Rare events in particle systems and stochastic PDE

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LSU

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Independent coin flips: independent random variables with

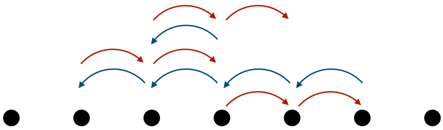
$$\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = {}^1\!/_2.$$

The simple random walk: the process $(S_n)_{n\in\mathbb{N}_0}$ with $S_0 = 0$ and

$$S_n = X_1 + \ldots + X_n.$$

A normal random variable: with mean μ and variance σ,

$$\mathbb{P}[a \leq Y < b] = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{|x-\mu|^2}{2\sigma^2}\Big).$$



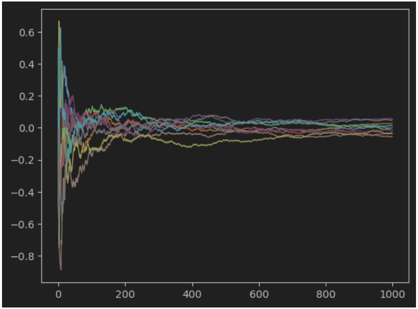
A realization of S_{11}.

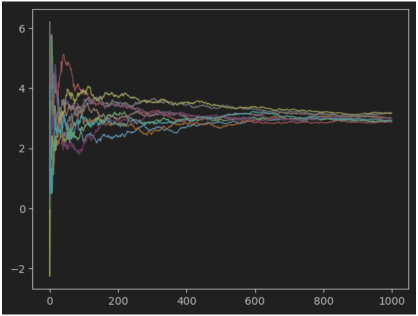
The Law of Large Numbers: for independent coin flips,

$$\lim_{n\to\infty} \frac{X_1 + \ldots + X_n}{n} = \mathbb{E}[X_1] = 0.$$

Normal random variables: independent with mean 3 and variance 25,

$$\lim_{n\to\infty} \frac{Y_1 + \ldots + Y_n}{n} = \mathbb{E}[Y_1] = 3.$$





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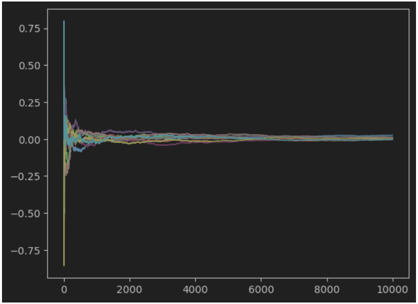

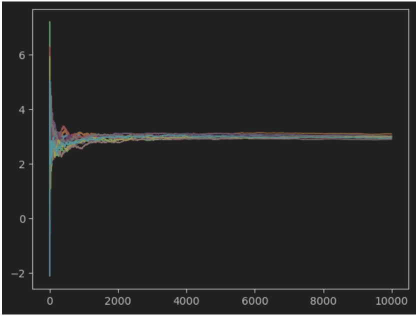

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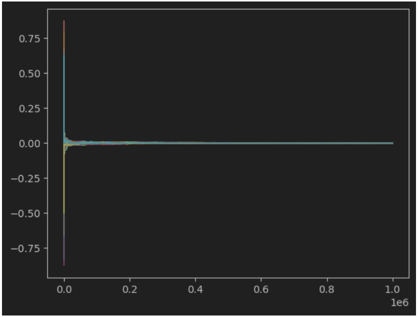
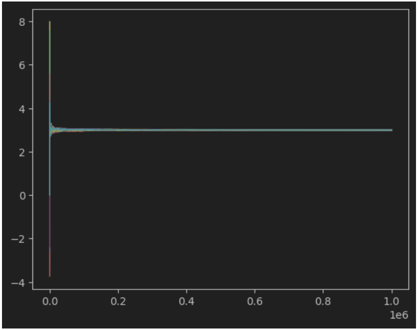


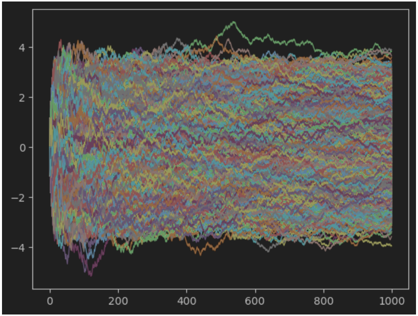


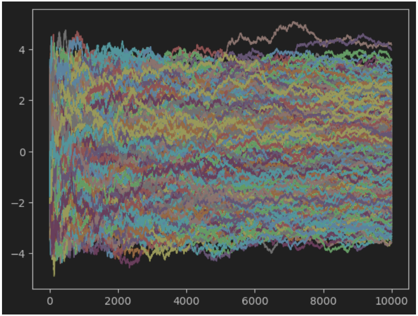
The Central Limit Theorem: for independent coin flips, for every $a \leq b \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left[a \leq \frac{X_1 + \ldots + X_n}{\sqrt{n}} < b\right] = \int_a^b (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x.$$

Normal random variables: for independent mean zero variance one variables,

$$\mathbb{P}\left[a \leq \frac{Y_1 + \ldots + Y_n}{\sqrt{n}} < b\right] = \int_a^b (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right).$$





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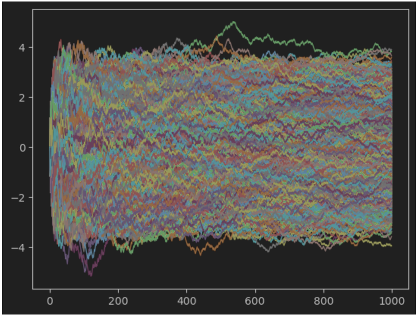
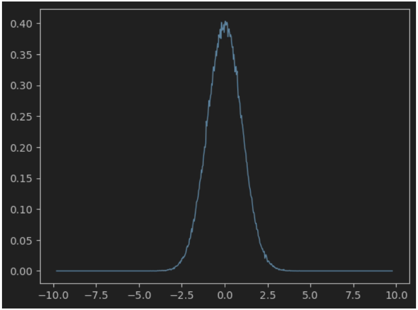




Linear central limit expansion: combining the LLN and CLT,

$$\frac{X_1+\ldots+X_n}{n} \simeq \mathbb{E}[X_1] + \frac{1}{\sqrt{n}}\mathcal{N}(0,1).$$

Understanding rare events: for $\delta \in (0,\infty)$,

$$\begin{aligned}\mathbb{P}\left[\frac{X_1+\ldots+X_n}{n} \geq \delta\right] &= \mathbb{P}\left[\frac{X_1+\ldots+X_n}{\sqrt{n}} \geq \sqrt{n}\delta\right] \\ &\simeq \int_{\sqrt{n}\delta}^{\infty} (2\pi)^{-1} \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \\ &\simeq \exp\left(-\frac{n\delta^2}{2}\right).\end{aligned}$$

Normal random variables: if Y_i are independent $\mathcal{N}(0,1)$,

$$\begin{aligned}\mathbb{P}\left[\frac{Y_1+\ldots+Y_n}{n} \geq \delta\right] &= \int_{\sqrt{n}\delta}^{\infty} (2\pi)^{-1} \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \\ &\text{“}=\text{”} \exp\left(-\frac{n\delta^2}{2}\right).\end{aligned}$$

I. The simple random walk

The simple random walk: for independent coin flips X_i,

$$\overline{X}_n = \frac{X_1 + \ldots X_n}{n}.$$

A large deviations principle: the $\overline{X}_n$ satisfy an LDP with rate function I if

$$-\inf_{x \in A^\circ} I(x) \leq \liminf_{n\to\infty} \left(\frac{1}{n} \log \left(\mathbb{P}(\overline{X}_n \in A) \right) \right) \text{ for all } A \subseteq \mathbb{R},$$

and

$$\limsup_{n\to\infty} \left(\frac{1}{n} \log \left(\mathbb{P}(\overline{X}_n \in A) \right) \right) \leq -\inf_{x \in \overline{A}} I(x) \text{ for all } A \subseteq \mathbb{R}.$$

The rate function: if such a lower semicontinuous rate function exists, then

$$I \colon \mathbb{R} \to [0, \infty] \text{ is unique.}$$

The informal LDP: we will write

$$\mathbb{P}(\overline{X}_n \simeq x) \simeq \exp(-nI(x)) \text{ or, } \mathbb{P}(\overline{X}_n \in A) \simeq \exp\left(-n(\inf_{x\in A} I(x)) \right).$$

The simple random walk: for independent coin flips X_i,

$$\overline{X}_n = \frac{X_1 + \ldots X_n}{n}.$$

Cramér's theorem: the simple random walk satisfies an LDP with rate function

$$\begin{aligned} I(x) &= \tanh^{-1}(x)x - \log\left(\frac{1}{2}\left(e^{-\tanh^{-1}(x)} + e^{\tanh^{-1}(x)}\right)\right) \\ &= \frac{x^2}{2} + o(x^2), \end{aligned}$$

and $I(x) = \infty$ if $|x| > 1$.

Linear central limit expansion: combining the LLN and CLT,

$$\overline{X}_n = \frac{X_1 + \ldots + X_n}{n} \simeq \mathbb{E}[X_1] + \frac{1}{\sqrt{n}}\mathcal{N}(0,1) = \frac{Y_1 + \ldots + Y_n}{n},$$

for independent normal random variables Y_i.

Large deviations of the CLT expansion: for $\overline{Y}_n = n^{-1}(Y_1 + \ldots + Y_n)$,

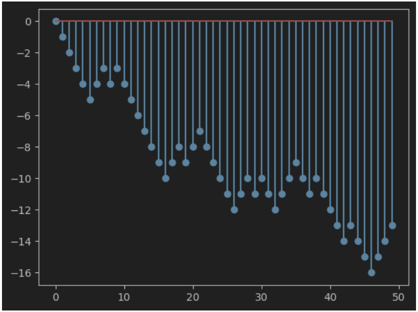
$$\mathbb{P}(\overline{Y}_n \simeq x) \simeq \exp\left(-\frac{nx^2}{2}\right) = \exp(-n\tilde{I}(x)) \text{ with rate function } \tilde{I}(x) = \frac{x^2}{2}.$$

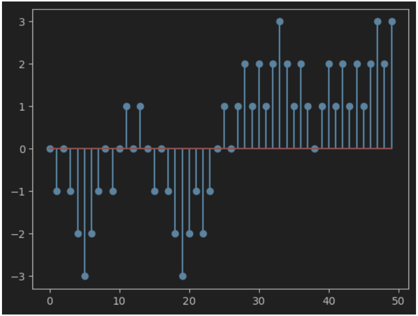
The random walk as a path: for the random walk

$$S_n = X_1 + X_2 + \dots X_n,$$

we define the path

$$W_t = S_{\lfloor t \rfloor}.$$





II. Brownian motion

The scaling limit: for the path $W_t = S_{\lfloor t \rfloor}$,

$$W_t^n = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}.$$

The increments: if $s < t$,

$$W_t^n - W_s^n = \frac{1}{\sqrt{n}} \Big(\sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor} X_i \Big) \simeq \frac{n(t-s) \text{ i.i.d. copies of } X_1}{\sqrt{n}}.$$

The central limit theorem: formally, as $n \to \infty$,

$$W_t^n - W_s^n \simeq \mathcal{N}(0, t-s),$$

a normal mean zero variance $t - s$ random variable.

Independent increments: if $0 \leq s_1 \leq s_2 \leq t_1 \leq t_2 < \infty$,

$$W_{s_2}^n - W_{s_1}^n \ \text{ and } \ W_{t_2}^n - W_{t_1}^n \ \text{ are independent.}$$

Time zero and jumps: we have that

$$W_0^n = 0 \ \text{ and the jump discontinuities are of order } \ \frac{1}{\sqrt{n}}.$$

II. Brownian motion

The scaling limit: for the path $W_t = S_{\lfloor t \rfloor}$,

$$W_t^n = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}.$$

Donsker's theorem: on path space, for a random path B_t,

$$W_t^n \rightharpoonup B_t \ \text{ in law.}$$

Brownian motion: the path B_t almost surely satisfies that

(i) $B_0 = 0$ and that **(ii)** the path $t \to B_t$ is continuous,

that, for every $s \leq t \in [0, \infty)$,

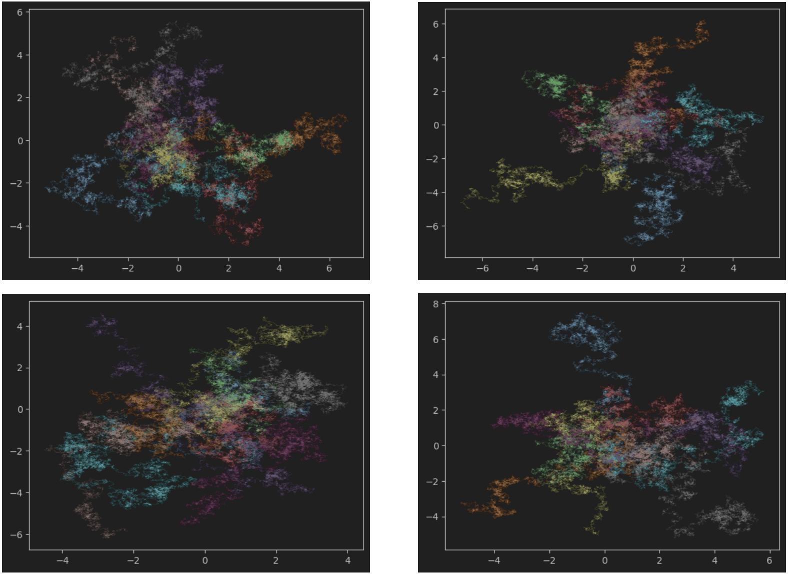
(iii) $(B_t - B_s) = \mathcal{N}(0, t-s)$,

and that, for every $0 \leq s_1 \leq s_2 \leq t_1 \leq t_2$,

(iv) $(B_{s_2} - B_{s_1})$ and $(B_{t_2} - B_{t_1})$ are independent.

Uniqueness: such properties uniquely characterize Brownian motion.

Brownian motion: 10 realizations on $[0, 10]$ in 2-dimensions,



Small noise limit: for a Brownian motion B_t,

$$W_t^\varepsilon = \sqrt{\varepsilon} B_t.$$

The law of large numbers: almost surely by continuity,

$$\lim_{\varepsilon \to 0} W_t^\varepsilon = 0.$$

The central limit theorem: we have that

$$\frac{1}{\sqrt{\varepsilon}}\left(W_t^\varepsilon - 0\right) = \frac{1}{\sqrt{\varepsilon}} \cdot \sqrt{\varepsilon} B_t = B_t.$$

Schilder's theorem: the paths W^ε satisfy an LDP with rate function

$$I(\pi) = \frac{1}{2} \inf \left\{ \int_0^t |g(s)|^2 \, \mathrm{d}s \colon \pi_\cdot = \int_0^\cdot g(s) \, \mathrm{d}s \right\}.$$

That is, for the minizer g,

$$\mathbb{P}(W^\varepsilon \simeq \pi) \simeq \exp\left(-\varepsilon^{-1} I(\pi) \right) \text{ and } \pi_t' = g_t.$$

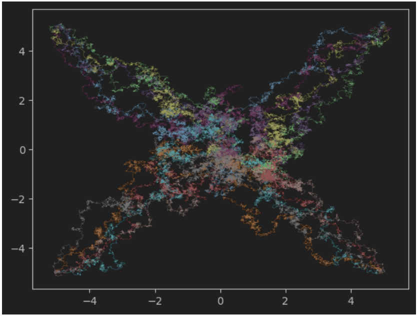
The Ornstein–Uhlenbeck Process: the random path X_t solving

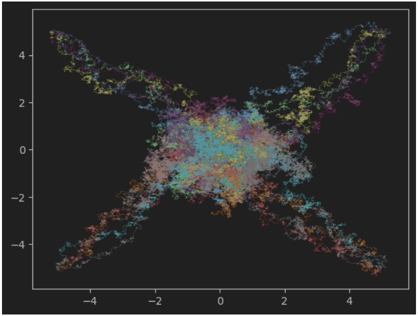
$$\mathrm{d}X_t = -X_t\,\mathrm{d}t + \mathrm{d}B_t \ \text{ in } \ (0,\infty).$$

An explicit representation: in terms of stochastic integration,

$$X_t = e^{-tX_0} + \int_0^t e^{s-t}\,\mathrm{d}B_s.$$

Example: 5 copies each on $[0,3]$ and $[0,10]$ from the four corners,





III. The Ornstein–Uhlenbeck process

The Ornstein–Uhlenbeck Process: the random path X_t solving

$$\mathrm{d}X_t = -X_t\,\mathrm{d}t + \mathrm{d}B_t \;\text{ in }\; (0,\infty).$$

Small noise limit: the process Y_t^ε solves

$$\mathrm{d}Y_t^\varepsilon = -Y_t^\varepsilon\,\mathrm{d}t - \sqrt{\varepsilon}B_t.$$

The law of large numbers: almost surely as $\varepsilon \to 0$,

$$Y_t^\varepsilon \to Y_t \;\text{ for } Y_t \text{ solving }\; Y_t' = -Y_t.$$

The central limit theorem: for every $\varepsilon \in (0,1)$,

$$Z_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}}\big(Y_t^\varepsilon - Y_t\big) \;\text{ solves }\; \mathrm{d}Z_t^\varepsilon = -Z_t^\varepsilon\,\mathrm{d}t + \mathrm{d}B_t,$$

and $Z_t^\varepsilon \simeq X_t$ in law.

Freidlin–Wentzell theorem: the paths Y^ε satisfy an LDP with rate function

$$I(\pi) = \frac{1}{2}\inf\left\{\int_0^t |g(s)|^2\,\mathrm{d}s \colon \pi_\cdot = \int_0^\cdot g(s) - \pi(s)\right\}.$$

For the minimizer g,

$$\pi_t' = -\pi_t + g_t.$$

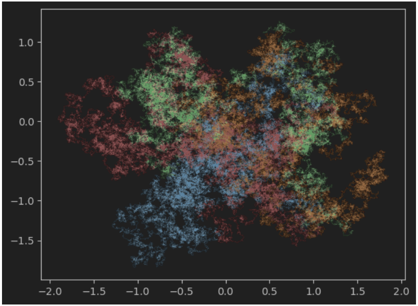
The Ornstein–Uhlenbeck Process: the random path X_t solving

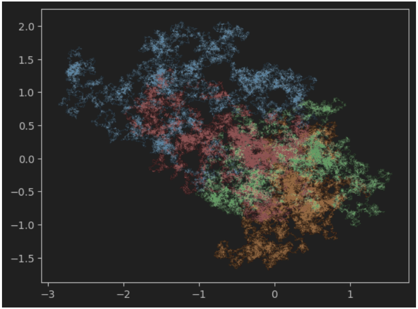
$$\mathrm{d}X_t = -X_t\,\mathrm{d}t + \mathrm{d}B_t \ \text{ in } \ (0,\infty).$$

Freidlin–Wentzell theorem: the paths Y^ε satisfy an LDP with rate function

$$I(\pi) = \frac{1}{2}\inf\left\{ \int_0^t |g(s)|^2\,\mathrm{d}s \colon \pi_\cdot = \int_0^\cdot g(s) - \pi(s) \right\}.$$

Example: four Ornstein–Uhlenbeck processes from the origin on $[0,3]$ and $[0,10]$,



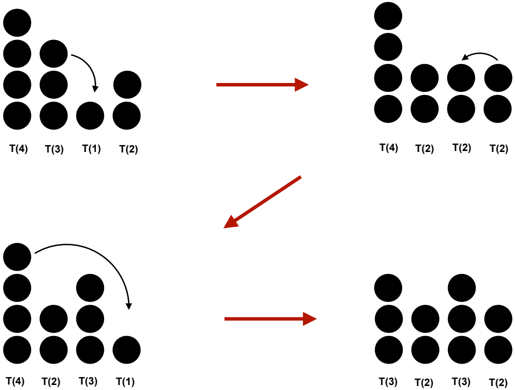


Random clocks: for $g\colon \mathbb{N}_0 \to \mathbb{N}_0$ be nondecreasing with $g(0)=0$,

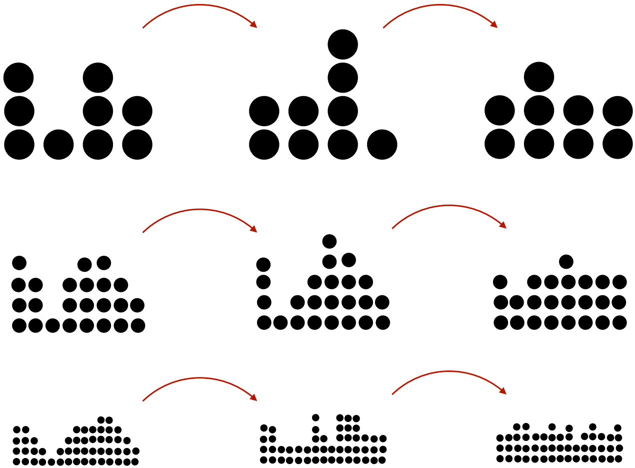
$$\mathbb{P}[0 \leq T(k) < a] = \int_0^a g(k)e^{-kt}\,\mathrm{d}t.$$

Independent particles: the case $g(k)=k$.

The zero range process: periodic with four sites,



The parabolically rescaled process: the cases $N = 4, 8, 15$,



The parabolically rescaled process: at scale N on $\mathbb{T}^1$,

$$\mu_t^N = \frac{1}{N^d}\sum_{x\in(\mathbb{Z}^d/N\mathbb{Z}^d)} \delta_{\frac{x}{N}} \cdot \eta^N_{N^2t}(x).$$

The hydrodynamic limit: for $f \in \mathrm{C}(\mathbb{T}^d \times [0,T])$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \mathbb{P}\left[|\langle f, \mu^N\rangle - \langle f, \overline{\rho}\rangle| > \delta\right] = 0,$$

where $\overline{\rho}\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ is the unique solution of

$$\partial_t \overline{\rho} = \Delta\phi(\overline{\rho}).$$

for the mean local jump rate ϕ.

Independent particles: in this case $\phi(\xi) = \xi$ and

$$\partial_t\overline{\rho} = \Delta\overline{\rho}.$$

Queuing: in the case that $g(k) = \mathbf{1}_{\{k>0\}}$, we have $\phi(\xi) = \frac{1}{1+\xi}$ and

$$\partial_t\overline{\rho} = \Delta\Big(\frac{\overline{\rho}}{1+\overline{\rho}}\Big).$$

IV. The zero range process

The parabolically rescaled process: at scale N on $\mathbb{T}^1$,

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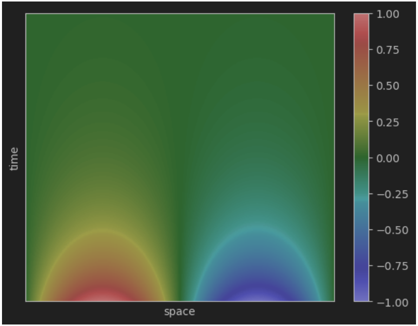
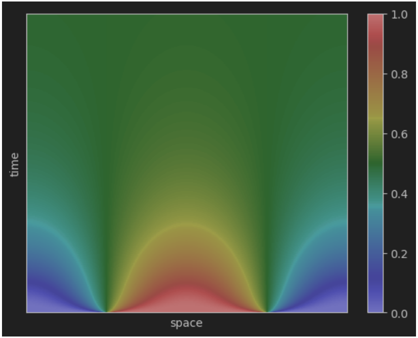
The hydrodynamic limit: for $\partial_t \overline{\rho} = \Delta \phi(\overline{\rho})$,

$$\mu_t^N \rightharpoonup \overline{\rho}\, \mathrm{d}x\, \mathrm{d}t.$$

Example: the case of $\phi(\xi) = \xi$ with $\rho_0(x) = \sin(2\pi x)$ and $\rho_0 = \mathbf{1}_{(1/4,3/4)}$,





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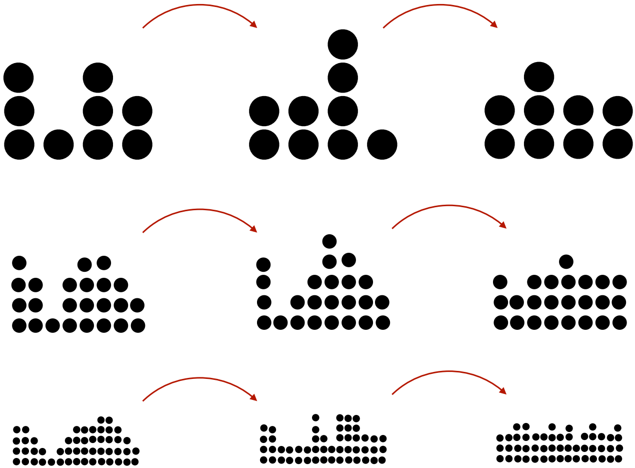


Space-time white noise: a random scalar $\mathrm{d}\xi$ on $\mathbb{T}^d \times [0,T]$ satisfying

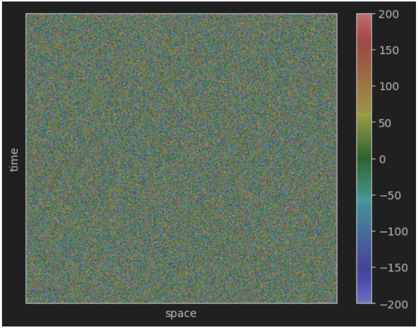
$$\mathbb{E}[\,\mathrm{d}\xi(x,s)\,\mathrm{d}\xi(y,t)] = \delta_0(x-y)\delta_0(s-t),$$

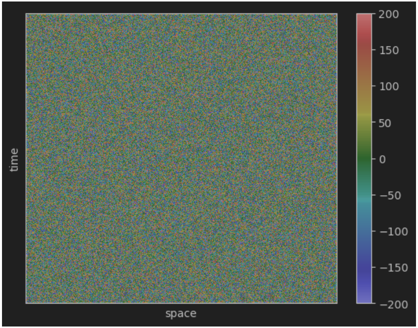
or, as a distribution, for $f,g \in L^2(\mathbb{T}^d \times [0,T])$,

$$\mathbb{E}\big[\langle f,\,\mathrm{d}\xi\rangle\langle g,\,\mathrm{d}\xi\rangle\big] = \langle f,g\rangle.$$

Fourier representation: for an orthonormal $L^2(\mathbb{T}^d)$-basis e_k,

$$\xi(x,t) = \sum_{k\in\mathbb{Z}^d} e_k(x)B_t^k \quad\text{and}\quad \mathrm{d}\xi(x,t) = \sum_{k\in\mathbb{Z}^d} e_k(x)\,\mathrm{d}B_t^k.$$





The parabolically rescaled process: at scale N on $\mathbb{T}^1$,

$$\mu_t^N = \frac{1}{N^d} \sum_{x \in (\mathbb{Z}^d / N\mathbb{Z}^d)} \delta_{\frac{x}{N}} \cdot \eta^N_{N^2 t}(x).$$

The hydrodynamic limit: for $\partial_t \overline{\rho} = \Delta \phi(\overline{\rho})$,

$$\mu_t^N \rightharpoonup \overline{\rho} \, \mathrm{d}x \, \mathrm{d}t.$$

The generalized Ornstein-Uhlenbeck process: as $N \to \infty$,

$$N^{\frac{d}{2}} \left(\mu_t^N - \overline{\rho} \, \mathrm{d}x \, \mathrm{d}t \right) \rightharpoonup v,$$

for v satisfying, for a d-dimensional space time white noise $\mathrm{d}\xi$,

$$\partial_t v = \nabla \cdot \phi'(\overline{\rho}) \nabla v - \nabla \cdot (\phi^{\frac{1}{2}}(\overline{\rho}) \, \mathrm{d}\xi).$$

The case of independent particles: we have that

$$\partial_t v = \Delta v - \nabla \cdot (\sqrt{\overline{\rho}} \, \mathrm{d}\xi).$$

The hydrodynamic limit: for $\partial_t \overline{\rho} = \Delta \phi(\overline{\rho})$,

$$\mu_t^N \rightharpoonup \overline{\rho}\, \mathrm{d}x\, \mathrm{d}t.$$

The central limit theorem: as $N \to \infty$,

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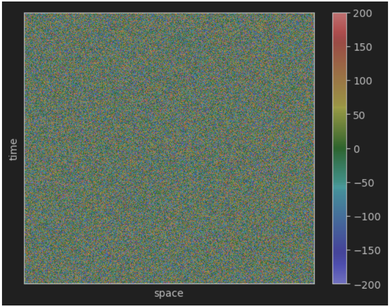
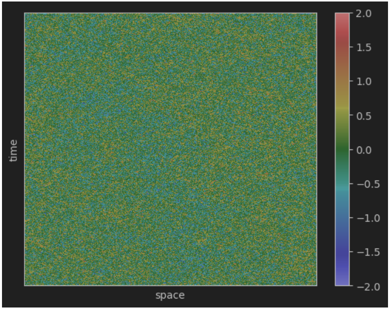
Example: the case $\phi(\xi) = \xi$,





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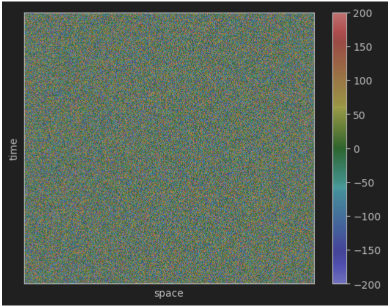
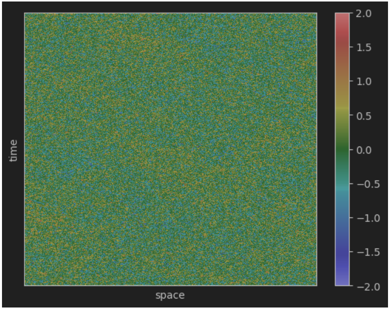
$$\partial_t v = \nabla\cdot\phi'(\overline{\rho})\nabla v - \nabla\cdot(\phi^{\frac{1}{2}}(\overline{\rho})\,\mathrm{d}\xi).$$

Example: the case $\phi(\xi)=\xi$,





The hydrodynamic limit: for $\partial_t \overline{\rho} = \Delta \phi(\overline{\rho})$,

$$\mu_t^N \rightharpoonup \overline{\rho}\, \mathrm{d}x\, \mathrm{d}t.$$

The central limit theorem: as $N \to \infty$,

$$N^{\frac{d}{2}} \left(\mu_t^N - \overline{\rho}\, \mathrm{d}x\, \mathrm{d}t \right) \rightharpoonup v,$$

for v satisfying, for a d-dimensional space time white noise $\mathrm{d}\xi$,

$$\partial_t v = \nabla \cdot \phi'(\overline{\rho}) \nabla v - \nabla \cdot (\phi^{\frac{1}{2}}(\overline{\rho})\, \mathrm{d}\xi).$$

The linear CLT expansion: does the expansion

$$\mu_t^N \simeq \overline{\rho} + N^{-\frac{d}{2}} v.$$

Space-time white noise: if $\xi = \sum_{k \in \mathbb{Z}^d} e_k B_t^k$, for the rate function

$$I(f) = \frac{1}{2} \inf\{ \|g\|_{L^2}^2 : \partial_t f(x,t) = g(x,t) \},$$

we have that

$$P\left(\varepsilon^{\frac{d}{2}} \xi \simeq f \right) \simeq \exp(-\varepsilon^{-d} I(f)).$$

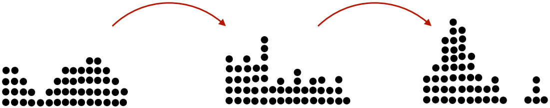
The Ornstein–Uhlenbeck rate function: the linear expansion suggests that, for

$$\tilde{I}(\pi) = \frac{1}{2}\inf\{\|g\|^2_{L^2} : \partial_t \pi = \Delta\phi(\pi) - \nabla \cdot (\phi^{\frac{1}{2}}(\overline{\rho})g)\},$$

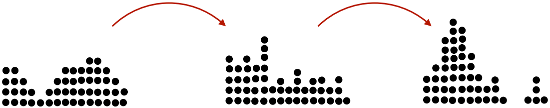
we have

$$\mathbb{P}\big(\mu^N_t \simeq \pi\big) \simeq \exp\big(-N^d \tilde{I}(\pi)\big).$$

The controlled OU-equation: $\partial_t \overline{p} = \Delta\overline{\rho} - \nabla \cdot (\sqrt{\overline{\rho}} g)$ on $\mathbb{T}^1 \times [0, T]$,



The controlled equation: $\partial_t \rho = \Delta\rho - \nabla \cdot (\sqrt{\rho} \cdot g)$ on $\mathbb{T}^1 \times [0, T]$,



The rate function: we have

$$I(\pi) = \frac{1}{2}\inf\left\{ \|g\|^2_{L^2_{t,x}} : m = \frac{\mathrm{d}\pi}{\mathrm{d}x} \text{ and } \partial_t m = \Delta\phi(m) - \nabla\cdot(\phi^{\frac{1}{2}}(m)g)\right\}.$$

Large Deviations Principle [Benois, Kipnis, Landim; 1995]

Let μ^N be the parabolically rescaled zero range process on $\mathbb{T}^d \times [0,T]$ (or, $\mathbb{R}\times[0,T]$) started from stationarity. For every closed subset $A \subseteq D([0,T];\mathcal{M}_+(\mathbb{T}^1))$,

$$\limsup_{N\to\infty}\left(\frac{1}{N}\log\left(\mathbb{P}[\mu^N \in U]\right)\right) \leq -\inf_{\pi\in A} I(\pi),$$

and, for open subset $U \subseteq D([0,T];\mathcal{M}_+(\mathbb{T}^1))$,

$$\liminf_{N\to\infty}\left(\frac{1}{N}\log\left(\mathbb{P}[\mu^N \in A]\right)\right) \geq -\inf_{\pi\in U} \overline{I|_{\mathcal{S}}}^{\mathrm{lsc}}(\pi),$$

for the space $\mathcal{S} = \left\{\pi\colon \partial_t\pi = \Delta\phi(\pi) - \nabla\cdot(\phi(\pi)\nabla H) \text{ for } H\in \mathrm{C}^{1,3}_{t,x}\right\}$.

Counterexample: for LDP of Kac's conservative particle system [Heydecker; 2022].

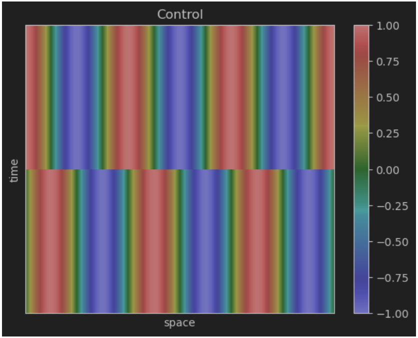
Fluctuating Boltzmann equation: [Bod., Gal., S.-Ray. Sim.; 2022].

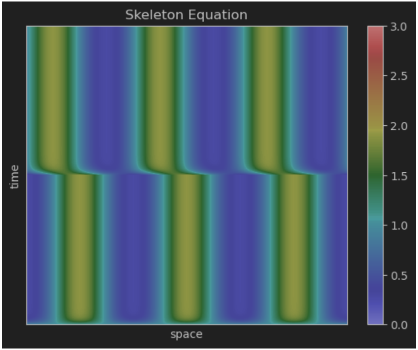
The skeleton equation: on $\mathbb{T}^1 \times [0,T]$,

$$\partial_t \rho = \Delta\rho - \nabla \cdot (\sqrt{\rho} g).$$

Example: for small times,

$$g(x,t) = \sin(6\pi x) \text{ and, for larger times } g(x,t) = \cos(6\pi x).$$





DiPerna–Lions theory [DiPerna, Lions; 1989], [Ambrosio; 2004]

Let $b \in (L^1_t BV_x)^d$ and $(\nabla \cdot b) \in (L^1_t L^\infty_x)^d$. Then, for every $\rho_0 \in L^\infty(\mathbb{T}^d)$,

$$\partial_t \rho = \nabla \cdot (\rho b),$$

has a unique weak solution in $(L^1 \cap L^\infty)(\mathbb{T}^d \times [0,T])$.

Counterexample: [Depauw; 2003] shows that this is essentially optimal.

Equivalence of weak and renormalized kinetic solutions [F., Gess; 2023], [F., Gess, Heydecker; 2025]

The skeleton equation

$$\partial_t \rho = \Delta \phi(\rho) - \nabla \cdot (\phi^{\frac{1}{2}}(\rho) g) \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ \rho(\cdot,0) = \rho_0,$$

is well-posed for L^2-controls and data with finite entropy.

Full large deviations principle [F., Gess; 2023], [F., Gess, Heydecker; 2025]

The rate functions I and $\overline{I_{|\mathcal{S}}}(\rho)$ are equal, and the zero range process satisfies a LDP with matching upper and lower bounds with rate function I.

The Dean–Kawasaki equation: for the empirical densities

$$m_n(x,t) = \frac{1}{n}\sum_{i=1}^{n}\delta_0(x - B_t^i),$$

we have formally in one-dimension that

$$\partial_t m_n = \frac{1}{2}\Delta m_n - N^{-\frac{d}{2}}\nabla\cdot(\sqrt{m_n}\,\mathrm{d}\xi).$$

Supercriticality: no renormalization theory, in general, for

$$\partial_t \rho_\varepsilon = \frac{1}{2}\Delta\phi(\rho_\varepsilon) - \varepsilon^{\frac{d}{2}}\nabla\cdot(\phi^{\frac{1}{2}}(\rho_\varepsilon)\xi).$$

Existence and uniqueness [F., Gess; 2024]

There exists a unique *stochastic kinetic solution* to the equation

$$\partial_t \rho_\varepsilon = \frac{1}{2}\Delta\phi(\rho_\varepsilon) - \varepsilon^{\frac{d}{2}}\nabla\cdot(\phi^{\frac{1}{2}}(\rho_\varepsilon)\circ\mathrm{d}\xi^\delta),$$

for nonnegative, integrable initial data. Furthermore, any two solutions $\mathbb{P}$-a.s. satisfy

$$\max_{t\in[0,T]}\|\rho_1 - \rho_2\|_{L^1} \leq \|\rho_{1,0} - \rho_{2,0}\|_{L^1}.$$

V. Stochastic PDE

Quantitative LLN and Central Limit Theorem [Clini, F.; 2025]

The scaling limit: let $\delta(\varepsilon)$ be any sequence satisfying, as $\varepsilon \to 0$,

$$\varepsilon\delta(\varepsilon)^{-(1+\frac{2}{d})} \to 0 \ \text{ and } \ \delta(\varepsilon) \to 0,$$

and for every $\varepsilon \in (0,1)$ let ρ^ε be the solution

$$\partial_t \rho^\varepsilon = \Delta\phi(\rho^\varepsilon) - \varepsilon^{\frac{d}{2}} \nabla \cdot (\phi^{\frac{1}{2}}(\rho^\varepsilon) \circ \xi^{\delta(\varepsilon)}).$$

Law of Large Numbers: then almost surely with a quantitative rate, as $\varepsilon \to 0$,

$$\rho^\varepsilon \to \overline{\rho} \ \text{ for } \ \partial_t \overline{\rho} = \Delta\phi(\overline{\rho}).$$

Central Limit Theorem: then as distributions, as $\varepsilon \to 0$,

$$\varepsilon^{-\frac{1}{2}}(\rho^\varepsilon - \overline{\rho}) \to v \ \text{ for } \ \partial_t v = \Delta(\phi'(\rho)v) - \nabla \cdot (\phi^{\frac{1}{2}}(\overline{\rho})\xi).$$

Universality: the central limit theorem is a statement about “universal” behavior:

$$\partial_t \tilde{\rho}^\varepsilon = \Delta\phi(\tilde{\rho}^\varepsilon) - \varepsilon^{\frac{d}{2}} \nabla \cdot (\phi^{\frac{1}{2}}(\overline{\rho})\, \mathrm{d}\xi^{\delta(\varepsilon)}).$$

Linear diffusions: [Cornalba, Fischer; 2023], [Cornalba, Fischer, Ingmanns, Raithel; 2023], [Djurdjevac, Kremp, Perkowski; 2022]

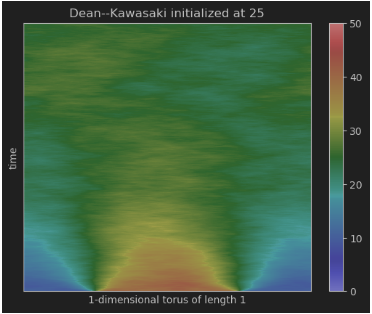
The Large Deviations Principle [F., Gess; 2024]

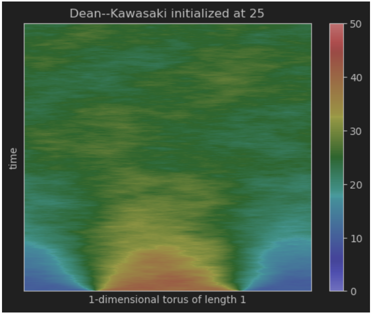
The LDP: along the scaling limit

$$\varepsilon\delta(\varepsilon)^{-(1+\frac{2}{d})} \to 0 \text{ and } \delta(\varepsilon) \to 0,$$

the solutions satisfy a LDP in $L^1_{t,x}$ and $\mathrm{C}_t\,\mathcal{M}_+$ with rate function

$$I(\rho) = \frac{1}{2}\inf\left\{ \|g\|^2_{L^2} : \partial_t\rho = \Delta\phi(\rho) - \nabla\cdot(\phi^{\frac{1}{2}}(\rho)g)\right\}.$$





The Dean–Kawasaki equation: along the scaling regime $\varepsilon\delta^{-(1+\frac{2}{d})} \to 0$,

$$\partial_t \rho^\varepsilon = \Delta \rho^\varepsilon - \varepsilon^{\frac{d}{2}} \nabla \cdot \left(\sqrt{\rho^\varepsilon} \circ \xi^{\delta(\varepsilon)}\right).$$