y = 0,$$

there exist unique solutions to the equations

$$\mathcal{L}z = \phi \;\text{ in }\; \mathbb{T}^d \;\text{ with }\; \int_{\mathbb{T}^d} z\,\mathrm{d}y = 0 \;\text{ and }\; \mathcal{L}^* w = \psi \;\text{ in }\; \mathbb{T}^d \;\text{ with }\; \int_{\mathbb{T}^d} w(y)\,\mathrm{d}y = 1.$$

- We can solve $\mathcal{L}z = \phi$ provided ϕ is *orthogonal* to the kernel of $\mathcal{L}^*$.
  - Orthogonality is *necessary*: if $\mathcal{L}z = \phi$ and $\mathcal{L}^* m = 0$ then

$$\langle \phi, m \rangle_{L^2(\mathbb{T}^d)} = \langle \mathcal{L}z, m \rangle_{L^2(\mathbb{T}^d)} = \langle z, \mathcal{L}^* m \rangle_{L^2(\mathbb{T}^d)} = 0.$$

  - *Sufficiency* relies strongly on *compactness*.

II. Ergodic properties of diffusions on the torus

Examples of invariant measures m:

- Divergence-form equations / reversible diffusions:

$$\mathcal{L}^* m = -\nabla \cdot a^t \nabla m = 0 \text{ implies that } m = 1.$$

- Pure diffusions in one-dimension: the case $b = 0$ and $d = 1$,

$$\mathcal{L}^* m = (am)_{xx} = 0 \text{ implies that } m = \langle a^{-1} \rangle^{-1} \frac{1}{a}$$

  for $\langle a^{-1} \rangle = \int_{\mathbb{T}^1} a^{-1}$.

- In general, for higher dimensions and nonzero drift, they are complicated.

Using the Green's function representation and Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{T}^d} \mathbb{E}_y \left[f(X_t) \right] m(y) \,\mathrm{d}y &= \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{p}_t(y,x) f(x) m(y) \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_{\mathbb{T}^d} f(x) \left(\int_{\mathbb{T}^d} \overline{p}_t(y,x) m(y) \,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{T}^d} f(x) m(x) \,\mathrm{d}x.\end{aligned}$$

We have that, for every $t \in [0, \infty)$ and $x \in \mathbb{T}^d$,

$$m(x) = \int_{\mathbb{T}^d} \overline{p}_t(y,x) m(y) \,\mathrm{d}y.$$

III. Homogenization of pure diffusions

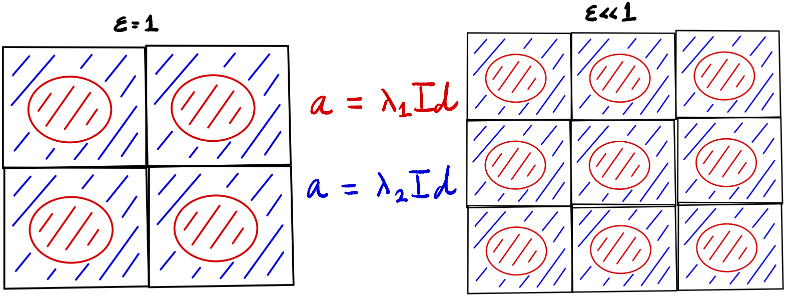
Consider the pure diffusion $\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t$ and the central limit scaling

$$\mathrm{d}X_t^\varepsilon = \sigma({}^{X_t}/_{\varepsilon})\,\mathrm{d}W_t^\varepsilon,$$

and the corresponding equation

$$\partial_t \rho^\varepsilon = \mathrm{tr}(a({}^{x}/_{\varepsilon})\nabla^2 \rho^\varepsilon),$$

for $a = \frac{1}{2}\sigma\sigma^2$.



What happens, for example, if $\lambda_1 \to 0$ and $\lambda_2 \to \infty$?

III. Homogenization of pure diffusions

Consider the pure diffusion $\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t$ and the central limit scaling

$$\mathrm{d}X_t^\varepsilon = \sigma({}^{X_t}/_{\varepsilon})\,\mathrm{d}W_t^\varepsilon$$

and the corresponding equation

$$\partial_t \rho^\varepsilon = \mathrm{tr}(a({}^{x}/_{\varepsilon})\nabla^2\rho^\varepsilon)$$

for $a = \frac{1}{2}\sigma\sigma^2$.

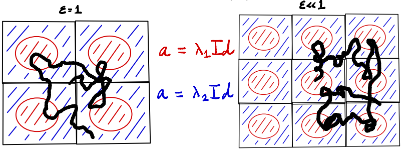
- *Homogenization*: identify $\overline{a} \in \mathbb{R}^{d\times d}$ such that

  $$\rho^\varepsilon \to \overline{\rho}$$

  for $\overline{\rho}$ the solution of

  $$\partial_t\overline{\rho} = \mathrm{tr}(\overline{a}\nabla^2\overline{\rho}).$$

- Equivalently, in law, $X_t^\varepsilon \to \overline{X}$ for $\mathrm{d}\overline{X}_t = \overline{\sigma}\,\mathrm{d}B_t$.
- a complicated, nonlinear averaging
  - — what is $\overline{a}$?
  - — it is very much not the case that $\overline{a} = \langle a \rangle$



- We have $\langle a \rangle \to \frac{1}{2}\lambda_2 I$ as $\lambda_1 \to 0$, while $\overline{a} \to 0$.

III. Homogenization of pure diffusions

The asymptotic expansion:

$$\tilde{\rho}^{\varepsilon}(x,t) = \overline{\rho}(x,t) + \varepsilon\rho_1(x, {}^{x}\!/\!_{\varepsilon}, t) + \varepsilon^2\rho_2(x, {}^{x}\!/\!_{\varepsilon}, t) + \dots$$

- Evaluating the equation, keeping terms of order ε^{-1}, ε^0, and ε,

$$\begin{aligned}\partial_t\tilde{\rho}^{\varepsilon} - \operatorname{tr}(a({}^{x}\!/\!_{\varepsilon})\nabla^2\tilde{\rho}^{\varepsilon}) = \varepsilon^{-1}\operatorname{tr}\left(a({}^{x}\!/\!_{\varepsilon})\nabla^2_y\rho_1\right)& \\ \partial_t\overline{\rho} - \operatorname{tr}\left(a({}^{x}\!/\!_{\varepsilon})\left(\nabla^2_x\overline{\rho} + \nabla^2_{xy}\rho_1 + \nabla^2_y\rho_2\right)\right)& \\ + \varepsilon\partial_t\rho_1 - \varepsilon\operatorname{tr}\left(a({}^{x}\!/\!_{\varepsilon})\left(\nabla^2_x\rho_1 + \nabla^2_{xy}\rho_2\right)\right).&\end{aligned}$$

- We conclude that $\rho_1 = 0$, which is very much related to the fact that

$$X_t = \sigma(X_t)\,\mathrm{d}B_t \text{ is a martingale,}$$

  and therefore have that

$$\partial_t\overline{\rho} = \operatorname{tr}\left(a({}^{x}\!/\!_{\varepsilon})\left(\nabla^2_x\overline{\rho} + \nabla^2_y\rho_2\right)\right) + O(\varepsilon).$$

- *Separation of scales*: we make the ansatz that $\rho_2(x,y,t) = \sum_{i,j=1}^{d} w_{ij}(y)\partial^2_{ij}\overline{\rho}$ so that

$$\partial_t\overline{\rho} = \operatorname{tr}\left(a({}^{x}\!/\!_{\varepsilon})\left(e_{ij} + \nabla^2 w_{ij}({}^{x}\!/\!_{\varepsilon})\right)\right)\partial^2_{ij}\overline{\rho} + O(\varepsilon).$$

- Solvablity / Fredholm alternative requires that

$$\operatorname{tr}\left(a(y)\left(e_{ij} + \nabla^2 w_{ij}(y)\right)\right) = \langle a_{ij}, m\rangle_{L^2(\mathbb{T}^d)} \text{ in } \mathbb{T}^d,$$

  and

$$\partial_t\overline{\rho} = \operatorname{tr}(\overline{a}\nabla^2\overline{\rho}) \text{ for } \overline{a} = \int_{\mathbb{T}^d} a(y)m(y)\,\mathrm{d}y.$$

III. Homogenization of pure diffusions

The asymptotic expansion:

$$\tilde{\rho}^\varepsilon(x,t) = \overline{\rho}(x,t) + \varepsilon^2 w_{ij}(x/\varepsilon)\nabla^2\overline{\rho}(x,t) + \dots$$

- We define the second-order correctors:

$$\operatorname{tr}(a(y)(e_{ij} + \nabla^2 w_{ij}(y))) = \langle a_{ij}, m\rangle_{L^2(\mathbb{T}^d)} \text{ in } \mathbb{T}^d.$$

- We define the homogenized solution

$$\partial_t \overline{\rho} = \operatorname{tr}(\overline{a}\nabla^2\overline{\rho}) \text{ for } \overline{a} = \int_{\mathbb{T}^d} a(y)m(y)\,\mathrm{d}y.$$

- The asymptotic expansion $\tilde{\rho}^\varepsilon$ satisfies

$$\partial_t\tilde{\rho}^\varepsilon = \operatorname{tr}(a(x/\varepsilon)\nabla^2\tilde{\rho}^\varepsilon) + O(\varepsilon) \text{ in } \mathbb{R}^d\times(0,\infty) \text{ with } \tilde{\rho}^\varepsilon = \rho_0 + O(\varepsilon^2).$$

- The difference $z^\varepsilon = \rho^\varepsilon - \tilde{\rho}^\varepsilon$ solves

$$\partial_t z^\varepsilon = \operatorname{tr}(a(x/\varepsilon)\nabla^2 z^\varepsilon) + O(\varepsilon) \text{ in } \mathbb{R}^d\times(0,\infty) \text{ with } z^\varepsilon = O(\varepsilon^2).$$

- Since $\tilde{\rho}^\varepsilon = \overline{\rho}(x,t) + O(\varepsilon^2)$, the comparison principle proves that, as $\varepsilon\to 0$,

$$\rho^\varepsilon \to \overline{\rho} \text{ for } \partial_t\overline{\rho} = \operatorname{tr}(\overline{a}\nabla^2\overline{\rho}).$$

  Or, equivalently, that in law the processes X^ε_t converges in law to $\overline{\sigma}B_t$ for $\overline{a} = \frac{1}{2}\overline{\sigma\sigma}^2$.

III. Homogenization of pure diffusions

Homogenization of pure diffusions

Let b and σ be sufficiently regular, and let a be uniformly elliptic. Then, for every $\rho_0 \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, the solutions

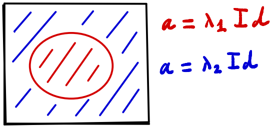
$$\partial_t \rho^\varepsilon = \operatorname{tr}(a(x/\varepsilon)\nabla^2 \rho^\varepsilon) \text{ in } \mathbb{R}^d \times (0,\infty) \text{ with } \rho^\varepsilon(\cdot,0) = \rho_0,$$

converge, as $\varepsilon \to 0$, to the solution

$$\partial_t \overline{\rho} = \operatorname{tr}(\overline{a}\nabla^2 \overline{\rho}) \text{ in } \mathbb{R}^d \times (0,\infty) \text{ with } \overline{\rho}(\cdot,0) = \rho_0,$$

for $\overline{a} = \int_{\mathbb{T}^d} a(y)m(y)\,\mathrm{d}y$.

- The *homogenized* / *effective* matrix is the average of the original matrix with respect to the invariant measure.
  - — uniformly elliptic with ellipticity constants determined by m
  - — emphasizes regions of *small diffusion* / *traps* / *degeneracies*
- In one-dimension, we have that $m(y) = \langle a^{-1}\rangle^{-1}a^{-1}$ and $\overline{a} = \langle a^{-1}\rangle^{-1}$.



If $0 < \lambda_1 \ll \lambda_2$,

$m \gg 1$

$0 < m \ll 1$.

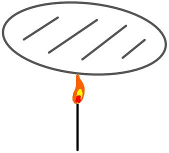
IV. Periodic homogenization of divergence form equations

A random uniformly elliptic, 1-periodic coefficient field $a\colon \mathbb{T}^d \to \mathbb{R}^{d\times d}$.

- *uniform ellipticity*: there exists $\lambda \le \Lambda \in (0,\infty)$ such that

$$\lambda\,|\xi|^2 \le \langle a(x)\xi,\xi\rangle \le \Lambda\,|\xi|^2 \;\text{ for every }\; x\in\mathbb{T}^d \;\text{ and }\; \xi\in\mathbb{R}^d.$$

- The solution ρ^ε of



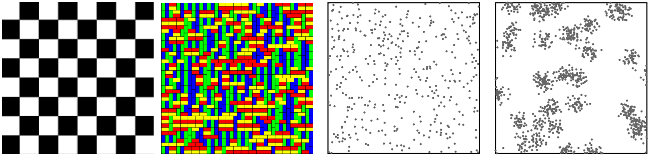
$$-\nabla\cdot a(x/\varepsilon)\nabla\rho^\varepsilon = f \;\text{ in }\; U \;\text{ with }\; \rho^\varepsilon = g \;\text{ on }\; \partial U.$$

describes the evolution of a system satisfying

$$\oint_{B_r(x)} a(y/\varepsilon,\omega)\nabla\rho^\varepsilon(y)\cdot\nu = \int_{B_r(x)} f(y).$$

- For symmetric a we have the variational formulation and Feynman-Kac formula

$$\inf_{v\in g+H^1_0(U)}\left(\int_U \langle a(x/\varepsilon)\nabla u,\nabla u\rangle - fu\,\mathrm{d}x\right) \;\text{ and }\; \rho^\varepsilon(x) = \mathbb{E}_x\left[g(X^\varepsilon_{\tau_U}) + \int_0^{\tau_U} f(X^\varepsilon_s)\,\mathrm{d}s\right].$$



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IV. Periodic homogenization of divergence form equations

A *weak solution* of the equation

$$-\nabla \cdot a(x/\varepsilon)\nabla\rho^\varepsilon = f \ \text{ in } \ U \ \text{ with } \ \rho^\varepsilon = g \ \text{ on } \ \partial U.$$

is a function $\rho^\varepsilon \in H^1(U)$ that satisfies, for every $\psi \in \mathrm{C}_c^\infty(U)$,

$$\int_U a(x/\varepsilon)\nabla\rho^\varepsilon \cdot \nabla\psi \,\mathrm{d}x = \int_U \psi f \ \text{ with } \ \rho^\varepsilon = g \ \text{ on } \ \partial U.$$

Consider the asymptotic expansion

$$\tilde{\rho}^\varepsilon(x) = \overline{\rho}(x) + \varepsilon\rho_1(x, x/\varepsilon) + \varepsilon^2\rho_2(x, x/\varepsilon) + \ldots$$

Ignoring terms of order ε and smaller, and evaluating the equation on $\tilde{\rho}^\varepsilon$,

$$\int_U a(x/\varepsilon)(\nabla_x\overline{\rho} + \nabla_y\rho_1) \cdot \nabla\psi \,\mathrm{d}x = \int_U \psi f.$$

- Identify the equation satisfied by $\overline{\rho}$.
- Separation of scales:

$$\rho_1(x, x/\varepsilon) = \phi_i(x/\varepsilon)\partial_i\overline{\rho}(x).$$

IV. Periodic homogenization of divergence form equations

Defined by *first-order correctors* ϕ_i, the asymptotic expansion

$$\tilde{\rho}^\varepsilon = \overline{\rho} + \varepsilon\phi_i(x/\varepsilon)\partial_i\overline{\rho}(x) + \ldots$$

satisfies, up to terms of order ε,

$$\begin{aligned}\int_U a(x/\varepsilon)(\nabla_x\overline{\rho} + \nabla_y\rho_1)\cdot\nabla\psi\,\mathrm{d}x &= \int_U a(x/\varepsilon)\left((e_i + \nabla\phi_i(x/\varepsilon))\partial_i\overline{\rho}\right)\cdot\nabla\psi\,\mathrm{d}x \\ &= -\int_U \psi a(x/\varepsilon)(e_i + \nabla\phi_i(x/\varepsilon))\cdot\nabla\partial_i\overline{\rho} = \int_U \psi f.\end{aligned}$$

As $\varepsilon \to 0$,

$$a(x/\varepsilon)(e_i + \nabla\phi_i(x/\varepsilon)) \rightharpoonup \langle a(e_i + \nabla\phi_i)\rangle =: \overline{a}e_i \;\text{ weakly in }\; L^2(\mathbb{T}^d),$$

and formally we have that

$$-\nabla\cdot a(y)(e_i + \nabla\phi_i(y)) = 0 \;\text{ in }\; \mathbb{T}^d,$$

and that $\overline{\rho}$ solves

$$-\nabla\cdot\overline{a}\nabla\overline{\rho} = f \;\text{ in }\; U \;\text{ with }\; \overline{\rho} = g \;\text{ on }\; \partial U.$$

- The functions $x_i + \phi_i(x)$ are a-harmonic
  — *A first order Liouville theorem*: any subquadratic solution of

$$-\nabla\cdot a\nabla\rho = 0 \;\text{ on }\; \mathbb{R}^d \;\text{ satisfies }\; \rho(x) = c + \xi\cdot x + \phi_\xi(x).$$

The perturbed test function method [See Section 3, notes]

Let $a\colon \mathbb{T}^d \to \mathbb{R}^{d\times d}$ be bounded, measurable, and uniformly elliptic and let $\rho_0 \in L^2(\mathbb{R}^d)$. For every $\varepsilon \in (0,1)$ let ρ^ε be the unique weak solution of

$$-\nabla \cdot a(x/\varepsilon)\nabla\rho^\varepsilon = f \;\text{ in }\; U \;\text{ with }\; \rho^\varepsilon = g \;\text{ on }\; \partial U.$$

Then, as $\varepsilon \to 0$,

$$\rho^\varepsilon \rightharpoonup \overline{\rho} \;\text{ weakly in }\; H^1(\mathbb{R}^d),$$

for $\overline{\rho}$ satisfying

$$-\nabla \cdot \overline{a}\nabla\overline{\rho} = f \;\text{ in }\; U \;\text{ with }\; \overline{\rho} = g \;\text{ on }\; \partial U,$$

for the effective coefficient field $\overline{a} \in \mathbb{R}^{d\times d}$ defined by

$$\overline{a}e_i = \langle a(e_i + \nabla\phi_i)\rangle \;\text{ for }\; -\nabla \cdot a(e_i + \nabla\phi_i) = 0 \;\text{ in }\; \mathbb{T}^d.$$

- compensated compactness methods
  - — the div-curl lemma
- existence / uniqueness of ϕ_i follows by Fredholm or the Lax-Milgram lemma

The solution ρ^ε of

$$-\nabla \cdot a(x/\varepsilon)\nabla\rho^\varepsilon = f \text{ in } U \text{ with } \rho^\varepsilon = g \text{ on } \partial U.$$

We have formally that

$$\rho^\varepsilon \simeq \tilde{\rho}^\varepsilon = \overline{\rho} + \varepsilon\phi_i(x/\varepsilon)\partial_i\overline{\rho}(x) + \dots$$

and therefore that

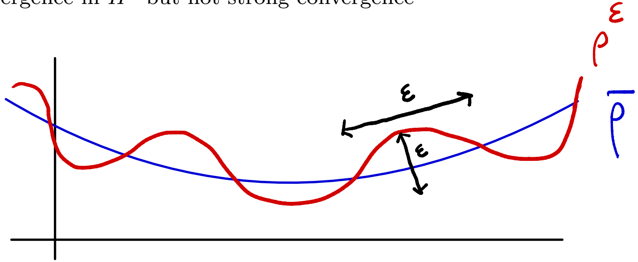
$$\nabla\rho^\varepsilon \simeq \nabla\overline{\rho} + \nabla\phi_i(x/\varepsilon)\partial_i\overline{\rho}.$$

As $\varepsilon \to 0$,

$$\nabla\rho^\varepsilon \rightharpoonup \nabla\overline{\rho} \text{ weakly in } L^2(\mathbb{R}^d \times [0,T];\mathbb{R}^d),$$

but not strongly.

- weak convergence in H^1 but not strong convergence



The *homogenization error*: the error in the two-scale expansion,

$$w^\varepsilon = \rho^\varepsilon - \overline{\rho} - \varepsilon\phi_i({}^x\!/\!_\varepsilon)\partial_i\overline{\rho},$$

for the correctors ϕ_i and the homogenized solution $\overline{\rho}$ satisfying

$$-\nabla\cdot\overline{a}\nabla\overline{\rho} = f \text{ in } U \text{ with } \overline{\rho} = g \text{ on } \partial U \text{ and } -\nabla\cdot a(e_i+\nabla\phi_i) = 0 \text{ in } \mathbb{T}^d.$$

The homogenization error w^ε satisfies

$$\begin{aligned}-\nabla\cdot a({}^x\!/\!_\varepsilon)\nabla w^\varepsilon = f &- \nabla\cdot(a({}^x\!/\!_\varepsilon)e_i\partial_i\overline{\rho}) - \nabla\cdot(a({}^x\!/\!_\varepsilon)\nabla\phi_i({}^x\!/\!_\varepsilon)\partial_i\overline{\rho}) \\ &- \varepsilon\nabla\cdot(a({}^x\!/\!_\varepsilon)\phi_i({}^x\!/\!_\varepsilon)\nabla(\partial_i\overline{\rho}))\,.\end{aligned}$$

After adding and subtracting

$$-\nabla\cdot\overline{a}\nabla\overline{\rho} = f,$$

we have that

$$\begin{aligned}-\nabla\cdot a({}^x\!/\!_\varepsilon)\nabla w^\varepsilon = \nabla\cdot&((a({}^x\!/\!_\varepsilon)(e_i+\nabla\phi_i({}^x\!/\!_\varepsilon)) - \overline{a}e_i)\,\partial_i\overline{\rho}) \\ &+ \varepsilon\nabla\cdot(a({}^x\!/\!_\varepsilon)\phi_i({}^x\!/\!_\varepsilon)(\nabla\partial_i\overline{\rho})).\end{aligned}$$

The energy estimate, for $c = c(d,f,g,\lambda,\Lambda)\in(0,\infty)$,

$$\int_U |\nabla w^\varepsilon|^2 \le c\left(\|(a({}^x\!/\!_\varepsilon)(e_i+\nabla\phi_i({}^x\!/\!_\varepsilon)) - \overline{a}e_i)\|^2_{L^2(\mathbb{T}^d)} + \varepsilon\,\|\phi_i\|^2_{L^2(\mathbb{T}^d)}\right).$$

IV. Periodic homogenization of divergence form equations

The homogenization error $w^\varepsilon = \rho^\varepsilon - \overline{\rho} - \varepsilon\phi_i(x/\varepsilon)\partial_i\overline{\rho}$ satisfies

$$-\nabla \cdot a(x/\varepsilon)\nabla w^\varepsilon = \nabla \cdot \left(\left(a(x/\varepsilon)(e_i + \nabla\phi_i(x/\varepsilon)) - \overline{a}e_i \right) \partial_i\overline{\rho} \right) \\ + \varepsilon\nabla \cdot (a(x/\varepsilon)\phi_i(x/\varepsilon)(\nabla\partial_i\overline{\rho})).$$

The essential observation is that the vectors

$$q_i = a(e_i + \nabla\phi_i) - \overline{a}e_i,$$

are mean zero:

$$\langle q_i \rangle = \langle a(e_i + \nabla\phi_i) \rangle - \overline{a}e_i = 0$$

and divergence free:

$$\nabla \cdot q_i = \nabla \cdot a(y)(e_i + \nabla\phi_i(y)) = 0.$$

- De Rham cohomology—every mean zero, divergence free field has a potential field
  - — there exists σ_i such that $\nabla \cdot \sigma_i = q_i$.
- σ is a $d \times d$ skew-symmetric matrix
  - — vector field is a $(d-1)$-form, skew symmetric matrix is a $(d-2)$-form
  - — the “stream matrix” from fluids

Let $q = (q_i)\colon \mathbb{T}^d \to \mathbb{R}^d$ be mean zero and divergence free.

There exists a skew-symmetric matrix $\sigma\colon \mathbb{T}^d \to \mathbb{R}^{d\times d}$ such that

$$\nabla \cdot \sigma = q \;\text{ where }\; (\nabla \cdot \sigma)_i = \partial_k \sigma_{ik}. \tag{3}$$

The solution to (3) is not unique—shift by any matrix with rows that are divergence free.

A variational problem: for $\sigma = (\sigma_{jk})$ minimize the energy

$$\int_{\mathbb{T}^d} \left|\nabla \sigma_{jk}\right|^2 \;\text{ subject to the constraint }\; \nabla \cdot \sigma = q.$$

This leads to the well-posed equation

$$-\Delta \sigma_{jk} = \partial_j q_k - \partial_k q_j \;\text{ on }\; \mathbb{T}^d \;\text{ with }\; \langle \sigma_{jk} \rangle = 0.$$

We define $\sigma = (\sigma_{jk})$ and observe that

$$\Delta(\nabla \cdot \sigma)_i = \Delta(\partial_k \sigma_{ik}) = \partial_k(\Delta \sigma_{ik}) = \Delta q_i - \partial_i (\nabla \cdot q) = \Delta q_i.$$

We have that

$$\Delta\left((\nabla \cdot \sigma)_i - q_i\right) = 0 \;\text{ and so }\; (\nabla \cdot \sigma)_i - q_i = \langle (\nabla \cdot \sigma)_i - q_i \rangle = 0,$$

and, therefore,

$$\nabla \cdot \sigma = q.$$

The *homogenization error*: the error in the two-scale expansion,

$$w^\varepsilon = \rho^\varepsilon - \overline{\rho} - \varepsilon\phi_i(x/\varepsilon)\partial_i\overline{\rho},$$

for the correctors ϕ_i and the homogenized solution $\overline{\rho}$ satisfying

$$-\nabla\cdot\overline{a}\nabla\overline{\rho} = f \text{ in } U \text{ with } \overline{\rho} = g \text{ on } \partial U \text{ and } -\nabla\cdot a(e_i+\nabla\phi_i) = 0 \text{ in } \mathbb{T}^d.$$

The homogenization error w^ε satisfies

$$\begin{aligned}-\nabla\cdot a(x/\varepsilon)\nabla w^\varepsilon = {} & \nabla\cdot\left(\left(a(x/\varepsilon)(e_i+\nabla\phi_i(x/\varepsilon)) - \overline{a}e_i\right)\partial_i\overline{\rho}\right) \\ & + \varepsilon\nabla\cdot\left(a(x/\varepsilon)\phi_i(x/\varepsilon)(\nabla\partial_i\overline{\rho})\right).\end{aligned}$$

The *fluxes* q_i: divergence free fields

$$q_i = a(e_i+\nabla\phi_i) - \overline{a}e_i.$$

The *flux correctors*: skew-symmetric matrices σ_i satisfying

$$\nabla\cdot\sigma_i = q_i \text{ fixed by } -\Delta\sigma_{ijk} = \partial_j q_{ik} - \partial_k q_{ij}.$$

It follows from the skew-symmetry that distributionally

$$\nabla\cdot\left(q_i(x/\varepsilon)\partial_i\overline{\rho}\right) = -\varepsilon\nabla\cdot\left(\sigma_i(x/\varepsilon)\nabla\partial_i\overline{\rho}\right),$$

and, therefore,

$$-\nabla\cdot a(x/\varepsilon)\nabla w^\varepsilon = \varepsilon\nabla\cdot\left(\left(a\phi_i(x/\varepsilon) - \sigma_i(x/\varepsilon)\right)\nabla\partial_i\overline{\rho}\right).$$

Periodic homogenization of divergence form equations

Let $a\colon \mathbb{T}^d \to \mathbb{R}^{d\times d}$ be bounded, measurable, and uniformly elliptic. Let $\rho_0 \in \mathrm{C}^\infty_c(\mathbb{R}^d)$ and for every $\varepsilon \in (0,1)$ let

$$-\nabla \cdot a(x/\varepsilon)\nabla\rho^\varepsilon = f \;\text{ in }\; U \;\text{ with }\; \rho^\varepsilon = g \;\text{ on }\; \partial U,$$

and let $\overline{\rho}$ solve

$$-\nabla \cdot \overline{a}\nabla\overline{\rho} = f \;\text{ in }\; U \;\text{ with }\; \overline{\rho} = g \;\text{ on }\; \partial U,$$

for $\overline{a}e_i = \langle a(e_i + \nabla\phi_i)\rangle$ defined by the correctors $-\nabla\cdot a(e_i+\nabla\phi_i) = 0$. Then there exists $c = c(\rho_0, d, f, \lambda, \Lambda) \in (0,\infty)$ such that the homogenization error

$$w^\varepsilon = \rho^\varepsilon - \overline{\rho} - \varepsilon\phi_i(x/\varepsilon)\partial_i\overline{\rho}$$

satisfies

$$\|w^\varepsilon\|_{H^1(U))} \leq c\varepsilon\left(\|\phi_i\|_{L^2(\mathbb{T}^d)} + \|\sigma_i\|_{L^2(\mathbb{T}^d)}\right).$$

- strong H^1-convergence of the two-scale expansion
  - the two-scale expansion corrects the function *and* the gradient
- the a-harmonic coordinates $x_i + \phi_i(x)$—sub-quadratic a-harmonic functions
  - Liouville theorems—these are the linear functions in the geometry of a

V. Periodic homogenization of non-divergence form equations

The diffusion equation

$$\mathrm{tr}(a(x/\varepsilon)\nabla^2\rho^\varepsilon) + \varepsilon^{-1}b(x/\varepsilon)\cdot\nabla\rho^\varepsilon = f \;\text{ in }\; U \;\text{ with }\; \rho^\varepsilon = g \;\text{ on }\; \partial U.$$

- Postulate an asymptotic expansion of the form

$$\tilde{\rho}^\varepsilon = \overline{\rho} + \varepsilon\phi_i(x/\varepsilon)\partial_i\overline{\rho} + \varepsilon^2\psi(x/\varepsilon) + \ldots$$

- The *corrector equation* at order ε^{-1}:

$$\mathrm{tr}(a(y)\nabla^2\phi_i) + b(y)\cdot\nabla\phi_i = -b_i \;\text{ in }\; \mathbb{T}^d.$$

  Solvability requires $\langle b, m\rangle = 0$ for the invariant measure $\mathcal{L}^*m = 0$.

- At order ε^0 the solvability condition for ψ requires that $\overline{\rho}$ solves

$$\partial_t\overline{\rho} = \mathrm{tr}(\overline{a}\nabla^2\overline{\rho}) \;\text{ for }\; \overline{a} = \int_{\mathbb{T}^d}(a(y)(1+\nabla\phi(y)) + \phi(y)\otimes b(y))\,m(y)\,\mathrm{d}y.$$

- For the process $\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t$ the process

$$M_t = X_t + \phi(X_t) \;\text{ for }\; \phi = (\phi_1,\ldots,\phi_d) \;\text{ is a martingale.}$$

- Form the decomposition

$$\varepsilon X_{t/\varepsilon^2} = \varepsilon M_{t/\varepsilon^2} - \varepsilon\phi(X_{t/\varepsilon^2}).$$

V. Periodic homogenization of non-divergence form equations

An alternate approach using the invariant measure: for $\mathcal{L}^*m = 0$,

$$\begin{aligned}&\operatorname{tr}(m(x/\varepsilon)a(x/\varepsilon)\nabla^2\rho^\varepsilon)) + \varepsilon^{-1}m(x/\varepsilon)b(x/\varepsilon)\cdot\nabla\rho^\varepsilon \\ &= \nabla\cdot(m(x/\varepsilon)a(x/\varepsilon)\nabla\rho^\varepsilon) - \varepsilon^{-1}\left(\nabla\cdot(m(x/\varepsilon)a(x/\varepsilon)) - m(x/\varepsilon)b(x/\varepsilon)\right)\cdot\nabla\rho^\varepsilon.\end{aligned}$$

- *Divergence-free drift*: the vector

  $$\tilde{b}(y) = \nabla\cdot(m(y)a(y)) - m(y)b(y) \ \text{ is divergenc-free,}$$

  since $\nabla\cdot\tilde{b} = \mathcal{L}^*m$.

- *Mean zero drift*: We have that

  $$\langle\tilde{b}\rangle = \langle\nabla\cdot(ma) - mb\rangle = -\langle mb\rangle = 0,$$

  if and only if b is perpendicular to m in the L^2-sense that $\langle b, m\rangle_{L^2(\mathbb{T}^d)} = 0$.

- If the solvability condition $\langle b, m\rangle_{L^2(\mathbb{T}^d)} = 0$ is satisfied, there exists a potential $\tilde{\sigma}$ with

  $$\nabla\cdot\tilde{\sigma} = \tilde{b},$$

  such that

  $$\operatorname{tr}(m(x/\varepsilon)a(x/\varepsilon)\nabla^2\rho^\varepsilon)) + \varepsilon^{-1}m(x/\varepsilon)b(x/\varepsilon)\cdot\nabla\rho^\varepsilon = \nabla\cdot((m(x/\varepsilon)a(x/\varepsilon) + \tilde{\sigma}(x/\varepsilon))\nabla\rho^\varepsilon)).$$

- For the new "diffusion matrix"

  $$\tilde{a} = am + \tilde{\sigma} \ \text{ we have } \ \nabla\cdot\tilde{a}(x/\varepsilon)\nabla\rho^\varepsilon = f(x)m(x/\varepsilon).$$

V. Periodic homogenization of non-divergence form equations

After multiplying the equation by $m(x/\varepsilon)$,

$$\begin{aligned}
&\operatorname{tr}(m(x/\varepsilon)a(x/\varepsilon)\nabla^2\rho^\varepsilon)) + \varepsilon^{-1}m(x/\varepsilon)b(x/\varepsilon)\cdot\nabla\rho^\varepsilon \\
&= \nabla\cdot(m(x/\varepsilon)a(x/\varepsilon)\nabla\rho^\varepsilon) - \varepsilon^{-1}\left(\nabla\cdot(m(x/\varepsilon)a(x/\varepsilon)) - m(x/\varepsilon)b(x/\varepsilon)\right)\cdot\nabla\rho^\varepsilon \\
&= \nabla\cdot((m(x/\varepsilon)a(x/\varepsilon) + \tilde{\sigma}(x/\varepsilon))\nabla\rho^\varepsilon) = fm(x/\varepsilon),
\end{aligned}$$

for $\langle b, m\rangle_{L^2(\mathbb{T}^d)} = 0$ and for the skew-symmetric matrix $\tilde{\sigma}$ satisfying

$$\nabla\cdot\tilde{\sigma} = \nabla\cdot(m(x/\varepsilon)a(x/\varepsilon)) - m(x/\varepsilon)b(x/\varepsilon).$$

The correctors

$$-\nabla\cdot(am+\tilde{\sigma})(e_i+\nabla\phi_i) = 0 \text{ in } \mathbb{T}^d.$$

Observe that by Hölder's inequality and Young's inequality that, fo $c = c(d) \in (0,\infty)$,

$$\int_{\mathbb{T}^d}\langle(am+\tilde{\sigma})\nabla\phi,\nabla\phi\rangle = \int_{\mathbb{T}^d}\langle a\nabla\phi,\nabla\phi\rangle m \le c\left(\|am\|^2_{L^2(\mathbb{T}^d)} + \|\tilde{\sigma}\|^2_{L^2(\mathbb{T}^d)}\right).$$

The homogenized matrix $\overline{a}$ is

$$\overline{a}e_i := \langle(am+\tilde{\sigma})(e_i+\nabla\phi_i)\rangle,$$

and $\overline{\rho}$ solves

$$\nabla\cdot\overline{a}\nabla\overline{\rho} = fm(x/\varepsilon) \text{ in } U \text{ with } \overline{\rho} = g \text{ on } \partial U.$$

Periodic homogenization of non-divergence form equations

Assume that $a \in \mathrm{C}^{1,\alpha}(\mathbb{T}^d;\mathbb{R}^{d\times d})$ is uniformly elliptic, assume that $b \in \mathrm{C}^{\alpha}(\mathbb{T}^d;\mathbb{R}^d)$, and assume that $\rho_0 \in \mathrm{C}^\infty_c(\mathbb{T}^d)$. Let m be the invariant measure $\mathcal{L}^*m = 0$ and assume that $\langle b, m\rangle_{L^2(\mathbb{T}^d)} = 0$. Let $\tilde{\sigma}$ be the vector potential satisfying

$$\nabla \cdot \tilde{\sigma} = \nabla \cdot (am) - bm,$$

and let $\tilde{a} \in C^\alpha(\mathbb{R}^d)$ be defined by $\tilde{a} = am + \tilde{\sigma}$. Define the homogenization correctors

$$-\nabla \cdot \tilde{a}(e_i + \nabla\phi_i) = 0 \;\text{ in }\; \mathbb{T}^d.$$

Then, for the effective matrix $\overline{a}e_i = \langle \tilde{a}(e_i + \nabla\phi_i)\rangle$, for $\overline{\rho}$ satisfying

$$\nabla \cdot \overline{a}\nabla\overline{\rho} = fm(x/\varepsilon) \;\text{ in }\; U \;\text{ with }\; \overline{\rho} = g \;\text{ on }\; \partial U,$$

the homogenization error

$$w^\varepsilon = \rho^\varepsilon - \overline{\rho} - \varepsilon\phi_i\partial_i\overline{\rho},$$

satisfies, for some $c = c(f, g, d, \lambda, \Lambda) \in (0, \infty)$,

$$\|w^\varepsilon\|_{L^2[0,T];H^1(\mathbb{T}^d)} \leq c\varepsilon\left(\|\phi_i\|_{L^2(\mathbb{T}^d)} + \|\sigma_i\|_{L^2(\mathbb{T}^d)}\right),$$

for the flux correctors $\nabla \cdot \sigma_i = \tilde{a}(e_i + \nabla\phi_i) - \overline{a}e_i$.

- Energy estimates for the correctors: there exists $c = c(a, m, \tilde{\sigma}) \in (0, \infty)$ such that

$$\int_{\mathbb{T}^d} |\nabla\phi_i(y)|^2\, m(y)\,\mathrm{d}y \leq c.$$

V. Periodic homogenization of non-divergence form equations

Important examples with $\overline{b}=0$:

- *Divergence-form*:

$$-\nabla\cdot a(x/\varepsilon)\nabla\rho^{\varepsilon}=\mathrm{tr}(a(\frac{x}{\varepsilon})\nabla^2\rho^{\varepsilon})+\varepsilon^{-1}(\nabla\cdot a^t(x/\varepsilon))\cdot\nabla\rho^{\varepsilon}.$$

  In this case $m=1$ and, as the integral of a periodic gradient,

$$\langle b,m\rangle_{L^2(\mathbb{T}^d)}=\int_{\mathbb{T}^d}(\nabla\cdot a^t(y))\,\mathrm{d}y=0.$$

- *Mean-zero divergence free drift*: for a potential s with $\nabla\cdot s=b$,

$$-\nabla\cdot a(x/\varepsilon)\nabla\rho^{\varepsilon}+\varepsilon^{-1}b(x/\varepsilon)\cdot\nabla\rho^{\varepsilon}=\nabla\cdot(a+s)(x/\varepsilon)\nabla\rho^{\varepsilon}.$$

  In this case $m=1$ and

$$\langle b,m\rangle_{L^2(\mathbb{T}^d)}=\int_{\mathbb{T}^d}b\,\mathrm{d}y=\int_{\mathbb{T}^d}\nabla\cdot s(y)\,\mathrm{d}y=0.$$

- *Brownian motion in a periodic potential*: we consider

$$-\Delta\rho^{\varepsilon}+\varepsilon^{-1}\nabla U(x/\varepsilon)\cdot\nabla\rho^{\varepsilon}=-e^{U(x/\varepsilon)}\nabla\cdot(e^{-U(x/\varepsilon)}\nabla\rho^{\varepsilon}).$$

  The invariant measure is the Gibbs measure $m=\langle e^{-U}\rangle^{-1}e^{-U}$ and

$$\langle b,m\rangle_{L^2(\mathbb{T}^d)}=\langle e^{-U}\rangle^{-1}\int_{\mathbb{T}^d}e^{-U}\nabla U\,\mathrm{d}y=-\langle e^{-U}\rangle^{-1}\int_{\mathbb{T}^d}\nabla e^{-U}\,\mathrm{d}y=0.$$

- *Symmetry*: restricted isotropy in law implies $\overline{b}=0$.

V. Periodic homogenization of non-divergence form equations

The case $\overline{b} = \langle b, m \rangle_{L^2(\mathbb{T}^d)} \neq 0$.

- Constant coefficients:
$$\mathrm{d}X_t = \mathrm{d}B_t + \overline{b}\,\mathrm{d}t.$$
In this case, for the diffusion beginning at zero,
$$X_t^\varepsilon = W_t^\varepsilon + \varepsilon^{-1} t\overline{b}.$$
Diffusive behavior after subtracting the "effective drift" $\overline{b}$:
$$X_t^\varepsilon - \varepsilon^{-1} t\overline{b} = W_t^\varepsilon.$$
- True in general: if $\overline{b} \neq 0$ then as $\varepsilon \to 0$ the process
$$X_t^\varepsilon \;\text{ is ballistic in the direction } \overline{b}.$$
However, after centering about this singular trajectory, as $\varepsilon \to 0$,
$$X_t^\varepsilon - \varepsilon^{-1}\overline{b}t \to \overline{\sigma} B_t \;\text{ in law.}$$
- Repeating the same proof:
$$\partial_t \rho^\varepsilon - \nabla \cdot \tilde{a}(x/\varepsilon)\nabla \rho^\varepsilon + \varepsilon^{-1}\overline{b} \cdot \nabla \rho^\varepsilon = f(x)m(x/\varepsilon),$$
and $\overline{\rho}^\varepsilon$ solves
$$\partial_t \overline{\rho}^\varepsilon - \nabla \cdot \overline{a}\nabla \overline{\rho}^\varepsilon + \varepsilon^{-1}\overline{b} \cdot \nabla \overline{\rho}^\varepsilon = f(x)m(x/\varepsilon).$$
Compare $\rho^\varepsilon(x - \varepsilon^{-1}\overline{b}t, t)$ to $\overline{\rho}^\varepsilon(x - \varepsilon^{-1}\overline{b}t, t)$.

The Poisson point process on $\mathbb{R}^d$:

- The probability space is the space of locally finite point measures

$$\Omega = \left\{ \omega = \sum_{i \in I} \delta_{x_i} : x_i \text{ are locally finite in } \mathbb{R}^d \right\},$$

with the sigma algebra $\mathcal{F}$ generated by all maps of the form

$$\omega \to \omega(B) = \#\{i \in I \colon x_i \in B\} \text{ for Borel subsets } B \subseteq \mathbb{R}^d.$$

- For $\lambda \in (0, \infty)$ there exists a unique probability measure $\mathbb{P}_\lambda$ on Ω satisfying:
  - For every Borel subset $B \subseteq \mathbb{R}^d$,

$$\mathbb{E}_\lambda[\omega(B)] = \lambda \, |B| \, .$$

  - For every collection of bounded, disjoint subsets $B_1, \ldots, B_N \subseteq \mathbb{R}^d$,

$$\text{the random variables } \omega \to \omega(B_k) \text{ are independent.}$$

  - For every $y \in \mathbb{R}^d$ and measurable set $A \in \mathcal{F}$,

$$\mathbb{P}_\lambda(A) = \mathbb{P}_\lambda(A + y) \text{ for } A + y = \{\omega(\cdot + y) \colon \omega \in A\}.$$

Fix a Poisson point process $(\Omega, \mathcal{F}, \mathbb{P}_\lambda)$. We define the random coefficient field $a(x,\omega)\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$, for $\omega = \sum_{i\in I} \delta_{x_i}$,

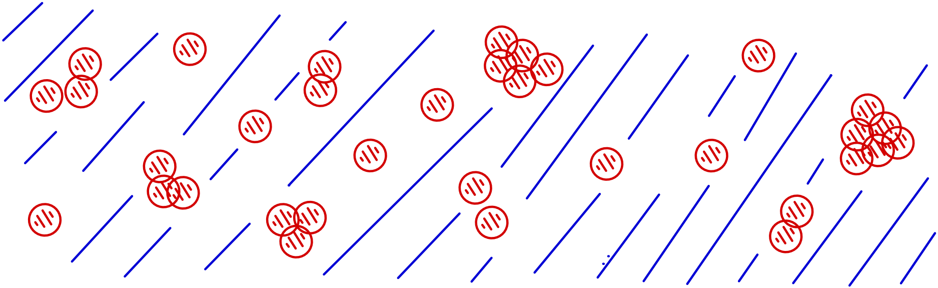
$$a(x,\omega) = \lambda_1 \mathbf{1}_{\{\cup_{i\in I} B_1(x_i)\}} + \lambda_2 \mathbf{1}_{\{\cup_{i\in I} B_1(x_i)\}^c}$$

For the measure-preserving transformation group $\{\tau_x\colon \mathbb{R}^d \to \mathbb{R}^d\}_{x\in\mathbb{R}^d}$ defined by

$$\tau_x(\omega)(\cdot) = \omega(\cdot - x),$$

we have

$$a(x+y,\omega) = a(y,\tau_x\omega) \text{ for every } x,y\in\mathbb{R}^d \text{ and } \omega\in\Omega.$$

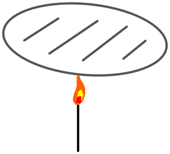


VII. Stochastic homogenization

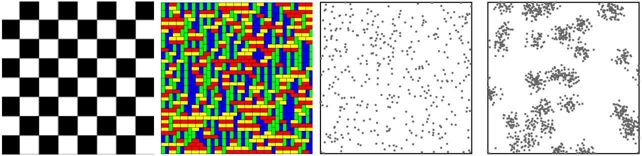
A random uniformly elliptic coefficient field $a(x,\omega)\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$.

- *stationary*: for a measure-preserving transformation group $\{\tau_x : \Omega \to \Omega\}_{x\in\mathbb{R}^d}$,
$$a(x+y,\omega) = a(x,\tau_y\omega).$$
- *ergodicity*: the transformation group is qualitatively mixing, for $g\colon \Omega \to \mathbb{R}$,
$$g(\tau_x\cdot) = g(\cdot) \ \text{ for every } \ x \in \mathbb{R}^d \ \text{ if and only if } g \text{ is constant.}$$

We are interested in the limiting behavior, as $\varepsilon \to 0$, of
$$-\nabla \cdot a(x/\varepsilon,\omega)\nabla\rho^\varepsilon = f \ \text{ in } \ U \ \text{ with } \ \rho^\varepsilon = g \ \text{ on } \ \partial U,$$



describes a system in equilibrium:
$$\oint_{B_r(x)} a(y/\varepsilon,\omega)\nabla\rho^\varepsilon(y)\cdot\nu = \int_{B_r(x)} f(y).$$



Periodic   Random tile   Poisson cloud   Cluster

The ergodic theorem [Becker]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with an ergodic measure preserving transformation group $\{\tau_x : \Omega \to \Omega\}_{x \in \mathbb{R}^d}$.

Then for every $f \in L^1(\Omega)$, for almost every $\omega \in \Omega$,

$$\lim_{R \to \infty} \fint_{B_R} f(\tau_x \omega)\, \mathrm{d}x = \mathbb{E}\,[f]\,.$$

And in the weak form, for every $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, as $\varepsilon \to 0$,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \psi(x) f(\tau_{x/\varepsilon}\omega)\, \mathrm{d}x \to \mathbb{E}[f] \int_{\mathbb{R}^d} \psi(x)\, \mathrm{d}x.$$

- A function $f \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$ is *stationary* and *ergodic* if
$$f(x,\omega) = f(0, \tau_x \omega) =: g(\tau_x \omega) \;\text{ for every }\; x \in \mathbb{R}^d \;\text{ and }\; \omega \in \Omega,$$
for some measurable $g \colon \Omega \to \mathbb{R}$ and $\{\tau_x\}_{x \in \mathbb{R}^d}$ is ergodic.
- *Ergodicity*: large-scale spatial averages almost surely approximate the expectation:
$$\fint_{B_R} f(x,\omega)\, \mathrm{d}x \simeq \mathbb{E}\,[f] \;\text{ as }\; R \to \infty.$$
- In the weak form, almost surely,
$$f(x/\varepsilon, \omega) \rightharpoonup \mathbb{E}[f] \;\text{ weakly as }\; \varepsilon \to 0.$$

VII. Stochastic homogenization

Stochastic homogenization: for the solutions

$$-\nabla \cdot a(x/\varepsilon, \omega)\rho^\varepsilon(x,\omega)) = f \;\text{ in }\; U \;\text{ with }\; \rho^\varepsilon(\cdot,\omega) = g \;\text{ on }\; \partial U,$$

there exists a deterministic $\overline{a} \in \mathbb{R}^{d\times d}$ such that

$$\rho^\varepsilon(\cdot,\omega) \to \overline{\rho} \;\text{ almost surely as } \varepsilon \to 0,$$

for the solution $\overline{\rho}$ of

$$-\nabla \cdot \overline{a}\nabla\overline{\rho} = f \;\text{ in }\; U \;\text{ with }\; \overline{\rho} = g \;\text{ on }\; \partial U.$$

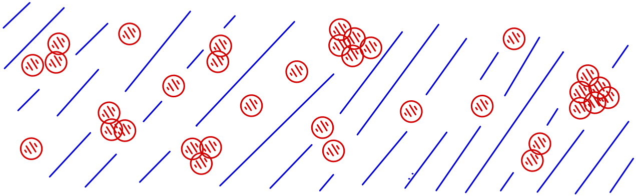
Diffusion in random environment: in the symmetric case, for the diffusion processes

$$\mathrm{d}X_t^\omega = \sigma(X_t^\omega, \omega)\,\mathrm{d}B_t + b(X_t^\omega,\omega)\,\mathrm{d}t,$$

for $a = \frac{1}{2}\sigma\sigma^t$, we have almost surely that

$$\varepsilon X^\omega_{t/\varepsilon^2} \to \overline{\sigma}B_t \;\text{ in law,}$$

for $\overline{a} = \frac{1}{2}\overline{\sigma\sigma}^t$.



VII. Stochastic homogenization

—The equation

$$-\nabla \cdot a(x/\varepsilon, \omega)\nabla \rho^\varepsilon = f \text{ in } U \text{ with } \rho^\varepsilon = g \text{ on } \partial U.$$

—The asymptotic expansion

$$\rho^\varepsilon(x,\omega) = \overline{\rho}(x) + \varepsilon \phi_i(x/\varepsilon, \omega)\partial_i \overline{\rho}(x) + \dots$$

—Almost surely by the ergodic theorem

$$a(x/\varepsilon, \omega)(e_i + \nabla \phi_i(x/\varepsilon, \omega)) \rightharpoonup \langle a(0,\omega)(e_i + \nabla \phi_i(0,\omega))\rangle =: \overline{a} e_i.$$

—By stationarity we have that $a(x,\omega) = A(\tau_x \omega)$ and $\nabla \phi(x,\omega) = \Phi_i(\tau_x \omega)$, so that

$$a(x/\varepsilon, \omega)(e_i + \nabla \phi_i(x/\varepsilon, \omega)) \rightharpoonup \langle a(0,\omega)(e_i + \nabla \phi_i(0,\omega))\rangle = \mathbb{E}\left[A(e_i + \Phi_i)\right].$$

—The first-order correctors ϕ_i almost surely satisfy on the whole space

$$-\nabla \cdot a(y,\omega)(e_i + \nabla \phi_i(y,\omega)) = 0 \text{ in } \mathbb{R}^d.$$

—For the homogenized coefficient $\overline{a}$ we have

$$-\nabla \cdot \overline{a} \nabla \overline{\rho} = f \text{ in } U \text{ with } \overline{\rho} = g \text{ on } \partial U.$$

VII. Stochastic homogenization

The corrector equation

$$-\nabla \cdot a(y,\omega)(e_i + \nabla \phi_i(y,\omega)) = 0 \ \text{ in } \ \mathbb{R}^d.$$

The periodic case:

- The probability space is the torus,

  $\Omega = \mathbb{T}^d$ with the Lebesgue sigma algebra and the normalized Lebesgue measure.

- The "random" variable

  $$A\colon \mathbb{T}^d \to \mathbb{R}^{d\times d} \ \text{ is 1-periodic.}$$

- The transformation group $\{\tau_x\}_{x\in\mathbb{R}^d}$ is defined by

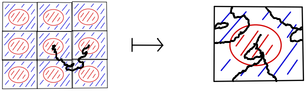
  $$\tau_x\omega = x + \omega \in \mathbb{T}^d \ \text{ for every } \ x \in \mathbb{R}^d \ \text{ and } \ \omega \in \mathbb{T}^d.$$

- The stationary "random" coefficient field is

  $$a(x,\omega) = A(x+\omega) = A(\tau_x\omega) \ \text{ for every } \ x \in \mathbb{R}^d \ \text{ and } \ \omega \in \mathbb{T}^d.$$

Lift the corrector equation to $\mathbb{T}^d$ using *the environment from the point of view of the particle*:

$$-\nabla \cdot A(y)(e_i + \nabla \phi_i(y)) = 0 \ \text{ in } \ \mathbb{T}^d.$$



The corrector equation

$$-\nabla \cdot a(y,\omega)(e_i + \nabla\phi_i(y,\omega)) = 0 \text{ in } \mathbb{R}^d.$$

and the asymptotic expansion $\rho^\varepsilon = \overline{\rho} + \varepsilon\phi_i(x/\varepsilon,\omega)\partial_i\overline{\rho} + \ldots$

- Validity of the asymptotic expansion requires *sublinearity*: almost surely,

$$\lim_{\varepsilon\to 0}\left(\sup_{B_1}\left(\varepsilon\left|\phi_i(x/\varepsilon,\omega)\right|\right)\right) = \lim_{R\to\infty}\left(R^{-1}\left(\sup_{B_R}|\phi_i(y,\omega)|\right)\right) = 0.$$

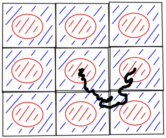
- In an L^2-sense, almost surely,

$$\lim_{R\to\infty}\left(R^{-1}\left(\fint_{B_R}\phi_i^2(y,\omega)\,\mathrm{d}y\right)^{\frac{1}{2}}\right) = 0.$$

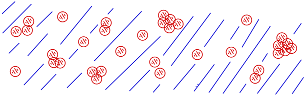
- A true *correction* of the diffusion process $\mathrm{d}X_t^\omega = \sigma(X_t^\omega,\omega)\,\mathrm{d}B_t + b(X_t^\omega,\omega)\,\mathrm{d}t$:

$$X_t^\omega = X_t^\omega + \phi(X_t^\omega) - \phi(X_t^\omega).$$

- *The environment from the point of view of the particle*:



$$\overline{\omega}_t = \tau_{X_t^\omega}\omega.$$



How to lift the corrector equation to Ω:

$$-\nabla \cdot a(y,\omega)(e_i + \nabla \phi_i(y,\omega)) = 0 \ \text{ in } \ \mathbb{R}^d.$$

- Differential operators on Ω:

$$D_i f(\omega) = \lim_{h \to 0} \frac{f(\tau_{he_i}\omega) - f(\omega)}{h} \ \text{ strongly in } \ L^2(\Omega).$$

- Smooth functions on Ω: for $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ and $f \in L^\infty(\Omega)$,

$$f_\psi(\omega) = \int_{\mathbb{R}^d} f(\tau_x \omega)\psi(\omega)\,\mathrm{d}x.$$

- Formally we have the H^1-space

$$\mathcal{H}^1(\Omega) = \cap_{i=1}^d \mathcal{D}(D_i).$$

- We can hope to solve

$$-D \cdot a(e_i + D\phi_i) = 0 \ \text{ in } \ \Omega,$$

  in the sense that

$$\mathbb{E}\left[a(e_i + D\phi_i) \cdot D\psi\right] = 0 \ \text{ for all } \ \psi \in \mathcal{H}^1(\Omega).$$

- No compactness, no Poincaré inequality, no Fredholm alternative.

VII. Stochastic homogenization

How to lift the corrector equation to Ω:

$$-\nabla \cdot a(y,\omega)(e_i + \nabla \phi_i(y,\omega)) = 0 \ \text{ in } \ \mathbb{R}^d.$$

- The differential operators and H^1-space:

$$D_i f(\omega) = \lim_{h\to 0} \frac{f(\tau_{he_i}\omega) - f(\omega)}{h} \ \text{ strongly in } \ L^2(\Omega) \ \text{ and } \ \mathcal{H}^1(\Omega) = \cap_{i=1}^d \mathcal{D}(D_i).$$

- The space of generalized gradients:

$$L^2_{\text{pot}}(\Omega) = \overline{\{D\psi \colon \psi \in \mathcal{H}^1(\Omega)\}}^{L^2(\Omega;\mathbb{R}^d)}.$$

- Every $\Phi \in L^2_{\text{pot}}(\Omega)$ is a gradient in the sense that it is distributionally curl free:

$$D_i\Phi_j = D_j\Phi_i \ \text{ for every } \ i,j \in \{1,\ldots,d\}.$$

- The Lax-Milgram lemma: there exists a unique $\Phi_i \in L^2_{\text{pot}}(\Omega)$ satisfying

$$-D \cdot A(e_i + \Phi_i) = 0 \ \text{ in } \ \Omega,$$

  in the sense that

$$\mathbb{E}\left[A(e_i + \Phi_i)\cdot \Psi\right] = 0 \ \text{ for every } \ \Psi \in L^2_{\text{pot}}(\Omega).$$

- We construct the *stationary gradient* of the corrector.

- We construct the stationary gradients of the correctors and flux correctors:

$$-D \cdot A(e_i + \Phi_i) = 0 \ \text{ and } \ -D \cdot \Sigma_{ijk} = D_j Q_{ik} - D_k Q_{ij},$$

for the fluxes $Q_i = A(e_i + \Phi_i) - \overline{a}e_i$.

- The correctors and flux correctors are almost surely defined by

$$\int_{B_1} \phi_i(y,\omega)\,\mathrm{d}y = 0 \ \text{ with } \ \nabla\phi_i(x,\omega) = \Phi_i(\tau_x\omega),$$

and

$$\int_{B_1} \sigma_{ijk}(y,\omega)\,\mathrm{d}y = 0 \ \text{ with } \ \nabla\sigma_{ijk}(x,\omega) = \Sigma_{ijk}(\tau_x\omega).$$

- For the fluxes $q_i = a(e_i + \nabla\phi_i) - \overline{a}e_i$ and for $\sigma_i = (\sigma_{ijk})$, almost surely,

$$-\nabla \cdot a(y,\omega)(e_i + \nabla\phi_i(y,\omega)) = 0 \ \text{ and } \ \nabla \cdot \sigma_i(y,\omega) = q_i(y,\omega) \ \text{ on } \ \mathbb{R}^d.$$

- Almost surely the homogenization error

$$w^\varepsilon(x,\omega) = \rho^\varepsilon(x,\omega) - \overline{\rho}(x) - \varepsilon\phi_i(x/\varepsilon,\omega)\partial_i\overline{\rho}$$

solves the equation

$$-\nabla \cdot a(x/\varepsilon,\omega)\nabla w^\varepsilon = \varepsilon\nabla \cdot \left(\left(a(x/\varepsilon,\omega)\phi_i(x/\varepsilon,o) - \sigma_i(x/\varepsilon,\omega)\right)\nabla\partial_i\overline{\rho}\right).$$

VII. Stochastic homogenization

- Almost surely the homogenization error

$$w^\varepsilon(x,\omega) = \rho^\varepsilon(x,\omega) - \overline{\rho}(x) - \varepsilon\phi_i(x/\varepsilon,\omega)\partial_i\overline{\rho}$$

  solves the equation

$$-\nabla\cdot a(x/\varepsilon,\omega)\nabla w^\varepsilon = \varepsilon\nabla\cdot\left(\left(a(x/\varepsilon,\omega)\phi_i(x/\varepsilon,\omega) - \sigma_i(x/\varepsilon,\omega)\right)\nabla\partial_i\overline{\rho}\right).$$

- The energy estimate, for some $c = c(\lambda,\Lambda,d,f,g) \in (0,\infty)$,

$$\int_U |\nabla w^\varepsilon|^2 \leq c\left(\int_U |\varepsilon\phi_i(x/\varepsilon,\omega)|^2 + |\varepsilon\sigma_i(x/\varepsilon,\omega)|^2\right).$$

- Homogenization requires L^2-*sublinearity*:

$$\lim_{\varepsilon\to 0}\left(\int_U |\varepsilon\phi_i(x/\varepsilon,\omega)|^2 \, \mathrm{d}x\right) = 0.$$

- It suffices to prove almost surely that

$$\lim_{\varepsilon\to 0}\left(\int_U |\varepsilon\phi_i(x/\varepsilon,\omega) - \langle\varepsilon\phi_i(\cdot/\varepsilon,\omega)\rangle_U|^2 \, \mathrm{d}y\right) = 0,$$

  for $\langle\varepsilon\phi_i(\cdot/\varepsilon,\omega)\rangle_U = \int_U \varepsilon\phi_i(x/\varepsilon,\omega)\,\mathrm{d}x$.

VII. Stochastic homogenization

To prove that:

$$\lim_{\varepsilon\to 0}\left(\int_U \left|\varepsilon\phi_i(x/\varepsilon,\omega) - \langle \varepsilon\phi_i(\cdot/\varepsilon,\omega)\rangle_U\right|^2 \,\mathrm{d}y\right) = 0.$$

- *The Poincaré inequality*: for every $\varepsilon \in (0,1)$, for $c = c(U) \in (0,\infty)$,

$$\int_U \left|\varepsilon\phi_i(x/\varepsilon,\omega) - \langle \varepsilon\phi_i(\cdot/\varepsilon,\omega)\rangle_U\right|^2 \,\mathrm{d}y \leq c\int_U \left|\nabla\phi_i(x/\varepsilon,\omega)\right|^2 \,\mathrm{d}y.$$

- *The ergodic theorem*: almost surely, for $c = c(\lambda,\Lambda) \in (0,\infty)$,

$$\lim_{\varepsilon\to 0}\int_U \left|\nabla\phi_i(x/\varepsilon,\omega)\right|^2 \,\mathrm{d}y = \mathbb{E}\left[\left|\Phi_i\right|^2\right] \leq c.$$

- *The Poincaré inequality*: almost surely,

$$\left\{\varepsilon\phi_i(x/\varepsilon,\omega) - \langle \varepsilon\phi_i(\cdot/\varepsilon,\omega)\rangle_U\right\}_{\varepsilon\in(0,1)} \text{ is bounded in } H^1(U).$$

- *The ergodic theorem*: almost surely,

$$\nabla\phi_i(x/\varepsilon,\omega) \rightharpoonup \mathbb{E}\left[\Phi_i\right] = 0 \text{ weakly in } H^1(U),$$

  and, therefore,

$$\varepsilon\phi_i(x/\varepsilon,\omega) - \langle \varepsilon\phi_i(\cdot/\varepsilon,\omega)\rangle_U \rightharpoonup c = 0 \text{ weakly in } H^1(U).$$

- *The Sobolev embedding theorem*: almost surely,

$$\varepsilon\phi_i(x/\varepsilon,\omega) - \langle \varepsilon\phi_i(\cdot/\varepsilon,\omega)\rangle_U \to 0 \text{ strongly in } L^2(U).$$

VII. Stochastic homogenization

Stochastic homogenization [Kozlov, Papanicolaou, Varadhan...]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $a \colon \Omega \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be a uniformly elliptic, stationary, and ergodic coefficient field and for every $\omega \in \Omega$ let ρ^ε solve the equation

$$-\nabla \cdot a(x/\varepsilon, \omega)\nabla \rho^\varepsilon = f \ \text{ in } \ U \ \text{ with } \ \rho^\varepsilon = g \ \text{ on } \ \partial U.$$

Let the homogenized coefficient $\overline{a} \in \mathbb{R}^{d\times d}$ be defined by

$$\overline{a} e_i := \mathbb{E}\left[A(e_i + \Phi_i)\right] \ \text{ for } \ \Phi_i \in L^2_{\text{pot}}(\Omega) \ \text{ satisfying } \ -D \cdot A(e_i + \Phi_i) = 0,$$

and let $\overline{\rho}$ be defined the homogenized solution

$$-\nabla \cdot \overline{a}\nabla\overline{\rho} = f \ \text{ in } \ U \ \text{ with } \ \overline{\rho} = g \ \text{ on } \ \partial U.$$

Then for the homogenization correctors ϕ_i defined by

$$\int_{B_1} \phi_i(y,\omega)\,\mathrm{d}y = 1 \ \text{ with } \ \nabla\phi_i(y,\omega) = \Phi_i(\tau_y\omega),$$

the two-scale expansion

$$w^\varepsilon(x,\omega) = \rho^\varepsilon(x,\omega) - \overline{\rho}(x) - \varepsilon\phi_i(x/\varepsilon,\omega)\partial_i\overline{\rho}(x),$$

almost surely satisfies

$$\lim_{\varepsilon\to 0} \|w^\varepsilon\|_{H^1(U)} = 0.$$

- *A regularity theory for random elliptic operators*; Gloria, Neukamm, Otto
- *Quantitative Stochastic Homogenization and Large-Scale Regularity*; Armstrong, et al.

VII. Stochastic homogenization

Divergence-free environments [Avelleneda, Komoroski, Majda, Olla, Kozma, Tóth, F...]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $a\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be a uniformly elliptic, stationary, and ergodic coefficient field and for every $\omega \in \Omega$ let ρ^ε solve the equation

$$-\nabla \cdot a(x/\varepsilon, \omega)\nabla \rho^\varepsilon + \varepsilon^{-1} b(x/\varepsilon, \omega) = f \ \text{ in } \ U \ \text{ with } \ \rho^\varepsilon = g \ \text{ on } \ \partial U,$$

for a stationary and ergodic, mean zero and divergence free drift $b(x,\omega) = B(\tau_x \omega)$. Assume that b admits a stationary L^p-integrable stream matrix S:

$$D \cdot S = B \ \text{ on } \ \Omega \ \text{ with } \ S \in L^p(\Omega; \mathbb{R}^{d\times d}).$$

Let the homogenized coefficient $\overline{a} \in \mathbb{R}^{d\times d}$ be defined by

$$\overline{a} e_i := \mathbb{E}\left[(A+S)(e_i + \Phi_i)\right] \ \text{ for } \ \Phi_i \in L^2_{\text{pot}}(\Omega) \ \text{ satisfying } \ -D\cdot (A+S)(e_i+\Phi_i) = 0,$$

and let $\overline{\rho}$ be the homogenized solution

$$-\nabla \cdot \overline{a} \nabla \overline{\rho} = f \ \text{ in } \ U \ \text{ with } \ \overline{\rho} = g \ \text{ on } \ \partial U.$$

If $p = 2$ then almost surely

$$\rho^\varepsilon \rightharpoonup \overline{\rho} \ \text{ weakly in } \ H^1(U).$$

If $p = d \wedge (2+\delta)$ then the two-scale expansion $w^\varepsilon = \rho^\varepsilon - \overline{\rho} - \varepsilon \phi_i \partial_i \overline{\rho}$ almost surely satisfies

$$\lim_{\varepsilon \to 0} \|w^\varepsilon\|_{H^1(U)} = 0.$$

- Also the case $b = \nabla U$ for a stationary potential U: Gibbs measure $m = \mathbb{E}\left[e^{-U}\right]^{-1} e^{-U}$

VII. Stochastic homogenization

The diffusion $\mathrm{d}X_t = \sigma(X_t, \omega)\,\mathrm{d}B_t$.

Homogenization of balanced environments [Papanicolaou, Varadhan]

Assume that $a\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ is uniformly elliptic, stationary, and ergodic. Then there exists a unique mutually absolutely continuous invariant measure π for the environment from the point of view of the particle on $(\Omega, \mathcal{F}, \mathbb{P})$: for every $f \in L^\infty(\Omega)$,

$$\mathbb{E}_\pi\left[E_{0,\omega}\left[f(\tau_{X_t}\omega)\right]\right] = \mathbb{E}_\pi\left[f\right].$$

The homogenized coefficient $\overline{a} \in \mathbb{R}^{d\times d}$ is defined by

$$\overline{a} = \mathbb{E}_\pi[a].$$

The solutions

$$\partial_t \rho^\varepsilon = \mathrm{tr}(a(x/\varepsilon, \omega)) \;\text{ on }\; \mathbb{R}^d \times (0,\infty) \;\text{ with }\; \rho^\varepsilon(\cdot, 0) = \rho_0,$$

converge almost surely as $\varepsilon \to 0$ to the solution

$$\partial_t \overline{\rho} = \mathrm{tr}(\overline{a}\nabla^2\overline{\rho}) \;\text{ on }\; \mathbb{R}^d \times (0,\infty) \;\text{ with }\; \rho(\cdot, 0) = \rho_0.$$

- The Aleksandrov-Bakelman-Pucci estimate: suppose that ρ^ε solves

  $$\mathrm{tr}(a(x/\varepsilon, \omega)\nabla^2\rho^\varepsilon) = f \;\text{ in }\; B_1 \;\text{ with }\; \rho^\varepsilon = 0 \;\text{ on }\; \partial B_1.$$

  Then, for $c = c(\lambda, \Lambda, d) \in (0,\infty)$ independent of $\varepsilon \in (0,1)$,

  $$\|\rho^\varepsilon\|_{L^\infty(B_1)} \leq c\,\|f\|_{L^d(B_1)}.$$

Consider the diffusion in random environment

$$\mathrm{d}X_t = \sigma(X_t, \omega)\,\mathrm{d}B_t + b(X_t, \omega)\,\mathrm{d}t.$$

In the periodic case, for the invariant measure m and in the central limit scaling,

$$\langle b, m \rangle_{L^2(\mathbb{T}^d)} = \bar{b} = 0 \text{ implies a diffusive behavior,}$$

and

$$\bar{b} \neq 0 \text{ implies ballistic behavior in direction } \bar{b}.$$

In the absence of an invariant measure try to rule out ballistic behavior using *symmetry*. Assume that, for every orthogonal transformation r that preserves the coordinate axis,

$$(r\sigma(x,\omega), rb(x,\omega))_{x\in\mathbb{R}^d} \text{ and } (\sigma(rx,\omega), b(rx,\omega))_{x\in\mathbb{R}^d} \text{ have the same law.}$$

Then, since for every orthogonal transformation r preserving the coordinate axis,

$$X_t \text{ and } rX_t \text{ have the same law under } \mathbb{P} \ltimes P_{0,\omega}.$$

In the *annealed sense* we have that

$$\mathbb{E}\left[E_{0,\omega}[X_t]\right] = 0.$$

Homogenization of isotropic diffusions [Sznitman, Zeitouni, F.]

Let $a\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ and $b\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ be uniformly elliptic, bounded, Lipschitz continuous, and stationary coefficient fields satisfying a finite range of dependence. Assume that for every orthogonal transformation r preserving the coordinate axis

$$(ra(x,\omega)r^t, rb(x,\omega))_{x\in\mathbb{R}^d} \text{ and } (a(rx,\omega), b(rx,\omega))_{x\in\mathbb{R}^d} \text{ have the same law.}$$

Then there exists $\eta \in (0,\infty)$ such that if

$$|a - I| \leq \eta \text{ and } |b| \leq \eta,$$

then there exists $\overline{a} \in \mathbb{R}$ such that the solutions

$$\partial_t \rho^\varepsilon = \operatorname{tr}(a(x/\varepsilon,\omega)\nabla^2\rho^\varepsilon) + \varepsilon^{-1} b(x/\varepsilon,\omega)\cdot\nabla\rho^\varepsilon \text{ in } \mathbb{R}^d\times(0,\infty) \text{ with } \rho^\varepsilon(\cdot,0) = \rho_0,$$

converge almost surely as $\varepsilon \to 0$ to the solution of

$$\partial_t\overline{\rho} = \overline{a}\Delta\overline{\rho} \text{ in } \mathbb{R}^d\times(0,\infty) \text{ with } \overline{\rho}(\cdot,0) = \rho_0.$$

Furthermore, there exists a unique mutually absolutely continuous invariant measure π on $(\Omega,\mathcal{F},\mathbb{P})$ for the process from the point of view of the particle: for every $f \in L^\infty(\Omega)$,

$$\mathbb{E}_\pi\left[E_{0,\omega}\left[f(\tau_{X_t}\omega)\right]\right] = \mathbb{E}_\pi\left[f\right].$$

- the perturbation says that for short times the process is like a Brownian motion
- an inductive renormalization argument controls traps / localization / coupling