STOCHASTIC HOMOGENIZATION

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ABSTRACT. We develop the theory of stochastic homogenization of divergence-form elliptic operators beginning from the periodic case. These notes are being written for a Spring 2020 lecture course at the University of Oxford.

CONTENTS

1. INTRODUCTION

The goal of homogenization is to understand materials or systems with complicated microstructures. Our model example will be the flow of heat or the conductance of charge through a metal with randomly deposited impurities. Examples of such are diverse. The simplest is a periodic composite, which is deterministic, such as a material consisting of a conductor in the black squares and an insulator in the white squares. However, the material may instead be random effectively random, with impurities deposited like a random tiling of space or like random points in the plane.

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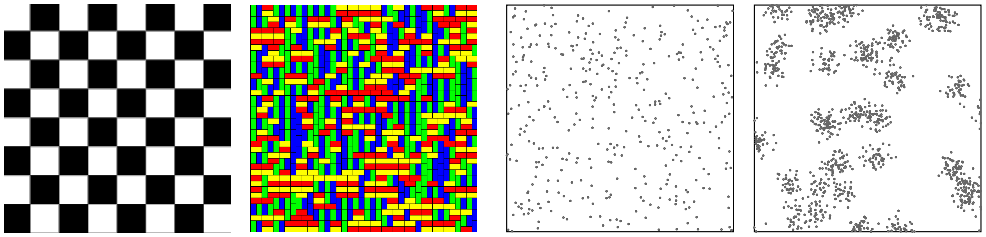

Periodic Random tile Poisson cloud Cluster

As explained in Section 1.1 below, the conductance is typically modeled using a parabolic or elliptic equation in divergence form. That is, for some diffusion matrix A, the density of heat/energy/charge either evolves according to the parabolic equation

$$\partial_t u = \nabla \cdot A\nabla u,$$

or its steady stead at equilibrium satisfies the elliptic equation

$$-\nabla \cdot A\nabla u = 0.$$

We will begin our study by considering a 1-periodic diffusion matrix $A(x)$ defined on $\mathbb{R}^d$, such as would describe the flow of heat through the checkerboard above. The heat flow through the periodic material above is then modeled by an equation of the form

$$\partial_t u^\varepsilon = \nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon \;\text{ or }\; -\nabla \cdot A(x/\varepsilon)\nabla v^\varepsilon = 0, \tag{1.1}$$

where $\varepsilon \in (0,1)$ is the scale of the periodic microstructure (since $A(x/\varepsilon)$ is ε-periodic). More generally, we will consider diffusion matrices $A(x,\omega)$ that are *stationary* and *ergodic* random variables; assumptions which state essentially that the random environment is statistically homogenous and weakly mixing. In this case, we will study equations of the form

$$\partial_t u^\varepsilon = \nabla \cdot A(x/\varepsilon,\omega)\nabla u^\varepsilon \;\text{ or }\; -\nabla \cdot A(x/\varepsilon,\omega)\nabla v^\varepsilon = 0, \tag{1.2}$$

where $\varepsilon \in (0,1)$ is the microscale of impurities. The aim is to characterize the behavior of the solutions u^ε as $\varepsilon \to 0$. This amounts to proving the existence of an effective environment, described by a constant coefficient diffusion matrix $\overline{A}$, such that, as $\varepsilon \to 0$, the solutions u^ε of (1.1) or *almost surely* the solutions of (1.2) converge to the solution $\overline{u}$ of the equation

$$\partial_t \overline{u} = \nabla \cdot \overline{A}\nabla \overline{u} \;\text{ or }\; -\nabla \cdot \overline{A}\nabla \overline{u} = 0. \tag{1.3}$$

The primary aims of *stochastic homogenization* are therefore to identify the effective environment $\overline{A}$ and to quantify the convergence of u^ε to $\overline{u}$.

Homogenization is a highly nonlinear form of averaging. In particular, we will see below that the effective matrix $\overline{A}$ is not equal to the average or expectation of A. From a mathematical point of view, the problems therefore present an interesting mix of analysis, probability, and ergodic theory and are of intrinsic interest in their own right. However, there are additional important practical applications of these problems. We will see below that in order to effectively model an equation like (1.1) or (1.2) it is necessary to take a discretization of scale much smaller than $\varepsilon \in (0,1)$. Therefore, already in three dimensions, their numerically simulation can be extremely costly. But computing the solution of (1.3), which is essentially the heat equation, is fast and straightforward. So, provided we can identify $\overline{A}$, we can well-approximate the behavior of the periodic or random environments for small values of $\varepsilon \in (0,1)$. Finally, the random setting is essential, in the sense that we are modeling materials with small-scale impurities or defects. We therefore cannot expect to know their positions exactly. In particular, there is on reason in general to expect that they will be periodic. But we can postulate properties of their random distribution, and then we can prove almost surely that every realization of the random environment, that is for almost every possible distribution of impurities, we have convergence to the deterministic, homogenized environment.

1.1. **The heat equation and the Laplace equation.** In this course, we will primarily be interested in elliptic and parabolic equations in divergence form, which can be used to model the conduction of heat or electricity. The heat equation, which is perhaps the simplest parabolic equation is defined by

$$\partial_t u = \Delta u + f \ \text{ on } \ U \times (0,\infty) \ \text{ with } \ u = g \ \text{ on } \ \partial U \times (0,\infty) \ \text{ and } \ u = u_0 \ \text{ on } \ U \times \{0\},$$

for some domain $U \subseteq \mathbb{R}^d$. The solution u as the density of heat, energy, or electric charge at time $t \in [0,\infty)$ and the point $x \in U$. The function f is a *source/sink* that quantifies the density of energy being put in or taken out of the system. Perhaps we are holding a constant flame to the plate. The heat along the boundary is held fixed according to the boundary data g. So, energy escaping to or from the boundary is either absorbed or replenished. The initial distribution of energy is u_0.

The derivation of the heat equation follows from the conservation of energy. Through every region of the domain, the total change of energy is equal to the amount of external energy provided/taken by f plus the total flux of energy through the boundary. Precisely, for every $x \in U$ and $r \in (0,\infty)$ satisfying $B_r(x) \subseteq U$, this is to say that

$$\partial_t \left(\int_{B_r(x)} u \right) = \oint_{\partial B_r(x)} \nabla u \cdot \nu + \int_{B_r(x)} f,$$

where ν is the unit normal. By the divergence theorem, since $\nabla \cdot (\nabla u) = \Delta u$, this implies that

$$\int_{B_r(x)} u = \int_{B_r(x)} \Delta u + \int_{B_r(x)} f,$$

and, after dividing by $|B_r(x)|$ and passing to the limit $r \to 0$, we recover the heat equation

$$\partial_t u = \Delta u + f.$$

The heat equation is a good model for diffusion in a homogenous material. Weighting the flux equally in all directions is tantamount to saying the heat/energy diffuses equally from every point in all directions.

We have seen that a parabolic equation models a system evolving in time. Divergence form elliptic equations model systems in equilibrium. Precisely, if u is a solution of the heat equation

$$\partial_t u = \Delta u + f \ \text{ on } \ U \times (0,\infty) \ \text{ with } \ u = g \ \text{ on } \ \partial U \times (0,\infty) \ \text{ and } \ u = u_0 \ \text{ on } \ U \times \{0\},$$

then we expect that, provided f and g are independent of time, as $t \to \infty$ the density approaches an equilibrium state and so $\partial_t u \to 0$. Therefore, as $t \to \infty$, we expect that $u \to v$ for v solving the Laplace equation

$$\Delta v + f = 0 \ \text{ in } \ U \ \text{ with } \ v = g \ \text{ on } \ \partial U.$$

Indeed, the Laplace equation is derived using the same logic based on conservation of energy. Since the system is in equilibrium, the rate of change in energy in any region must be zero, which means that the energy provided/taken by f must be balanced by the flux of energy through that region. That is, for each $x \in U$ and $r \in (0,\infty)$ satisfying $B_r(x) \subseteq U$,

$$\oint_{\partial B_r(x)} \nabla v \cdot \nu + \int_{B_r(x)} f = 0.$$

By the divergence theorem, this implies that

$$\int_{B_r(x)} \Delta v + \int_{B_r(x)} f = 0,$$

and therefore, after dividing by $|B_r|$ and passing to the limit $r \to 0$,

$$\Delta v + f = 0.$$

As with the heat equation, the Laplace equation is a good model for homogenous systems in equilibrium. The density diffuses equally from every point in all directions.

1.2. **Diffusion processes.** The heat equation and the Laplace equation are intricately related to Brownian motion. Indeed, the law of a standard Brownian motion $(B_t^x)_{t\in[0,\infty)}$ beginning from $x\in\mathbb{R}^d$ is simply the heat kernel. That is, for each $t\in[0,\infty)$, for every Borel measurable subset $A\subseteq\mathbb{R}^d$,

$$\mathbb{P}[B_t^x\in A]=\int_A(2\pi t)^{-\frac{d}{2}}\exp\left(-\frac{|y-x|^2}{2t}\right)\,\mathrm{d}y.$$

Therefore, if u is a solution of the heat equation

$$\partial_t u=\Delta u \text{ on } \mathbb{R}^d\times(0,\infty) \text{ with } u=u_0 \text{ on } \mathbb{R}^d\times\{0\},$$

we have that

$$u(x,t)=\int_{\mathbb{R}^d}(2\pi t)^{-\frac{d}{2}}u_0(y)\exp\left(-\frac{|y-x|^2}{2t}\right)\,\mathrm{d}y=\mathbb{E}[u_0(B_t^x)],$$

which is one version of the *Feynman-Kac formula*. That is, the solution of the heat equation is simply the initial condition averaged with respect to Brownian motion. This is why the heat equation is regularizing, and the greater the diffusion the regularity you see. That is, for each $\alpha\in(0,\infty)$ let u_α solve the equation

$$\partial_t u_\alpha=\alpha\Delta u_\alpha \text{ on } \mathbb{R}^d\times(0,\infty) \text{ with } u_\alpha=u_0 \text{ on } \mathbb{R}^d\times\{0\}.$$

It follows that $u_\alpha(x,t)=u(x,\alpha t)$ and therefore, by Brownian scaling,

$$u_\alpha(x,t)=\mathbb{E}[u_0(B_{\alpha t}^x)]=\mathbb{E}[u_0(\sqrt{\alpha}B_t^x)].$$

So, we see a greater regularizing effect as $\alpha\to\infty$, and a vanishing regularizing effect as $\alpha\to 0$.

Note that the constant coefficient diffusion matrix need not be isotropic. That is, it need not be rotationally invariant. Consider for $\lambda_1,\lambda_2\in(0,\infty)$ the two-dimensional matrix

$$A=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}. \tag{1.4}$$

The solution of the equation

$$\partial_t u=\nabla\cdot A\nabla u \text{ on } \mathbb{R}^d\times(0,\infty) \text{ with } u=u_0 \text{ on } \mathbb{R}^d\times\{0\}, \tag{1.5}$$

is then related to the diffusion process, defined for independent Brownian motions $(B_t^1)_{t\in[0,\infty)}$ and $(B_t^2)_{t\in[0,\infty)}$,

$$\mathrm{d}X_t=\mathrm{d}\begin{pmatrix}X_t^1\\ X_t^2\end{pmatrix}=\mathrm{d}\begin{pmatrix}\sqrt{2\lambda_1}B_t^1\\ \sqrt{2\lambda_2}B_t^2\end{pmatrix}, \tag{1.6}$$

by the Feynman-Kac formula

$$u(x,t)=\mathbb{E}_x[(X_t)],$$

where $\mathbb{E}_x$ denotes the expectation of the solution to (1.6) starting from $x\in\mathbb{R}^2$. Suppose that $\lambda_1>>\lambda_2$. It is then the case that the solutions $(X_t)_{t\in[0,\infty)}$ diffuses more rapidly in the x_1-direction than in the x_2-direction. This is to say that the solution to (1.5) observes more averaging in the x_1-direction than in the x_2-direction. And, indeed, this is reflected in the natural energy estimate

$$\frac{1}{2}\int_{\mathbb{R}^d}u^2(x,t)\,\mathrm{d}x+\int_0^t\int_{\mathbb{R}^d}\lambda_1(\partial_{x_1}u(x,s))^2+\lambda_2(\partial_{x_2}u(x,s))^2\,\mathrm{d}x\,\mathrm{d}s\leq\frac{1}{2}\int_{\mathbb{R}^d}u_0^2(x)\,\mathrm{d}x.$$

The extreme case follows from taking $\lambda_1\to\infty$ and $\lambda_2\to 0$, for which we see that u becomes constant in the x_1-direction and for which we have no regularity in the x_2-direction. The important principle to keep in mind is that diffusion implies regularity because diffusion implies averaging.

1.3. **General elliptic and parabolic equations in divergence form.** More generally, a parabolic or an elliptic equation in divergence form is described by a non-constant diffusion matrix A that describes the diffusion of heat away from each point. In the parabolic case, the equation

$$\partial_t u = \nabla \cdot A \nabla u + f \ \text{ on } \ U \times (0,\infty) \ \text{ with } \ u = g \ \text{ on } \ \partial U \times (0,\infty) \ \text{ and } \ u = u_0 \ \text{ on } \ U \times \{0\},$$

is derived again using conservation of energy. However, in this case the flux is described by the matrix A. That is, for every $x \in U$ and $r \in (0,\infty)$ satisfying $B_r(x) \subseteq U$, we assert that

$$\partial_t \left(\int_{B_r(x)} u(y,t)\,\mathrm{d}y \right) = \oint_{\partial B_r(x)} A(y) \nabla u(y) \cdot \nu \,\mathrm{d}S(y) + \int_{B_r(x)} f(y)\,\mathrm{d}y,$$

from which it follows formally follows by the divergence theorem that

$$\partial_t u = \nabla \cdot A \nabla u + f. \tag{1.7}$$

In equilibrium we expect have that

$$\oint_{\partial B_r(x)} A(y) \nabla u(y) \cdot \nu \,\mathrm{d}S(y) + \int_{B_r(x)} f(y)\,\mathrm{d}y = 0,$$

from which we derive the equation

$$-\nabla \cdot A \nabla u = f. \tag{1.8}$$

Think again of the example (1.5) above. If $\lambda_2 = 0$ then there is no diffusion of energy in the x_2-direction, which is reflected by the fact that we no longer consider the flux of ∇u but of $A\nabla u$.

If A is symmetric, then it is again the case that the solutions (1.7) and (1.8) are related to a diffusion process. Let σ be a matrix satisfying $\sigma\sigma^t = 2a$ and let $(X_t)_{t\in[0,\infty)}$ be the solution to the stochastic differential equation

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + (\nabla \cdot A)\,\mathrm{d}t.$$

Then, for the exit time τ from the domain U we that the solution of (1.7) satisfies

$$u(x,t) = \mathbb{E}_x\left[g(X_\tau) | \tau \le t\right] + \mathbb{E}_x\left[u_0(X_t) | \tau > t\right] + \mathbb{E}_x\left[\int_0^{\tau\wedge t} f(X_s)\,\mathrm{d}s\right]$$

and the solution of (1.8) satisfies

$$u(x,t) = \mathbb{E}_x\left[g(X_\tau) + \int_0^{\tau} f(X_s)\,\mathrm{d}s\right].$$

Such formulas will not play a significant role in this course, but it is important to keep in mind the relationship between the partial differential equation and the diffusion process. Homogenization of PDEs is formally equivalent to proving scaling limits in law for diffusion processes, as we will describe below.

1.4. **A motivating computation.** Let $A\colon \mathbb{R}^d \to \mathbb{R}^d$ be a one-periodic, bounded, and strictly positive function. We are interested in the asymptotic behavior of the solution

$$-\nabla \cdot A(x/\varepsilon) \nabla u^\varepsilon = f \ \text{ in } \ (0,1) \ \text{ with } \ u^\varepsilon = 0 \ \text{ on } \ \{0,1\}.$$

We can then integrate to see that, for some $c \in (0,\infty)$,

$$\nabla u^\varepsilon(x) = A(x/\varepsilon)^{-1}\left(c - \int_0^x f(y)\,\mathrm{d}y\right).$$

Then, since as $\varepsilon \to 0$,

$$A(x/\varepsilon)^{-1} \rightharpoonup \langle A^{-1} \rangle = \int_0^1 A^{-1}(y)\,\mathrm{d}y \ \text{ weakly in } \ L^2((0,1)),$$

we see that, as $\varepsilon \to 0$,

$$\nabla u^\varepsilon(x) \rightharpoonup \langle A^{-1}\rangle(c - \int_0^x f(y)\,\mathrm{d}y) \text{ weakly in } L^2((0,1)).$$

Therefore, if we define $\overline{u} \in H_0^1((0,1))$ by

$$\nabla\overline{u} = \langle A^{-1}\rangle(c - \int_0^x f(y)\,\mathrm{d}y),$$

it follows that $\overline{u}$ solves

$$-\nabla\cdot\langle A^{-1}\rangle^{-1}\nabla\overline{u} = f \text{ in } (0,1) \text{ with } \overline{u} = 0 \text{ on } \{0,1\}.$$

This shows that the homogenized coefficient is not simply the average of the original coefficients, due to the fact that if A is not constant then in general

$$\langle A\rangle \neq \langle A^{-1}\rangle^{-1}. \tag{1.9}$$

Furthermore, we see that while as $\varepsilon \to 0$ we have

$$\nabla u^\varepsilon \rightharpoonup \nabla\overline{u} \text{ weakly in } L^2((0,1)),$$

but that, as $\varepsilon \to 0$,

$$\nabla u^\varepsilon \not\to \nabla\overline{u} \text{ strongly in } L^2((0,1)). \tag{1.10}$$

That is, the oscillations of the solution cancel in a weak sense but on in a strong sense.

1.5. **A remark on numerics.** Suppose that we are interested in numerically solving the one-dimensional problem

$$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon = -\nabla\cdot f \text{ in } (0,1) \text{ with } u^\varepsilon = 0 \text{ on } \{0,1\}. \tag{1.11}$$

We can reduce this to a problem in linear algebra after restricting to the solution space $\mathcal{V}_h$ of H_0^1-functions on $(0,1)$ that are piecewise-linear on the partition $[0, 1/h, 2/h, \ldots, h-1/h, 1]$, where $h \in \mathbb{N}$. That is, when taking into account the boundary conditions, an element of $\mathcal{V}_h$ is uniquely represented by a vector $(x_1, x_2, \ldots, x_h) \in \mathbb{R}^h$ satisfying the property that $x_1 + x_2 + \ldots + x_h = 0$.

The Lax-Milgram theorem allows to solve (1.11) in the space $\mathcal{V}_h$. In this case, the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, \ldots, u_h^\varepsilon)$ satisfies for every $v = (v_1, v_2, \ldots, v_h)$ that

$$\sum_{k=1}^h \left(\int_{\frac{k-1}{h}}^{\frac{k}{h}} A(y/\varepsilon)\,\mathrm{d}y\right) u_k^\varepsilon v_k = \sum_{k=1}^h \left(\int_{\frac{k-1}{h}}^{\frac{k}{h}} f(y)\,\mathrm{d}y\right) v_k.$$

At this point we see that if $h = \varepsilon^{-1}$ then for every $k \in \{1, 2, \ldots, h\}$ we have that

$$\int_{\frac{k-1}{h}}^{\frac{k}{h}} A(y/\varepsilon)\,\mathrm{d}y = \varepsilon\langle A\rangle,$$

and more generally if $h \geq \varepsilon$ as $\varepsilon \to 0$ we have that, for each $k \in \{1, 2, \ldots, h\}$, as $\varepsilon \to 0$,

$$\int_{\frac{k-1}{h}}^{\frac{k}{h}} A(y/\varepsilon)\,\mathrm{d}y \to \frac{1}{h}\langle A\rangle.$$

This is to say that, if the discretization parameter h is not chosen sufficiently smaller than ε, then the numerical scheme is approximating the equation

$$-\nabla\cdot\langle A\rangle\nabla\tilde{u} = -\nabla\cdot f \text{ in } (0,1) \text{ with } \tilde{u} = 0 \text{ on } \{0,1\},$$

which fails to capture the correct behavior of the solution. That is, as we observed in (1.9), the homogenized coefficient is not simply the average of the original coefficient.

We therefore conclude that, in order to capture the correct behavior of the solutions u^ε, it is necessary to take discretization with order ε^{-1} elements. In higher dimension, the same argument proves that the mesh size h of the grid must be taken of order less than ε. This leads to a discretization with order ε^{-d} elements, and already in dimension three this is computationally too expensive in practice. Conversely, estimating the solution of the homogenized equation is cheap and relies only on understanding the homogenized coefficient $\overline{A}$. So, provided we can effectively compute $\overline{A}$, we can well approximate solutions of the oscillating equation (1.11) by computing the solution of the constant coefficient, homogenized equation.

2. Periodic homogenization

A matrix $A \in \mathbb{R}^{d\times d}$ is *uniformly elliptic* if, for some constants $\lambda, \Lambda \in (0,\infty)$,

$$|A\xi| \leq \Lambda\xi \quad \text{and} \quad A\xi\cdot\xi \geq \lambda\,|\xi|^2 \quad \text{for every} \quad \xi \in \mathbb{R}^d. \tag{2.1}$$

We say that a non-constant matrix $A\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ if there exist constants $\lambda, \Lambda \in (0,\infty)$ such that $A(x)$ satisfies (2.1) for every $x \in \mathbb{R}^d$. For a 1-periodic, uniformly elliptic matrix $A\colon \mathbb{R} \to \mathbb{R}^d$, we will study the limiting behavior, as $\varepsilon \to 0$, of the solutions u^ε to the problem

$$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon = f \quad \text{in} \quad U \quad \text{with} \quad u^\varepsilon = 0 \quad \text{on} \quad \partial U. \tag{2.2}$$

We restrict to zero boundary conditions and consider the elliptic problem only for simplicity. The methods of this section apply readily to nonzero boundary conditions and the parabolic problem as well. The well-posedness of (2.2) in $H^1_0(U)$is a consequence of the uniform ellipticity and the Lax-Milgram theorem.

Definition 2.1. Let $U \subseteq \mathbb{R}^d$ be a bounded open set, let $A\colon U \to \mathbb{R}^{d\times d}$ be uniformly elliptic, an d let $f \in H^{-1}(U)$. We say that a function $u \in H^1_0(U)$ is a weak solution of the equation

$$-\nabla\cdot A\nabla u = f \quad \text{in} \quad U \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial U,$$

if, for every $v \in H^1_0(U)$,

$$\int_U A\nabla u\cdot\nabla v = \langle f, v\rangle_{H^{-1}(U)},$$

where $\langle\cdot,\cdot\rangle_{H^{-1}(U)}$ denotes the pairing between $H^1_0(U)$ and its dual $H^{-1}(U)$.

Proposition 2.2. *Let $U \subseteq \mathbb{R}^d$ be a bounded open set and let $A\colon U \to \mathbb{R}^{d\times d}$ be uniformly elliptic. Then for every $f \in H^{-1}(U)$ there exists a unique weak solution $u_f \in H^1_0(U)$ of the equation*

$$-\nabla\cdot A\nabla u_f = f \quad \textit{in} \quad U \quad \textit{with} \quad u_f = 0 \quad \textit{on} \quad \partial U.$$

Furthermore,

$$\|u_f\|_{H^1_0(U)} \leq \frac{1}{\lambda}\,\|f\|_{H^{-1}(U)}\,.$$

Proof. The boundedness of U and the Poincaré inequality prove that there exists $c \in (0,\infty)$ such that, for every $v \in H^1_0(U)$,

$$\|\nabla v\|_{L^2(U)} \leq \|v\|_{H^1(U)} \leq c\,\|\nabla v\|_{L^2(U;\mathbb{R}^d)}\,.$$

Therefore, for every $v, w \in H^1(U)$, the bilinear form

$$\langle v, w\rangle_{H^1_0(U)} = \int_U \nabla v\cdot\nabla w$$

defines a positive definite inner product on $H^1_0(U)$ and we take $\|u\|_{H^1_0(U)} = \langle u, u\rangle^{\frac{1}{2}}$. It then follows from the uniform ellipticity and Hölder's inequality that, for every $v, w \in H^1_0(U)$,

$$\left|\int_U A\nabla v\cdot\nabla w\right| \leq \|v\|_{H^1_0(U)}\,\|w\|_{H^1_0(U)}\,,$$

and that, for every $v \in H^1_0(U)$,

$$\int_U A\nabla v\cdot\nabla v \geq \lambda\int_U |\nabla v|^2 = \lambda\left\|v\right\|^2_{H^1_0(U)}.$$

The Lax-Milgram theorem therefore proves that for every $f \in H^{-1}(U)$ there exists a unique $u_f \in H^1_0(U)$ which satisfies

$$\int_U A\nabla u_f\cdot\nabla v = \langle f, v\rangle_{H^{-1}(U)} \text{ for every } v\in H^1_0(U).$$

It then follows after choosing $v = u_f$ that

$$\lambda\left\|u\right\|^2_{H^1_0(U)} \leq \left\|f\right\|_{H^{-1}(U)}\left\|u\right\|_{H^1_0(U)},$$

and therefore that

$$\left\|u\right\|_{H^1_0(U)} \leq \frac{1}{\lambda}\left\|f\right\|_{H^{-1}(U)}.$$

This completes the proof. ☐

Remark 2.3. An essential conclusion of Proposition 2.2 is that for each $f \in H^{-1}(U)$ the solutions

$$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon = f \text{ in } U \text{ with } u^\varepsilon = 0 \text{ on } \partial U,$$

are uniformly bounded in $\varepsilon \in (0,1)$. That is,

$$\sup_{\varepsilon\in(0,1)}\left\|u^\varepsilon\right\|_{H^1_0(U)} \leq \frac{1}{\lambda}\left\|f\right\|_{H^{-1}(U)}.$$

We therefore know a priori that the solutions remain in a relatively weakly compact subset of $H^1_0(U)$ and a relatively compact subset of $L^p(U)$ for every $p \in [1, 2_*)$ for the Sobolev exponent $1/2_* = 1/2 - 1/d$ if $d \geq 3$, with $2_* = \infty$ if $d = 2$, and with the solutions remaining in a relatively compact subset of $\mathrm{C}^\alpha(U)$ for every $\alpha \in (0, 1/2)$ if $d = 1$. This is far from proving that the solutions converge along the full sequence $\varepsilon \to 0$, however.

2.1. **The asymptotic expansion.** We will approach the homogenization problem (2.2) by separating the macroscopic scale (that is, scale 1) from the microscopic scale (that is, scale ε). For this we formally postulate that the solution u^ε admits an asymptotic expansion of the form

$$u^\varepsilon(x) \simeq u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \dots, \tag{2.3}$$

where the functions $u_i\colon U\times\mathbb{T}^d \to \mathbb{R}$ are periodic in the second variable. An expansion of this type is not a priori justified, so the following computations will proceed on a formal level. We will write x for the slow variable and y for the fast variable, so that we have, for instance,

$$\nabla u_1(x,y) = \nabla_x u_1(x,y) + \varepsilon^{-1}\nabla_y u_1(x,y),$$

where the variable y stands in for x/ε.

We first exploit the divergence form structure of the equation to argue that the terms of order two and higher will not effect the $\varepsilon \to 0$ limit. Precisely, suppose that $u^\varepsilon \in H^1_0(U)$ is the solution of the equation

$$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon = f \text{ in } U \text{ with } u^\varepsilon = 0 \text{ on } \partial U.$$

Then, for each $\psi \in \mathrm{C}^\infty_c(U)$ we have

$$\begin{aligned}\int_U A(x/\varepsilon)\nabla u^\varepsilon(x)\cdot\nabla\psi(x)\,\mathrm{d}x &= \int_U A(x/\varepsilon)\left(\nabla_x u_0 + \varepsilon^{-1}\nabla_y u_0 + \nabla_y u_1 + \varepsilon\nabla_x u_1 + O(\varepsilon)\right)\cdot\nabla\psi(x)\,\mathrm{d}x\\ &= \int_U f\psi\,\mathrm{d}x.\end{aligned}$$

So, as $\varepsilon \to 0$, we see formally that the higher order terms $u_2, u_3, \ldots$ do not effect the equation's weak formulation. We therefore postulate an asymptotic expansion of the form

$$u^\varepsilon(x) \simeq u_0(x,y) + \varepsilon u_1(x, x/\varepsilon). \tag{2.4}$$

This divergence form structure is essential for this simplification to be valid. If we instead considered the non-divergence form equation

$$-\operatorname{tr}(A(x/\varepsilon \nabla^2 u^\varepsilon)) + 1/\varepsilon b(x/\varepsilon) \cdot \nabla u^\varepsilon = f,$$

it would be necessary to additionally consider the higher order term u_2.

Returning to (2.4), we evaluate the equation to find that

$$-(\nabla_x + \varepsilon^{-1}\nabla_y) \cdot \left[A(y)(\varepsilon^{-1}\nabla_y u_0 + \nabla_x u_0 + \nabla_y u_1 + \varepsilon \nabla_x u_1)\right] = f.$$

We proceed by equating powers of ε. The equation of order ε^{-2} is

$$-\nabla_y \cdot A(y)\nabla_y u_0(y,x) = 0 \;\text{ in }\; \mathbb{T}^d \;\text{ for every }\; x \in U.$$

It follows from the weak maximum principle, or the standard energy estimate, that $u_0(x,y) = u_0(x)$ is independent of the fact variable $y \in \mathbb{T}^d$.

The equation of order ε^{-1} is

$$-\nabla_y \cdot A(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y)) = 0. \tag{2.5}$$

Based on the intuition that the solution is determined by the underlying diffusion process, and the fact that the underlying diffusion process sees on the fast scale, or based on PDE considerations, we postulate here a separation of scales. That is, we make the ansatz that

$$u_1(x,y) = \phi_i(y)\partial_i u_0(x), \tag{2.6}$$

for periodic functions $\phi_i\colon \mathbb{T}^d \to \mathbb{R}$. Here and throughout the notes we will use Einstein's summation convention over repeated indices. Returning to (2.5) and applying the ansatz (2.6), we have that

$$-\nabla_y \cdot A(y)\left(e_i + \nabla_y \phi_i(y)\right)) \, \partial_i u_0(x) = 0.$$

Since the functions $\partial_i u_0$ are effectively arbitrary, and we will see that they can be fixed to be almost anything by changing the righthand side f, we have for each $i \in \{1, \ldots, d\}$ that

$$-\nabla_y \cdot A(y)(e_i + \nabla_y \phi_i) = 0. \tag{2.7}$$

Equation (2.7) is the so-called *corrector equation* or *cell problem.* The solutions ϕ_i are called *homogenization correctors*. We will say more about these solutions in the next section.

The equation of order 1 is then

$$-\nabla_x \cdot [A(y)(e_i + \nabla\phi_i(y))\partial_i u_0(x)] = f.$$

Or, if we return to the original scaling, we have that

$$-\nabla_x \cdot [A(x/\varepsilon)(e_i + \nabla\phi_i(x/\varepsilon))\partial_i u_0(x)] = f(x).$$

Since, as $\varepsilon \to 0$,

$$A(x/\varepsilon)(e_i + \nabla\phi_i(x/\varepsilon)) \rightharpoonup \int_{\mathbb{T}^d} A(y)(e_i + \nabla\phi_i(y))\,\mathrm{d}y = \langle A(e_i + \nabla\phi_i)\rangle,$$

we conclude that, since u_0 and f are independent of ε,

$$-\nabla_x \cdot [\langle A(e_i + \nabla\phi_i)\rangle \partial_i u_0(x)] = f(x). \tag{2.8}$$

We use (2.8) to define the homogenized coefficient $\overline{A}$. That is, for each $i \in \{1, \ldots, d\}$, we define

$$\overline{A}e_i = \langle A(e_i + \nabla\phi_i)\rangle \;\text{ so that }\; \overline{A}_{ji} = \langle A(e_i + \nabla\phi_i)\rangle \cdot e_j, \tag{2.9}$$

and conclude from (2.8) that u_0 solves

$$-\nabla \cdot \overline{A}\nabla u_0 = f \;\text{ in }\; U \;\text{ with }\; u_0 = 0 \;\text{ on }\; \partial U. \tag{2.10}$$

We can then justify the formal expansion a posteriori in the sense that after defining the ϕ_i according to (2.7), the homogenized coefficient $\overline{A}$ according to (2.9), and u_0 according to (2.10) we have

$$u^\varepsilon(x) \simeq u_0(x) + \varepsilon\phi^i(x/\varepsilon)\partial_i u_0(x). \tag{2.11}$$

We will make this precise in the next two sections.

2.2. **The homogenization corrector.** We will write $H_0^1(\mathbb{T}^d)$ for the space of mean zero H^1-functions on the torus. Thanks to the Poincaré inequality, the mean zero condition guarantees that the inner product

$$\langle u, v\rangle_{H_0^1(\mathbb{T}^d)} = \int_{\mathbb{T}^d} \nabla u \cdot \nabla v,$$

defines a positive definite inner product on $H_0^1(\mathbb{T}^d)$. We will now show that there exists a unique $\phi_i \in H_0^1(\mathbb{T}^d)$ satisfying (2.7). We emphasize that correctors are clearly not unique, in the sense that if ϕ_i solves (2.7) then so too does $\phi_i + c$ for any $c \in \mathbb{R}$. What the following proposition proves is that this is the only source of non-uniqueness, and that correctors are unique up to the addition of a constant.

Proposition 2.4. *For every $i \in \{1, \dots, d\}$ there exists a unique weak solution $\phi_i \in H_0^1(\mathbb{T}^d)$ of the equation*

$$-\nabla_y \cdot A(y)(e_i + \nabla\phi_i(y)) = 0 \;\text{ in }\; \mathbb{T}^d.$$

Proof. The proof is virtually identical to Proposition 2.2, and is an immediate consequence the uniform ellipticity of A, the Poincaré inequality, and the Lax-Milgram theorem. □

The correctors some to define the intrinsic geometry of the space in the following sense. Suppose that $u \in \mathrm{C}^2(\mathbb{R}^d)$ is a sub-quadratic harmonic function in the sense that

$$-\Delta u = 0 \;\text{ in }\; \mathbb{R}^d \;\text{ and }\; \limsup_{|x|\to 0} \frac{|u(x)|}{|x|^2} = 0.$$

Then the first-order Liouville theorem proves that u is linear. That is, there exists $c \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$ such that $u(x) = c + \xi \cdot x$. We can view the correctors as correcting the linear functions x_i so as to make them A-harmonic in the sense that

$$-\nabla \cdot A(y) \cdot (e_i + \nabla\phi_i) = -\nabla \cdot A(y)\nabla(x_i + \phi_i) = 0.$$

In this way, the functions $x_i + \phi_i$ define the natural coordinates in the geometry defined by A. And indeed we will see that this can be made precise in the sense that if $u\colon \mathbb{R}^d \to \mathbb{R}$ is A-harmonic and strictly sub-quadratic in the sense that, for some $\alpha \in (0, 1)$,

$$-\nabla \cdot A\nabla u = 0 \;\text{ in }\; \mathbb{R}^d \;\text{ and }\; \limsup_{|x|\to 0} \frac{|u(x)|}{|x|^{1+\alpha}} = 0,$$

then there exists $c \in \mathbb{R}$ and $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ such that $u(x) = c + \xi \cdot x + \phi_\xi(x)$ where $\phi_\xi = \xi_i\phi_i$.

The role of the corrector can also been seen on the level of the diffusion

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + (\nabla \cdot A)(X_t)\,\mathrm{d}t,$$

for $\sigma\sigma^t = 2A$. Indeed, for $\phi = (\phi_1, \dots, \phi_d)$, we see that the corrector ϕ modifies the solution X_t so as to make it a martingale. That is, if we define $M_t = X_t + \phi(X_t)$, then it follows from Itô's formula that M_t is a martingale. And, thus,

$$\varepsilon X_{t/\varepsilon^2} = \varepsilon M_{t/\varepsilon^2} + \varepsilon\phi(X_{t/\varepsilon^2}),$$

where the first term on the righthand side can be handled using martingale convergence arguments and the second term formally vanishes in the $\varepsilon \to 0$ limit.

2.3. **The homogenized coefficient.** Henceforth, for each $i \in \{1, \ldots, d\}$, let ϕ_i be the solution of the correctors equation

$$-\nabla \cdot A(y)(e_i + \nabla\phi_i(y)) = 0 \ \text{ in } \ \mathbb{T}^d.$$

We recall from (2.9) that the homogenized coefficient $\overline{A}$ is defined by the equation, for each $\xi \in \mathbb{R}^d$,

$$\overline{A}\xi = \int_{\mathbb{T}^d} A(y)(\xi + \nabla\phi_\xi(y))\,\mathrm{d}y = \langle A(\xi + \nabla\phi_\xi)\rangle,$$

for $\phi_\xi = \xi_i\phi_i$. The following proposition proves that $\overline{A}$ is uniformly elliptic if A is uniformly elliptic, and that $\overline{A}$ is symmetric if A is symmetric.

Proposition 2.5. *Assume that A is periodic and satisfies* (2.1) *for constants $\lambda, \Lambda \in (0,1)$ and let $\{\phi_i\}_{i\in\{1,\ldots,d\}}$ be as in Proposition* 2.4. *Then $\overline{A} \in \mathbb{R}^{d\times d}$ defined by*

$$\overline{A}e_i = \langle A(e_i + \nabla\phi_i)\rangle,$$

satisfies, for every $\xi \in \mathbb{R}^d$,

$$\left|\overline{A}\xi\right| \leq \Lambda\,|\xi|\,\langle\sum_{i=1}^d |e_i + \nabla\phi_i|^2\rangle^{\frac{1}{2}} \ \textit{ and } \ \overline{A}\xi\cdot\xi \geq \lambda\,|\xi|^2\,.$$

Proof. The uniform ellipticity of A proves that, for every $\xi \in \mathbb{R}^d$,

$$\left|\overline{A}\xi\right| = \left|\int_{\mathbb{T}^d} A(y)\xi_i(e_i + \nabla\phi_i(y))\,\mathrm{d}y\right| \leq \Lambda\int_{\mathbb{T}^d} |\xi_i|\,|e_i + \nabla\phi_i(y)|\,\mathrm{d}y,$$

where $|\cdot|$ denotes the usual Euclidean metric on $\mathbb{R}^d$ and the absolute value on $\mathbb{R}$. It then follows from Hölder's inequality on $\mathbb{R}^d$ and Jensen's inequality that

$$\left|\overline{A}\xi\right| \leq \Lambda\,|\xi|\left(\sum_{i=1}^d \langle|e_i + \nabla\phi_i|\rangle^2\right)^{\frac{1}{2}} \leq \Lambda\,|\xi|\left(\langle\sum_{i=1}^d |e_i + \nabla\phi_i|^2\rangle\right)^{\frac{1}{2}}.$$

For the uniform ellipticity, observe by linearity that for every $\xi \in \mathbb{R}^d$ the corrector $\phi_\xi = \xi_i\phi_i$ satisfies the equation

$$-\nabla\cdot A(\xi + \nabla\phi_\xi) = 0 \ \text{ in } \ \mathbb{T}^d. \tag{2.12}$$

By definition of $\overline{A}$ and since ϕ_ξ solves (2.12), we have for every $\xi \in \mathbb{R}^d$ that

$$A\xi\cdot\xi = \langle A(\xi + \nabla\phi_\xi)\rangle\cdot\xi = \langle A(\xi + \nabla\phi_\xi)\cdot\xi\rangle = \langle A(\xi + \nabla\phi_\xi)\cdot(\nabla\phi_\xi + \xi)\rangle.$$

Therefore, by the uniform ellipticity of A and Jensen's inequality,

$$A\xi\cdot\xi \geq \langle\lambda\,|\xi + \nabla\phi_\xi|^2\rangle \geq \lambda\,|\langle\xi + \nabla\phi_\xi\rangle|^2 = \lambda\,|\xi|^2\,,$$

since the integral of the gradient $\nabla\phi_\xi$ vanishes. This completes the proof. □

The essential role of Proposition 2.5 is to show that the equation (2.10) defining u_0 of the asymptotic expansion is well-posed. What we have guaranteed is that if we start with a periodic, uniformly elliptic environment then the homogenized environment remains uniformly elliptic. From the point of view of the diffusions this is in some sense obvious. If at every point we have a lower bound for the diffusion in all directions, then we have to retain this lower bound in the limit. And, indeed, we see that the ellipticity constant of the homogenized matrix from below is at least as good as for the original matrix. Of course, in practice it will always be somewhat better, unless the periodic matrix is a constant.

In the final proposition of this section, we will prove that the homogenized matrix corresponding to the transpose A^t is the transpose of the homogenized matrix. That is, if A is a uniformly elliptic matrix then so too is A^t. And so for each $i \in \{1,\ldots,d\}$ we can define the corrector ϕ_i^t corresponding to the transposed problem

$$-\nabla\cdot A^t(e_i+\nabla\phi_i^t)=0 \ \text{ in } \ \mathbb{T}^d, \tag{2.13}$$

and we can define the corresponding homogenized coefficient $\tilde{A}\in\mathbb{R}^{d\times d}$ for each $i\in\{1,\ldots,d\}$ by

$$\tilde{A}e_i=\langle A^t(e_i+\nabla\phi_i^t)\rangle. \tag{2.14}$$

We will prove that for $\overline{A}$ defined in Proposition 2.5 we have $\tilde{A}=\overline{A}^t$.

Proposition 2.6. *Assume that A is periodic and uniformly elliptic. Let $\overline{A}\in\mathbb{R}^{d\times d}$ be defined by Proposition 2.5 and let $\tilde{A}\in\mathbb{R}^{d\times d}$ be defined by (2.14). Then $\tilde{A}=\overline{A}^t$. So, in particular, if A is symmetric then $\overline{A}$ is symmetric.*

Proof. For each $i\in\{1,\ldots,d\}$ let ϕ_i be defined by Proposition 2.4 for the matrix A and let ϕ_i^t be defined by Proposition 2.4 for the matrix A^t. Then, for each $i,j\in\{1,\ldots,d\}$, since ϕ_i satisfies (2.12) for $\xi=e_i$ and since ϕ_j^t satisfies (2.13),

$$\begin{aligned}\overline{A}_{ji}&=\langle A(e_i+\nabla\phi_i)\rangle\cdot e_j\\&=\langle A(e_i+\nabla\phi_i)\cdot(e_j+\nabla\phi_j^t)\rangle\\&=\langle(e_i+\nabla\phi_i)\cdot A^t(e_j+\nabla\phi_j^t)\rangle\\&=\langle A^t(e_j+\nabla\phi_j^t)\rangle\cdot e_i\\&=\tilde{A}_{ij}.\end{aligned}$$

We therefore have that $\tilde{A}=\overline{A}^t$. Finally, if A is symmetric, Proposition 2.4 proves that $\phi_i=\phi_i^t$ for every $i\in\{1,\ldots,d\}$ and therefore that $\tilde{A}=\overline{A}$. Thus, if A is symmetric then $\overline{A}=\overline{A}^t$ is symmetric. □

2.4. **The perturbed test function method.** The perturbed test function method is a classical technique to prove the weak convergence in $H_0^1(U)$ of the solutions u^ε of

$$-\nabla\cdot A(x/\varepsilon)\nabla u^\varepsilon=f \ \text{ in } \ U \ \text{ with } \ u^\varepsilon=0 \ \text{ on } \ \partial U, \tag{2.15}$$

to the solution v of the homogenized equation

$$-\nabla\cdot\overline{A}\nabla v=f \ \text{ in } \ U \ \text{ with } \ u^\varepsilon=0 \ \text{ on } \ \partial U, \tag{2.16}$$

for $\overline{A}$ defined in Proposition 2.5. The technique is almost too clever for its own good, and is based on a compensated compactness argument using the div-curl lemma.

We first recall the Helmholtz decomposition for vector fields defined on a smooth, bounded open set $U\subseteq\mathbb{R}^d$. For a vector field $V=(V_i)_{i\in\{1,\ldots,d\}}\in L^2(U;\mathbb{R}^d)$ we will understand derivatives of the type ∂_iV_j, which formally denote the ith distributional derivative of the jth component, in the distributional sense. That is, we say that $\partial_iV_j=f$ if for every $\psi\in C_c^\infty(U)$ we have that

$$-\int_U V_j\partial_i\psi=\int_U f\psi.$$

In this way, it makes sense to talk about non-smooth vector fields being curl free or divergence free. We say that a vector field $V\in L^2(U;\mathbb{R}^d)$ is curl free if $\partial_iV_j=\partial_jV_i$ in the distributional sense for every $i,j\in\{1,\ldots,d\}$ and we say that V is divergence free if the distributional divergence $\partial_iV_i=0$. These are the *potential* and *solenoidal* vector fields:

$$L^2_{\text{pot}}(U)=\left\{V\in L^2(U;\mathbb{R}^d)\colon V=\nabla v \ \text{ for some } \ v\in H_0^1(U)\right\},$$

and

$$L^2_{\text{sol}}(U)=\left\{V\in L^2(U;\mathbb{R}^d)\colon \partial_iV_i=0\right\}.$$

In particular, $L^2_{\text{sol}}(U)$ contains the constant vector fields. The Helmholtz decomposition proves that $L^2(U;\mathbb{R}^d)$ splits as a direct sum of solenoidal and potential fields. The proof of the div-curl lemma then follows immediately.

Proposition 2.7. *Let $U \subseteq \mathbb{R}^d$ be a smooth, bounded open set. Then*

$$L^2(U;\mathbb{R}^d) = L^2_{pot}(U) \oplus L^2_{sol}(U).$$

Proof. Let $V \in L^2(U;\mathbb{R}^d)$ and let $\psi \in H^1_0(U)$ denote the unique weak solution of

$$\Delta\psi = \nabla\cdot V \ \text{ in } \ U \ \text{ with } \ \psi = 0 \ \text{ on } \ \partial U.$$

Then $V = \nabla\psi + (V - \nabla\psi)$ for $\nabla\psi \in L^2_{\text{pot}}(U)$ and $(V - \nabla\psi) \in L^2_{\text{sol}}(U)$. The uniqueness follows from the fact that if $v \in H^1_0(U)$ and $V \in L^2_{\text{sol}}(U)$ then

$$\int_U \nabla v \cdot V = 0,$$

which also proves the direct sum decomposition and completes the proof. □

Proposition 2.8. *Let $U \subseteq \mathbb{R}^d$ be a smooth, bounded open set and assume that $\{p^\varepsilon\}_{\varepsilon\in(0,1)} \subseteq L^2(U;\mathbb{R}^d)$ and $\{V^\varepsilon\}_{\varepsilon\in(0,1)} \in L^2_{pot}(U)$ satisfy, as $\varepsilon \to 0$,*

$$p^\varepsilon \rightharpoonup p_0 \ \text{ weakly in } \ L^2(U;\mathbb{R}^d) \ \text{ and that } \ V^\varepsilon \rightharpoonup V_0 \ \text{ weakly in } \ L^2_{pot}(U).$$

Assume in addition that, as $\varepsilon \to 0$, $\nabla\cdot p^\varepsilon \to f$ strongly in $H^{-1}(U)$. Then, as $\varepsilon \to 0$,

$$p^\varepsilon V^\varepsilon \to p_0 V_0 \ \text{ in } \ D'(U),$$

in the sense that, for $\psi \in \mathrm{C}^\infty_c(U)$,

$$\lim_{\varepsilon\to 0}\int_U p^\varepsilon \cdot v^\varepsilon\psi = \int_U p_0 \cdot v_0\psi.$$

Proof. We may assume without loss of generality that, as $\varepsilon \to 0$,

$$p^\varepsilon \rightharpoonup 0 \ \text{ weakly in } \ L^2(U;\mathbb{R}^d), \ \text{ that } \ V^\varepsilon \rightharpoonup 0 \ \text{ weakly in } \ L^2_{\text{pot}}(U), \tag{2.17}$$

and that $\nabla\cdot p^\varepsilon \to 0$ strongly in $H^{-1}(U)$. This follows from the equality, for every $\psi \in \mathrm{C}^\infty_c(U)$,

$$\int_U (p^\varepsilon - p_0)\cdot(V^\varepsilon - V_0)\psi = \int_U p^\varepsilon \cdot V^\varepsilon\psi + \int p_0 \cdot V_0\psi - \int_U p^\varepsilon \cdot V_0\psi - \int_U p_0 \cdot V^\varepsilon\psi$$

The weak convergence guarantees that the final three terms on the righthand side converge as $\varepsilon \to 0$ to $-\int_U p_0 \cdot V_0\psi$. The result therefore follows by proving that the lefthand side converges to zero, which follows from the case that both weak limits are zero.

Assume (2.17). By assumption and by definition of $L^2_{\text{pot}}(U)$, for every $\varepsilon \in (0,1)$ there exists $v^\varepsilon \in H^1_0(U)$ such that $V^\varepsilon = \nabla v^\varepsilon$. Therefore, for every $\psi \in \mathrm{C}^\infty_c(U)$, by the chain rule

$$\int_U p^\varepsilon \cdot V^\varepsilon\psi = \int_U p^\varepsilon \cdot \nabla v^\varepsilon\psi = \int_U p^\varepsilon \cdot \nabla(f^\varepsilon\psi) - \int_U p^\varepsilon \cdot \nabla\psi v^\varepsilon.$$

Since the v^ε are converging weakly to zero in $H^1_0(U)$ they are uniformly bounded in $H^1_0(Y)$. Therefore, since ψ is smooth, we have that $v^\varepsilon\psi$ is uniformly bounded in $H^1_0(U)$. Thus, since $\nabla\cdot p^\varepsilon \to 0$ strongly in $H^{-1}(U)$, we have that

$$\lim_{\varepsilon\to 0}\int_U p^\varepsilon \cdot \nabla(f^\varepsilon\psi) = 0.$$

Finally, since the v^ε are bounded and converging weakly to zero along the full sequence $\varepsilon \to 0$, the Sobolev embedding theorem proves that, as $\varepsilon \to 0$, we have $\varepsilon \to 0$ strongly in $L^2(U)$. Therefore, since $p^\varepsilon \cdot \nabla\psi$ is uniformly bounded in $L^2(U)$, Hölder's inequality proves that

$$\lim_{\varepsilon\to 0}\int_U p^\varepsilon \cdot \nabla\psi v^\varepsilon = 0,$$

which completes the proof. □

We will now prove the homogenization of (2.15) weakly in $H^1_0(U)$. The proof is based on the idea that we want to exploit the regularity of the solution to the constant coefficient equation (2.16). And, on the level of the equation's weak formulation, the test functions are a proxy for the limit. We therefore perturb the test function using the homogenization corrector, in a manner analogous to the asymptotic expansion (2.11). However, because we are perturbing the test function and not the solution itself, we use the adjoint correctors (2.13) as opposed to the original correctors (2.12).

Lemma 2.9. *Let $f \in L^2_0(\mathbb{T}^d)$ be one-periodic on $\mathbb{R}^d$. Then, as $\varepsilon \to 0$,*

$$f({}^x\!/\!_\varepsilon) \rightharpoonup \langle f\rangle \;\; \textit{weakly in} \;\; L^2_{loc}(\mathbb{R}^d).$$

Proof. By density of smooth functions in $L^2_{\text{loc}}(\mathbb{R}^d)$ it suffices to prove that, for every $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^d)$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\psi(x)f({}^x\!/\!_\varepsilon)\,\mathrm{d}x = \langle f\rangle\int_{\mathbb{R}^d}\psi.$$

Let $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^d)$. Then there exists $c \in (0,\infty)$ depending on $\|\nabla\psi\|_{L^\infty(\mathbb{R}^d)}$ such that, for each $\varepsilon \in (0,1)$,

$$\left|\int_{\mathbb{R}^d}\psi(x)f({}^x\!/\!_\varepsilon)\,\mathrm{d}x - \varepsilon^d\sum_{x\in\varepsilon\mathbb{Z}^d}\psi(x)\langle f\rangle\right| \le c\varepsilon.$$

Since there exists $c \in (0,\infty)$ depending on $\|\nabla\psi\|_{L^\infty(\mathbb{R}^d)}$ such that, for each $\varepsilon \in (0,1)$,

$$\left|\varepsilon^d\sum_{x\in\varepsilon\mathbb{Z}^d}\psi(x) - \int_{\mathbb{R}^d}\psi\right| \le c\varepsilon,$$

we conclude that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\psi(x)f({}^x\!/\!_\varepsilon)\,\mathrm{d}x = \langle f\rangle\int_{\mathbb{R}^d}\psi.$$

This completes the proof. □

Theorem 2.10. *Let $U \subseteq \mathbb{R}^d$ be a smooth, bounded domain, let $A\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be periodic and uniformly elliptic, and let $f \in L^2(U)$. For every $\varepsilon \in (0,1)$ let $u^\varepsilon \in H^1_0(U)$ be the unique weak solution of*

$$-\nabla\cdot A({}^x\!/\!_\varepsilon)\nabla u^\varepsilon = f \;\; \textit{in} \;\; U \;\; \textit{with} \;\; u^\varepsilon = 0 \;\; \textit{on} \;\; \partial U. \tag{2.18}$$

Then, as $\varepsilon \to 0$,

$$u^\varepsilon \rightharpoonup v \;\; \textit{weakly in} \;\; H^1_0(U),$$

for $v \in H^1_0(U)$ the unique weak solution of

$$-\nabla\cdot\overline{A}\nabla v = f \;\; \textit{in} \;\; U \;\; \textit{with} \;\; v = 0 \;\; \textit{on} \;\; \partial U,$$

for $\overline{A} \in \mathbb{R}^{d\times d}$ defined in Proposition 2.5.

Proof. By Proposition 2.2 the u^ε are uniformly bounded in $H^1_0(U)$. Therefore, after passing to a subsequence, there exists $v \in H^1_0(U)$ such that, as $\varepsilon \to 0$,

$$u^\varepsilon \rightharpoonup v \text{ weakly in } H^1_0(U).$$

In particular, this implies that, as $\varepsilon \to 0$,

$$\nabla u^\varepsilon \rightharpoonup \nabla v \text{ weakly in } L^2_{\text{pot}}(U).$$

Let $\psi \in \mathrm{C}^\infty_c(U)$ and for each $i \in \{1, \ldots, d\}$ let ϕ^t_i be the adjoint corrector defined in (2.13). For each $\varepsilon \in (0,1)$ we define

$$\psi^\varepsilon(x) = \psi(x) + \varepsilon\phi^t_i(x/\varepsilon)\partial_i\psi(x).$$

Since $\psi^\varepsilon \in H^1_0(U)$ it is an admissible test function and we have that, for each $\varepsilon \in (0,1)$,

$$\nabla\psi^\varepsilon = (e_i + \nabla\phi^t_i(x/\varepsilon))\partial_i\psi + \varepsilon\phi^t_i(x/\varepsilon)\nabla\partial_i\psi,$$

and, after testing with (2.22),

$$\int_U A(x/\varepsilon)\nabla u^\varepsilon \cdot (e_i + \nabla\phi^t_i(x/\varepsilon))\partial_i\psi + \int_U A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla(\partial_i\psi)\varepsilon\phi^i_t(x/\varepsilon) = \int_U f\psi^\varepsilon.$$

And, after transposing A,

$$\int_U \nabla u^\varepsilon \cdot A^t(x/\varepsilon)(e_i + \nabla\phi^t_i(x/\varepsilon))\partial_i\psi + \int_U A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla(\partial_i\psi)\varepsilon\phi^i_t(x/\varepsilon) = \int_U f\psi^\varepsilon. \tag{2.19}$$

Since the $\phi^t_i(x/\varepsilon)$ are uniformly bounded in $H^1_0(U)$ since they are converging weakly to zero, it follows from Hölder's inequality that, as $\varepsilon \to 0$,

$$\psi^\varepsilon \to \psi \text{ strongly in } L^2(U),$$

and that

$$\varepsilon\phi^i_t(x/\varepsilon) \to 0 \text{ strongly in } L^2(U).$$

Therefore, by Hölder's inequality

$$\lim_{\varepsilon\to 0}\int_U A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla(\partial_i\psi)\varepsilon\phi^i_t(x/\varepsilon) = 0 \text{ and } \lim_{\varepsilon\to 0}\int_U f\psi^\varepsilon = \int_U f\psi. \tag{2.20}$$

Finally, since the $p^\varepsilon = A^t(x/\varepsilon)(e_i + \nabla\phi^t_i(x/\varepsilon))$ are divergence free, since the p^ε are converging weakly as $\varepsilon \to 0$ to $\langle A^t(y)(e_i + \nabla\phi^t_i\rangle$, and since the $\nabla u^\varepsilon \in L^2_{\text{pot}}(U)$ are converging weakly along a subsequence to ∇v, it follows from Proposition 2.6 and Proposition 2.8 that, along a subsequence,

$$\lim_{\varepsilon\to 0}\int_U \nabla u^\varepsilon \cdot A^t(x/\varepsilon)(e_i + \nabla\phi^t_i(x/\varepsilon))\partial_i\psi = \int_U \nabla v \cdot \overline{A}^t\nabla\psi. \tag{2.21}$$

And therefore, after transposing $\overline{A}$ it follows from (2.19), (2.20), and (2.21) that $v \in H^1_0(U)$ satisfies, for every $\psi \in \mathrm{C}^\infty_c(U)$,

$$\int_U \overline{A}\nabla v \cdot \nabla\psi = \int_U f\psi.$$

This is to say exactly that $v \in H^1_0(U)$ is a weak solution of the equation

$$-\nabla \cdot \overline{A}\nabla v = f \text{ in } U \text{ with } u^\varepsilon = 0 \text{ on } \partial U.$$

However, since $\overline{A}$ is uniformly elliptic by Proposition 2.5 weak solutions are unique by Proposition 2.2. Therefore, the weak limit $v \in H^1_0(U)$ is unique and we conclude that, along the full sequence $\varepsilon \to 0$,

$$u^\varepsilon \rightharpoonup v \text{ weakly in } H^1_0(U).$$

This completes the proof. □

Recall as well that divergence form elliptic and parabolic equations are defined by the flux. That is, the solution $u^\varepsilon \in H^1_0(U)$ of the equation solution of

$$-\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = f \text{ in } U,$$

is characterized in the interior of the domain by the equality

$$\oint_{B_r(x)} A(y/\varepsilon)\nabla u^\varepsilon(y) \cdot \nu = \int_{B_r(x)} f,$$

for any $\overline{B}_r(x) \subseteq U$ and for ν the outward unit normal. So, when proving homogenization, we are also interested in the convergence of the flux $A(x/\varepsilon)\nabla u^\varepsilon$. In the following proposition, using a variant of the perturbed test function method, we prove that the flux converges weakly to the flux of the homogenized equation.

Theorem 2.11. *Let $U \subseteq \mathbb{R}^d$ be a smooth, bounded domain, let $A\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be periodic and uniformly elliptic, and let $f \in L^2(U)$. For every $\varepsilon \in (0,1)$ let $u^\varepsilon \in H^1_0(U)$ be the unique weak solution of*

$$-\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = f \text{ in } U \text{ with } u^\varepsilon = 0 \text{ on } \partial U. \tag{2.22}$$

Then, as $\varepsilon \to 0$,

$$A(x/\varepsilon)u^\varepsilon \rightharpoonup \overline{A}\nabla v \text{ weakly in } L^2(U;\mathbb{R}^d),$$

for $v \in H^1_0(U)$ the unique weak solution of

$$-\nabla \cdot \overline{A}\nabla v = f \text{ in } U \text{ with } v = 0 \text{ on } \partial U,$$

for $\overline{A} \in \mathbb{R}^{d\times d}$ defined in Proposition 2.5.

Proof. The uniform ellipticity of A and the $H^1_0(U)$-boundedness of the solutions $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ prove that, after passing to a subsequence, there exists $F_0 \in L^2(U;\mathbb{R}^d)$ such that, as $\varepsilon \to 0$,

$$A(x/\varepsilon)\nabla u^\varepsilon \rightharpoonup F_0 \text{ weakly in } L^2(U;\mathbb{R}^d). \tag{2.23}$$

The convergence (2.24) proves that, after passing to a subsequence, for every $\psi \in \mathrm{C}^\infty_c(U)$,

$$\int_U F_0 \cdot \nabla\psi = \lim_{\varepsilon\to 0}\int_U A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla\psi = \int_U f\psi. \tag{2.24}$$

Let $\xi \in \mathbb{R}^d$ and define

$$w^\varepsilon_\xi(x) = \xi \cdot x + \varepsilon\phi^t_\xi(x/\varepsilon),$$

for $\phi^t_\xi = \xi_i\phi^t_i$ defined by the solutions to the transposed corrector equation (2.13). Let $\psi \in \mathrm{C}^\infty_c(U)$ be arbitrary. We alternately test the equation satisfied by u^ε by the admissible test function $w^\varepsilon_\xi\psi$ and we will test the equation satisfied by w^ε_ξ by the admissible test function $u^\varepsilon\psi$. That is, using the equation satisfied by u^ε, we have that

$$\int_U A(x/\varepsilon)\nabla u^\varepsilon \cdot \big((\xi + \nabla\phi^t_\xi(x/\varepsilon))\psi + w^\varepsilon_\xi(x)\nabla\psi\big) = \int_U f w^\varepsilon_\xi\psi. \tag{2.25}$$

Alternately we have that

$$\int_U A^t(x/\varepsilon)(\xi + \nabla\phi^t_\xi(x/\varepsilon)) \cdot (\nabla u^\varepsilon\psi + u^\varepsilon\nabla\psi) = 0. \tag{2.26}$$

After transposing the matrix A and subtracting (2.26) from (2.25), we have that

$$\int_U A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla\psi w^\varepsilon_\xi(x) - A^t(x/\varepsilon)(\xi + \nabla\phi^t_\xi(x/\varepsilon)) \cdot \nabla\psi u^\varepsilon = \int_U f w^\varepsilon_\xi\psi. \tag{2.27}$$

Since as $\varepsilon \to 0$ $w^\varepsilon_\xi(x) \to (\xi \cdot x)$ and $u^\varepsilon \to v$ strongly in $L^2(U)$, and since after passing to a subsequence it follows from (2.23) that, as $\varepsilon \to 0$,

$$A(x/\varepsilon)\nabla u^\varepsilon \rightharpoonup F_0 \;\text{ and }\; A^t(x/\varepsilon)(\xi + \nabla\phi^t_\xi(x/\varepsilon)) \rightharpoonup \overline{A}^t\xi \;\text{ weakly in }\; L^2(U),$$

we have that, after passing to the limit $\varepsilon \to 0$ in (2.27),

$$\int_U F_0 \cdot \nabla\psi(\xi \cdot x) - \int_U \overline{A}^t\xi \cdot \nabla\psi v = \int_U f\psi(x \cdot \xi). \tag{2.28}$$

Since we have from (2.24) that

$$\int_U F_0 \cdot \nabla\psi(\xi \cdot x) = \int_U f\psi(x \cdot \xi) - \int_U F_0 \cdot \xi\psi,$$

and since after integrating by parts

$$\int_U \overline{A}^t\xi \cdot \nabla\psi v = -\int_U \overline{A}\nabla v \cdot \xi\psi,$$

we conclude from (2.28) that, for every $\xi \in \mathbb{R}^d$ and $\psi \in \mathrm{C}^\infty_c(U)$,

$$\int_U F_0 \cdot \xi\psi = \int_U \overline{A}\nabla v \cdot \xi\psi.$$

Since ξ and ψ were arbitrary, we conclude that $F_0 = \overline{A}\nabla v$ in $L^2(U)$. This proves uniqueness of the weak limit, and therefore that the full sequence $A(x/\varepsilon)\nabla u^\varepsilon$ converges weakly as $\varepsilon \to 0$ to $\overline{A}\nabla v$ in $L^2(U)$, which completes the proof. □

2.5. **The homogenization error.** In this section, we will analyze the error in the two-scale expansion

$$w^\varepsilon = u^\varepsilon - v - \varepsilon\phi^\varepsilon_i\partial_i v, \tag{2.29}$$

for u^ε the solution of (2.15), for $\phi^\varepsilon_i = \phi_i(x/\varepsilon)$ defined by the correctors ϕ_i constructed in Proposition 2.4, and for v the solution of (2.16) with $\overline{A}$ defined in Proposition 2.5. The two-scale expansion suggests that the conclusion of Theorem 2.10 is the best that can be obtained on the level of the solution. The solution u^ε simply does not converges strongly as $\varepsilon \to 0$ to v in $H^1_0(U)$.

Formally we expect the solution u^ε will have oscillations of order ε on scale ε about the homogenized solution v. This means that the gradient will be of order one and, indeed, the gradient of the two-scale expansion is

$$\nabla\left(v + \varepsilon\phi^\varepsilon_i\partial_i v\right) = \nabla v + \nabla\phi_i(x/\varepsilon)\partial_i v + \varepsilon\phi_i(x/\varepsilon)\nabla(\partial_i v).$$

The final term on the righthand side converges strongly to zero as $\varepsilon \to 0$, but the second term on the righthand side only converges weakly to zero as $\varepsilon \to 0$. The weak convergence here explains the weak convergence obtained in Theorem 2.10.

The purpose of this section will be to prove that the homogenization error defined by the two-scale expansion (2.34) converges strongly to zero in $H^1(U)$. That is, after we subtract the oscillations described by the correctors, we obtain strong convergence on the level of the gradient. We will do this by studying the equation satisfied by (2.34). We see that

$$\nabla w^\varepsilon = \nabla u^\varepsilon - \nabla v - \nabla\phi_i(x/\varepsilon)\partial_i v - \varepsilon\phi^\varepsilon_i\nabla\partial_i v.$$

Then, using the fact that $-\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = f$,

$$-\nabla \cdot A(x/\varepsilon)\nabla w^\varepsilon = f + \nabla \cdot A(x/\varepsilon)\nabla v + \nabla \cdot A(x/\varepsilon)\left(\nabla\phi_i(x/\varepsilon)\partial_i v + \varepsilon\phi^\varepsilon_i\nabla\partial_i v\right).$$

Then, using the fact that $-\nabla \cdot \overline{A}\nabla v = f$,

$$-\nabla \cdot A(x/\varepsilon)\nabla w^\varepsilon = \nabla \cdot A(x/\varepsilon)\nabla v - \nabla \cdot \overline{A}\nabla v + \nabla \cdot A(x/\varepsilon)\left(\nabla\phi_i(x/\varepsilon)\partial_i v + \varepsilon\phi^\varepsilon_i\nabla\partial_i v\right).$$

The righthand side can be equivalently written in the form

$$-\nabla\cdot A(x/\varepsilon)\nabla w^\varepsilon=\nabla\cdot\left(\left(A(x/\varepsilon)\left(e_i+\nabla\phi_i(x/\varepsilon)\right)-\overline{A}e_i\right)\partial_i v\right)+\nabla\cdot A(x/\varepsilon)\left(\varepsilon\phi_i^\varepsilon\nabla\partial_i v\right). \tag{2.30}$$

The second term appearing on the righthand side of (2.35) is formally a good term, since the solution v of the homogenized equation is expected to be quite smooth and the correction $\varepsilon\phi_i^\varepsilon$ vanishes at $\varepsilon\to 0$. The first term, however, does not clearly vanish as $\varepsilon\to 0$. It does so weakly in the sense that, as $\varepsilon\to 0$,

$$A(x/\varepsilon)\left(e_i+\nabla\phi_i(x/\varepsilon)\right)\rightharpoonup\langle A\left(e_i+\nabla\phi_i\right)\rangle=\overline{A}e_i\;\text{ weakly in }\;L^2(U;\mathbb{R}^d),$$

but the necessary strong convergence fails.

The essential observation is that the first term on the righthand side of (2.35) is measuring the oscillations of the flux with respect to the A-harmonic coordinates $(x_i+\phi_i)$ about the flux $\overline{A}e_i$ in the homogenized environment. We therefore define, for each $i\in\{1,\ldots,d\}$,

$$q_i=A(e_i+\nabla\phi_i)\;\text{ in }\;\mathbb{T}^d, \tag{2.31}$$

and $q_i^\varepsilon(x)=q_i(x/\varepsilon)$. Since by definition $\langle q_i\rangle=\overline{A}e_i$, it follows from (2.35) and (2.32) that

$$-\nabla\cdot A(x/\varepsilon)\nabla w^\varepsilon=\nabla\cdot\left(\left(q_i^\varepsilon-\langle q_i\rangle\right)\partial_i v\right)+\nabla\cdot A(x/\varepsilon)\left(\varepsilon\phi_i^\varepsilon\nabla\partial_i v\right). \tag{2.32}$$

In the next section, we will construct a corrector σ_i^ε that accounts for the fluctuations of q_i^ε about its expectation.

2.6. **The flux corrector.** In this section, we will construct a flux correction that corrects the oscillation of the flux

$$q_i=A(e_i+\nabla\phi_i), \tag{2.33}$$

about its expectation $\langle q_i\rangle=\overline{A}e_i$. The essential observation is that the flux q_i is by definition divergence free, since the corrector equation states exactly that

$$-\nabla\cdot q_i=-\nabla\cdot A(e_i+\nabla\phi_i)=0.$$

The construction is motivated by differential forms and De Rham cohomology. The flux q_i defines a *closed* $(d-1)$-form as a divergence free vector field. This means that, due to the simple geometry of Euclidean space, there exists a $(d-2)$-form σ_i satisfying the property that $d\sigma_i=q_i$ where d denotes the exterior derivative. A $(d-2)$-form an be expressed as a skew-symmetric matrix. This is, for each $i\in\{1,\ldots,d\}$, we expect that there exists a skew-symmetric matrix $\sigma_i=(\sigma_{ijk})$ satisfying

$$\nabla\cdot\sigma_i=q_i-\langle q_i\rangle\;\text{ for }\;(\nabla\cdot\sigma_i)_j=\partial_k\sigma_{jk}. \tag{2.34}$$

However solutions to (2.34) are not unique. We will therefore identify a solution by fixing a choice of gauge.

We aim to find the least energy solution to (2.34). That is, for each $i\in\{1,\ldots,d\}$, we aim to minimize the energy

$$E(\sigma_i)=\frac{1}{2}\sum_{j,k=1}^d\int_{\mathbb{T}^d}\left|\nabla\sigma_{ijk}\right|^2\;\text{ subject to the constraint }\;\nabla\cdot\sigma_i=q_i.$$

For this we take motivation from the theory of Lagrange multipliers, and consider the function

$$F(\sigma_i)=\frac{1}{2}\int_{\mathbb{T}^d}\left|\nabla\cdot\sigma_i\right|^2.$$

On the set $\{\nabla\cdot\sigma_i=q_i\}$ a necessary but not sufficient for the energy E to be minimized by a skew-symmetric σ_i is that, for any skew-symmetric $v\in\mathbb{R}^{d\times d}$,

$$\lim_{h\to 0}\frac{E(\sigma_i+hv)-E(\sigma_i)}{h}=\lim_{h\to 0}\frac{F(\sigma_i+hv)-F(\sigma_i)}{h}.$$

Formally, this implies that the gradients of E and F are parallel, which is the condition required by Lagrange multipliers when optimizing a function subject to constraints. Let $j,k\in\{1,2,\ldots,d\}$ be arbitrary. We choose a matrix v of the form $v_{jk}=\tilde v$ and $v_{kj}=-\tilde v$ for some $\tilde v\in H^1(\mathbb{T}^d)$ and $v_{rs}=0$ for every other $r,s\in\{1,\ldots,d\}$. The variational relationship then yields, for $q_i=(q_{i1},q_{i2},\ldots,q_{id})$,

$$\int_{\mathbb{T}^d}\nabla\sigma_{ijk}\cdot\nabla\tilde v=\int_{\mathbb{T}^d}q_{ij}\partial_k\tilde v-q_{ik}\partial_j\tilde v.$$

This is to say that, for each $i,j,k\in\{1,\ldots,d\}$,

$$-\Delta\sigma_{ijk}=\partial_j q_{ik}-\partial_k q_{ij}\ \text{ in }\ \mathbb{T}^d. \tag{2.35}$$

We will take (2.35) as the defining equation for the components of σ_i. It will then follow by definition of $\sigma_{ijk}=-\sigma_{ikj}$ and we will show that after defining $\sigma_i=(\sigma_{ijk})$ we have $\nabla\cdot\sigma_i=q_i-\langle q_i\rangle$. The following proposition proves the existence of solutions to (2.35)

Proposition 2.12. *Let $q\in L^2(\mathbb{T}^d;\mathbb{R}^d)$. Then there exists a unique weak solution $\sigma\in H^1_0(\mathbb{T}^d)$ of the equation*

$$-\Delta\sigma=\nabla\cdot q\ \text{ in }\ \mathbb{T}^d.$$

Proof. The proof is an immediate consequence of the Poincaré inequality and the Lax-Milgram theorem, as in the case of Proposition 2.4 with A the identity matrix. □

Remark 2.13. We use Proposition 2.12 to obtain a solution of (2.35) by choosing $q=(q_1,\ldots,q_d)$ with $q_j=q_{ik}$, $q_k=-q_{ij}$ and $q_s=0$ for every $s\in\{1,\ldots,d\}\setminus\{i,j\}$.

Proposition 2.14. *Let $A\colon\mathbb{T}^d\to\mathbb{R}^{d\times d}$ be periodic and uniformly elliptic. For each $i\in\{1,2,\ldots,d\}$ let q_i be defined by (2.33) and for each $i,j,k\in\{1,2,\ldots,d\}$ be defined by (2.35). Then for each $i\in\{1,2,\ldots,d\}$ the matrix $\sigma_i=(\sigma_{ijk})\in L^2(\mathbb{T}^d;\mathbb{R}^{d\times d})$ is skew-symmetric and satisfies*

$$\nabla\cdot\sigma_i=q_i-\langle q_i\rangle. \tag{2.36}$$

Proof. The skew-symmetry is an immediate consequence of (2.35) and the uniqueness of Proposition 2.12, which proves that for every $i,j,k\in\{1,2,\ldots,d\}$ we have $\sigma_{ijk}=-\sigma_{ikj}$. It remains to prove the equality (2.36). We will prove that, for each $i,j\in\{1,\ldots,d\}$, as distributions,

$$\Delta\left((\nabla\cdot\sigma_i)_j-q_{ij}\right)=0.$$

Indeed, it follows from (2.35) that

$$\begin{aligned}\partial_s\partial_s\left(\partial_k\sigma_{ijk}-q_{ij}\right)&=\partial_k(\partial_s\partial_s\sigma_{ijk})-\partial_s\partial_s q_{ij}\\&=\partial_k\partial_k q_{ij}-\partial_k\partial_j q_{ik}+\partial_s\partial_s q_{ij}\\&=\Delta q_{ij}-\partial_j(\nabla\cdot q_i)-\Delta q_{ij}\\&=0,\end{aligned}$$

where the final equality relies on the fact that q_i is divergence free. What this implies is that, for every $\psi\in\mathrm{C}^\infty(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d}\left((\nabla\cdot\sigma_i)_j-q_{ij}\right)\Delta\psi=0. \tag{2.37}$$

For every $\varepsilon\in(0,1)$ let ρ^ε be a standard convolution kernel on the torus of scale ε. Then, for each $\varepsilon\in(0,1)$, we conclude from (2.37) that

$$\Delta\left(\rho^\varepsilon*((\nabla\cdot\sigma_i)_j-q_{ij})\right)(x)=\int_{\mathbb{T}^d}\left((\nabla\cdot\sigma_i)_j-q_{ij}\right)(y)\Delta_y\rho^\varepsilon(y-x)\,\mathrm{d}y=0,$$

from which it follows that $(\rho^\varepsilon*((\nabla\cdot\sigma_i)_j-q_{ij}))$ is constant for every $\varepsilon\in(0,1)$. Therefore, for every $i\in\{1,2,\ldots,d\}$, we have that $(\nabla\cdot\sigma_i)-q_i$ is a constant vector and so

$$(\nabla\cdot\sigma_i)-q_i=\langle(\nabla\cdot\sigma_i)-q_i\rangle=-\langle q_i\rangle,$$

where the final equality follows from the fact that the integral of a gradient is zero. We therefore have

$$\nabla \cdot \sigma_i = q_i - \langle q_i \rangle,$$

for every $i \in \{1, 2, \ldots, d\}$, which completes the proof. □

2.7. **Strong convergence of the two-scale expansion.** We recall the two-scale expansion

$$w^\varepsilon = u^\varepsilon - v - \varepsilon \phi_i^\varepsilon \partial_i v,$$

for u^ε the solution of (2.15), for $\phi_i^\varepsilon = \phi_i(x/\varepsilon)$ defined by the correctors ϕ_i constructed in Proposition 2.4, and for v the solution of (2.16) with $\overline{A}$ defined in Proposition 2.5. Returning to (2.32), we have that the homogenization error w^ε satisfies the equation

$$-\nabla \cdot A(x/\varepsilon)\nabla w^\varepsilon = \nabla \cdot \left(\left(q_i^\varepsilon - \langle q_i \rangle\right) \partial_i v\right) + \nabla \cdot A(x/\varepsilon) \left(\varepsilon \phi_i^\varepsilon \nabla \partial_i v\right) \ \text{ in } \ U. \tag{2.38}$$

For every $i \in \{1, \ldots, d\}$ let $\sigma_i = (\sigma_{ijk})$ be defined in Proposition 2.14 and let $\sigma_i^\varepsilon(x) = \sigma(x/\varepsilon)$. We will use σ_i^ε to control the oscillations of the flux on the righthand side of (2.38). Precisely, since we have that $\nabla \cdot (\varepsilon \sigma_i^\varepsilon) = q_i^\varepsilon - \langle q^i \rangle$, for every $\psi \in \mathrm{C}_c^\infty(U)$,

$$\int_U \left(q_i^\varepsilon - \langle q_i \rangle\right) \partial_i v \cdot \nabla \psi = \int_U \left(\nabla \cdot (\varepsilon \sigma_i^\varepsilon)\right) \partial_i v \cdot \nabla \psi.$$

Therefore, after integrating by parts and using the skew-symmetry of σ_i,

$$\int_U \left(q_i^\varepsilon - \langle q_i \rangle\right) \partial_i v \cdot \nabla \psi = -\int_U \varepsilon \sigma_i^\varepsilon \nabla (\partial_i v) \cdot \nabla \psi.$$

This is to say that, as distributions,

$$\nabla \cdot \left(\left(q_i^\varepsilon - \langle q_i \rangle\right) \partial_i v\right) = -\nabla \cdot \left(\varepsilon \sigma_i^\varepsilon \nabla (\partial_i v)\right). \tag{2.39}$$

Hence, returning to (2.38), we have that

$$-\nabla \cdot A(x/\varepsilon)\nabla w^\varepsilon = \nabla \cdot \left(\left(A(x/\varepsilon)\varepsilon \phi_i^\varepsilon - \varepsilon \sigma_i^\varepsilon\right) \nabla (\partial_i v)\right) \ \text{ in } \ U, \tag{2.40}$$

which explains the introduction of the flux correctors σ_i. It is now the case that the righthand side of (2.40) converges strongly to zero in $L^2(U)$ as $\varepsilon \to 0$. This is the content of the following theorem.

Theorem 2.15. *Let $U \subseteq \mathbb{R}^d$ be a smooth bounded domain, let $A \colon \mathbb{R}^d \to \mathbb{R}^{d \times d}$ be periodic and uniformly elliptic, and let $f \in \mathrm{C}^\alpha(U)$ for some $\alpha \in (0,1)$. For every $\varepsilon \in (0,1)$ let $u^\varepsilon \in H_0^1(U)$ be the unique weak solution of*

$$-\nabla \cdot A(x/\varepsilon)\nabla u^\varepsilon = f \ \text{ in } \ U \ \text{ with } \ u^\varepsilon = 0 \ \text{ on } \ \partial U,$$

and let $v \in H_0^1(U)$ be the unique weak solution of

$$-\nabla \cdot \overline{A}\nabla u^\varepsilon = f \ \text{ in } \ U \ \text{ with } \ u^\varepsilon = 0 \ \text{ on } \ \partial U,$$

for $\overline{A} \in \mathbb{R}^{d \times d}$ defined in Proposition 2.5. Then, for the correctors $\phi_i \in H_0^1(\mathbb{T}^d)$ defined in Proposition 2.4, as $\varepsilon \to 0$,

$$u^\varepsilon - v - \varepsilon \phi_i(x/\varepsilon)\partial_i v \to 0 \ \text{ strongly in } \ H^1(U).$$

Proof. We aim to use the energy identity (2.40), which requires the introduction of a cutoff function due to the fact that the homogenization error

$$w^\varepsilon = u^\varepsilon - v - \varepsilon \phi_i^\varepsilon \partial_i v,$$

for $\phi_i^\varepsilon(x) = \phi_i(x/\varepsilon)$ does not vanish along the boundary. For every $\rho \in (0,1)$ let $U_\rho \subseteq U$ be defined by

$$U_\rho = \{x \in U \colon \ \mathrm{d}(x, \partial U) < \rho\},$$

and using the smoothness of the domain for every $\rho\in(0,1)$ let $\eta_\rho\colon \overline{U}\to\mathbb{R}$ satisfy $\eta_\rho=1$ on U_ρ, $\eta_\rho=0$ on $\overline{U}\setminus U_{\rho/2}$, and $|\nabla\eta_\rho|\leq c/\rho$ for some $c\in(0,\infty)$ independent of $\rho\in(0,1)$. For every $\varepsilon,\rho\in(0,1)$ let

$$w^{\varepsilon,\rho}=u^\varepsilon-v-\varepsilon\phi_i^\varepsilon\partial_i.$$

The uniform ellipticity of $\overline{A}$ proves that there exists $c\in(0,\infty)$ such that

$$\|v\|_{C^{2,\alpha}(U)}\leq c\,\|f\|_{C^\alpha(U)}\,. \tag{2.41}$$

Therefore, it follows from the definition of η_ρ, from the fact that $\phi_i\in H^1_0(\mathbb{T}^d)$, and from the fact that $u,v\in H^1_0(U)$ that $w^{\varepsilon,\rho}\in H^1_0(U)$. We will analyze the equation satisfied by $w^{\varepsilon,\rho}$.

Let $\varepsilon,\rho\in(0,1)$. We observe that

$$\nabla w^{\varepsilon,\rho}=\nabla u^\varepsilon-\nabla v-\nabla\left(\eta_\rho\varepsilon\phi_i^\varepsilon\partial_i v\right).$$

Therefore, using the equations satisfied by u^ε and v,

$$-\nabla\cdot A^\varepsilon\nabla w^{\varepsilon,\rho}=\nabla\cdot(A^\varepsilon-\overline{A})\nabla v+\nabla\cdot\left(A^\varepsilon\nabla\phi_i^\varepsilon\left(\eta_\rho\partial_i v\right)\right)+\nabla\cdot\left(A^\varepsilon\varepsilon\phi_i^\varepsilon\nabla\left(\eta_\rho\partial_i v\right)\right).$$

It then follows from the definitions of q_i and $\overline{A}$ that

$$-\nabla\cdot A^\varepsilon\nabla w^{\varepsilon,\rho}=\nabla\cdot\left[(1-\eta_\rho)(A^\varepsilon-\overline{A})\nabla v\right]+\nabla\cdot\left(\left(q_i-\langle q_i\rangle\right)\left(\eta_\rho\partial_i v\right)\right)+\nabla\cdot\left(A^\varepsilon\varepsilon\phi_i^\varepsilon\nabla\left(\eta_\rho\partial_i v\right)\right),$$

for $q_i^\varepsilon=q_i(x/\varepsilon)$. Finally, using the flux corrections σ_i and the distributional equality (2.39),

$$-\nabla\cdot A^\varepsilon\nabla w^{\varepsilon,\rho}=\nabla\cdot\left[(1-\eta_\rho)(A^\varepsilon-\overline{A})\nabla v\right]+\nabla\cdot\left(\left(A^\varepsilon\varepsilon\phi_i^\varepsilon-\varepsilon\sigma_i^\varepsilon\right)\nabla\left(\eta_\rho\partial_i v\right)\right), \tag{2.42}$$

for $\sigma_i^\varepsilon=\sigma_i(x/\varepsilon)$. After testing equation (2.42) with $w^{\varepsilon,\rho}$,

$$\begin{aligned}\int_U A^\varepsilon\nabla w^{\varepsilon,\rho}\cdot\nabla w^{\varepsilon,\rho}=&-\int_U\left[(1-\eta_\rho)(A^\varepsilon-\overline{A})\nabla v\right]\cdot\nabla w^{\varepsilon,\rho}\\&-\int_U\left(\left(A^\varepsilon\varepsilon\phi_i^\varepsilon-\varepsilon\sigma_i^\varepsilon\right)\nabla\left(\eta_\rho\partial_i v\right)\right)\cdot\nabla w^{\varepsilon,\rho},\end{aligned}$$

and after applying Hölder's inequality and Young's inequality and using the uniform ellipticity, for some $c\in(0,\infty)$ independent of $\varepsilon\in(0,1)$,

$$\int_U|\nabla w^{\varepsilon,\rho}|^2\leq c\left(\int_U(1-\eta)^2|\nabla v|^2+\int_U\left(\varepsilon^2|\phi_i^\varepsilon|^2+\varepsilon^2|\sigma_i^\varepsilon|^2\right)|\nabla(\eta_\rho\partial_i v)|^2\right).$$

Estimate (2.41), the boundedness of U, and the definition of η_ρ prove that, for some $c\in(0,\infty)$ independent of $\varepsilon,\rho\in(0,1)$,

$$\int_U|\nabla w^{\varepsilon,\rho}|^2\leq c\left(\rho+\frac{\varepsilon^2}{\rho^2}\int_U\left(|\phi_i^\varepsilon|^2+|\sigma_i^\varepsilon|^2\right)\right). \tag{2.43}$$

We now return to the original homogenization error w^ε defined by

$$w^\varepsilon=u^\varepsilon-v-\varepsilon\phi_i^\varepsilon\partial_i v,$$

for which we have that

$$\nabla w^\varepsilon=\nabla w^{\varepsilon,\rho}+\nabla\left((1-\eta_\rho)\varepsilon\phi_i^\varepsilon\partial_i v\right).$$

Since it follows the definition η_ρ and estimate (2.41) that, for some $c\in(0,\infty)$ independent of $\varepsilon,\rho\in(0,1)$,

$$\int_U|\nabla\left((1-\eta_\rho)\varepsilon\phi_i^\varepsilon\partial_i v\right)|^2\leq c\left(\frac{\varepsilon^2}{\rho^2}\int_U|\phi_i^\varepsilon|^2+\int_U(1-\eta_\rho)^2|\nabla\phi_i^\varepsilon|^2+\varepsilon^2\int_U(1-\eta_\rho)^2|\phi_i^\varepsilon|^2\right),$$

the triangle inequality and (2.43) prove that, for some $c\in(0,\infty)$ independent of $\varepsilon,\rho\in(0,1)$,

$$\int_U|\nabla w^\varepsilon|^2\leq c\left(\rho+\frac{\varepsilon^2}{\rho^2}\int_U\left(|\phi_i^\varepsilon|^2+|\sigma_i^\varepsilon|^2\right)+\int_U(1-\eta_\rho)^2|\nabla\phi_i^\varepsilon|^2\right). \tag{2.44}$$

Since, as $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0}\int_U \left(|\phi_i^\varepsilon|^2 + |\sigma_i^\varepsilon|^2\right) + \int_U (1-\eta_\rho)^2\,|\nabla\phi_i^\varepsilon|^2 = |U|\int_{\mathbb{T}^d}|\phi_i|^2 + |\sigma_i|^2 + \int_{\mathbb{T}^d}|\nabla\phi_i|^2\int_U (1-\eta_\rho)^2,$$

it follows from the definition of η_ρ and the boundedness of U that, for some $c \in (0,\infty)$ independent of $\rho \in (0,1)$,

$$\limsup_{\varepsilon\to 0}\int_U |\nabla w^\varepsilon|^2 \leq c\rho\left(1 + \int_{\mathbb{T}^d}|\nabla\phi_i|^2\right).$$

Finally, after passing to the limit $\rho \to 0$, we conclude that

$$\lim_{\varepsilon\to 0}\int_U |\nabla w^\varepsilon|^2 = 0,$$

which completes the proof. □

3. Stochastic Homogenization

In this section, we will prove the homogenization of random environments described by a random coefficient field $A(x,\omega)\colon \mathbb{R}^d \to \Omega \to \mathbb{R}^{d\times d}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The random environment is indexed by $\omega \in \Omega$. We will see that the random framework strictly generalizes the periodic framework above, and our analysis will be strongly motivated by the methods from periodic homogenization. Given a uniformly elliptic, *stationary*, and *ergodic* coefficient field $A\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ we will identify a deterministic, constant coefficient field $\overline{A}$ such that the solutions

$$-\nabla\cdot A(x/\varepsilon,\omega)\nabla u^\varepsilon = f \;\text{ in }\; U \;\text{ with }\; u^\varepsilon = 0 \;\text{ on }\; \partial U,$$

almost surely converge as $\varepsilon \to 0$ to the solution

$$-\nabla\cdot\overline{A}\nabla v = f \;\text{ in }\; U \;\text{ with }\; v = 0 \;\text{ on }\; \partial U.$$

This is to say that for almost every $\omega \in \Omega$ the random medium described by $A(\cdot/\varepsilon,\omega)$ is well approximated by the homogenous medium $\overline{A}$. An essential difference in this case, however, is that the scale $\varepsilon \in (0,1)$ for which this approximation becomes valid is itself random. And for this reason, the quantitative theory of stochastic homogenization is more complicated than in the periodic case.

3.1. **A random environment.** In this section, we will construct a random environment that models a material with randomly deposited impurities. We could equally construct a random tiling of the plane, as shown in the second picture of the introduction. Let Ω denote the space of all locally finite point measures on $\mathbb{R}^d$. That is, Ω is the collection of all measures of the form

$$\omega = \sum_{j\in\mathcal{J}}\delta_{x_j}$$

where $\mathcal{J} \subseteq \mathbb{N}$ is a countable index set, $x_j \in \mathbb{R}^d$ for every $j \in \mathcal{J}$, δ_{x_j} denotes the Dirac distribution centered at x_j, and for every bounded Borel subset $B \subseteq \mathbb{R}^d$ the set

$$\omega(B) = \#\{j \in \mathcal{J}\colon x_j \in B\} \;\text{ is finite.}$$

Let $\mathcal{F}$ denote the sigma algebra generated by all maps of the form $\omega \in \Omega \mapsto \omega(B)$ for some bounded Borel subset $B \subseteq \mathbb{R}^d$. We will now construct a probability measure on $(\Omega, \mathcal{F})$. We construct a *Poisson point process* on $\mathbb{R}^d$ with intensity $\lambda \in (0,\infty)$ by equipping Ω with probability measure $\mathbb{P}_\lambda$ satisfying the following three properties:

- For every Borel subset $B \subseteq \mathbb{R}^d$,

$$\mathbb{E}_\lambda\left[\omega(B)\right] = \lambda\,|B|\,, \tag{3.1}$$

  for the Lebesgue measure $|B|$ of B.

- For every collection of bounded, disjoint subsets $B_1, \ldots, B_N$ the random variables

$$\omega \in \Omega \mapsto \omega(B_k) \text{ for } k \in \{1, \ldots, N\} \text{ are independent.} \tag{3.2}$$

- For every $y \in \mathbb{R}^d$ and measurable set $A \in \mathcal{F}$,

$$\mathbb{P}_\lambda(A) = \mathbb{P}_\lambda(A + y), \tag{3.3}$$

where $A + y = \{\omega(\cdot + y) \colon \omega \in A\}$.

The measure $\mathbb{P}_\lambda$ is uniquely characterized by these properties, and the resulting process is a Poission point process with intensity λ. Property (3.1) asserts that on average there exist λ points in a subset of measure one. Property 3.2 is a strong form of mixing which states that the behavior of the process is disjoint sets is independent. This is a quantified version of ergodicity. Lastly, property (3.3) is a version of stationarity that asserts that the environment is statistically homogenous. You are as likely to see a cluster of points near that origin, as you are at some point a thousand miles away.

Henceforth let $(\Omega, \mathcal{F}, \mathbb{P}_\lambda)$ be a Poission point process with intensity $\lambda \in (0, \infty)$. Then, for some constants $\alpha_1, \alpha_2, \delta \in (0, \infty)$ we define a random coefficient field

$$A(x, \omega) = \left(\alpha_1 \mathbf{1}_{\cup_{j \in \mathcal{J}} B_\delta(x_j)} + \alpha_2 \mathbf{1}_{\mathbb{R}^d \setminus \cup_{j \in \mathcal{J}} B_\delta(x_j)}\right) I_{d \times d}. \tag{3.4}$$

That is, in the δ-neighborhood of each point x_j in the realization of the point process we see a diffusion coefficient α_1, and away from these points we see a diffusion coefficient α_2. So, if $\alpha_1 \simeq 0$ and $\alpha_2 \simeq 1$ it is as though we are modeling a model with randomly deposited, non-conducting impurities. For a bounded, nonnegative, compactly supported function $W \colon \mathbb{R}^d \to \mathbb{R}$ we could similarly define

$$A(x, \omega) = \left(1 + \sum_{j \in \mathcal{J}} W(x - x_j)\right) I_{d \times d}.$$

In this case, the matrix A is not globally bounded from above, and so our techniques would not immediately apply. Nonetheless, the theory can be extended to degenerate environments and perforated domains.

The essential statistical properties of A are its *stationarity* and *ergodicity*. That is, the probability space $(\Omega, \mathcal{F}, \mathbb{P}_\lambda)$ comes equipped with a transformation group $\{\tau_x\}_{x \in \mathbb{R}^d}$ defined by

$$\tau_x \omega = \omega(\cdot - x),$$

which extends to a transformation group on the space of coefficient fields A defined for every $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$ by

$$A(x + y, \omega) = A(y, \tau_x \omega).$$

Property (3.3) proves that transformation group preserves the measure $\mathbb{P}_\lambda$ in the sense that, for every $y \in \mathbb{R}^d$ and $A \in \mathcal{F}$,

$$\mathbb{P}_\lambda[A] = \mathbb{P}_\lambda[\tau_y A],$$

Property (3.3) proves that the ensemble is stationary in the sense that, for every $x_1, \ldots, x_n, y \in \mathbb{R}^d$,

$$(A(x_1, \cdot), \ldots, A(x_n, \cdot)) \text{ and } (A(y + x_1, \cdot), \ldots, A(y + x_n, \cdot)) \text{ have the same law on } \left(\mathbb{R}^{d \times d}\right)^n. \tag{3.5}$$

And property (3.3) proves that $A \in \mathcal{F}$ satisifes

$$\mathbb{P}_\lambda[(\tau_x A) \triangle (A)] = 0 \text{ for every } x \in \mathbb{R}^d \text{ if and only if } \mathbb{P}_\lambda[A] = 0 \text{ or } \mathbb{P}_\lambda[A] = 1, \tag{3.6}$$

where $(\tau_x A) \triangle (A) = (\tau_x A \setminus A) \cup (A \setminus \tau_x A)$ is the symmetric difference. Indeed, Property (3.3) is a stronger form of mixing than is generally implied by (3.6). In particular, it follows from (3.3) and (3.4) that the matrix A satisfies a finite range of dependence. That is, whenever subsets $A, B \subseteq \mathbb{R}^d$ satisfy $\delta(A, B) > 2\delta$,

$$\sigma(A(x, \cdot) \colon x \in A) \text{ and } \sigma(A(x, \cdot) \colon x \in B) \text{ are indepedent.} \tag{3.7}$$

Conditions like the finite range of dependence (3.7) are quantified forms of ergodicity and always imply the weaker condition (3.6). For this reason, assumption (3.6) is sometimes referred to as a qualitative form of mixing.

We will take this framework as our starting point. We will assume that the coefficient field $A\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a measure-preserving transformation group $\{\tau_x\}_{x\in\mathbb{R}^d}$ that satisfy the following properties:

- *uniform ellipticity*: there exist $\lambda, \Lambda \in (0,\infty)$ such that almost surely, for every $x, \xi \in \mathbb{R}^d$,

$$|A(x,\omega)\xi| \leq \Lambda\,|\xi| \;\text{ and }\; A(x,\omega)\xi\cdot\xi \geq \lambda\,|\xi|^2\,. \tag{3.8}$$

- *stationarity*: almost surely, for every $x, y \in \mathbb{R}^d$,

$$A(x+y,\omega) = A(x,\tau_y\omega). \tag{3.9}$$

- *ergodicity*: for every $A \in \mathcal{F}$,

$$\mathbb{P}_\lambda[(\tau_x A)\triangle(A)] = 0 \;\text{ for every }\; x \in \mathbb{R}^d \;\text{ if and only if }\; \mathbb{P}_\lambda[A] = 0 \;\text{ or }\; \mathbb{P}_\lambda[A] = 1. \tag{3.10}$$

- *stochastic continuity*: for every $\delta \in (0,1)$,

$$\lim_{|x|\to 0} \mathbb{P}\left[|A(0,\omega) - A(x,\omega)| > \delta\right] = 0. \tag{3.11}$$

The final condition is a satisfied by the environments constructed using the Poisson point process above, as is a technical condition that will allow us to regularize functions defined on the probability space using convolutions defined by the transformation group. These conditions are the most general for which we can expect to prove homogenization. If the transformation group is not ergodic, then we do not expect to a see a deterministic limit defined by $\overline{A}$, and if the environment is not stationary then we do not expect to see the averaging require for homogenization to occur.

3.2. **The ergodic theorem.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\tau_x\}_{x\in\mathbb{R}^d}$ be a measure preserving group of transformation on Ω in the sense that, for every $x, y \in \mathbb{R}^d$ and $A \in \mathcal{F}$,

$$\tau_x \circ \tau_y = \tau_{x+y} \;\text{ and }\; \mathbb{P}[A] = \mathbb{P}[\tau_x A].$$

We will assume that the transformation group is ergodic in the sense that

$$A \in \mathcal{F} \;\text{ satisfies }\; \mathbb{P}[(A)\triangle(\tau_x A)] = 0 \;\text{ for every }\; x \in \mathbb{R}^d \;\text{ if and only if }\; \mathbb{P}[A] = 0 \;\text{ or }\; \mathbb{P}[A] = 1.$$

The essential role of ergodicity is to almost surely replace averages in expectation by large-scale averages in space. That is, if $f \in L^1(\Omega)$ then almost surely

$$\mathbb{E}[f] \simeq \fint_{B_R} f(\tau_x\omega)\,\mathrm{d}x \;\text{ for }\; R \in (0,\infty) \;\text{ sufficiently large.}$$

Intuitively this means that almost surely every individual environment is representative of the family as a whole in the sense that after averaging over the whole space we recover the global expectation. This is the content of the following ergodic theorem.

Theorem 3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\tau_x\}_{x\in\mathbb{R}^d}$ be an ergodic, measure preserving group of transformation on Ω. Then for every $f \in L^1(\Omega)$ there exists a subset $\Omega' \subseteq \Omega$ of full probability such that, every bounded open subset $U \subseteq \mathbb{R}^d$ containing the origin,*

$$\mathbb{E}[f] = \lim_{R\to\infty} \fint_{U_R} f(\tau_x\omega)\,\mathrm{d}x \;\text{ for every }\; \omega \in \Omega',$$

for $U_R = \{Rx\colon x \in U\}$.

Proof. The details will be added later. See Becker [1981] provided in the lecture. □

A corollary of the ergodic theorem is the following version of Lemma 2.9. In a stationary and ergodic environments the rescaled random variables $f(\tau_{x/\varepsilon}\omega)$ for $f \in L^2(\Omega)$ almost surely converge weakly to the expectation of f.

Corollary 3.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\tau_x\}_{x\in\mathbb{R}^d}$ be an ergodic, measure preserving group of transformation on Ω. Then, for every $f \in L^2(\Omega)$, almost surely as $\varepsilon \to 0$,*

$$f(\tau_{x/\varepsilon}\omega) \rightharpoonup \mathbb{E}\left[f\right] \;\; \textit{weakly in} \;\; L^2_{loc}(\mathbb{R}^d).$$

Proof. By density of linear combinations of indicator functions of bounded open subsets containing the origin in $L^2_{\text{loc}}(\mathbb{R}^d)$ it suffices to prove that for every bounded open subset $A \subseteq \mathbb{R}^d$ containing the origin we have almost surely that

$$\lim_{\varepsilon\to 0}\int_A f(\tau_{x/\varepsilon}\omega)\,\mathrm{d}x = |A|\,\mathbb{E}\left[f\right].$$

But, after rescaling,

$$\lim_{\varepsilon\to 0}\int_A f(\tau_{x/\varepsilon}\omega)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\varepsilon^d\int_{A_\varepsilon} f(\tau_x\omega)\,\mathrm{d}x = |A|\lim_{\varepsilon\to 0}\fint_{A_\varepsilon} f(\tau_x\omega)\,\mathrm{d}x,$$

for $A_\varepsilon = \{ve^{-1}x\colon x \in A\}$. After relabeling $R = \varepsilon^{-1}$, the ergodic theorem proves almost surely that

$$\lim_{\varepsilon\to 0}\int_A f(\tau_{x/\varepsilon}\omega)\,\mathrm{d}x = |A|\lim_{R\to\infty}\fint_{A_R} f(\tau_x\omega)\,\mathrm{d}x = |A|\,\mathbb{E}\left[f\right],$$

for $A_R = \{Rx\colon x \in A\}$. This completes the proof. □

An important consequence of Corollary 3.2 is the local L^2-boundedness of random variables $f^\varepsilon = f(\tau_{x/\varepsilon}\omega)$ defined by random variables $f \in L^2(\Omega)$ for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a measure-preserving ergodic transformation group $\{\tau_x\}_{x\in\mathbb{R}^d}$. Precisely because as $\varepsilon \to 0$ we have almost surely that $f^\varepsilon \rightharpoonup \mathbb{E}[f]$ weakly in $L^2_{\text{loc}}(\mathbb{R}^d)$ we have almost surely that, for each $R \in (0,\infty)$,

$$\sup_{\varepsilon\in(0,1)}\left\|f(\tau_{x/\varepsilon}\omega)\right\|_{L^2(B_R)} < \infty.$$

That is, while the value of the supremum on the lefthand side is itself is random and need not be uniformly bounded in $\omega \in \Omega$, it is almost surely finite.

3.3. **The random homogenization corrector.** Motivated by the coefficient field (3.4) defined by the Poisson point process, we will henceforth consider a uniformly elliptic coefficient field $a\colon \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with an ergodic, measure-preserving transformation group $\{\tau_x\}_{x\in\mathbb{R}^d}$ such that almost surely

$$a(x+y,\omega) = a(y,\tau_x\omega) \;\text{ for every }\; x,y \in \mathbb{R}^d.$$

Observe that the stationarity implies almost surely that $a(x,\omega) = A(\tau_x\omega)$ for $A(\omega) = a(0,\omega) \in L^\infty(\Omega;\mathbb{R}^{d\times d})$. More precisely, we assume that the environment satisfies (3.8), (3.9), (3.10), and (3.11). These are the foundational assumptions of the theory, and are the most general for which we would expect to see homogenization.

The goal of stochastic homogenization is to characterize almost surely the asymptotic behavior as $\varepsilon \to 0$ of the solutions

$$-\nabla\cdot a(x/\varepsilon,\omega)\nabla u^\varepsilon = f \;\text{ in }\; U \;\text{ with }\; u^\varepsilon = 0 \;\text{ on }\; \partial U.$$

We take our motivation from periodic homogenization and postulate an asymptotic expansion of the form

$$u^\varepsilon(x,\omega) = u_0(x, x/\varepsilon, \omega) + \varepsilon u_1(x, x/\varepsilon, \omega) + \varepsilon^2 u_2(x, x/\varepsilon, \omega) + \ldots.$$

The same heuristics from the periodic case suggest that the asymptotic expansion reduces to an expansion of the form

$$u^\varepsilon(x,\omega) = v(x) + \varepsilon\phi_i(x/\varepsilon,\omega)\partial_i v(x),$$

where for each $i \in \{1,\ldots,d\}$ the corrector ϕ_i almost surely satisfies

$$-\nabla\cdot a(y,\omega)(\nabla\phi_i(y,\omega) + e_i) = 0 \;\text{ on }\; \mathbb{R}^d, \tag{3.12}$$

and were v solves

$$-\nabla \cdot \overline{a}\nabla v = f \text{ in } U \text{ with } v = 0 \text{ on } \partial U,$$

for the homogenized coefficient $\overline{A}$ defined for each $i \in \{1, \ldots, d\}$ by

$$\overline{a} = \mathbb{E}\left[A(\omega)(\nabla\phi_i(0,\omega) + e_i)\right] = \mathbb{E}\left[a(0,\omega)(\nabla\phi_i(0,\omega) + e_i)\right].$$

This is in direct analogy with the periodic case, where the expectation in the random case replaces the average over the torus.

Indeed, the periodic case can be placed into the random framework. The probability space is $\Omega = \mathbb{T}^d = [0,1]^d$ and the probability measure $\mathbb{P} = \mathrm{d}x$ is the Lebesgue measure. Given a one-periodic coefficient field $A\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ we define the "random" coefficient field $a(x,\omega) = A(x+\omega)$ where the "randomness" here simply describes the point ω in the periodic cell at which the coefficient field $a(x,\omega)$ is centered. That is, $a(x,\omega)$ is describes the original environment A shifted by $\omega \in \mathbb{T}^d$ and the transformation group $\{\tau_x\}_{x\in\mathbb{R}^d}$ is simply $\tau_x\omega = \omega + x$ for which we have

$$a(x+y,\omega) = A(x+y+\omega) = a(y,\tau_x\omega).$$

The point in the periodic case is that recentering the environment does not fundamentally change anything. This is very much not the case for generic random environments. Taking (3.4) for example, recentering the environment may transfer you from a region of high diffusivity to a region of virtually no diffusivity. In terms of the analysis, we will see that moving from the periodic case to the stochastic case results in a fundamental loss of compactness.

Our first aim will be to construct the correctors ϕ_i solving (3.12). You would correct to say that (3.12) is trivial solvable by choosing $\phi_i(x,\omega) = -x_i$. In the periodic case, we avoided this solution by insisting that ϕ_i be one periodic. In the random setting, we take our motivation from the asymptotic expansion

$$u^\varepsilon(x,\omega) = v(x) + \varepsilon\phi_i(x/\varepsilon,\omega)\partial_i v(x).$$

If we expect almost surely that, as $\varepsilon \to 0$,

$$u^\varepsilon \to v,$$

then we essentially require that, for each $i \in \{1,\ldots,d\}$, after rescaling,

$$\lim_{\varepsilon\to 0} |\varepsilon\phi_i(x/\varepsilon)| = \lim_{|x|\to\infty} \frac{|\phi(x)|}{|x|} = 0.$$

This is *sublinearity* which we will measure below in an L^2-sense. That is, we will construct a solution of (3.12) that is sublinear in the sense that, almost surely for each $i \in \{1,\ldots,d\}$,

$$\lim_{\varepsilon\to 0} \varepsilon \left(\fint_{B_1} |\phi_i(x/\varepsilon,\omega)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} = \lim_{R\to\infty} \frac{1}{R} \left(\fint_{B_R} |\phi_i(x,\omega)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} = 0.$$

It is this condition that rules out the trivial solution $\phi_i(x) = -x_i$, and it is exactly this condition that justifies the asymptotic expansion. Indeed, we will see that the convergence of u^ε to v is controlled by the sublinearity of the corrector and analogous flux corrector.

Constructing a sublinear solution of (3.12) requires that we incorporate the probabilistic structure of the environment. That is, due to the stationarity and ergodicity of the environment, we expect equation (3.12) to be exhibit more cancellations and averaging than would be generically expected for a deterministic equation of the type (3.12). To do this, we will use the transformation group to lift the equation to the probability space and to construct a solution of (3.12) by constructing its random gradient as a mean zero, curl-free random vector Φ_i. We then almost surely obtain the solution ϕ_i by integration, where the sublinearity follows from the fact that $\mathbb{E}[\Phi_i] = 0$. That is, from the fact that since the gradient is mean zero it exhibits cancellations that force ϕ_i to grow sublinearly.

To lift the equation to the probability space, we define the so-called horizontal derivatives using the transformation group $\{\tau_x\}_{x\in\mathbb{R}^d}$. For this we observe that there exists a natural class of test functions on Ω. For each $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ and for each $f \in L^\infty(\Omega)$ we define $\psi_f \in L^\infty(\Omega)$ by

$$\psi_f(\omega) = \int_{\mathbb{R}^d} f(\tau_x\omega)\psi(x)\,\mathrm{d}x,$$

and we will write $\mathcal{D}(\Omega)$ for the space of all such functions. If $f \in L^\infty(\Omega)$ and $\rho^\varepsilon \in \mathrm{C}_c^\infty(\Omega)$ for each $\varepsilon \in (0,1)$ is a standard convolution kernel of scale $\varepsilon \in (0,1)$, it follows from (3.11) that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} f(\tau_x\omega)\rho^\varepsilon(x)\,\mathrm{d}x = f \;\text{ in }\; L^p(\Omega) \;\text{ for every }\; p\in[1,\infty).$$

In this way, from the density of $L^\infty(\Omega)$ in $L^p(\Omega)$ for every $p\in[1,\infty)$, we see that $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$ for every $p\in[1,\infty)$.

We will now define the horizontal derivatives. For every $i\in\{1,\ldots,d\}$ let $\mathcal{D}(D_i)\subseteq L^(\Omega)$ be the space

$$\mathcal{D}(D_i) = \left\{ f\in L^2(\Omega)\colon \lim_{h\to 0}\frac{f(\tau_{he_i}\omega)-f(\omega)}{h} \;\text{ exists strongly in }\; L^2(\Omega)\right\},$$

and define $D_i\colon \mathcal{D}(D_i)\to L^2(\Omega)$ by

$$D_i f = \lim_{h\to 0}\frac{f(\tau_{he_i}\omega)-f(\omega)}{h}. \tag{3.13}$$

The following proposition proves that the operators $\mathcal{D}(D_i)$ are densely defined and closed.

Proposition 3.3. *The operators* $\{D_i\}_{i\in\{1,\ldots,d\}}$ *defined in* (3.13) *are densely defined and closed.*

Proof. Let $\psi_f\in\mathcal{D}(\Omega)$. Then, for each $i\in\{1,\ldots,d\}$ the dominated convergence theorem proves that

$$D_i\psi_f(\omega) = -\int_{\mathbb{R}^d} f(\tau_x\omega)\partial_i\rho^\varepsilon(x)\,\mathrm{d}x \in L^2(\Omega).$$

The density of $\mathcal{D}(\Omega)$ in $L^2(\Omega)$ proves that the operators D_i are densely defined. Let $f,g\in\mathcal{D}(D_i)$ and $g\in L^2(\Omega)$. Then, it follows by definition of D_i, the fact that the transformation group preserves the measure, and Hölder's inequality that

$$\mathbb{E}[D_i f g] = \mathbb{E}\left[\left(\lim_{h\to 0}\frac{f(\tau_{he_i}\omega)-f(\omega)}{h}\right)g(\omega)\right] = \mathbb{E}\left[f(\omega)\left(\lim_{h\to 0}\frac{g(\tau_{-he_i}\omega)-g(\omega)}{h}\right)\right] = -\mathbb{E}[fD_i g].$$

We therefore conclude that the adjoint $(D_i)^* = -D_i$. Or, equivalently, that $D_i = (-D_i)^*$. Therefore, D_i is closed since it is the adjoint of a densely defined operator. □

We now define the analogue of H^1 on the probability space: let $\mathcal{H}^1(\Omega)\subseteq L^2(\Omega)$ denote the space

$$\mathcal{H}^1(\Omega) = \cap_{i=1}^d \mathcal{D}(D_i),$$

for which we have $\mathcal{D}(\Omega)\subseteq\mathcal{H}^1(\Omega)$ as above. Here we must emphasize an essential point. We will *not* construct the solutions ϕ_i of (3.12) as elements of $\mathcal{H}^1(\Omega)$. That is, in general, there do not exist random variables $\overline{\phi}_i\in\mathcal{H}^1(\Omega)$ such that almost surely we have $\phi_i(x,\omega) = \overline{\phi}_i(\tau_x\omega)$. The correctors ϕ_i themselves *do not* in general exist as stationary functions. This is not totally surprising in the sense that only the gradient of the corrector comes to define the homogenized coefficient in both the periodic and random cases and in the sense that, from the point of view of the asymptotic expansion, it is the gradient of the corrector that accounts for the oscillations of ∇u^ε in the sense that, as $\varepsilon\to 0$,

$$\nabla u^\varepsilon \simeq \nabla v + \nabla\phi_i^\varepsilon\partial_i v.$$

We will therefore construct *the gradient* of the corrector as a stationary random variable. That is, we will show that there exists a mean zero, curl free vector field $\Phi_i\in L^2(\Omega;\mathbb{R}^d)$ such that almost surely we have $\nabla\phi_i(x,\omega) = \Phi_i(\tau_x\omega)$.

We will construct the random gradient in the space of potential vector fields. For every $\psi \in \mathcal{H}^1(\Omega)$ we define

$$D\psi = (D_1\psi, \ldots, D_d\psi) \in L^2(\Omega; \mathbb{R}^d).$$

Given a vector field $V \in L^2(\Omega; \mathbb{R}^d)$ we will understand distributional equalities in $\mathcal{D}'(O)$. That is, we say that $D \cdot V = D_i V_i = 0$ if $\mathbb{E}\left[V \cdot D\psi\right] = 0$ for every $\psi \in \mathcal{D}(\Omega)$. Observe that for every $\psi \in \mathcal{H}^1(\Omega)$ the gradient is curl-free in the sense that, for every $i, j \in \{1, \ldots, d\}$,

$$D_i D_j \psi = D_j D_i \psi. \tag{3.14}$$

Furthermore, for each $i \in \{1, \ldots, d\}$, it follows by definition of D_i and the fact that the transformation group preserves the measure that

$$\mathbb{E}\left[D_i\psi\right] = \lim_{h\to 0} \frac{1}{h}\mathbb{E}\left[\psi(\tau_{he_i}\omega) - \psi(\omega)\right] = \lim_{h\to 0} \frac{1}{h}\left(\mathbb{E}\left[\psi(\omega)\right] - \mathbb{E}\left[\psi(\omega)\right]\right) = 0. \tag{3.15}$$

In combination, (3.14) and (3.15) prove that the gradient is mean zero and curl free. We then define the space of potential vector fields

$$L^2_{\mathrm{pot}}(\Omega) = \overline{\{D\psi \colon \psi \in \mathcal{H}^1(\Omega)\}}^{L^2(\Omega;\mathbb{R}^d)}$$

to be the strong $L^2(\Omega; \mathbb{R}^d)$ closure of the space of $\mathcal{H}^1(\Omega)$ gradients. Since the distributional equalities and expectation are stable with respect to strong convergence, we see that for every $V = (V_i)_{i\in\{1,\ldots,d\}} \in L^2_{\mathrm{pot}}(\Omega)$,

$$D_j V_i = D_i V_j \ \text{ and } \ \mathbb{E}\left[V\right] = 0. \tag{3.16}$$

Indeed, we will see below that $L^2(\Omega; \mathbb{R}^d)$ admits a Helmholtz decomposition similar to what we say in the deterministic case, and that we could have equivalently defined $L^2_{\mathrm{pot}}(\Omega)$ to be the space of mean zero, curl free fields

$$L^2_{\mathrm{pot}}(\Omega) = \{V = (V_i)_{i\in\{1,\ldots,d\}} \in L^2(\Omega; \mathbb{R}^d) \colon D_i V_j = D_j V_i \ \forall\, i, j \in \{1, \ldots, d\} \ \text{ and } \ \mathbb{E}[V] = 0\}.$$

It follows by definition that $L^2_{\mathrm{pot}}(\Omega)$ is a Hilbert space when equipped with the inner product

$$\langle V, W\rangle_{L^2_{\mathrm{pot}}(\Omega)} = \mathbb{E}\left[V \cdot W\right].$$

We will now construct the random gradients Φ_i in the space $L^2_{\mathrm{pot}}(\Omega)$.

Proposition 3.4. *Let $A \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ be uniformly elliptic in the sense that there exist constants $\lambda, \Lambda \in (0, \infty)$ such that almost surely, for each $\xi \in \mathbb{R}^d$,*

$$|A(\omega)\xi| \le \Lambda\,|\xi| \ \text{ and } \ A(\omega)\xi \cdot \xi \ge \lambda\,|\xi|^2.$$

Then for each $i \in \{1, \ldots, d\}$ there exists unique $\Phi_i \in L^2_{pot}(\Omega)$ satisfying

$$\mathbb{E}\left[A(\Phi_i + e_i) \cdot \Psi\right] = 0 \ \text{ for every } \ \Psi \in L^2_{pot}(\Omega).$$

Proof. The proof is an immediate consequence of the uniform ellipticity of A, the definition of $L^2_{\mathrm{pot}}(\Omega)$, and the Lax-Milgram theorem. □

We can use the transformation group to lift the potential fields Φ_i to the physical space. The following proposition proves that these fields are almost surely locally L^2-bounded and curl free. This is to say that the lift of Φ_i to the physical space almost surely defines the gradient of a function.

Proposition 3.5. *For each $i \in \{1, \ldots, d\}$ let $\Phi_i = (\Phi_{ik})_{k\in\{1,\ldots,d\}} \in L^2_{pot}(\Omega)$ be defined in Proposition 3.4. Then it holds almost surely for each $i \in \{1, \ldots, d\}$ that the vector field $\overline{\Phi}_i(x, \omega) = \Phi_i(\tau_x\omega)$ satisfies*

$$\overline{\Phi}_i(\cdot, \omega) \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^d) \ \text{ and } \ \partial_j\overline{\Phi}_{ik}(\cdot, \omega) = \partial_k\overline{\Phi}_{ij}(\cdot, \omega) \ \text{ as distributions.}$$

Proof. Let $i \in \{1,\ldots,d\}$. It follows from Fubini's theorem and the stationarity that, for each $R \in (0,\infty)$,

$$\mathbb{E}\left[\fint_{B_R} |\Phi_i(\tau_x\omega|^2 \,\mathrm{d}x\right] = \fint_{B_R} \mathbb{E}\left[|\Phi_i|^2\right]\,\mathrm{d}x = \mathbb{E}\left[|\Phi_i|^2\right],$$

it follows almost surely that $\Phi_i(\tau_x\omega) \in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. Now let $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^d)$ and let $A \subset \Omega$ be measurable. It follows that

$$\begin{aligned}&\mathbb{E}\left[\int_{\mathbb{R}^d} \Phi_{ik}(\tau_x\omega)\partial_j\psi(x) - \Phi_{ij}(\tau_x\omega)\partial_k\psi(x)\,\mathrm{d}x\mathbf{1}_A(\omega)\right]\\ &= \mathbb{E}\left[\Phi_{ik}(\omega)\int_{\mathbb{R}^d}\partial_j\psi(x)\mathbf{1}_A(\tau_{-x}\omega)\,\mathrm{d}x - \Phi_{ij}(\omega)\int_{\mathbb{R}^d}\partial_k\psi(x)\mathbf{1}_A(\tau_{-x}\omega)\,\mathrm{d}x\right]\\ &= \mathbb{E}\left[\Phi_{ik}(\omega)D_j\psi_A(\omega) - \Phi_{ij}(\omega)D_k\psi_A(\omega)\right],\end{aligned}$$

for $\psi_A(\omega) = \int_{\mathbb{R}^d}\psi(x)\mathbf{1}_A(\tau_{-x}\omega)\,\mathrm{d}x \in \mathcal{D}(\Omega)$. Therefore, since Φ_i is curl-free, we conclude that

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\Phi_{ik}(\tau_x\omega)\partial_j\psi(x) - \Phi_{ik}(\tau_x\omega)\partial_j\psi(x)\,\mathrm{d}x\mathbf{1}_A(\omega)\right] = 0.$$

Since $A \subseteq \Omega$ was arbitrary, it follows that there exists a subset $\Omega' \subseteq \Omega$ of full probability depending on ψ such that, for every $\omega \in \Omega'$,

$$\int_{\mathbb{R}^d}\Phi_{ik}(\tau_x\omega)\partial_j\psi(x) - \Phi_{ij}(\tau_x\omega)\partial_k\psi(x)\,\mathrm{d}x = 0.$$

Finally, since the space of smooth, compactly supported functions is separable, we conclude that there exists a subset of full probability such that for every $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\Phi_{ik}(\tau_x\omega)\partial_j\psi(x) - \Phi_{ij}(\tau_x\omega)\partial_k\psi(x)\,\mathrm{d}x = 0.$$

This completes the proof. □

In the following proposition, we construct almost surely solutions $\phi_i(\cdot,\omega)$ to (3.12) on the physical space $\mathbb{R}^d$. The solutions $\phi_i(\cdot,\omega)$ are defined by their gradient $\nabla\phi_i(\cdot,\omega) = \Phi_i(\tau.\omega)$, using the fact that Proposition 3.5 proves that the $\Phi_i(\tau.\omega)$ are almost surely curl-free on $\mathbb{R}^d$.

Proposition 3.6. *Let $A \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be uniformly elliptic, let $a\colon \mathbb{R}^d\times\Omega \to \mathbb{R}^{d\times d}$ be defined by $a(x,\omega) = A(\tau_x\omega)$, and for each $i \in \{1,\ldots,d\}$ let $\Phi_i \in L^2_{pot}(\Omega)$ be defined in Proposition 3.4. Then for each $i \in \{1,\ldots,d\}$ there almost surely exists a unique $\phi_i \in H^1_{loc}(\mathbb{R}^d)$ satisfying the properties*

$$\fint_{B_1}\phi_i(x,\omega)\,\mathrm{d}x = 0 \ \text{ and } \ \nabla\phi_i(x,\omega) = \Phi_i(\tau_x\omega) \ \text{ in } \ L^2_{loc}(\mathbb{R}^d;\mathbb{R}^d).$$

Furthermore, almost surely for every $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} a(x,\omega)(\nabla\phi_i(x,\omega)+e_i)\cdot\nabla\psi\,\mathrm{d}x = 0.$$

Proof. For each $\varepsilon \in (0,1)$ let $\rho^\varepsilon \in \mathrm{C}^\infty_c(\mathbb{R}^d)$ be a standard convolution kernel of scale $\varepsilon \in (0,1)$ and for each $i \in \{1,\ldots,d\}$ let $\overline{\Phi}_i\colon \mathbb{R}^d\times\Omega \to \mathbb{R}^d$ be defined by $\overline{\Phi}_i(x,\omega) = \Phi_i(\tau_x\omega)$. Since Proposition 3.5 proves almost surely that $\overline{\Phi}_i(\cdot,\omega) \in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ is curl free, $\overline{\Phi}^\varepsilon_i(\cdot,\omega) = (\Phi^i * \rho^\varepsilon)(\cdot,\omega)$ is almost surely smooth, curl free, and almost surely satisfies for every $R \in (0,\infty)$ that

$$\sup_{\varepsilon\in(0,1)}\left\|\overline{\Phi}^\varepsilon_i\right\|_{L^2(B_R;\mathbb{R}^d)} < \infty. \tag{3.17}$$

We can therefore define for each $i \in \{1,\ldots,d\}$ and $\varepsilon \in (0,1)$,

$$\tilde{\phi}^\varepsilon_i(x) = \int_0^1 \overline{\Phi}^\varepsilon_i(sx,\omega)\cdot x\,\mathrm{d}s \ \text{ and } \ \phi^\varepsilon_i = \tilde{\phi}_i - \fint_{B_1}\tilde{\phi}^\varepsilon_i. \tag{3.18}$$

It follows from the Poincaré inequality, (3.17), and (3.18) that, almost surely for each $R \in (0,\infty)$ and $i \in \{1,\ldots,d\}$,

$$\sup_{\varepsilon\in(0,1)} \|\phi_i^\varepsilon\|_{H^1(B_R)} < \infty. \tag{3.19}$$

Since as $\varepsilon \to 0$ the gradients almost surely satisfy $\overline{\Phi}_i^\varepsilon \to \overline{\Phi}_i$ strongly in $L^2_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, and on the subset of full probability satisfying (3.19) after passing to a random diagonal subsequence there almost surely exists $\phi \in H^1_{\text{loc}}(\mathbb{R}^d)$ such that $\phi_i^\varepsilon \rightharpoonup \phi_i$ weakly in $H^1_{\text{loc}}(\mathbb{R}^d)$, we conclude from the weak convergence, the strong convergence of the gradients, and (3.18) that almost surely $\phi_i \in H^1_{\text{loc}}(\mathbb{R}^d)$ satisifes

$$\fint_{B_1} \phi(x,\omega)\,\mathrm{d}x = 0 \;\text{ and }\; \nabla\phi_i(x,\omega) = \overline{\Phi}_i(\tau_x\omega).$$

Uniqueness follows from the linearity and the fact that the only $H^1_{\text{loc}}(\mathbb{R}^d)$ function ψ satisfying $\fint_{B_1}\psi = 0$ and $\nabla\psi = 0$ is the zero function.

It remains only to prove that the ϕ_i almost surely satisfy the equation. Let $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ and let $A \subseteq \Omega$ be measurable. Then, for each $i \in \{1,\ldots,d\}$, using the fact that the transformation group preserves the measure,

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_A(\omega)\int_{\mathbb{R}^d} a(x,\omega)(\nabla\phi_i(x,\omega)+e_i)\cdot\nabla\psi\right] \\
&= \mathbb{E}\left[\mathbf{1}_A(\omega)\int_{\mathbb{R}^d} A(\tau_x\omega)(\Phi_i(\tau_x\omega)+e_i)\cdot\nabla\psi\right] \\
&= \mathbb{E}\left[A(\omega)(\Phi_i(\omega)+e_i)\cdot\int_{\mathbb{R}^d}\mathbf{1}_A(\tau_{-x}\omega)\nabla\psi(x)\,\mathrm{d}x\right] \\
&= -\mathbb{E}\left[A(\omega)(\Phi_i(\omega)+e_i)\cdot\int_{\mathbb{R}^d}\mathbf{1}_A(\tau_x\omega)\nabla\psi(-x)\,\mathrm{d}x\right] \\
&= \mathbb{E}\left[A(\omega)(\Phi_i(\omega)+e_i)\cdot D\left(\int_{\mathbb{R}^d}\mathbf{1}_A(\tau_x\omega)\psi(-x)\,\mathrm{d}x\right)\right] \\
&= 0,
\end{aligned}$$

where the final inequality follows from the fact that $-D\cdot A(\Phi_i+e_i) = 0$ from Proposition 3.4. Since $A \subseteq \Omega$ was arbitrary, we conclude that there exists a subset of full probability $\Omega' \subseteq \Omega$ depending on ψ such that, for every $\omega \in \Omega'$ and $i \in \{1,\ldots,d\}$,

$$\int_{\mathbb{R}^d} a(x,\omega)(\nabla\phi_i(x,\omega)+e_i)\cdot\nabla\psi = 0.$$

The separability of the space of smooth functions then proves that there exists a subset $\Omega' \subseteq \Omega$ of full probability such that, for every $\omega \in \Omega'$, $i \in \{1,\ldots,d\}$, and $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} a(x,\omega)(\nabla\phi_i(x,\omega)+e_i)\cdot\nabla\psi = 0.$$

This completes the proof. □

Observe that the correctors ϕ_i constructed in Proposition 3.6 are manifestly not stationary. Indeed, since the correctors are constructed to satisfy the property $\mathbb{E}\left[\fint_{B_1}\phi_i\right] = 0$ then stationarity would imply that $\mathbb{E}\left[\fint_{B_1(x)}\phi_i\right] = 0$ for every $x \in \mathbb{R}^d$. This will not in general be true, and the construction of stationary correctors is in general a difficult problem. They have been shown to exist, for instance, in dimensions $d \geq 3$ assuming that the environment satisfies strong mixing assumptions. However, in dimension $d = 2$ it is known that stationary correctors do not exist in general. In the next section, however, we will how that the corrector constructed in (3.6) is sublinear in the L^2-sense.

3.4. **The sublinearity of the corrector.** In this section, we will prove that the correctors constructed in Proposition 3.6 are sublinear in an L^2-sense. The proof is a consequence of the ergodic theorem, using the fact that the stationary gradient constructed in Proposition 3.4 has mean zero.

Proposition 3.7. *For each $i \in \{1, \ldots, d\}$ let $\Phi_i \in L^2_{pot}(\Omega)$ be defined in Proposition 3.4 and let $\phi_i(\cdot, \omega) \in H^1_{loc}(\mathbb{R}^d)$ be almost surely defined in Proposition 3.6. Then almost surely*

$$\lim_{R\to\infty} \frac{1}{R} \left(\fint_{B_R} |\phi_i|^2 \right)^{\frac{1}{2}} = 0.$$

Proof. For each $\varepsilon \in (0,1)$ let $\phi_i^\varepsilon(x,\omega) = \varepsilon \phi^i(x/\varepsilon, \omega)$ and observe by rescaling that

$$\limsup_{R\to\infty} \frac{1}{R} \left(\fint_{B_R} |\phi_i|^2 \right)^{\frac{1}{2}} = \limsup_{\varepsilon\to 0} \left(\fint_{B_1} |\phi_i^\varepsilon|^2 \right)^{\frac{1}{2}}. \tag{3.20}$$

We will first prove that

$$\limsup_{\varepsilon\to 0} \left(\fint_{B_1} \left| \phi_i^\varepsilon - \fint_{B_1} \phi_i^\varepsilon \right|^2 \right)^{\frac{1}{2}} = 0. \tag{3.21}$$

Since almost surely

$$\nabla \left(\phi_i^\varepsilon - \fint_{B_1} \phi_i^\varepsilon \right) = \Phi_i(\tau_{x/\varepsilon}\omega),$$

and since almost surely the ergodic theorem proves that, as $\varepsilon \to 0$,

$$\Phi_i(\tau_{x/\varepsilon}\omega) \rightharpoonup \mathbb{E}\left[\Phi_i\right] = 0 \text{ weakly in } L^2(B_1; \mathbb{R}^d),$$

it follows from the weak convergence that

$$\sup_{\varepsilon\in(0,1)} \left\| \nabla \left(\phi_i^\varepsilon - \fint_{B_1} \phi_i^\varepsilon \right) \right\|_{L^2(B_1;\mathbb{R}^d)} < \infty,$$

and therefore from the Poincaré inequality that

$$\sup_{\varepsilon\in(0,1)} \left\| \left(\phi_i^\varepsilon - \fint_{B_1} \phi_i^\varepsilon \right) \right\|_{H^1(B_1)} < \infty.$$

The boundedness of the functions $\left(\phi_i^\varepsilon - \fint_{B_1} \phi_i^\varepsilon \right)$ in $H^1(B_1)$, the fact that $\fint_{B_1} \left(\phi_i^\varepsilon - \fint_{B_1} \phi_i^\varepsilon \right) = 0$, and the weak convergence of the gradient to zero prove that, as $\varepsilon \to 0$,

$$\left(\phi_i^\varepsilon - \fint_{B_1} \phi_i^\varepsilon \right) \rightharpoonup 0 \text{ weakly in } H^1(B_1).$$

Therefore, the Sobolev embedding theorem proves that, as $\varepsilon \to 0$,

$$\left(\phi_i^\varepsilon - \fint_{B_1} \phi_i^\varepsilon \right) \to 0 \text{ strongly in } L^2(B_1),$$

and hence that

$$\limsup_{\varepsilon\to 0} \left(\fint_{B_1} \left| \phi_i^\varepsilon - \fint_{B_1} \phi_i^\varepsilon \right|^2 \right)^{\frac{1}{2}} = 0.$$

This fact is sufficient for our purposes, since the correctors are only defined up to an additive constant. However, the statement (3.21) is in fact equivalent to the stronger statement

$$\limsup_{\varepsilon\to 0}\left(\fint_{B_1}|\phi_i^\varepsilon|^2\right)^{\frac{1}{2}}=0. \tag{3.22}$$

Indeed, it follows from (3.20) and (3.21) that for every $\delta\in(0,1)$ there exists $R_0\in(0,\infty)$ such that, for every $R\geq R_0$,

$$\left(\fint_{B_R}\left|\phi-\fint_{B_R}\phi\right|^{p_*}\right)^{\frac{1}{p_*}}\leq R\delta.$$

By the triangle inequality, for every $R\in[R_0,2R_0]$,

$$\begin{aligned}\left|\fint_{B_R}\phi-\fint_{B_{R_0}}\phi\right| &\leq \left(\fint_{B_{R_0}}\left|\phi-\fint_{B_R}\phi\right|^2\right)^{\frac{1}{2}}+\left(\fint_{B_{R_0}}\left|\phi-\fint_{B_{R_0}}\phi\right|^2\right)^{\frac{1}{2}}\\ &\leq \left(\frac{R}{R_0}\right)^{\frac{d}{2}}R\delta+R_0\delta\leq\left(2^{\frac{d+2}{2}}+1\right)R_0\delta=cR_0\delta.\end{aligned}$$

Therefore, for every $R\in[R_0,2R_0]$,

$$\left|\frac{1}{R}\fint_{B_R}\phi\right|\leq\left(\frac{R_0}{R}\right)\left|\frac{1}{R_0}\fint_{B_{R_0}}\phi\right|+c\left(\frac{R_0}{R}\right)\delta.$$

It then follows inductively that, for every $R\in[2^{k-1}R_0,2^kR_0]$,

$$\begin{aligned}\fint_{B_R}\phi &\leq \left(\frac{2^{k-1}R_0}{R}\right)\left|\frac{1}{2^{k-1}R_0}\fint_{B_{2^{k-1}R_0}}\phi\right|+c\left(\frac{2^{k-1}R_0}{R}\right)\delta\\ &\leq \left|\frac{1}{R}\fint_{B_{R_0}}\phi\right|+c\left(\sum_{j=0}^\infty 2^{-j}\right)\delta=\left|\frac{1}{R}\fint_{B_{R_0}}\phi\right|+2c\delta.\end{aligned}$$

Since $\delta\in(0,1)$ was arbitrary, we have almost surely that

$$\limsup_{R\to\infty}\left|\frac{1}{R}\fint_{B_{R_0}}\phi\right|=0. \tag{3.23}$$

The triangle inequality, (3.20), (3.21), and (3.23) then prove that

$$\limsup_{R\to\infty}\frac{1}{R}\left(\fint_{B_R}|\phi|^2\right)^{\frac{1}{2}}\leq\limsup_{R\to\infty}\frac{1}{R}\left(\fint_{B_R}\left|\phi-\int_{B_R}\phi\right|^2\right)^{\frac{1}{2}}+\limsup_{R\to\infty}\left|\frac{1}{R}\fint_{B_{R_0}}\phi\right|=0,$$

which completes the proof. □

3.5. **The homogenized coefficient.** The random potential fields defined in Proposition 3.4 can now be used to define the homogenized coefficient $\overline{A}\in\mathbb{R}^{d\times d}$ by the rule

$$\overline{A}e_i=\mathbb{E}\left[A(\Phi_i+e_i)\right]\quad\text{for each}\ \ i\in\{1,\ldots,d\},$$

for $A(\omega)=a(0,\omega)$. We will show that the homogenized coefficient behaves as it did in the periodic case. It is uniformly elliptic, symmetric if A symmetric, and it satisfies the same transpose relation. This is the content of the next two propositions.

Proposition 3.8. *Let $A \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be uniformly elliptic with constants $\lambda,\Lambda\in(0,\infty)$: almost surely for every $\xi\in\mathbb{R}^d$,*

$$|A\xi| \leq \Lambda\,|\xi| \;\; \text{and} \;\; A\xi\cdot\xi \geq \lambda\,|\xi|^2.$$

Let $\overline{a}\in\mathbb{R}^{d\times d}$ be defined for each $i\in\{1,\ldots,d\}$ by

$$\overline{A}e_i = \mathbb{E}\left[A(\Phi_i + e_i)\right],$$

for $\Phi_i \in L^2_{pot}(\Omega)$ defined in Proposition 3.4. Then, for every $\xi\in\mathbb{R}^d$,

$$\left|\overline{A}\xi\right| \leq \lambda\left(\sum_{i=1}^d \mathbb{E}\left[|\Phi_i+e_i|^2\right]\right)^{\frac{1}{2}}|\xi| \;\; \text{and} \;\; \overline{A}\xi\cdot\xi \geq \lambda\,|\xi|^2.$$

Proof. It follows from Hölder's inequality, the uniform ellipticity, and the definition of $\overline{A}$ that, for every $\xi=(\xi_i)\in\mathbb{R}^d$,

$$\left|\overline{A}\xi\right| \leq \Lambda\,|\xi_i|\,\mathbb{E}\left[|\Phi_i+e_i|\right] \leq \lambda\left(\sum_{i=1}^d \mathbb{E}\left[|\Phi_i+e_i|^2\right]\right)^{\frac{1}{2}}|\xi|.$$

Alternately, the uniform ellipticity, the definition of $\overline{A}$, the equation satisfied by Φ_i, the fact that Φ_i is mean zero, and Jensen's inequality prove that, for every $\xi\in\mathbb{R}^d$,

$$\overline{A}\xi\cdot\xi = \mathbb{E}\left[A(\Phi_\xi+\xi)\cdot\xi\right] = \mathbb{E}\left[A(\Phi_\xi+\xi)\cdot(\Phi_\xi+\xi)\right] \geq \lambda\mathbb{E}\left[|\Phi_\xi+\xi|^2\right] \geq \lambda\left|\mathbb{E}[\Phi_\xi+\xi]\right|^2 = \lambda\,|\xi|^2,$$

for $\Phi_\xi = \xi_i\Phi_i$. This completes the proof. ☐

Proposition 3.9. *Let $A \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be uniformly elliptic with constants $\lambda,\Lambda\in(0,\infty)$: almost surely for every $\xi\in\mathbb{R}^d$,*

$$|A\xi| \leq \Lambda\,|\xi| \;\; \text{and} \;\; A\xi\cdot\xi \geq \lambda\,|\xi|^2.$$

For each $i\in\{1,\ldots,d\}$ let $\Phi_i,\Phi_i^t\in L^2_{pot}(\Omega)$ be the unique solutions of

$$-D\cdot A(\Phi_i+e_i) = 0 \;\; \text{and} \;\; -D\cdot A^t(\Phi_i^t+e_i) = 0 \;\; \text{in} \;\; L^2_{pot}(\Omega).$$

Let $\overline{A},\tilde{A}\in L^\infty(\Omega)$ be defined for each $i\in\{1,\ldots,d\}$ by

$$\overline{A}e_i = \mathbb{E}\left[A(\Phi_i+e_i)\right] \;\; \text{and} \;\; \tilde{A}e_i = \mathbb{E}\left[A^t(\Phi_i^t+e_i)\right].$$

Then $\tilde{A}=\overline{A}^t$ and $\overline{A}$ is symmetric if A is symmetric.

Proof. Let $\overline{A}=(\overline{a}_{ij})$ and $\tilde{A}=(\tilde{a}_{ij})$ for $i,j\in\{1,\ldots,d\}$. Then, for each $i,j\in\{1,\ldots,d\}$, the equations satisfied by Φ_k and Φ_k^t and the definitions prove that

$$\begin{aligned}\overline{a}_{ij} &= \mathbb{E}\left[A(\Phi_i+e_i)\right]\cdot e_j\\ &= \mathbb{E}\left[A(\Phi_i+e_i)(\Phi_j^t+e_j)\right]\\ &= \mathbb{E}\left[(\Phi_i+e_i)A^t(\Phi_j^t+e_J)\right]\\ &= \mathbb{E}\left[A^t(\Phi_j^t+e_j)\right]\cdot e_i\\ &= \tilde{a}_{ji}.\end{aligned}$$

Therefore $\tilde{A}=\overline{A}^t$. Finally, if A is symmetric then by uniqueness $\Phi_i=\Phi_i^t$ and $\tilde{A}=\overline{A}$ and $\overline{A}=\overline{A}^t$. This completes the proof. ☐

3.6. **The perturbed test function method.** Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\{\tau_x\}_{x\in\mathbb{R}^d}$ be an ergodic measure-preserving transformation group on Ω, and let $A\in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be uniformly elliptic. We recall that, for each $i\in\{1,\ldots,d\}$, the corrector gradient fields $\Phi_i\in L^2_{\text{pot}}(\Omega)$ solve the equation

$$\mathbb{E}\left[A(\Phi_i+e_i)\cdot\Psi\right]=0\;\;\text{for every}\;\;\Psi\in L^2_{\text{pot}}(\Omega),$$

and the homogenized coefficient $\overline{a}\in\mathbb{R}^{d\times d}$ is defined for each $i\in\{1,\ldots,d\}$ by

$$\overline{a}e_i=\mathbb{E}\left[A(\Phi_i+e_i)\right].$$

Let $v\in H^1_0(U)$ be the unique solution of

$$-\nabla\cdot\overline{a}\nabla v=f\;\;\text{in}\;\;U\;\;\text{with}\;\;v=0\;\;\text{on}\;\;\partial U.$$

We will now use the perturbed test function method to prove the stochastic homogenization of the equation

$$-\nabla\cdot a^\varepsilon\nabla u^\varepsilon=f\;\;\text{in}\;\;U\;\;\text{with}\;\;u^\varepsilon=0\;\;\text{on}\;\;\partial U,$$

for $a^\varepsilon(x,\omega)=A(\tau_{x/\varepsilon}\omega)$. The proof is essentially the same as in the periodic case, relying on the sublinearity of the correctors. We similarly prove the homogenization of the flux.

Theorem 3.10. *Let $U\subseteq\mathbb{R}^d$ be a bounded domain and let $f\in L^2(U)$. Then, almost surely as $\varepsilon\to 0$,*

$$u^\varepsilon\rightharpoonup v\;\;\textit{weakly in}\;\;H^1_0(U).$$

Proof. For each $i\in\{1,\ldots,d\}$ let $\Phi^t_i\in L^2_{\text{pot}}(\Omega)$ solve

$$\mathbb{E}\left[A^t(\Phi^t_i+e_i)\Psi\right]=0\;\;\text{for every}\;\;\Psi\in L^2_{\text{pot}}(\Omega),$$

and almost surely for each $i\in\{1,\ldots,d\}$ let $\phi^t_i(\cdot,\omega)\in H^1_{\text{loc}}(\mathbb{R}^d)$ be the unique function satisfying $\fint_{B_1}\phi^t_i=0$, satisfying $\nabla\phi^t_i(x,\omega)=\Phi^t_i(\tau_x\omega)$, and satsifying, for each $\psi\in \mathrm{C}^\infty_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}a^t(y,\omega)(\nabla\phi_i(x,\omega)+e_i)\cdot\nabla\psi=0.$$

The uniform ellipticity and the Poincaré inequality prove almost surely that

$$\sup_{\varepsilon\in(0,1)}\|u^\varepsilon\|_{H^1_0(U)}<\infty,$$

and therefore, after passing to a subsequence, there exists $\tilde{v}\in H^1_0(U)$ such that, as $\varepsilon\to 0$,

$$u^\varepsilon\rightharpoonup\tilde{v}\;\;\text{weakly in}\;\;H^1_0(U).$$

We will prove that, for every $\psi\in\mathrm{C}^\infty_c(U)$,

$$\int_U\overline{a}\nabla\tilde{v}\cdot\nabla\psi=\int_U f\psi,$$

which by uniqueness implies that $\tilde{v}=v$ and therefore that the full sequence converges to this unique limit.

Let $\psi\in\mathrm{C}^\infty_c(\mathbb{R}^d)$ and for each $\varepsilon\in(0,1)$ define the perturbed test function

$$\psi^\varepsilon=\psi+\varepsilon\phi^t_i(x/\varepsilon,\omega)\partial_i\psi.$$

Testing the equation satisfied by u^ε with ψ^ε then yields

$$\begin{aligned}\int_U a^\varepsilon\nabla u^\varepsilon\cdot\nabla\psi^\varepsilon&=\int_U a^\varepsilon\nabla u^\varepsilon\cdot(\nabla\phi^t_i(x/\varepsilon,\omega)+e_i)\partial_i\psi+\int_U a^\varepsilon\nabla u^\varepsilon\cdot\nabla(\partial_i v)\varepsilon\phi^t_i(x/\varepsilon)\\&=\int_U f\left(\psi+\varepsilon\phi^t_i(x/\varepsilon)\partial_i v\right).\end{aligned}$$

Therefore, after transposing the matrix,

$$\int_U \nabla u^\varepsilon \cdot (a^\varepsilon)^t(\nabla\phi_i^t(x/\varepsilon,\omega)+e_i)\partial_i\psi + \int_U a^\varepsilon\nabla u^\varepsilon\cdot\nabla(\partial_i v)\varepsilon\phi_i^t(x/\varepsilon) = \int_U f\left(\psi+\varepsilon\phi_i^t(x/\varepsilon)\partial_i v\right).$$

Since the ergodic theorem and Proposition 3.9 prove almost surely that, for each $i\in\{1,\ldots,d\}$ as $\varepsilon\to 0$,

$$(a^\varepsilon)^t(\nabla\phi_i^t(x/\varepsilon,\omega)+e_i)\rightharpoonup \overline{a}^t e_i \;\text{ weakly in }\; L^2_{\text{loc}}(U;\mathbb{R}^d),$$

and since along a subsequence $\nabla u^\varepsilon\rightharpoonup\nabla\tilde{v}$ weakly in $L^2(U;\mathbb{R}^d)$ the div-curl lemma can be applied to the first term on the lefthand side to prove that

$$\lim_{\varepsilon\to 0}\int_U \nabla u^\varepsilon\cdot(a^\varepsilon)^t(\nabla\phi_i^t(x/\varepsilon,\omega)+e_i)\partial_i\psi = \int_U\nabla\tilde{v}\cdot\overline{a}^t\nabla\psi = \int_U \overline{a}\nabla\tilde{v}\cdot\nabla\psi.$$

The uniform ellipticity and Hölder's inequality prove that the second term on the lefthand side satisfies

$$\left|\int_U a^\varepsilon\nabla u^\varepsilon\cdot\nabla(\partial_i v)\varepsilon\phi_i^t(x/\varepsilon)\right| \le \|\nabla(\partial_i v)\|_{L^\infty(U;\mathbb{R}^d)}\|\nabla u^\varepsilon\|_{L^2(U;R^d)}\left(\int_U\left|\varepsilon\phi_i^t(x/\varepsilon)\right|^2\right)^{\frac{1}{2}},$$

and similarly, for the final term on the righthand side,

$$\left|\int_U f\varepsilon\phi_i^t(x/\varepsilon)\partial_i v\right| \le \|\partial_i v\|_{L^\infty(U)}\|f\|_{L^2(U)}\left(\int_U\left|\varepsilon\phi_i^t(x/\varepsilon,\omega)\right|^2\right)^{\frac{1}{2}}.$$

Therefore, the almost sure sublinearity of the ϕ_i^t proved in Proposition 3.7 proves that almost surely

$$\limsup_{\varepsilon\to 0}\left(\left|\int_U a^\varepsilon\nabla u^\varepsilon\cdot\nabla(\partial_i v)\varepsilon\phi_i^t(x/\varepsilon)\right| + \left|\int_U f\varepsilon\phi_i^t(x/\varepsilon)\partial_i v\right|\right) = 0.$$

We therefore conclude that

$$\int_U \overline{a}\nabla\tilde{v}\cdot\nabla\psi = \int_U f\psi,$$

which proves that

$$-\nabla\cdot\overline{a}\nabla\tilde{v} = f \;\text{ in }\; U \;\text{ with }\; \tilde{v}=0 \;\text{ on }\; \partial U.$$

Therefore, by uniqueness, we conclude that $\tilde{v}=v$. This completes the proof. □

Theorem 3.11. *Let $U\subseteq\mathbb{R}^d$ be a bounded domain and let $f\in L^2(U)$. Then, almost surely as $\varepsilon\to 0$,*

$$a^\varepsilon\nabla u^\varepsilon\rightharpoonup\overline{a}\nabla v \;\text{ weakly in }\; L^2(U;R^d).$$

Proof. For each $i\in\{1,\ldots,d\}$ let $\Phi_i^t\in L^2_{\text{pot}}(\Omega)$ solve

$$\mathbb{E}\left[A^t(\Phi_i^t+e_i)\Psi\right]=0 \;\text{ for every }\; \Psi\in L^2_{\text{pot}}(\Omega),$$

and almost surely for each $i\in\{1,\ldots,d\}$ let $\phi_i^t(\cdot,\omega)\in H^1_{\text{loc}}(\mathbb{R}^d)$ be the unique function satisfying $⨍_{B_1}\phi_i^t=0$, satisfying $\nabla\phi_i^t(x,\omega)=\Phi_i^t(\tau_x\omega)$, and satsifying, for each $\psi\in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} a^t(y,\omega)(\nabla\phi_i(x,\omega)+e_i)\cdot\nabla\psi = 0.$$

The uniform ellipticity and the Poincaré inequality prove almost surely that

$$\sup_{\varepsilon\in(0,1)}\|u^\varepsilon\|_{H^1_0(U)}<\infty,$$

and therefore due to the uniform ellipticity, after passing to a subsequence, there exists $F_0\in L^2(U;\mathbb{R}^d)$ such that, as $\varepsilon\to 0$,

$$a^\varepsilon\nabla u^\varepsilon\rightharpoonup F_0 \;\text{ weakly in }\; L^2(U;\mathbb{R}^d),$$

for F_0 solving, for every $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\int_U F_0 \cdot \nabla\psi = \int_U f\psi.$$

We will prove that, for each $\xi \in \mathbb{R}^d$ and $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\int_U (F_0 \cdot \xi)\psi = \int_U (\overline{a}\nabla v \cdot \xi)\psi,$$

from which it follows that $F_0 = \overline{a}\nabla v$ in $L^2(U;\mathbb{R}^d)$. Let $\xi \in \mathbb{R}^d$ and $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. We introduce a perturbed version of the linear function $w_\xi(x) = \xi \cdot x$ for each $\varepsilon \in (0,1)$ defined by

$$w_\xi^\varepsilon(x) = \xi \cdot x + \varepsilon\phi_\xi^t(x/\varepsilon, \omega),$$

for $\phi_\xi^t = \xi_i\phi_i^t$. After testing the equation satisfied by u^ε with the test function ϕw_ξ^ε,

$$\int_U a^\varepsilon\nabla u^\varepsilon \cdot (\nabla\phi_\xi^t + \xi)\psi + a^\varepsilon\nabla u^\varepsilon \cdot \nabla\psi w_\xi^\varepsilon = \int_U f w_\xi^\varepsilon \psi.$$

Similarly, testing the equation satisfied by ϕ_ξ^t with $u^\varepsilon\psi$,

$$\int_U (a^\varepsilon)^t(\nabla\phi_\xi^t + \xi) \cdot \nabla\psi u^\varepsilon + (a^\varepsilon)^t(\nabla\phi_\xi^t + \xi) \cdot \nabla u^\varepsilon\psi = 0.$$

After transposing the matrix and subtracting these two equations,

$$\int_U a^\varepsilon\nabla u^\varepsilon \cdot \nabla\psi w_\xi^\varepsilon - (a^\varepsilon)^t(\nabla\phi_\xi^t + \xi) \cdot \nabla\psi u^\varepsilon = \int_U f w_\xi^\varepsilon \psi.$$

Since the Sobolev embedding theorem and Theorem 3.10 prove that, almost surely as $\varepsilon \to 0$,

$$u^\varepsilon \to v \text{ strongly in } L^2(U),$$

since the sublinearity of Proposition 3.7 proves that, almost surely as $\varepsilon \to 0$,

$$w_\xi^\varepsilon \to (\xi \cdot x) \text{ strongly in } L^2(U),$$

and since the ergodic theorem and Proposition 3.9 prove almost surely that, as $\varepsilon \to 0$,

$$(a^\varepsilon)^t(\nabla\phi_\xi^t + \xi) \rightharpoonup \overline{a}^t e_i \text{ weakly in } L^2_{\mathrm{loc}}(U;\mathbb{R}^d),$$

after passing to the limit $\varepsilon \to 0$ along a subsequence we have almost surely from the weak convergence of $a^\varepsilon\nabla u^\varepsilon$ that

$$\int_U F_0 \cdot \nabla\psi(\xi \cdot x) - \overline{a}^t\xi \cdot \nabla\psi v = \int_U f(\xi \cdot x)\psi.$$

Therefore, after integrating by parts and using the equation satisfied by F_0,

$$\int_U (F_0 \cdot \xi)\psi = \int_U (\overline{a}\nabla v \cdot \xi)\psi.$$

We therefore conclude that $F_0 = \overline{a}\nabla v$, which completes the proof. □

3.7. **The random flux-corrector.** Returning to (2.35), the equation satisfied by the homogenization error

$$w = u^\varepsilon - v - \varepsilon\partial_i\phi_i^\varepsilon\partial_i v,$$

remains the same in the random case. We have that

$$-\nabla \cdot A(x/\varepsilon, \omega)\nabla w^\varepsilon = \nabla \cdot \left(\left(A(x/\varepsilon, \omega)\left(e_i + \nabla\phi_i(x/\varepsilon, \omega)\right) - \overline{A}e_i\right)\partial_i v\right) + \nabla \cdot A(x/\varepsilon)\left(\varepsilon\phi_i^\varepsilon\nabla\partial_i v\right), \tag{3.24}$$

for $\nabla\phi_i(x/\varepsilon, \omega) = \Phi_i(\tau_{x/\varepsilon}\omega)$ and for $A(x/\varepsilon, \omega) = A(\tau_{x/\varepsilon}\omega)$. In this case, for each $i \in \{1, \ldots, d\}$, the flux q_i^ε is the translation of the stationary quantity $Q_i \in L^2(\Omega;\mathbb{R}^d)$ deifned by

$$Q_i = A(\Phi_i + e_i) - \mathbb{E}\left[A(\Phi_i + e_i)\right] = A(\Phi_i + e_i) - \overline{A}e_i, \tag{3.25}$$

for which we have

$$q_i^\varepsilon = \left(A(x/\varepsilon,\omega)\,(e_i + \nabla\phi_i(x/\varepsilon,\omega)) - \overline{A}e_i\right) = Q_i(\tau_{x/\varepsilon}\omega).$$

The fact that the fluxes q_i^ε are defined by the stationary quantities Q_i allows us to lift the equation defining the flux-corrections σ_i to the probability space in exactly the way we lifted the equation defining the homogenization corrector to the probability space. The following two propositions almost surely define $\sigma_i = (\sigma_{ijk}) \in H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d})$ by the gradients of its components. The final proposition proves that the flux correction is sublinear.

Proposition 3.12. *Let $A \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be uniformly elliptic, for each $i \in \{1,\ldots,d\}$ let $\Phi_i \in L^2_{pot}(\Omega)$ be the unique solution of $-D\cdot A(\Phi_i + e_i) = 0$ in $L^2_{pot}(\Omega)$, and let $Q_i = (Q_{ik}) \in L^2(\Omega;\mathbb{R}^d)$ be defined by* (3.25)*. Then for every $i,j,k \in \{1,\ldots,d\}$ there exists a unique $\Sigma_{ijk} \in L^2_{pot}(\Omega)$ that satisfies*

$$\mathbb{E}\left[\Sigma_{ijk}\cdot\Psi\right] = \mathbb{E}\left[Q_{ij}\Psi_k - Q_{ik}\Psi_j\right] \text{ for every } \Psi \in L^2_{pot}(\Omega). \tag{3.26}$$

Proof. The proof is a consequence of the Lax-Milgram theorem applied to the Hilbert space $L^2_{\mathrm{pot}}(\Omega)$. Indeed equation (3.26) is the lift of the equation $-\Delta\phi_{ijk} = \partial_j q_{ik} - \partial_k q_{ij}$ to the space $L^2_{\mathrm{pot}}(\Omega)$ of stationary gradient fields. □

Proposition 3.13. *For every $i,j,k \in \{1,\ldots,d\}$ let $\Sigma_{ijk} \in L^2_{pot}(\Omega)$ be defined in Proposition* 3.12 *and almost surely let $\sigma_{ijk} \in H^1_{loc}(\mathbb{R}^d)$ be the unique function satisfying $\fint_{B_1}\sigma_{ijk}(x,\omega)\,\mathrm{d}x = 0$ and $\nabla\sigma_{ijk}(x,\omega) = \Sigma_{ijk}(\tau_x\omega)$. Then for each $i \in \{1,\ldots,d\}$ the matrix $\sigma_i = (\sigma_{ijk}) \in H^1_{loc}(\mathbb{R}^d;\mathbb{R}^{d\times d})$ is almost surely skew-symmetric and satisfies*

$$\nabla\cdot\sigma_i = q_i \text{ in } L^2_{loc}(\mathbb{R}^d;\mathbb{R}^d),$$

for $q_i(x,\omega) = Q_i(\tau_x\omega)$.

Proof. Proposition 3.12 and a repetition of the proof of Proposition 3.6 prove that for each $i,j,k \in \{1,\ldots,d\}$ the function $\sigma_{ijk} \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ that is almost surely defined uniquely by the properties $\fint_{B_1}\sigma_{ijk} = 0$ and $\nabla\sigma_{ijk}(x,\omega) = \Sigma_{ijk}(\tau_x\omega)$ almost surely satisfies, for every $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\nabla\sigma_{ijk}\cdot\nabla\psi = \int_{\mathbb{R}^d}\partial_j\psi q_{ik} - \partial_k\psi q_{ij}.$$

It then follows from uniqueness of the Σ_{ijk} that $\Sigma_{ijk} = -\Sigma_{ikj}$ for every $i,j,k \in \{1,\ldots,d\}$ and therefore from the almost sure uniqueness of the σ_{ijk} that $\sigma_{ijk} = -\sigma_{ikj}$ for every $i,j,k \in \{1,\ldots,d\}$. Almost surely define $\sigma_i = (\sigma_{ijk}) \in H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d})$. The σ_i are then skew-symmetric by definition. Fix $i,j \in \{1,\ldots,d\}$. As in the periodic case, we will show that in the sense of distributions

$$\Delta\left[\left(\nabla\cdot\sigma_i\right)_j - q_{ij}\right] = 0. \tag{3.27}$$

Indeed, the computation is identical using the equation satisfied by the σ_{ijk},

$$\partial_s\partial_s\left[\partial_k\sigma_{ijk} - q_{ij}\right] = \partial_k\left[\partial_k q_{ij} - \partial_j q_{ik} - \partial_s\partial_s q_{ij}\right] = -\partial_j(\nabla\cdot q_i) = 0,$$

where the final equality follows from the fact that q_i is divergence-free. For every $\varepsilon \in (0,1)$ let $\rho^\varepsilon \in \mathrm{C}^\infty_c(\mathbb{R}^d)$ be a standard convolution kernel of scale $\varepsilon \in (0,1)$. It follows from (3.27) that, for every $\varepsilon \in (0,1)$ and $i,j \in \{1,\ldots,d\}$,

$$\Delta\left[\left(\left(\nabla\cdot\sigma_i\right)_j - q_{ij}\right) * \rho^\varepsilon\right] = 0 \text{ in } \mathbb{R}^d. \tag{3.28}$$

Let $\eta\colon \mathbb{R}^d \to [0,1]$ be a smooth function satisfying $\eta = 1$ on $\overline{B}_1$ and $\eta = 0$ on $\mathbb{R}^d\setminus B_2$ and for each $R \in (0,\infty)$ let $\eta_R(x) = \eta(x/R)$. After testing (3.28) with the admissible test function $\left(\left(\left(\nabla\cdot\sigma_i\right)_j - q_{ij}\right) * \rho^\varepsilon\right)\eta_R^2$, it follows from Hölder's inequality, Young's inequality, and the definition

of η_R that, for each $R \in (0,\infty)$ and $\varepsilon \in (0,1)$, for some $c \in (0,\infty)$ independent of $R \in (0,\infty)$ and $\varepsilon \in (0,1)$,
$$\fint_{B_R} \left|\nabla\left(\left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon\right)\right|^2 \leq \frac{c}{R}\fint_{B_{2R}} \left|\left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon\right|^2.$$
This is a version of the Caccioppoli inequality. Observe that both quantities are stationary in the sense that
$$\left((\nabla\cdot\sigma_i)_j - q_{ij}\right)(x,\omega) = \left((\Sigma_{ijk})_k - Q_{ij}\right)(\tau_x\omega),$$
that
$$\left(\left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon\right)(x,\omega) = \int_{\mathbb{R}^d} \left((\Sigma_{ijk})_k - Q_{ij}\right)(\tau_{x+y}\omega)\rho^\varepsilon(y)\,\mathrm{d}y,$$
and that
$$\left(\nabla\left(\left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon\right)\right)(x,\omega) = -\int_{\mathbb{R}^d} \left((\Sigma_{ijk})_k - Q_{ij}\right)(\tau_{x+y}\omega)\nabla\rho^\varepsilon(y)\,\mathrm{d}y.$$
Therefore, it follows almost surely from the ergodic theorem and $\Sigma_{ijk}, Q_i \in L^2(\Omega;\mathbb{R}^d)$ that
$$\lim_{R\to\infty}\fint_{B_R} \left|\nabla\left(\left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon\right)\right|^2 = -\mathbb{E}\left[\left|\int_{\mathbb{R}^d} \left((\Sigma_{ijk})_k - Q_{ij}\right)(\tau_y\omega)\nabla\rho^\varepsilon(y)\,\mathrm{d}y\right|^2\right] < \infty,$$
that
$$\lim_{R\to\infty}\fint_{B_{2R}} \left|\left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon\right|^2 = \mathbb{E}\left[\left|\int_{\mathbb{R}^d} \left((\Sigma_{ijk})_k - Q_{ij}\right)(\tau_y\omega)\rho^\varepsilon(y)\,\mathrm{d}y\right|^2\right] < \infty,$$
and therefore that
$$\lim_{R\to\infty}\frac{1}{R}\fint_{B_{2R}} \left|\left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon\right|^2 = 0.$$
Therefore, for each $\varepsilon \in (0,1)$, on a subset of full probability,
$$\nabla\left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon = 0 \;\text{ on }\; \mathbb{R}^d$$
and
$$\left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon \;\text{ is constant.}$$
It then follows from stationarity, the ergodic theorem, Fubini's theorem, and the fact that for each $i,j,k \in \{1,\ldots,d\}$ we ahve $\mathbb{E}\left[\Sigma_{ijk}\right] = \mathbb{E}\left[Q_i\right] = 0$,
$$\lim_{R\to\infty}\fint_{B_R} \left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon = \mathbb{E}\left[\int_{\mathbb{R}^d} \left((\Sigma_{ijk})_k - Q_{ij}\right)(\tau_y\omega)\rho^\varepsilon(y)\,\mathrm{d}y\right] = 0.$$
Hence, for every $\varepsilon \in (0,1)$, on a subset of full probability,
$$\left((\nabla\cdot\sigma_i)_j - q_{ij}\right) * \rho^\varepsilon = 0 \;\text{ on }\; \mathbb{R}^d,$$
from which it follows almost surely that, for each $i,j \in \{1,\ldots,d\}$,
$$(\nabla\cdot\sigma_i)_j - q_{ij} = 0 \;\text{ on }\; \mathbb{R}^d,$$
which completes the proof. □

Proposition 3.14. *For every $i \in \{1,\ldots,d\}$ let $\sigma_i \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be almost surely defined by Proposition* 3.13*. Then, almost surely for each $i \in \{1,\ldots,d\}$,*
$$\lim_{R\to\infty}\frac{1}{R}\left(\fint_{B_R} |\sigma_i|^2\right)^{\frac{1}{2}} = 0.$$

Proof. The proof is identical to Proposition 3.7 and relies only on the stationarity and L^2-integrability of the stationary gradients Σ_{ijk} defining the gradients of the σ_i. □

3.8. **Strong convergence of the two-scale expansion.** In this section, for a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with an ergodic measure-preserving transformation group $\{\tau_x\}_{x\in\mathbb{R}^d}$, for a uniformly elliptic $A \in L^\infty(\Omega;\mathbb{R}^{d\times d})$, we will prove almost surely that the two-scale expansion associated to the equation

$$-\nabla\cdot a^\varepsilon\nabla u^\varepsilon = f \text{ in } U \text{ with } u^\varepsilon = 0 \text{ on } \partial U,$$

converges strongly to zero in $H^1(U)$. Precisely, for each $i \in \{1,\ldots,d\}$ almost surely $\phi_i \in H^1_{\text{loc}}(\mathbb{R}^d)$ be the corrector constructed in Proposition 3.6 and let $\sigma \in H^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d})$ be the skew-symmetric flux corrector constructed in Proposition 3.13. The uniformly elliptic homogenized coefficient field $\overline{a} \in \mathbb{R}^{d\times d}$ is defined in Proposition 3.8 and $v \in H^1_0(U)$ is the unique solution of

$$-\nabla\cdot\overline{a}\nabla v = f \text{ in } U \text{ with } v = 0 \text{ on } \partial U.$$

The two-scale expansion is

$$w^\varepsilon = u^\varepsilon - v - \varepsilon\phi(x/\varepsilon,\omega)\partial_i v.$$

In the following theorem, using the homogenization flux correctors, we prove almost surely that w^ε converges strongly to zero in $H^1(U)$ as $\varepsilon \to 0$.

Theorem 3.15. *For some $\alpha \in (0,1)$ let $U \subseteq \mathbb{R}^d$ be a bounded $\mathrm{C}^{2,\alpha}$-domain and let $f \in \mathrm{C}^\alpha(U)$. Then, almost surely as $\varepsilon \to 0$,*

$$u^\varepsilon - v - \varepsilon\phi_i(x/\varepsilon,\omega)\partial_i v \to 0 \text{ strongly in } H^1(U).$$

Proof. For every $\rho \in (0,1)$ let $\eta_\rho\colon \mathbb{R}^d \to [0,1]$ be a smooth cutoff function satisfying $\eta_\rho(x) = 1$ if $\mathrm{d}(x,\partial U) \geq \rho$, satisfying $\eta_\rho(x) = 0$ if $\mathrm{d}(x,\partial U) < \rho/2$, and satisfying $|\nabla\eta_\rho| \leq c/\rho$ for some $c \in (0,\infty)$ independent of $\rho \in (0,1)$. For each $\varepsilon, \rho \in (0,1)$ we define

$$w^{\varepsilon,\rho} = u^\varepsilon - v - \varepsilon\phi_i(x/\varepsilon,\omega)\eta_\rho\partial_i v,$$

The reason for introducing the cutoff η_ρ is to guarantee that $w^{\varepsilon,\rho}$ vanishes along the boundary. A repetition of the computation in the computation from the periodic case then proves that

$$-\nabla a^\varepsilon\nabla w^{\varepsilon,\rho} = \nabla\cdot[(1-\eta_\rho)(a^\varepsilon - \overline{a})\nabla(\partial_i v)] + \nabla\cdot[(\varepsilon\phi_i^\varepsilon - \varepsilon\sigma_i^\varepsilon)\nabla(\eta_\rho\partial_i v)].$$

Since the regularity of the domain proves that, for some $c \in (0,\infty)$

$$\|v\|_{\mathrm{C}^{2,\alpha}(U)} \leq c\,\|f\|_{\mathrm{C}^\alpha(U)}, \tag{3.29}$$

it follows from the uniform ellipticity, the definition of η_ρ, Hölder's inequality, and Young's inequality that, for some $c \in (0,\infty)$ depending on U but independent of ε and ρ,

$$\begin{aligned}\int_U |\nabla w^{\varepsilon,\rho}|^2 &\leq c\left(\int_U (1-\eta_\rho)^2|\nabla(\partial_i v)|^2 + \int_U \left(|\varepsilon\phi_i^\varepsilon|^2 + |\varepsilon\sigma_i^\varepsilon|^2\right)|\nabla(\eta_\rho\partial_i v)|^2\right)\\ &\leq c\,\|f\|_{\mathrm{C}^\alpha(U)}\left(\rho + \frac{1}{\rho}\int_U \left(|\varepsilon\phi_i^\varepsilon|^2 + |\varepsilon\sigma_i^\varepsilon|^2\right)\right).\end{aligned}$$

It then follows almost surely from the sublinearity of Propositions 3.7 and 3.14 that, for each $\rho \in (0,1)$, for $c \in (0,\infty)$ independent of ρ,

$$\limsup_{\varepsilon\to 0}\int_U |\nabla w^{\varepsilon,\rho}|^2 \leq c\rho\,\|f\|_{\mathrm{C}^\alpha(U)}. \tag{3.30}$$

We then write, for each $\varepsilon, \rho \in (0,1)$,

$$\nabla w^\varepsilon = \nabla w^{\varepsilon,\rho} + \nabla\left(\varepsilon\phi_i^\varepsilon(1-\eta_\rho)\partial_i v\right) = \nabla w^{\varepsilon,\rho} + \nabla\phi_i^\varepsilon(1-\eta_\rho)\partial_i v + \varepsilon\phi_i^\varepsilon\nabla[(1-\eta_\rho)\partial_i v]. \tag{3.31}$$

Since Proposition 3.7, the definition of η_ρ, and (3.29) prove almost surely that, as $\varepsilon \to 0$,

$$\varepsilon\phi_i^\varepsilon\nabla[(1-\eta_\rho)\partial_i v] \to 0 \text{ strongly in } L^2(U;\mathbb{R}^d),$$

the ergodic theorem, Hölder's inequality, Young's inequality, (3.30), and (3.31) prove that, for some $c \in (0,\infty)$ depending on U but independent of ρ,

$$\limsup_{\varepsilon\to 0}\int_U |\nabla w^\varepsilon|^2 \le c\rho\, \|f\|_{\mathrm{C}^\alpha(U)}\left(1+\sum_{i=1}^d \mathbb{E}\left[|\Phi_i|^2\right]\right).$$

Then, since $\rho \in (0,1)$ was arbitrary, we conclude that

$$\limsup_{\varepsilon\to 0}\int_U |\nabla w^\varepsilon|^2 = 0. \tag{3.32}$$

Since Theorem 3.10 and the Sobolev embedding theorem prove almost surely that, as $\varepsilon \to 0$,

$$u^\varepsilon \to v \ \text{ strongly in } \ L^2(U),$$

and since Proposition 3.7 proves almost surely that, aas $\varepsilon \to 0$,

$$\varepsilon\phi_i^\varepsilon \to 0 \ \text{ strongly in } \ L^2(U),$$

we conclude almost surely that, as $\varepsilon \to 0$,

$$w^\varepsilon \to 0 \ \text{ strongly in } \ L^2(U). \tag{3.33}$$

In combination, (3.32) and (3.33) complete the proof. □