Non-equilibrium fluctuations, the skeleton equation, and SPDEs with conservative noise

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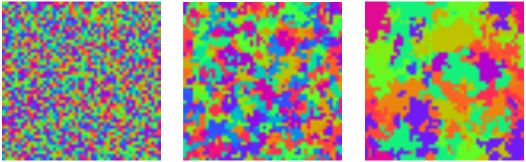
I. Interacting particle systems

- Statistical physics
  - — zero range process
  - — Ising and Potts models
- Belief/infection propagation
  - — voter model
  - — contact process
- Traffic models
  - — exclusion processes
- Neural networks as interacting particle systems

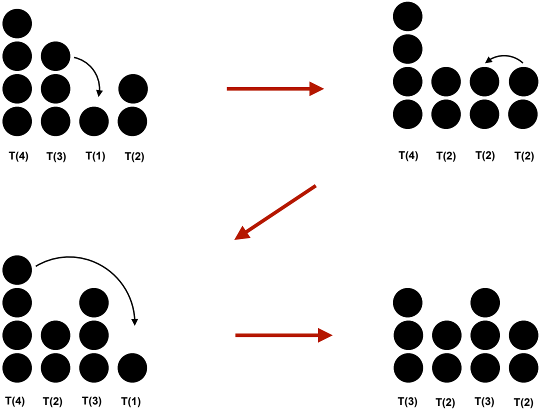


The voter model [Swart; 2020]

The zero range process:

- let $g\colon \mathbb{N}_0 \to \mathbb{N}_0$ be nondecreasing
  — $g(0) = 0$ and $g(k) > 0$ if $k \neq 0$
- independent random clocks $T(k)$ with distribution

$$T(k) \sim g(k)\exp(-g(k)t) \;\text{ on }\; [0, \infty).$$



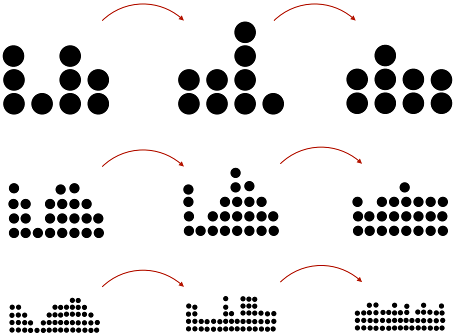
I. Interacting particle systems

A zero range process: on $\mathbb{T}_N^d = (\mathbb{Z}^d/N\mathbb{Z}^d)$ with generator

$$(\mathcal{L}_N f)(\eta) = \sum_{x\in\mathbb{T}_N^d}\sum_{z\in\mathbb{T}_N^d} p_N(z)g(\eta(x))\big(f(\eta^{x,x+z}) - f(\eta)\big),$$

The transition kernel: $p_N(z) = \sum_{y\in\mathbb{Z}^d} p(z+y_N)$ for a compactly supported p with zero mean $\sum_{z\in\mathbb{T}^d} zp(z) = 0$.

Parabolic rescalings: for $N = 4, 8, 15$,



The zero range process η_t^N on $(\mathbb{Z}^d/N\mathbb{Z}^d)$ and the scaled empirical density

$$\mu_t^N = \frac{1}{N^d} \sum_{x \in (\mathbb{Z}^d/N\mathbb{Z}^d)} \delta_{\frac{x}{N}} \cdot \eta_{N^2 t}^N(x).$$

Hydrodynamic limit [Ferrari, Presutti, Vares; 1988]

For every continuous $f\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \mathbb{P}\left[|\langle f, \mu^N \rangle - \langle f, \overline{\rho} \rangle| > \delta \right] = 0,$$

where $\overline{\rho}\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ is the unique solution of the equation

$$\partial_t \overline{\rho} = \frac{1}{2} \Delta \Phi(\overline{\rho}),$$

for the mean local jump rate Φ [Kipnis, Landim; 1999].

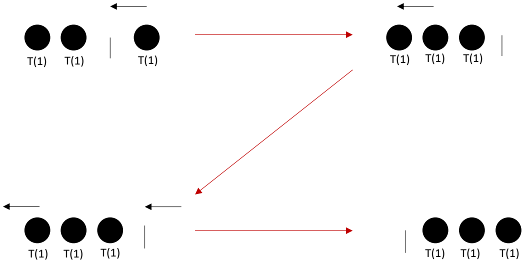
- $\langle f, \mu^N \rangle = \int f \mu^N$ and $\langle f, \overline{\rho} \rangle = \int f \overline{\rho}$
- if $T(k) \sim e^{-t}$ then $\partial_t \overline{\rho} = \frac{1}{2} \Delta \left(\frac{\overline{\rho}}{1+\overline{\rho}} \right)$
- if $T(k) \sim k e^{-kt}$ then $\partial_t \overline{\rho} = \frac{1}{2} \Delta \overline{\rho}$

The symmetric simple exclusion process:

- independent exponentially distributed clocks $T(1)$ with rate 1 — $T(1) \sim \exp(-t)$ on $[0, \infty)$
- the generator on $\mathbb{T}^d_N$,

$$\mathcal{L}_N f(\eta) = \sum_{x \in \mathbb{T}^d_N} \sum_{z \in \mathbb{T}^d_N} p_N(z)\eta(x)(1 - \eta(x+z))(f(\eta^{x,z}) - f(\eta)),$$

- the transition kernel $p_N(z) = \sum_{y \in \mathbb{Z}^d} p(z + Ny)$ for a compactly supported p satisfying $\sum_{z \in \mathbb{Z}^d} zp(z) = 0$.



The symmetric simple exclusion process η_t^N on $(\mathbb{Z}^d/N\mathbb{Z}^d)$ and the scaled density

$$\mu_t^N = \frac{1}{N^d} \sum_{x \in (\mathbb{Z}^d/N\mathbb{Z}^d)} \delta_{\frac{x}{N}} \cdot \eta_{N^2 t}^N(x).$$

Hydrodynamic limit [Kipnis, Olla, Varadhan; 1989]

For every continuous $f\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \mathbb{P}\left[|\langle f, \mu^N \rangle - \langle f, \overline{\rho} \rangle| > \delta \right] = 0,$$

where $\overline{\rho}\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ is the unique solution of the equation

$$\partial_t \overline{\rho} = \frac{1}{2} \Delta \overline{\rho}.$$

For initial data $0 \leq \rho_0 \leq 1$, the hydrodynamic limit of the symmetric simple exclusion process and zero range process with jump rates $g(k) = k$ are the same.

Mean-field limit of independent brownian motions: for B^i_t on $\mathbb{T}^d$,

$$m^N_t = \frac{1}{N}\sum_{i=1}^N \delta_{B^i_t} \rightharpoonup \overline{\rho}\,\mathrm{d}x \;\text{ for }\; \partial_t\overline{\rho} = \frac{1}{2}\Delta\overline{\rho}.$$

Under the parabolic rescaling $\varepsilon B^i_{\varepsilon^{-2}t} \sim B^i_t$.

Forced brownian motions: let $\mathrm{d}X^i_t = \mathrm{d}B^i_t + b(X^i_t)\,\mathrm{d}t$ and $X^{\varepsilon,i}_t = \varepsilon X^i_{\varepsilon^{-1}t}$,

$$m^{N,\varepsilon}_t = \frac{1}{N}\sum_{i=1}^N \delta_{X^{\varepsilon,t}_i} \rightharpoonup \overline{\rho}^\varepsilon\,\mathrm{d}x \;\text{ for }\; \partial_t\overline{\rho}^\varepsilon = \frac{\varepsilon}{2}\Delta\overline{\rho}^\varepsilon - \nabla\cdot(\overline{\rho}^\varepsilon b),$$

for the flux $j(\overline{\rho}^\varepsilon) = \frac{\varepsilon}{2}\nabla\overline{\rho}^\varepsilon - \overline{\rho}^\varepsilon b$. The drift diverges in the parabolic scaling.

The hyperbolic scaling limit: as $\varepsilon \to 0$, the law of the $X^{\varepsilon,i}_t$ satisfies

$$\overline{\rho}^\varepsilon \to \overline{\rho} \;\text{ for }\; \partial_t\overline{\rho} + \nabla\cdot(\overline{\rho}b) = 0,$$

for the flux $j(\overline{\rho}) = -\overline{\rho}b$.

A notion of mobility: the mobility of the system is $m(\overline{\rho}) = \overline{\rho}$.

I. Interacting particle systems

The zero range process with nonzero mean: let η_t^N be the zero range process on $\mathbb{T}_N^d$ with transition kernel p satisfying $\sum_{z\in\mathbb{Z}^d} zp(z) = \gamma$.

The hyperbolic rescaling: let μ_t^N be the hyperbolically rescaled

$$\mu_t^N = \frac{1}{N^d} \sum_{x\in(\mathbb{Z}^d/N\mathbb{Z}^d)} \delta_{\frac{x}{N}} \cdot \eta_{Nt}^N(x).$$

Hydrodynamic limit [Rezakhanlou; 1991]

For every continuous $f\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \mathbb{P}\left[|\langle f, \mu^N\rangle - \langle f, \overline{\rho}\rangle| > \delta \right] = 0,$$

where $\overline{\rho}\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ is the unique solution of the equation

$$\partial_t \overline{\rho} = \nabla \cdot (\Phi(\overline{\rho})\gamma),$$

for the mean local jump rate Φ [Kipnis, Landim; 1999].

Mobility: the mobility of the zero range process is $m(\overline{\rho}) = \Phi(\overline{\rho})$

The exclusion process with nonzero mean: let η_t^N be the exclusion process on $\mathbb{T}_N^d$ with transition kernel p satisfying $\sum_{z\in\mathbb{Z}^d} zp(z) = \gamma$.

The hyperbolic rescaling: let μ_t^N be the hyperbolically rescaled

$$\mu_t^N = \frac{1}{N^d} \sum_{x\in(\mathbb{Z}^d/N\mathbb{Z}^d)} \delta_{\frac{x}{N}} \cdot \eta_{Nt}^N(x).$$

Hydrodynamic limit [Rezakhanlou; 1991]

For every continuous $f\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \mathbb{P}\left[|\langle f, \mu^N\rangle - \langle f, \overline{\rho}\rangle| > \delta\right] = 0,$$

where $\overline{\rho}\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ is the unique solution of the equation

$$\partial_t\overline{\rho} = \nabla \cdot (\overline{\rho}(1-\overline{\rho})\gamma)$$

Mobility: the mobility of the exclusion process is $m(\overline{\rho}) = \overline{\rho}(1-\overline{\rho})$

I. Interacting particle systems

The hydrodynamic limit: the parabolically rescaled, mean zero particle process μ_t^N on $\mathbb{T}_N^d$, as $N \to \infty$,

$$\mu_t^N \rightharpoonup \overline{\rho}\,\mathrm{d}x \ \text{ for } \ \partial_t \overline{\rho} = \Delta\sigma(\overline{\rho}) = \nabla \cdot J(\overline{\rho}),$$

for $J(\overline{\rho}) = \nabla\sigma(\overline{\rho})$.

Macroscopic fluctuation theory: the probability of observing a space-time fluctuation (ρ, j) satisfying

$$\partial_t \rho = \nabla \cdot j$$

satisfies the large deviations bound [Bertini et al.; 2014]

$$\mathbb{P}[\mu^N \simeq \rho] \simeq \exp\big(-NI(\rho,j)\big) \ \text{ for } \ I(\rho,j) = \int_0^T \int_{\mathbb{T}^d} (j - J(\rho)) \cdot m(\rho)^{-1}(j - J(\rho)).$$

The skeleton equation: if $(j - J(\rho)) = \sqrt{m(\rho)}g$ then $I(\rho,j) = \int_0^T \int_{\mathbb{T}^d} |g|^2$ and

$$\partial_t \rho = \nabla \cdot \big(J(\rho) + (j - J(\rho))\big) = \Delta\sigma(\rho) - \nabla \cdot (\sqrt{m(\rho)}g).$$

The zero range process: $\sigma(\rho) = \Phi(\rho)$ and $m(\rho) = \Phi(\rho)$ and

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho)g).$$

The exclusion process: $\sigma(\rho) = \rho$ and $m(\rho) = \rho(1-\rho)$ and

$$\partial_t \rho = \Delta\rho - \nabla \cdot (\sqrt{\rho(1-\rho)}g).$$

II. Stochastic PDE with conservative noise

Space-time white noise: a Gaussian noise ξ on $\mathbb{T}^d$ defined by

$$\mathrm{d}\xi = \sum_{k\in\mathbb{Z}^d} \big(\sqrt{2}\sin(k\cdot x)\,\mathrm{d}B^k_t + \sqrt{2}\cos(k\cdot x)\,\mathrm{d}W^k_t\big),$$

for independent Brownian motions $(B^k, W^k)_{k\in\mathbb{Z}^d}$. Distributionally, we have that

$$\langle \xi(x,t)\xi(y,s)\rangle = \delta_0(x-y)\delta_0(t-s).$$

Schilder's theorem: for a Brownian motion B and $A \subseteq \mathrm{C}([0,T])$,

$$\mathbb{P}[\sqrt{\varepsilon}B \in A] \simeq \exp\big(-\varepsilon^{-1}\inf_{x\in A} I(x)\big) \ \text{ for } \ I(x) = \frac{1}{2}\int_0^T |\dot{x}(s)|^2\,\mathrm{d}s.$$

The contraction principle: for the solutions

$$\partial_t\rho^\varepsilon = \Delta\Phi(\rho^\varepsilon) - \sqrt{\varepsilon}\nabla\cdot(\Phi^{\frac{1}{2}}(\rho^\varepsilon)\xi),$$

we have formally that, for $A \subseteq L^1_t L^1_x$,

$$\mathbb{P}[\rho^\varepsilon \in A] \simeq \exp\big(-\varepsilon^{-1}\inf_{\rho\in A} I(\rho)\big),$$

for the rate function

$$I(\rho) = \frac{1}{2}\inf\Big\{\int_0^T\int_{\mathbb{T}^d}|g|^2 : \partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot\big(\Phi^{\frac{1}{2}}(\rho)g\big)\Big\}.$$

II. Stochastic PDE with conservative noise

The mean behavior: the hydrodynamic limit

$$\partial_t \overline{\rho} = \Delta\sigma(\overline{\rho}) = \nabla \cdot \nabla\sigma(\overline{\rho}),$$

for the flux $J(\overline{\rho}) = \nabla\sigma(\overline{\rho})$.

Fluctuating hydrodynamics: the isotropic non-equilibrium fluctuations ρ described by the continuity equation

$$\partial_t = \nabla \cdot j(\rho) \;\text{ with }\; j(\rho) = J(\rho) + \alpha,$$

for the mobility m and a Gaussian noise α satisfying [Spohn; 1991]

$$\langle \alpha_i(x,t)\alpha_j(y,s)\rangle = m(\rho)\delta_{ij}\delta_0(x-y)\delta_0(y-s).$$

The formal SPDE: the noise $\alpha = \sqrt{m(\rho)}\xi$ for ξ a space-time white noise,

$$\partial_t\rho = \Delta\sigma(\rho) - \nabla \cdot (\sqrt{m(\rho)}\xi).$$

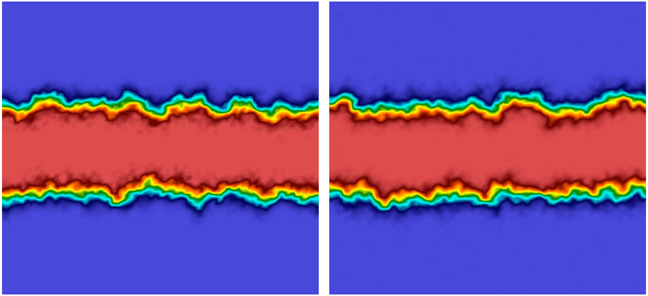
The zero range process: $\sigma(\rho) = \Phi(\rho)$ and $m(\rho) = \Phi(\rho)$ and

$$\partial_t\rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho)\xi).$$

The exclusion process: $\sigma(\rho) = \rho$ and $m(\rho) = \rho(1-\rho)$ and

$$\partial_t\rho = \Delta\rho - \nabla \cdot (\sqrt{\rho(1-\rho)}\xi).$$

II. Stochastic PDE with conservative noise



- a miscible mixture developing a rough diffusive interface due to the effect of thermal fluctuations [Donev; 2018]
- Fluctuating hydrodynamics, for example, [Spohn; 1991]
  — in the zero range case, the formal SPDE

$$\partial_t \rho^\varepsilon = \Delta \Phi(\rho^\varepsilon) - \sqrt{\varepsilon} \nabla \cdot (\Phi^{\frac{1}{2}}(\rho^\varepsilon)\xi).$$

  — fluctuation-dissipation relation, for the free energy $\Psi'_\Phi(\xi) = \log(\Phi(\xi))$,

$$\Phi'(\rho) = \Phi(\rho)\Psi''_\Phi(\rho).$$

  — coarse-graining and correlated noise

II. Stochastic PDE with conservative noise

The empirical density: let m_n denote the measure

$$m_n(x,t) = \frac{1}{n}\sum_{k=1}^{n} \delta(x - B_t^k)$$

for independent Brownian motions B^k on $\mathbb{T}^d$.

The derivation: for every $f \in \mathrm{C}^\infty(\mathbb{T}^d)$,

$$\begin{aligned}\partial_t\left(\int_{\mathbb{T}^d} f(x) m_n\right) &= \partial_t\left(\frac{1}{n}\sum_{k=1}^{n} f(B_t^k)\right)\\ &= \frac{1}{2}\int_{\mathbb{T}^d} \Delta f\, m_n + \text{“Gaussian noise”}\\ &= \frac{1}{2}\int_{\mathbb{T}^d} \Delta f\, m_n + \frac{1}{\sqrt{n}}\int_{\mathbb{T}^d} \nabla f \cdot \sqrt{m_n}\,\xi,\end{aligned}$$

for ξ an $\mathbb{R}^d$-valued space-time white noise.

The Dean–Kawasaki equation:

$$\partial_t m_n = \frac{1}{2}\Delta m_n - \frac{1}{\sqrt{n}}\nabla \cdot (\sqrt{m_n}\xi).$$

II. Stochastic PDE with conservative noise

The Dean–Kawasaki equation: we have,

$$\partial_t \rho^\varepsilon = \frac{1}{2}\Delta\rho^\varepsilon - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho^\varepsilon}\xi).$$

The Zero Range Process: the formal SPDE describing non-equilibrium behavior,

$$\partial_t \rho^\varepsilon = \Delta\Phi(\rho^\varepsilon) - \sqrt{\varepsilon}\nabla\cdot(\Phi^{\frac{1}{2}}(\rho^\varepsilon)\xi).$$

- Supercritical in the language of regularity structures [Hairer; 2014]
  — no solution theory
- Ill-posedness vs. triviality
  — for example, [Konarovskyi, Lehmann, von Renesse; 2019]
- Degenerate diffusions
  — porous media and fast diffusions, $\Phi(\xi) = \xi^m$ for every $m \in (0,\infty)$
- Irregular noise coefficients

The Dean–Kawasaki equation: for independent Brownian motions,

$$\partial_t \rho = \frac{1}{2}\Delta\rho - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho}\xi).$$

- White noise is too singular (particles systems, course graining, and function-valued large deviations):

- Spatially correlated noise:

$$\xi^\delta = \xi * \kappa^\delta \text{ for a convolution kernel } \kappa^\delta \text{ of scale } \delta\in(0,1).$$

The Dean–Kawasaki equation with correlated noise: the Stratonovich equation,

$$\partial_t \rho = \frac{1}{2}\Delta\rho - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho}\circ\xi^\delta).$$

Fluctuations and large deviations formally the same for Itô vs. Stratonovich.

The Stratonovich-to-Itô correction: we consider the Stratonovich SPDE

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho) \circ f(x)\,\mathrm{d}B_t),$$

for the d-dimensional noise $\mathrm{d}\xi = f\,\mathrm{d}B_t$. The Stratonovich integral

$$\begin{aligned}
\int_0^t \int_{\mathbb{T}^d} \sigma(\rho_s) \circ f\,\mathrm{d}B_s &= \int_{\mathbb{T}^d} f \sum_{|\mathcal{P}|\to 0} \frac{\sigma(\rho_{t_{i+1}}) + \sigma(\rho_{t_i})}{2}(B_{t_{i+1}} - B_{t_i}) \\
&= \int_{\mathbb{T}^d} f\Big(\frac{1}{2}\sum_{|\mathcal{P}|\to 0}(\sigma(\rho_{t_{i+1}}) - \sigma(\rho_{t_i}))(B_{t_{i+1}} - B_{t_i}) + \sum_{|\mathcal{P}|\to 0}\sigma(\rho_{t_i})(B_{t_{i+1}} - B_{t_i}) \\
&= \frac{1}{2}\int_0^t \int_{\mathbb{T}^d} f\sigma'(\rho)\,\mathrm{d}\langle \partial_t\rho, B\rangle_s + \int_0^t \int_{\mathbb{T}^d} f\sigma(\rho)\,\mathrm{d}B_s \\
&= \frac{1}{2}\int_0^t \int_{\mathbb{T}^d} f\sigma'(\rho)\nabla(\sigma(\rho)f)\,\mathrm{d}s + \int_0^t \int_{\mathbb{T}^d} f\sigma(\rho)\,\mathrm{d}B_s.
\end{aligned}$$

The Itô-form of the SPDE: we have that

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho) f(x)\,\mathrm{d}B_t) + \frac{1}{2}\nabla \cdot (\sigma'(\rho) f \nabla(\sigma(\rho) f)).$$

III. Stochastic kinetic solutions

A general SPDE with conservative noise: for consider the Stratonovich SPDE

$$\partial_t \rho = \Delta\Phi(\rho) - \sqrt{\varepsilon}\nabla \cdot (\sigma(\rho) \circ \mathrm{d}\xi^\delta),$$

for probabilistically stationary noise $\xi^\delta = (\xi * \kappa^\delta)$ and scalar σ.

The Itô-formulation for the spatially constant quadratic variation $\langle \xi^\delta \rangle$,

$$\partial_t \rho = \Delta\Phi(\rho) - \sqrt{\varepsilon}\nabla \cdot (\sigma(\rho)\, \mathrm{d}\xi^\delta) + \frac{\varepsilon\langle \xi^\delta \rangle}{2}\nabla \cdot (\sigma'(\rho)\nabla\sigma(\rho)).$$

Logarithmic divergence of the correction: if $\sigma(\rho) = \sqrt{\rho}$ then

$$\frac{\varepsilon\langle \xi^\delta \rangle}{2}\nabla \cdot (\sigma'(\rho)\nabla\sigma(\rho)) = \frac{\varepsilon\langle \xi^\delta \rangle}{8}\nabla \cdot \left(\frac{1}{\rho}\nabla\rho\right) = \frac{\varepsilon\langle \xi^\delta \rangle}{8}\Delta \log(\rho),$$

and we have, in the Dean–Kawasaki case,

$$\partial_t \rho = \Delta\rho - \sqrt{\varepsilon}\nabla \cdot (\sqrt{\rho}\, \mathrm{d}\xi^\delta) + \frac{\varepsilon\langle \xi^\delta \rangle}{8}\Delta \log(\rho).$$

A general SPDE with conservative noise: for the Itô-equation

$$\partial_t \rho = \Delta\Phi(\rho) + \eta\Delta\rho - \sqrt{\varepsilon}\nabla\cdot(\sigma(\rho)\,\mathrm{d}\xi^\delta) + \frac{\varepsilon\langle\xi^\delta\rangle}{2}\nabla\cdot(\sigma'(\rho)\nabla\sigma(\rho)),$$

we have using Itô's formula, for smooth S and ψ,

$$\partial_t\int\psi S(\rho) = \int\psi S'(\rho)\,\mathrm{d}\rho + \frac{1}{2}\int\psi S''(\rho)\,\mathrm{d}\langle\rho\rangle =$$

$$-\int\Phi'(\rho)S'(\rho)\nabla\rho\cdot\nabla\psi - \sqrt{\varepsilon}\int\psi S'(\rho)\nabla\cdot(\sigma(\rho)\,\mathrm{d}\xi^\delta) - \frac{\varepsilon\langle\xi^\delta\rangle}{2}\int(\sigma'(\rho))^2\nabla\rho\cdot S'(\rho)\nabla\psi$$

$$-\int\psi S''(\rho)\Phi'(\rho)|\nabla\rho|^2 - \eta\int\psi S''(\rho)|\nabla\rho|^2 - \frac{\varepsilon\langle\xi^\delta\rangle}{2}\int\psi S''(\rho)|\nabla\sigma(\rho)|^2$$

$$+\frac{\varepsilon\langle\xi^\delta\rangle}{2}\int\psi S''(\rho)|\nabla\sigma(\rho)|^2 + \frac{\varepsilon\langle\nabla\xi^\delta\rangle}{2}\int\psi S''(\rho)\sigma(\rho)^2.$$

Stochastic coercivity: identical techniques treat the Itô equation

$$\partial\rho = \Delta\rho - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho}\,\mathrm{d}\xi^\delta),$$

provided that $(\sigma')^2 \lesssim \Phi'$. In the Dean–Kawasaki case, this means controlling $|\nabla\sqrt{\rho}|^2$ by $|\nabla\rho|^2$ [F., Gess, Gvalani; 2022].

A general SPDE with conservative noise: for the Itô-equation

$$\partial_t \rho = \Delta\Phi(\rho) - \sqrt{\varepsilon}\nabla\cdot(\sigma(\rho)\,\mathrm{d}\xi^\delta) + \frac{\varepsilon\langle\xi^\delta\rangle}{2}\nabla\cdot(\sigma'(\rho)\nabla\sigma(\rho)).$$

The kinetic formulation: for the kinetic function $\chi = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$, and for a nonnegative measure q, we have for every $\phi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times (0,\infty))$,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{T}^d}\phi\chi_t = \int_{\mathbb{R}}\int_{\mathbb{T}^d}\phi\chi_0 &- \int_0^t\int_{\mathbb{T}^d}\Phi'(\rho)\nabla\rho(x,\rho)\cdot(\nabla\phi)(x,\rho)\\ &- \sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\phi(x,\rho)\nabla\cdot(\sigma(\rho)\,\mathrm{d}\xi^\delta) - \frac{\varepsilon\langle\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d}(\sigma'(\rho))^2\nabla\rho\cdot(\nabla\phi)(x,\rho)\\ &- \int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\Phi'(\rho)|\nabla\rho|^2 - \int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\partial_\xi\phi\,\mathrm{d}q\\ &+ \frac{\varepsilon\langle\nabla\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\sigma(\rho)^2.\end{aligned}$$

Or, distributionally, for $\delta_\rho = \delta(\xi-\rho)$ and for the measure $p = \delta_\rho\Phi'(\xi)|\nabla\rho|^2$,

$$\begin{aligned}\partial_t\chi = \Phi'(\xi)\Delta_x\chi - \sqrt{\varepsilon}\sigma'(\xi)\,\mathrm{d}\xi^\delta\cdot\nabla\chi + \sqrt{\varepsilon}\sigma(\xi)\partial_\xi\chi\nabla\cdot\,\mathrm{d}\xi^\delta + \frac{\varepsilon\langle\xi^\delta\rangle}{2}\nabla\cdot((\sigma'(\xi))^2\nabla\chi)\\ + \partial_\xi p + \partial_\xi q - \partial_\xi\left(\delta_\rho\frac{\varepsilon\langle\nabla\xi^\delta\rangle}{2}\sigma(\xi)^2\right).\end{aligned}$$

Stochastic kinetic solutions [F. Gess; 2021]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration on $(\Omega, \mathcal{F})$, let the noise ξ^δ be $\mathcal{F}_t$-adapted, and let $\rho_0 \in L^1$ be nonnegative and $\mathcal{F}_0$-measurable. A *stochastic kinetic solution* is a continuous $L^1(\mathbb{T}^d)$-valued, $\mathcal{F}_t$-predictable process ρ that satisfies the following five properties.

(i) *Preservation of mass*: for every $t \in [0,T]$, $\mathbb{E}[\|\rho(\cdot,t)\|_{L^1(\mathbb{T}^d)}] = \mathbb{E}[\|\rho_0\|_{L^1(\mathbb{T}^d)}]$.

(ii) *Integrability of the flux*: we have $\sigma(\rho) \in L^2(\Omega \times \mathbb{T}^d \times [0,T])$.

(iii) *Local regularity*: for every $K \in \mathbb{N}$, $(\rho \wedge K) \vee (1/K) \in L^2(\Omega \times [0,T]; H^1(\mathbb{T}^d))$.

(iv) *Vanishing at infinity*: $\liminf_{M\to\infty} \mathbb{E}[(p+q)(\mathbb{T}^d \times [M, M+1] \times [0,T])] = 0$.

(v) *The equation*: for a nonnegative measure q, for every $\phi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times (0,\infty))$,

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d} \phi\chi_t = \int_{\mathbb{R}}\int_{\mathbb{T}^d} \phi\chi_0 - \int_0^t\int_{\mathbb{T}^d} \Phi'(\rho)\nabla\rho\cdot(\nabla\phi)(x,\rho)$$
$$- \sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^d} \phi(x,\rho)\nabla\cdot(\sigma(\rho)\,\mathrm{d}\xi^\delta) - \frac{\varepsilon\langle\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d} (\sigma'(\rho))^2\nabla\rho\cdot(\nabla\phi)(x,\rho)$$
$$- \int_0^t\int_{\mathbb{T}^d} (\partial_\xi\phi)(x,\rho)\Phi'(\rho)|\nabla\rho|^2 - \int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d} \partial_\xi\phi\,\mathrm{d}q + \frac{\varepsilon\langle\nabla\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d} (\partial_\xi\phi)(x,\rho)\sigma(\rho)^2.$$

III. Stochastic kinetic solutions

Extensions: we consider general equations of the type

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot \Big(\sigma(\rho) \circ \xi^\delta + \nu(\rho)\Big) + \lambda(\rho) + \phi(\rho)\xi^\delta,$$

including non-equilibrium fluctuations of asymmetric systems, mean-field games, stochastic geometric PDEs, and branching interacting diffusions.

- The generalized Dean-Kawasaki equation with correlated noise

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot \Big(\Phi(\rho) + \Phi^{\frac{1}{2}}(\rho) \circ \xi^\delta\Big).$$

- Nonlinear Dawson-Watanabe equation

$$\partial_t \rho = \Delta\Phi(\rho) + \sqrt{\rho}\xi^\delta.$$

- Fluctuating mean-curvature equation

$$\partial_t \rho = \nabla \cdot \left(\frac{\nabla \rho}{1+\rho^2}\right) + \nabla \cdot \Big((1+\rho^2)^{\frac{1}{4}} \circ \xi^\delta\Big).$$

- Fast diffusion and porous media: $\Phi(\xi) = \xi^m$ for any $m \in (0, \infty)$.
- ϕ is globally $1/2$-Hölder continuous, λ is globally Lipschitz continuous.

The Dean–Kawasaki equation: we consider the Dean–Kawasaki equation

$$\partial_t \rho = \Delta\Phi(\rho) - \sqrt{\varepsilon}\nabla\cdot(\Phi^{\frac{1}{2}}(\rho)\circ \,\mathrm{d}\xi^{\delta}).$$

for which we have, almost surely for every $\phi \in \mathrm{C}_c^\infty(\mathbb{T}^d\times(0,\infty))$,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{T}^d}\phi\chi_t &= \int_{\mathbb{R}}\int_{\mathbb{T}^d}\phi\chi_0 - \int_0^t\int_{\mathbb{T}^d}\Phi'(\rho)\nabla\rho\cdot(\nabla\phi)(x,\rho)\\ &- \sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\phi(x,\rho)\nabla\cdot(\Phi^{\frac{1}{2}}(\rho)\,\mathrm{d}\xi^{\delta}) - \frac{\varepsilon\langle\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d}(\Phi^{\frac{1}{2}})'(\rho)^2\nabla\rho\cdot(\nabla\phi)(x,\rho)\\ &- \int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\Phi'(\rho)|\nabla\rho|^2 - \int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\partial_\xi\phi\,\mathrm{d}q + \frac{\varepsilon\langle\nabla\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\Phi(\rho).\end{aligned}$$

Entropy estimate: let $\Psi_\Phi(\xi) = \int_0^\xi \log(\Phi(\xi'))\,\mathrm{d}x'$ and $\phi(\xi) = \log(\Phi(\xi))$,

$$\begin{aligned}\int_{\mathbb{T}^d}\Psi_\Phi(\rho_t) &= \int_{\mathbb{T}^d}\Psi_\Phi(\rho_0) - \sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\log(\Phi(\rho))\nabla\cdot(\Phi^{\frac{1}{2}}(\rho)\,\mathrm{d}\xi^\delta)\\ &\quad - \int_0^t\int_{\mathbb{T}^d}\frac{\Phi'(\rho)^2}{\Phi(\rho)}|\nabla\rho|^2 - \int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\frac{\Phi'(\xi)}{\Phi(\xi)}\,\mathrm{d}q + \frac{\varepsilon\langle\nabla\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d}\Phi'(\rho).\end{aligned}$$

and, using the definition of p,

$$\int_0^t\int_{\mathbb{T}^d}\frac{\Phi'(\rho)^2}{\Phi(\rho)}|\nabla\rho|^2 = \int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\frac{\Phi'(\xi)}{\Phi(\xi)}\,\mathrm{d}p.$$

The entropy estimate: we consider the Dean–Kawasaki equation

$$\partial_t \rho = \Delta\Phi(\rho) - \sqrt{\varepsilon}\nabla\cdot(\Phi^{\frac{1}{2}}(\rho)\circ \mathrm{d}\xi^{\delta}),$$

and for $\Psi_\Phi = \int_0^\xi \log(\Phi(\xi'))\,\mathrm{d}x'$ and $\phi(\xi) = \log(\Phi(\xi))$,

$$\mathbb{E}\sup_{t\in[0,T]}\int_{\mathbb{T}^d}\Psi_\Phi(\rho_t) + \mathbb{E}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\frac{\Phi'(\xi)}{\Phi(\xi)}(\,\mathrm{d}p + \,\mathrm{d}q) \leq$$

$$\mathbb{E}\int_{\mathbb{T}^d}\Psi_\Phi(\rho_0) + \mathbb{E}\sup_{t\in[0,T]}|\sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\log(\Phi(\rho))\nabla\cdot(\Phi^{\frac{1}{2}}(\rho)\,\mathrm{d}\xi^{\delta})| + \frac{\varepsilon\langle\nabla\xi^{\delta}\rangle}{2}\mathbb{E}\int_0^T\int_{\mathbb{T}^d}\Phi'(\rho).$$

Using $\nabla\log(\Phi(\rho)) = \frac{\Phi'(\rho)}{\Phi(\rho)}\nabla\rho$ and the Burkholder–Davis–Gundy inequality,

$$\mathbb{E}\sup_{t\in[0,T]}|\sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\log(\Phi(\rho))\nabla\cdot(\Phi^{\frac{1}{2}}(\rho)\,\mathrm{d}\xi^{\delta})| \leq c\sqrt{\varepsilon}\langle\xi^{\delta}\rangle^{\frac{1}{2}}\mathbb{E}\Big(\int_0^T\int_{\mathbb{R}}\int_{\mathbb{T}^d}\frac{\Phi'(\xi)}{\Phi(\xi)}\,\mathrm{d}p\Big)^{\frac{1}{2}},$$

and using Hölder's and Young's inequality, assuming $\frac{\Phi(\xi)}{\Phi'(\xi)} \leq c\xi$ so that $\frac{\Phi'(\xi)}{\Phi(\xi)} \geq \frac{1}{c\xi}$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\int_{\mathbb{T}^d}\Psi_\Phi(\rho_t) + \int_0^t\int_0^\infty\int_{\mathbb{T}^d}\frac{1}{\xi}(\,\mathrm{d}p + \,\mathrm{d}q)\Big)$$

$$\leq c\mathbb{E}\Big(\int_{\mathbb{T}^d}\Psi_\Phi(\rho_0) + \varepsilon\langle\xi^{\delta}\rangle + \frac{\varepsilon\langle\nabla\xi^{\delta}\rangle}{2}\int_0^T\int_{\mathbb{T}^d}\Phi'(\rho)\Big).$$

III. Stochastic kinetic solutions

The equation: $\partial_t \rho = \Delta\Phi(\rho) - \sqrt{\varepsilon}\nabla \cdot \left(\Phi^{\frac{1}{2}}(\rho) \circ \xi^F\right)$.

The kinetic measure vanishes at zero [F. Gess; 2021]

Let $\rho_0 \in L^1(\Omega; L^1(\mathbb{T}^d))$ be nonnegative and $\mathcal{F}_0$-measurable, and let ρ be a stochastic kinetic solution with initial data ρ_0 with kinetic measure q. Then,

$$\liminf_{\beta \to 0} \left(\beta^{-1}\mathbb{E}\left[(p+q)(\mathbb{T}^d \times [\beta/2, \beta] \times [0,T]\right]\right) = 0.$$

- Essentially equivalent to the preservation of the L^1-norm.

Existence and uniqueness [F. Gess; 2021]

Let $\rho_0 \in L^1(\Omega; L^1(\mathbb{T}^d))$ be nonnegative and $\mathcal{F}_0$-measurable. Then, there exists a unique stochastic kinetic solution with initial data ρ_0. Furthermore, two solutions ρ^1 and ρ^2 almost surely satisfy, for every $t \in [0,T]$,

$$\left\|\rho^1(\cdot,) - \rho^2(\cdot, t)\right\|_{L^1(\mathbb{T}^d)} \leq \left\|\rho_0^1 - \rho_0^2\right\|_{L^1(\mathbb{T}^d)}.$$

- Stochastic dynamics, random dynamical systems, and invariant measures [F., Gess, Gvalani; 2022].

The kinetic equation: for test functions $\phi \in C_c^\infty(\mathbb{T}^d \times (0,\infty))$,

$$\partial_t \chi = \Phi'(\xi)\Delta_x \chi - \sqrt{\varepsilon}\delta_\rho \nabla \cdot (\Phi^{\frac{1}{2}}(\rho)\,\mathrm{d}\xi^\delta) + \frac{\varepsilon\langle\xi^\delta\rangle}{2}\nabla \cdot ((\sigma'(\xi))^2 \nabla\chi)$$
$$+ \partial_\xi p + \partial_\xi q - \partial_\xi\Big(\delta_\rho \frac{\varepsilon\langle\nabla\xi^\delta\rangle}{2}\Phi(\xi)^2\Big).$$

Let ζ_M be a cutoff of $[\frac{1}{M}, M]$ supported on $[\frac{1}{2M}, M+1]$ so that

$$|\zeta_M'| \leq cM\mathbf{1}_{(\frac{1}{2M},\frac{1}{M})} + c\mathbf{1}_{(M,M+1)}.$$

The uniqueness proof: the proof is based on differentiating the identity

$$\partial_t \int_{\mathbb{T}^d} |\rho_t^1 - \rho_t^2| = \partial_t \int_{\mathbb{R}}\int_{\mathbb{T}^d} |\chi^1 - \chi^2|^2 = \partial_t \int_{\mathbb{R}}\int_{\mathbb{T}^d} \chi^1 \mathrm{sgn}(\xi) + \chi^2 \mathrm{sgn}(\xi) - 2\chi^1\chi^2,$$

for which we introduce the cutoff and differentiate

$$\partial_t \int_{\mathbb{R}}\int_{\mathbb{T}^d} (\chi^1\mathrm{sgn}(\xi) + \chi^2\mathrm{sgn}(\xi) - 2\chi^1\chi^2)\zeta_M = \int_R\int_{\mathbb{T}^d} (\text{deterministic terms})\zeta_M$$
$$+ \int_{\mathbb{R}}\int_{\mathbb{T}^d} (2\chi^2 - 1)\delta_{\rho^1}\nabla\cdot(\Phi^{\frac{1}{2}}(\rho^1)\,\mathrm{d}\xi^\delta)\zeta_M + (2\chi^1 - 1)\delta_{\rho^2}\nabla\cdot(\Phi^{\frac{1}{2}}\,\mathrm{d}\xi^\delta)\zeta_M$$
$$- \sum_{i=1}^{2}\int_{\mathbb{R}}\int_{\mathbb{T}^d} \zeta_M'(\xi)(\,\mathrm{d}p^i + \mathrm{d}q^i) + \sum_{i=1}^{2}\frac{\varepsilon\langle\nabla\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\zeta_M'(\rho^i)\Phi(\rho^i)^2.$$

The stochastic term: for the term

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d}(2\chi^2-1)\delta_{\rho^1}\nabla\cdot(\Phi^{\frac{1}{2}}(\rho^1)\,\mathrm{d}\xi^\delta)\zeta_M+(2\chi^1-1)\delta_{\rho^2}\nabla\cdot(\Phi^{\frac{1}{2}}\,\mathrm{d}\xi^\delta)\zeta_M$$

we have that, without the cutoff ζ_M,

$$\int_{\mathbb{R}}(2\chi^2-1)\delta_{\rho^1}=\int_{\mathbb{R}}\mathrm{sgn}(\rho^2-\xi)\delta_{\rho^1}=\mathrm{sgn}(\rho^1-\rho^2).$$

Therefore, ignoring the cutoff ζ_M (a bad idea),

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{T}^d}(2\chi^2-1)\delta_{\rho^1}\nabla\cdot(\Phi^{\frac{1}{2}}(\rho^1)\,\mathrm{d}\xi^\delta)\zeta_M+(2\chi^1-1)\delta_{\rho^2}\nabla\cdot(\Phi^{\frac{1}{2}}\,\mathrm{d}\xi^\delta)\\&=\int_{\mathbb{T}^d}\mathrm{sgn}(\rho^1-\rho^2)\nabla\cdot((\Phi^{\frac{1}{2}}(\rho^1)-\Phi^{\frac{1}{2}}(\rho^2))\,\mathrm{d}\xi^\delta)\\&=-2\int_{\mathbb{T}^d}\delta_0(\rho^1-\rho^2)(\nabla\rho^1-\nabla\rho^2)\cdot(\Phi^{\frac{1}{2}}(\rho^1)-\Phi^{\frac{1}{2}}(\rho^2))\,\mathrm{d}\xi^\delta=0?\end{aligned}$$

- $\Phi^{\frac{1}{2}}$ is not Lipschitz continuous and ρ^i is not regular
- exploit the cutoff ζ_M, local regularity of ρ^i, and local Lipschitz continuity of $\Phi^{\frac{1}{2}}$

The uniqueness proof: we have that

$$\partial_t \int_{\mathbb{R}}\int_{\mathbb{T}^d} (\chi^1 \mathrm{sgn}(\xi) + \chi^2 \mathrm{sgn}(\xi) - 2\chi^1\chi^2)\zeta_M = \int_R \int_{\mathbb{T}^d} (\text{deterministic terms})\zeta_M$$
$$+ \int_{\mathbb{R}}\int_{\mathbb{T}^d} (2\chi^2 - 1)\delta_{\rho^1}\nabla\cdot(\Phi^{\frac{1}{2}}(\rho^1)\,\mathrm{d}\xi^\delta)\zeta_M + (2\chi^1 - 1)\delta_{\rho^2}\nabla\cdot(\Phi^{\frac{1}{2}}\,\mathrm{d}\xi^\delta)\zeta_M$$
$$- \sum_{i=1}^2 \int_{\mathbb{R}}\int_{\mathbb{T}^d} \zeta_M'(\xi)(\mathrm{d}p^i + \mathrm{d}q^i) + \sum_{i=1}^2 \frac{\varepsilon\langle\nabla\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\zeta_M'(\rho^i)\Phi(\rho^i)^2.$$

The cutoff terms: for the cutoff terms, we have that

$$|\int_{\mathbb{R}}\int_{\mathbb{T}^d} \zeta_M'(\xi)(\mathrm{d}p^1 + \mathrm{d}q^1)| + |\frac{\varepsilon\langle\nabla\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\zeta_M'(\rho^1)\Phi(\rho^1)^2|$$
$$\leq c(p^1+q^1)(\mathbb{T}^d \times (M, M+1)\times\{t\}) + c\int_{\mathbb{T}^d}\mathbf{1}_{\{M<\rho^1<M+1\}}\Phi(\rho^1)$$
$$+ cM(p^1+q^1)\left(\mathbb{T}^d \times \left(\frac{1}{2M}, \frac{1}{M}\right)\times\{t\}\right) + cM\int_{\mathbb{T}^d}\mathbf{1}_{\{\frac{1}{2M}<\rho^1<\frac{1}{M}\}}\Phi(\rho^1).$$

Vanishes as $M \to \infty$ due to singular moments and decay of the measures.

The Large Deviations Principle [F., Gess; 2022]

The scaling limit: let $\delta(\varepsilon)$ be any sequence satisfying, as $\varepsilon \to 0$,

$$\varepsilon\delta(\varepsilon)^{-(d+2)} \to 0 \text{ and } \delta(\varepsilon) \to 0,$$

and for every $\varepsilon \in (0,1)$ let ρ^ε be the solution

$$\partial_t \rho^\varepsilon = \Delta\Phi(\rho^\varepsilon) - \sqrt{\varepsilon}\nabla \cdot (\Phi^{\frac{1}{2}}(\rho^\varepsilon) \circ \xi^{\delta(\varepsilon)}).$$

The large deviations principle: the solutions ρ^ε satisfy a large deviations principle with rate function

$$I(\rho) = \frac{1}{2} \inf \left\{ \|g\|_{L^2}^2 : \partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \right\}.$$

The linear fluctuating hydrodynamics: the linear fluctuating hydrodynamics

$$\partial_t \tilde{\rho}^\varepsilon = \Delta\Phi(\tilde{\rho}^\varepsilon) - \sqrt{\varepsilon}\nabla \cdot (\Phi^{\frac{1}{2}}(\overline{\rho})\xi^{\delta(\varepsilon)}),$$

for the hydrodynamics limit $\partial_t \overline{\rho} = \Delta\Phi(\overline{\rho})$ satisfy an LDP with rate function

$$\tilde{I}(\rho) = \frac{1}{2} \inf \left\{ \|g\|_{L^2}^2 : \partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\overline{\rho}) g) \right\}.$$

The Large Deviations Principle [F., Gess; 2022]

The scaling limit: let $\delta(\varepsilon)$ be any sequence satisfying, as $\varepsilon \to 0$,

$$\varepsilon\delta(\varepsilon)^{-(d+2)} \to 0 \text{ and } \delta(\varepsilon) \to 0,$$

and for every $\varepsilon \in (0,1)$ let ρ^ε be the solution

$$\partial_t \rho^\varepsilon = \Delta\Phi(\rho^\varepsilon) - \sqrt{\varepsilon}\nabla \cdot (\Phi^{\frac{1}{2}}(\rho^\varepsilon) \circ \xi^{\delta(\varepsilon)}).$$

The large deviations principle: the solutions ρ^ε satisfy a large deviations principle with rate function

$$I(\rho) = \frac{1}{2} \inf \left\{ \|g\|_{L^2}^2 : \partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \right\}.$$

The controlled SPDE: for weakly convergent controls $g^\varepsilon \rightharpoonup g$ the solutions

$$\partial_t \rho^\varepsilon = \Delta\Phi(\rho^\varepsilon) - \sqrt{\varepsilon}\nabla \cdot (\Phi^{\frac{1}{2}}(\rho^\varepsilon)) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho^\varepsilon) g^\varepsilon),$$

converge in the scaling limit $\varepsilon\langle\nabla\xi^\delta\rangle \simeq \varepsilon\delta(\varepsilon)^{-(d+2)} \to 0$ to the solution

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g).$$

Weak approach to large deviations [Budhiraja, Dupuis, Maroulas; 2008].

IV. The large deviations principle

The rate function: for $\rho \in L^1([0,T];L^1(\mathbb{T}^d))$,

$$I(\rho) = \frac{1}{2}\inf\left\{\|g\|_{L^2}^2 : \partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g)\right\}.$$

The Hilbert space: $H^1_{\Phi(\rho)}$ is the strong closure w.r.t. the inner product

$$\langle\nabla\psi,\nabla\phi\rangle = \int_0^T\int_{\mathbb{T}^d}\Phi(\rho)\nabla\psi\cdot\nabla\phi \;\text{ for }\; \phi,\psi\in \mathrm{C}^\infty.$$

Unique minimizer: if $I(\rho)<\infty$ then the minimizer $g = \Phi^{\frac{1}{2}}(\rho)\nabla H$ for $H\in H^1_{\Phi(\rho)}$,

$$I(\rho) = \frac{1}{2}\int_0^T\int_{\mathbb{T}^d}\Phi(\rho)|\nabla H|^2 = \frac{1}{2}\|H\|^2_{H^1_{\Phi(\rho)}} = \frac{1}{2}\|\partial_t\rho - \Delta\Phi(\rho)\|^2_{H^{-1}_{\Phi}(\rho)},$$

where the equation defines $\partial_t\rho - \Delta\Phi(\rho) = -\nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g)\in H^{-1}_{\Phi(\rho)}$.

The “ill-posed” equation: we have the formally “supercritical” equation

$$\partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi(\rho)\nabla H).$$

IV. The large deviations principle

The space of smooth fluctuations $\mathcal{S}$: we define the space

$\mathcal{S} = \{\rho\colon \partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi(\rho)\nabla H)$ for $\rho_0 \in \mathrm{C}^\infty(\mathbb{T}^d)$ and $H \in \mathrm{C}^{3,1}(\mathbb{T}^d\times[0,T])\}$.

Recovery sequence: suppose that $I(\rho) < \infty$ and, for the minimizer g,

$$\partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g).$$

Let ρ_n solve, for cutoff functions σ_n on $(0,\infty)$,

$$\partial_t\rho_n = \Delta\Phi(\rho_n) - \nabla\cdot(\sigma_n(\rho_n)\Phi^{\frac{1}{2}}(\rho_n)g).$$

Then, there exists H_n with $\int\Phi(\rho_n)|\nabla H_n|^2 \leq \int\sigma_n(\rho_n)^2|g|^2$ such that

$$-\nabla\cdot(\Phi(\rho_n)\nabla H_n) = \partial_t\rho_n - \Delta\Phi(\rho_n).$$

[Kipnis, Olla, Varadhan; 1989], [Benois, Kipnis, Landim; 1995]

The zero range process satisfies a large deviations upper bound with rate function I and a large deviations lower bound with rate function $\overline{I_{|\mathcal{S}}}(\rho)$, the l.s.c. envelope of I restricted to $\mathcal{S}$.

[F., Gess; 2022]

These rate functions coincide and are equal to I.

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