The kinetic formulation of the skeleton equation

Benjamin Fehrman

University of Oxford

27 February 2023

The equation: the scalar conservation law

$$\partial_t \rho + \nabla \cdot A(\rho) = 0 \text{ in } \mathbb{R}^d \times (0,\infty) \text{ with } \rho(\cdot,0) = \rho_0,$$

where ρ is the scalar *density* and A is the $\mathbb{R}^d$-valued *flux* satisfying

$$\partial_t \int_U \rho = -\oint_{\partial U} A(\rho) \cdot \nu,$$

for the unit outer normal ν to U.

Weak formulation: for every $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d \times [0,\infty))$,

$$\int_{\mathbb{R}^d} \psi(x,0)\rho_0(x) + \int_0^\infty \int_{\mathbb{R}^d} \rho \partial_t \psi = -\int_0^\infty \int_{\mathbb{R}^d} A(\rho) \cdot \nabla \psi.$$

A *weak solution* is an integrable ρ satisfying this equation.

Uniqueness of smooth solutions: let ρ^i solve $\partial_t \rho_i + \nabla \cdot A(\rho^i) = 0$ and let $f^\delta(\xi) = |\xi|^\delta$ so that $(f^\delta)'(\xi) = \text{sgn}^\delta(\xi)$ and $(f^\delta)''(\xi) \simeq \frac{4}{\delta}\mathbf{1}_{\{-\delta<\xi<\delta\}}$. Then,

$$\partial_t \int f^\delta(\rho^1 - \rho^2) = \int (f^\delta)'(\rho^1 - \rho^2)\nabla \cdot (A(\rho^1) - (\rho^2))$$
$$= -\int (f^\delta)''(\rho^1 - \rho^2)(\nabla\rho^1 - \nabla\rho^2) \cdot (A(\rho^1) - A(\rho^2)).$$

We have that

$$|\int (f^\delta)''(\rho^1 - \rho^2)(\nabla\rho^1 - \nabla\rho^2) \cdot (A(\rho^1) - A(\rho^2))|$$
$$\leq \frac{c}{\delta}\int \mathbf{1}_{\{|\rho^1-\rho^2|<\delta\}}|\nabla\rho^1 - \nabla\rho^2||A(\rho^1) - A(\rho^2)|$$
$$\leq c\,\|A\|_{\text{Lip}}\int \mathbf{1}_{\{|\rho^1-\rho^2|<\delta\}}(|\nabla\rho^1| + |\nabla\rho^2|).$$

Passing $\delta \to 0$ using dominated convergence, for $\rho^i_t = \rho^i(\cdot, t)$,

$$\sup_t \int |\rho^1_t - \rho^2_t| \leq \int |\rho^1_0 - \rho^2_0|.$$

Lipschitz continuity justifies

$$\partial_t \int |\rho^1 - \rho^2| = -2\int \delta_0(\rho^1_t - \rho^2_t)(\nabla\rho^1 - \nabla\rho^2) \cdot (A(\rho^1) - A(\rho^2)) = 0.$$

Nonlinear transport: we have the "transport" equation, for $A = (A_1, \ldots, A_d)$,

$$\partial_t \rho + \nabla \cdot (A(\rho)) = \partial_t \rho + \sum_{i=1}^{d} A_i'(\rho)\partial_i \rho = 0.$$

Method of characteristics: we formally solve the ODE, for $A' = (A_1', \ldots, A_d')$,

$$\dot{X}_t^x = A'(\rho(X_t^x, t)) \text{ with } X_0^x = x,$$

and observe that, on the trajectories X_t^x,

$$\begin{aligned}\partial_t \rho(X_t^x, t) &= \partial_t \rho(X_t^x, t) + \dot{X}_t^x \cdot \nabla \rho(X_t^x, t) \\ &= \partial_t \rho(X_t^x, t) + A'(\rho(X_t^x, t)) \cdot \nabla \rho(X_t^x, t) = 0.\end{aligned}$$

The solution is constant on the trajectories X_t^x,

$$\rho(X_t^x, t) = \rho_0(x) \text{ and } \dot{X}_t^x = A'(\rho_0(x)) \text{ with } X_0^x = x.$$

Representation formula: we have, for the inverse characteristics Y_t^x,

$$\rho(x,t) = \rho_0(Y_t^x) \text{ on } \mathbb{R}^d \times [0, \infty),$$

which is a local in time smooth solution [Evans; 2010].

Burger's equation: in one-dimension,

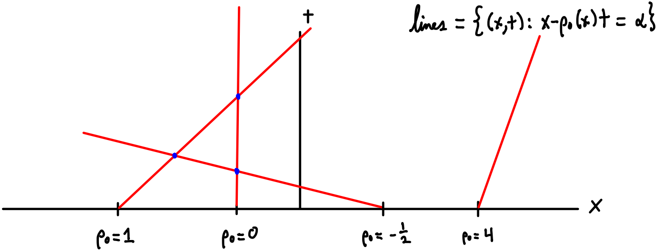
$$\partial_t \rho + \partial_x \left(\frac{1}{2} \rho^2 \right) = \partial_t \rho + \rho \partial_x \rho = 0.$$

The characteristics: In this case, $A'(\rho) = \rho$ and the characteristic equations are

$$\dot{X}_t^x = A'(\rho_0(x)) = \rho_0(x) \text{ with } X_t^x = x + \rho_0(x)t.$$

We therefore have, for the inverse characteristics Y_t^x,

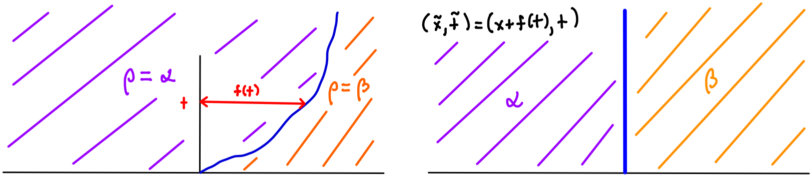
$$Y_t^x = x - \rho_0(x)t \text{ and } \rho(x,t) = \rho_0(x - \rho_0(x)t).$$



Scalar conservation law: in one-dimension,

$$\partial_t \rho + \nabla \cdot (A(\rho)) = 0,$$

and there exists a *shock* on graph $(f(t), t)$.



Rankine-Hugoniot condition: since $A(\rho(x + f(t), t))$ is constant in time,

$$\partial_t \Big(A(\rho(x + f(t), t)) \Big) = \partial_x (A(\rho)) f'(t) + \partial_t (A(\rho)) = 0,$$

and from the equation

$$\partial_t \left(\rho - \frac{1}{f'(t)} A(\rho) \right) = 0.$$

Hence, by equating the jump,

$$f'(t) = \frac{A(\beta) - A(\alpha)}{\beta - \alpha}.$$

Burger's equation: in one-dimension,

$$\partial_t \rho + \rho \partial_x \rho = 0,$$

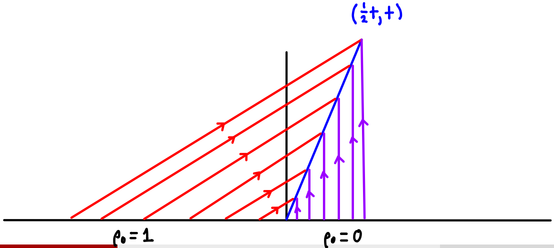
and we have, for the inverse characteristics Y_t^x,

$$Y_t^x = x - \rho_0(x)t \text{ and } \rho(x,t) = \rho_0(x - \rho_0(x)t).$$

Rankine-Hugoniot condition: for $\rho_0(x) = 1$ if $x \leq 0$ and $\rho_0(x) = 0$ if $x > 0$,

$$f'(t) = \frac{A(0) - A(1)}{0 - 1} = \frac{1}{2},$$

for the shock line $(f(t), t)$ with $t \in [0, \infty)$.



Burger's equation: in one-dimension,

$$\partial_t \rho + \rho \partial_x \rho = 0,$$

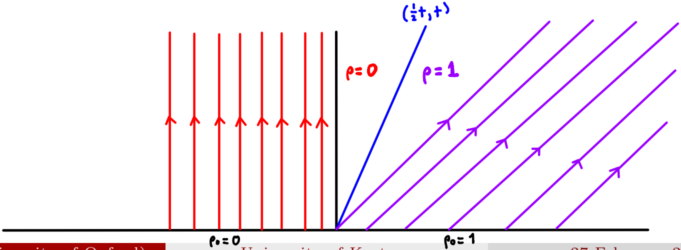
and we have, for the inverse characteristics Y_t^x,

$$Y_t^x = x - \rho_0(x)t \;\text{ and }\; \rho(x,t) = \rho_0(x - \rho_0(x)t).$$

Rankine-Hugoniot condition: for $\rho_0(x) = 0$ if $x \leq 0$ and $\rho_0(x) = 1$ if $x > 0$,

$$f'(t) = \frac{A(1) - A(0)}{1 - 0} = \frac{1}{2}.$$

Shock: a weak solution is $\rho(x,t) = 0$ if $x \leq \frac{1}{2}t$ and $\rho(x,t) = 1$ if $x > \frac{1}{2}t$.



Burger's equation: in one-dimension,

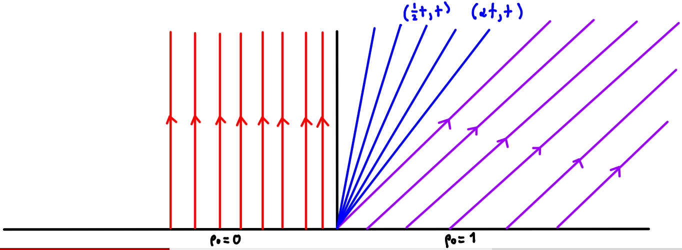
$$\partial_t \rho + \rho \partial_x \rho = 0,$$

with $\rho_0(x) = 0$ if $x \leq 0$ and $\rho_0(x) = 1$ if $x > 0$.

Rarefaction wave: we define $\rho(x,t) = 0$ if $x \leq 0$, $\rho(x,t) = 1$ if $x \geq t$, and $\rho(x,t) = \alpha$ on the line $(\alpha t, t)$ for $\alpha \in (0,1)$. Since ρ is constantly α on the line $(\alpha t, t)$,

$$0 = \partial_t(\rho(\alpha t, t)) = \partial_t \rho(\alpha t, t) + \alpha \partial_x \rho(\alpha t, t) = \partial_t \rho + \rho \partial_x \rho.$$

Infinitely many weak solutions: shock vs. rarefaction wave vs. combination.



The regularized equation: for $\eta \in (0,1)$, the equation

$$\partial_t \rho^\eta - \eta \Delta \rho^\eta + \nabla \cdot (A(\rho^\eta)) = 0 \text{ in } \mathbb{R}^d \times (0,\infty) \text{ with } \rho^\eta(\cdot,0) = \rho_0,$$

is classically well-posed for general A.

A selection principle as $\eta \to 0$: if S is convex, for the composition $S(\rho^\eta)$,

$$\begin{aligned}\partial_t S(\rho^\eta) &= \eta S'(\rho^\eta)\Delta\rho^\eta - S'(\rho^\eta)\nabla \cdot A(\rho^\eta) \\ &= \eta \Delta S(\rho^\eta) - S'(\rho^\eta)\nabla \cdot A(\rho^\eta) - \eta S''(\rho^\eta)|\nabla \rho^\eta|^2 \\ &\leq \eta \Delta S(\rho^\eta) - S'(\rho^\eta)\nabla \cdot A(\rho^\eta).\end{aligned}$$

The entropy inequality: arguing that, for all smooth and compactly supported ψ,

$$\lim_{\eta \to 0} \int \eta \Delta S(\rho^\eta)\psi = \lim_{\eta \to 0} \int \eta S(\rho^\eta)\Delta\psi = 0,$$

if $\rho^\eta \to \rho$ as $\eta \to 0$ then, for all convex S,

$$\partial_t S(\rho) + S'(\rho)\nabla \cdot A(\rho) = \partial_t S(\rho) + \nabla \cdot \beta(\rho) \leq 0,$$

for $\beta = (\beta_1, \ldots, \beta_d)$ satisfying $\beta_i' = S' A_i'$.

II. Entropy solutions

The regularized equation: for $\eta \in (0,1)$, the equation

$$\partial_t \rho^\eta - \eta \Delta \rho^\eta + \nabla \cdot (A(\rho^\eta)) = 0 \text{ in } \mathbb{R}^d \times (0,\infty) \text{ with } \rho^\eta(\cdot, 0) = \rho_0.$$

A particular choice of entropy: for $K \in \mathbb{R}$ we formally differentiate

$$\begin{aligned}\partial_t |\rho^\eta - K| &= \eta \mathrm{sgn}(\rho^\eta - K)\Delta\rho^\eta - \mathrm{sgn}(\rho^\eta - K)\nabla \cdot (A(\rho^\eta))\\ &= \eta\Delta|\rho^\eta - K| - \mathrm{sgn}(\rho^\eta - K)\nabla \cdot (A(\rho^\eta) - A(K)) - 2\eta\delta_0(\rho^\eta - K)|\nabla\rho^\eta|^2\\ &\leq \eta\Delta|\rho^\eta - K| - \nabla \cdot \big(\mathrm{sgn}(\rho^\eta - K)(A(\rho^\eta) - A(K))\big)\\ &\quad + 2\delta_0(\rho^\eta - K)\nabla\rho^\eta \cdot (A(\rho^\eta) - A(K))\\ &= \eta\Delta|\rho^\eta - K| - \nabla \cdot \big(\mathrm{sgn}(\rho^\eta - K)(A(\rho^\eta) - A(K))\big).\end{aligned}$$

Passing $\eta \to 0$ as before, if $\rho^\eta \to \rho$,

$$\partial_t|\rho - K| + \nabla \cdot \big(\mathrm{sgn}(\rho - K)(A(\rho) - A(K))\big) \leq 0.$$

An entropy solution: we say that ρ is an *entropy solution* of the equation

$$\partial_t \rho + \nabla \cdot (A(\rho)) = 0,$$

if for every $K \in \mathbb{R}$, distributionally on $\mathbb{R}^d \times [0,\infty)$,

$$\partial_t|\rho - K| + \nabla \cdot \big(\mathrm{sgn}(\rho - K)(A(\rho) - A(K))\big) \leq 0.$$

II. Entropy solutions

An entropy solution: we say that ρ is an *entropy solution* of the equation

$$\partial_t \rho + \nabla \cdot (A(\rho)) = 0,$$

if for every $K \in \mathbb{R}$, distributionally on $\mathbb{R}^d \times [0, \infty)$,

$$\partial_t |\rho - K| + \nabla \cdot \big(\mathrm{sgn}(\rho - K)(A(\rho) - A(K))\big) \leq 0.$$

Uniqueness of entropy solutions: following the variable doubling technique of [Kružkov; 1970], we define $\Phi(x, y, s, t) = |u(x,t) - v(y,s)|$ and observe that

$$\partial_t \Phi \leq -\nabla_x \cdot (\mathrm{sgn}(u - v)(A(u) - A(v))) \text{ and } \partial_s \Phi \leq -\nabla_y \cdot (\mathrm{sgn}(v - u)(A(v) - A(u))).$$

That is, $(\partial_t + \partial_s)\Phi \leq -(\nabla_x + \nabla_y) \cdot (\mathrm{sgn}(u - v)(A(u) - A(v)))$.

Convolution trick: let $\kappa^\varepsilon = \kappa^\varepsilon_d(x - y)\kappa^\varepsilon_1(t - s)$ for standard scale ε convolution kernels κ^ε_d on $\mathbb{R}^d$ and κ^ε_1 on $\mathbb{R}$, for which $(\partial_t + \partial_s)\kappa^\varepsilon = (\nabla_x + \nabla_y)\kappa^\varepsilon = 0$.

L^1**-contraction**: we conclude that, for every $\varepsilon \in (0, 1)$,

$$(\partial_t + \partial_s) \int_{(\mathbb{R}^d)^2} \Phi(x, y, t, s)\kappa^\varepsilon(x, y, t, s) \leq -\int_{(\mathbb{R}^d)^2} \Phi(x, y, t, s)(\nabla_x + \nabla_y)\kappa^\varepsilon = 0,$$

which, after taking $\varepsilon \to 0$, yields $\partial_t \int_{\mathbb{R}^d} |u - v| \leq 0$ and $\|u - v\|_{L^1} \leq \|u_0 - v_0\|_{L^1}$.

Uniqueness of entropy solutions [Kružkov; 1970]

Let $A\colon \mathbb{R}^d \to \mathbb{R}^d$ be locally Lipschitz continuous and let $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then there exists a unique entropy solution of the equation

$$\partial_t \rho + \nabla \cdot A(\rho) = 0 \;\text{ in }\; \mathbb{R}^d \times (0,\infty) \;\text{ with }\; \rho(\cdot,0) = \rho_0.$$

Furthermore, if ρ^1 and ρ^2 are two solutions with initial data ρ_0^1 and ρ_0^2,

$$\sup_{t\in[0,\infty)} \left\|\rho^1 - \rho^2\right\|_{L^1(\mathbb{R}^d)} \leq \left\|\rho_0^1 - \rho_0^2\right\|_{L^1(\mathbb{R}^d)}.$$

Burger's equation: in one-dimension,

$$\partial_t \rho + \rho \partial_x \rho = 0,$$

with $\rho_0(x) = 0$ if $x \leq 0$ and $\rho_0(x) = 1$ if $x > 0$.

The entropy solution: the rarefaction wave is a continuous and smooth (away the lines $\{x = 0\}$ and $\{x = t\}$) solution, and is hence the entropy solution.

Burger's equation: $\partial_t \rho + \rho \partial_x \rho = 0$ with $\rho_0(x) = 0$ if $x \leq 0$ and $\rho_0(x) = 1$ if $x > 0$.

Shock: a weak solution is $\rho(x,t) = 0$ if $x \leq \frac{1}{2}t$ and $\rho(x,t) = 1$ if $x > \frac{1}{2}t$.

Failure of the entropy condition: formally since $|\rho - K|(x + \frac{1}{2}t, t)$ is constant in time,

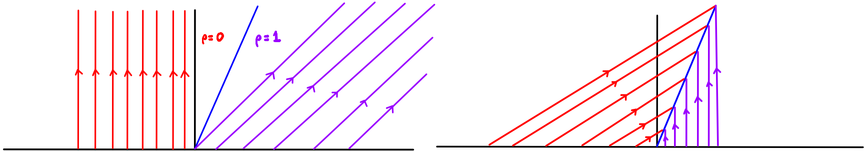
$$\partial_t \left(|\rho - K|(x + \frac{1}{2}t, t)\right) = \frac{1}{2}\partial_x |\rho - K| + \partial_t |\rho - K| = 0,$$

and so the entropy condition becomes

$$\partial_t |\rho - K| + \partial_x (\mathrm{sgn}(\rho - K)(\frac{\rho^2}{2} - \frac{K^2}{2})) = \frac{1}{2}\partial_x \left(\mathrm{sgn}(\rho - K)(\rho^2 - K^2) - |\rho - K|\right) \leq 0.$$

Equating the jumps requires $1 - 2K^2 \leq 1 - 2K$ for $K \in [0,1]$, a contradiction.

If $\rho_0(x) = 1$ for $x \leq 0$ and $\rho_0(x) = 0$ for $x > 0$, the condition is $2K^2 - 1 \leq 2K - 1$.



Degenerate parabolic-hyperbolic PDE: for a nondecreasing Φ,

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (A(\rho, x)),$$

and the regularization

$$\partial_t \rho^\eta = \Delta\Phi(\rho^\eta) + \eta\Delta\rho^\eta - \nabla \cdot (A(\rho^\eta, x)).$$

The entropy formulation: for a smooth, convex S,

$$\begin{aligned}\partial_t S(\rho^\eta) &= S'(\rho^\eta)\Delta\Phi(\rho^\eta) + \eta S'(\rho^\eta)\Delta\rho^\eta - S'(\rho^\eta)\nabla \cdot (A(\rho^\eta, x)) \\ &= \nabla \cdot (\Phi'(\rho^\eta)\nabla S(\rho^\eta)) + \eta\Delta S(\rho^\eta) - S'(\rho^\eta)\nabla \cdot (A(\rho^\eta, x)) \\ &\quad - S''(\rho^\eta)\Phi'(\rho^\eta)|\nabla\rho^\eta|^2 - \eta S''(\rho^\eta)|\nabla\rho^\eta|^2.\end{aligned}$$

If $\rho^\eta \to \rho$ strongly and $\nabla\rho^\eta \rightharpoonup \nabla\rho$ weakly as $\eta \to 0$,

$$S''(\rho)\Phi'(\rho)|\nabla\rho|^2 \leq \liminf_{\eta\to 0} S''(\rho^\eta)\Phi'(\rho^\eta)|\nabla\rho^\eta|^2 \ \text{ in the sense of measures,}$$

and

$$\partial_t S(\rho) \leq \nabla \cdot (\Phi'(\rho)\nabla S(\rho)) - S'(\rho)\nabla \cdot (A(\rho, x)) - S''(\rho)\Phi'(\rho)|\nabla\rho|^2.$$

Entropy solutions: an *entropy solution* of the equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (A(\rho, x)),$$

is a function ρ that satisfies, distributionally for every convex S,

$$\partial_t S(\rho) \le \nabla \cdot (\Phi'(\rho)\nabla S(\rho)) - S'(\rho)\nabla \cdot (A(\rho, x)) - S''(\rho)\Phi'(\rho)|\nabla\rho|^2.$$

The porous media equation: in the case $\partial_t \rho = \Delta\rho^{[m]}$ for $\xi^{[m]} = \xi|\xi|^{m-1}$,

$$\begin{aligned}\partial_t \int |u - v| &= \int \mathrm{sgn}(u - v)\Delta\big(u^{[m]} - v^{[m]}\big) \\ &= \int \mathrm{sgn}\big(u^{[m]} - v^{[m]}\big)\Delta\big(u^{[m]} - v^{[m]}\big) \\ &= -2\int \delta_0\big(u^{[m]} - v^{[m]}\big)|\nabla u^{[m]} - \nabla v^{[m]}|^2 \le 0.\end{aligned}$$

- formally requires H^1-regularity of $u^{[m]}$ and therefore $\rho_0 \in L^{m+1}$
- the entropy inequality requires convex S
- renormalized entropy solutions [Bénilan, Carrillo, Wittbold; 2000]

III. The kinetic formulation

The regularized equation: for the equation

$$\partial_t \rho^\eta = \Delta\Phi(\rho^\eta) + \eta\Delta\rho^\eta - \nabla\cdot(A(\rho^\eta, x)),$$

we have, for a smooth S,

$$\partial_t S(\rho^\eta) = \nabla\cdot(\Phi'(\rho^\eta)\nabla S(\rho^\eta)) + \eta\Delta S(\rho^\eta) - S'(\rho^\eta)\nabla\cdot(A(\rho^\eta, x)) \\ - S''(\rho^\eta)\Phi'(\rho^\eta)|\nabla\rho^\eta|^2 - \eta S''(\rho^\eta)|\nabla\rho^\eta|^2.$$

For a smooth test function ψ and $U \subseteq \mathbb{R}^d$,

$$\partial_t \int_U S(\rho^\eta)\psi = -\int_U \Phi'(\rho^\eta)S'(\rho^\eta)\nabla\psi\cdot\nabla\rho^\eta + \eta\int_U S(\rho^\eta)\Delta\psi \\ -\int_U \psi S'(\rho^\eta)\nabla\cdot(A(\rho^\eta, x)) - \int_U \psi S''(\rho^\eta)\Phi'(\rho^\eta)|\nabla\rho^\eta|^2 - \int_U \psi S''(\rho^\eta)\eta|\nabla\rho^\eta|^2,$$

we "factor out" the dependence on the "test function" $S'(\xi)\psi(x)$,

$$\int_U \Phi'(\rho^\eta)S'(\rho^\eta)\nabla\psi\cdot\nabla\rho^\eta = \int_{\mathbb{R}}\int_U \Phi'(\xi)S'(\xi)\nabla\psi\cdot\delta_0(\xi-\rho^\eta)\nabla\rho^\eta.$$

The regularized equation: for the equation

$$\partial_t \rho^\eta = \Delta\Phi(\rho^\eta) + \eta\Delta\rho^\eta - \nabla\cdot(A(\rho^\eta, x)),$$

we have, for a smooth S and ψ and $\delta_{\rho^\eta} = \delta_0(\xi - \rho^\eta)$,

$$\begin{aligned}\int_{\mathbb{R}}\int_U S'(\xi)\psi\delta_{\rho^\eta}\partial_t\rho^\eta = &-\int_{\mathbb{R}}\int_U \Phi'(\xi)S'(\xi)\nabla\psi\cdot\delta_{\rho^\eta}\nabla\rho^\eta + \eta\int_U S(\rho^\eta)\Delta\psi \\ &-\int_{\mathbb{R}}\int_U \psi S'(\xi)(\partial_\xi A)(x,\xi)\cdot\delta_{\rho^\eta}\nabla\rho^\eta - \int_{\mathbb{R}}\int_U \psi S'(\xi)(\nabla\cdot A)(x,\xi)\delta_{\rho^\eta} \\ &-\int_{\mathbb{R}}\int_U \psi S''(\xi)\delta_{\rho^\eta}\Phi'(\xi)|\nabla\rho^\eta|^2 - \int_{\mathbb{R}}\int_U \psi S''(\xi)\delta_{\rho^\eta}\eta|\nabla\rho^\eta|^2.\end{aligned}$$

The defect measures: we define the *parabolic* and *entropy* defect measures

$$p^\eta = \delta_{\rho^\eta}\Phi'(\xi)|\nabla\rho^\eta|^2 \ \text{ and } \ q^\eta = \delta_{\rho^\eta}\eta|\nabla\rho^\eta|^2.$$

The "energy inequality" for $\partial_t\rho = \Delta\Phi(\rho)$ encodes the nonlinear regularity,

$$\frac{1}{2}\partial_t\left(\int \rho^2\right) + \int \Phi'(\rho)|\nabla\rho|^2 = 0,$$

and the entropy measure is analogous to the "shocks" from before.

The regularized equation: for a smooth S and ψ and $\delta_{\rho^\eta}=\delta_0(\xi-\rho^\eta)$,

$$\int_{\mathbb{R}}\int_U S'(\xi)\psi\delta_{\rho^\eta}\partial_t\rho^\eta = -\int_{\mathbb{R}}\int_U \Phi'(\xi)S'(\xi)\nabla\psi\cdot\delta_{\rho^\eta}\nabla\rho^\eta + \eta\int_U S(\rho^\eta)\Delta\psi$$
$$-\int_{\mathbb{R}}\int_U \psi S'(\xi)(\partial_\xi A)(x,\xi)\cdot\delta_{\rho^\eta}\nabla\rho^\eta - \int_{\mathbb{R}}\int_U \psi S'(\xi)(\nabla\cdot A)(x,\xi)\delta_{\rho^\eta}$$
$$-\int_{\mathbb{R}}\int_U \psi S''(\xi)\,\mathrm{d}p^\eta - \int_{\mathbb{R}}\int_U \psi S''(\xi)\,\mathrm{d}q^\eta,$$

for the parabolic and entropy defect measures

$$p^\eta=\delta_{\rho^\eta}\Phi'(\xi)|\nabla\rho^\eta|^2 \text{ and } q^\eta=\delta_{\rho^\eta}\eta|\nabla\rho^\eta|^2.$$

The kinetic function: the kinetic function $\chi^\eta : U\times\mathbb{R}\times[0,\infty)\to\{-1,0,1\}$ of ρ^η,

$$\chi^\eta(x,\xi,t)=\mathbf{1}_{\{0<\xi<\rho^\eta(x,t)\}}-\mathbf{1}_{\{\rho^\eta(x,t)<\xi<0\}}.$$

We observe the distributional equalities, for $\partial_\xi F(x,\xi)=f(x,\xi)$ and $F(x,0)=0$,

$$\int_{\mathbb{R}}\int_U \chi^\eta(x,\xi,t)\nabla f(x,\xi) = \int_U\int_0^{\rho^\eta}\nabla f(x,\xi) = \int_U (\nabla F)(x,\rho^\eta) = -\int_U f(x,\rho^\eta)\nabla\rho^\eta.$$

That is, $\nabla\chi^\eta=\delta_{\rho^\eta}\nabla\rho^\eta$, $\partial_t\chi^\eta=\delta_{\rho^\eta}\partial_t\rho^\eta$, and $\partial_\xi\chi^\eta=\delta_0-\delta_{\rho^\eta}$.

III. The kinetic formulation

The regularized equation: for a smooth S and ψ and for $(\nabla \cdot A)(x,0) = 0$,

$$\int_{\mathbb{R}}\int_U S'(\xi)\psi\partial_t\chi^\eta = -\int_{\mathbb{R}}\int_U \Phi'(\xi)S'(\xi)\nabla\psi\cdot\nabla\chi + \eta\int_{\mathbb{R}}\int_U \chi^\eta S'(\xi)\Delta\psi$$
$$-\int_{\mathbb{R}}\int_U \psi S'(\xi)(\partial_\xi A)(x,\xi)\cdot\nabla\chi + \int_{\mathbb{R}}\int_U \psi S'(\xi)(\nabla\cdot A)(x,\xi)\partial_\xi\chi$$
$$-\int_{\mathbb{R}}\int_U \psi S''(\xi)\,\mathrm{d}p^\eta - \int_{\mathbb{R}}\int_U \psi S''(\xi)\,\mathrm{d}q^\eta,$$

for the parabolic and entropy defect measures

$$p^\eta = \delta_{\rho^\eta}\Phi'(\xi)|\nabla\rho^\eta|^2 \text{ and } q^\eta = \delta_{\rho^\eta}\eta|\nabla\rho^\eta|^2,$$

and for the kinetic function

$$\chi^\eta(x,\xi,t) = \mathbf{1}_{\{0<\xi<\rho^\eta(x,t)\}} - \mathbf{1}_{\{\rho^\eta(x,t)<\xi<0\}}.$$

That is, for the test function $\phi(x,\xi) = \psi(x)S'(\xi)$,

$$\int_{\mathbb{R}}\int_U \phi\partial_t\chi^\eta = \int_{\mathbb{R}}\int_U \Phi'(\xi)\chi^\eta\Delta\phi + \eta\int_{\mathbb{R}}\int_U \chi^\eta\Delta\phi - \int_{\mathbb{R}}\int_U \phi(\partial_\xi A)(x,\xi)\cdot\nabla\chi^\eta$$
$$+\int_{\mathbb{R}}\int_U \phi(\nabla\cdot A)(x,\xi)\partial_\xi\chi^\eta - \int_{\mathbb{R}}\int_U \partial_\xi\phi\,\mathrm{d}p^\eta - \int_{\mathbb{R}}\int_U \partial_\xi\phi\,\mathrm{d}q^\eta.$$

III. The kinetic formulation

The regularized kinetic equation: for smooth ϕ and $(\nabla \cdot A)(x,0) = 0$,

$$\int_{\mathbb{R}}\int_U \phi\partial_t\chi^\eta = \int_{\mathbb{R}}\int_U \Phi'(\xi)\chi^\eta\Delta\phi + \eta\int_{\mathbb{R}}\int_U \chi^\eta\Delta\phi - \int_{\mathbb{R}}\int_U \phi(\partial_\xi A)(x,\xi)\cdot\nabla\chi^\eta$$
$$+ \int_{\mathbb{R}}\int_U \phi(\nabla\cdot A)(x,\xi)\partial_\xi\chi^\eta - \int_{\mathbb{R}}\int_U \partial_\xi\phi\,\mathrm{d}p^\eta - \int_{\mathbb{R}}\int_U \partial_\xi\phi\,\mathrm{d}q^\eta,$$

or, in the sense of distributions,

$$\partial_t\chi^\eta = \Phi'(\xi)\Delta\chi^\eta + \eta\Delta\chi^\eta - (\partial_\xi A)\cdot\nabla\chi^\eta + (\nabla\cdot A)\partial_\xi\chi^\eta + \partial_\xi p^\eta + \partial_\xi q^\eta.$$

The $\eta \to 0$ limit: suppose that $\rho^\eta \to \rho$, $\nabla\rho^\eta \rightharpoonup \nabla\rho$, $q^\eta \rightharpoonup \tilde{q}$, and $p^\eta \rightharpoonup \tilde{p}$. Then, $\chi^\eta \to \chi$ and, in the sense of measures, the parabolic defect measure p of ρ satisfies

$$p = \delta_\rho\Phi'(\xi)|\nabla\rho|^2 \leq \liminf_{n\to\infty}\delta_{\rho^\eta}\Phi'(\xi)|\nabla\rho^\eta|^2 = \liminf_{n\to\infty} p^\eta = \tilde{p},$$

and we define the nonnegative entropy defect measure $q = \tilde{q} + (\tilde{p} - p)$.

The kinetic equation: we have the kinetic equation, in the sense of distributions,

$$\partial_t\chi = \Phi'(\xi)\Delta\chi - (\partial_\xi A)\cdot\nabla\chi + (\nabla\cdot A)\partial_\xi\chi + \partial_\xi p + \partial_\xi q.$$

III. The kinetic formulation

The kinetic equation: for the original equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (A(\rho, x)),$$

the kinetic formulation is

$$\partial_t \chi = \Phi'(\xi)\Delta\chi - (\partial_\xi A) \cdot \nabla\chi + (\nabla \cdot A)\partial_\xi \chi + \partial_\xi p + \partial_\xi q,$$

for the kinetic function $\chi = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$, for a nonnegative entropy defect measure q, and for the parabolic defect measure

$$p = \delta_0(\xi - \rho)\Phi'(\xi)|\nabla\rho|^2.$$

Measures decay at infinity: for $\phi(\xi) = \mathbf{1}_{\{\xi \geq K\}}$ and $A(x,\xi) = A(\xi)$, since

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d} \chi(x,\xi,t)\phi(\xi) = \int_{\mathbb{T}^d} (\rho - K)_+$$

we have that

$$\int_{\mathbb{T}^d} (\rho(x,T) - K)_+ + p\big(\mathbb{T}^d \times \{K\} \times [0,T]\big) + q\big(\mathbb{T}^d \times \{K\} \times [0,T]\big) = \int_{\mathbb{T}^d} (\rho_0 - K)_+.$$

If $\rho_0 \in L^1$ then $p, q \to 0$ as $|\xi| \to \infty$.

The equation: if $A(x,\xi)=A(\xi)$ we have $\partial_t\chi=\Phi'(\xi)\Delta\chi-(\partial_\xi A)\cdot\nabla\chi+\partial_\xi p+\partial_\xi q$.

Uniqueness of kinetic solutions: let ζ be a smooth cutoff function and let ρ^1 and ρ^2 be two kinetic solutions with kinetic functions χ^1 and χ^2. Then,

$$\int_{\mathbb{T}^d}|\rho^1-\rho^2|=\int_{\mathbb{R}}\int_{\mathbb{T}^d}|\chi^1-\chi^2|^2=\int_{\mathbb{R}}\int_{\mathbb{T}^d}\chi^1\mathrm{sgn}(\xi)+\chi^2\mathrm{sgn}(\xi)-2\chi^1\chi^2,$$

we observe that

$$\begin{aligned}\partial_t\int_{\mathbb{R}}\int_{\mathbb{T}^d}|\chi^1-\chi^2|^2\zeta&=\int_{\mathbb{R}}\int_{\mathbb{T}^d}\left(\partial_t\chi^1\mathrm{sgn}(\xi)+\partial_t\chi^2\mathrm{sgn}(\xi)-2\partial_t\chi^1_t\chi^2_t-2\chi^1_t\partial_t\chi^2_t\right)\\
&=-2\zeta(0)\sum_{i=1}^2\int_{\mathbb{R}}\int_{\mathbb{T}^d}\left(\delta_0\,\mathrm{d}p^i+\delta_0\,\mathrm{d}q^i\right)+2\zeta(0)\sum_{i=1}^2\int_{\mathbb{R}}\int_{\mathbb{T}^d}\left(\delta_0\,\mathrm{d}p^i+\delta_0\,\mathrm{d}q^i\right)\\
&\quad+4\int_{\mathbb{R}}\int_{\mathbb{T}^d}\Phi'(\xi)\delta_{\rho^1}\delta_{\rho^2}\nabla\rho^1\cdot\nabla\rho^2\zeta(\rho^1)\\
&\quad-2\int_{\mathbb{R}}\int_{\mathbb{T}^d}\zeta(\rho^1)\left(\delta_{\rho^2}\,\mathrm{d}p^1+\delta_{\rho^2}\,\mathrm{d}q^1\right)-2\int_{\mathbb{R}}\int_{\mathbb{T}^d}\zeta(\rho^1)\left(\delta_{\rho^1}\,\mathrm{d}p^2+\delta_{\rho^1}\,\mathrm{d}q^2\right)\\
&\quad+\sum_{i=1}^2\int_{\mathbb{R}}\int_{\mathbb{T}^d}(2\chi^{i+1}-1)\zeta'(\xi)(\,\mathrm{d}p^i+\,\mathrm{d}q^i).\end{aligned}$$

Uniqueness of kinetic solutions: we have using the nonnegativity and definitions of the defect measures and Hölder's inequality,

$$\begin{aligned}
\partial_t \int_{\mathbb{R}} \int_{\mathbb{T}^d} |\chi^1 - \chi^2|^2 \zeta &= 4 \int_{\mathbb{R}} \int_{\mathbb{T}^d} \Phi'(\xi) \delta_{\rho^1} \delta_{\rho^2} \nabla \rho^1 \cdot \nabla \rho^2 \zeta(\rho^1) \\
&\quad - 2 \int_{\mathbb{R}} \int_{\mathbb{T}^d} \zeta(\rho^1) \big(\delta_{\rho^2}\, \mathrm{d}p^1 + \delta_{\rho^2}\, \mathrm{d}q^1\big) - 2 \int_{\mathbb{R}} \int_{\mathbb{T}^d} \zeta(\rho^1) \big(\delta_{\rho^1}\, \mathrm{d}p^2 + \delta_{\rho^1}\, \mathrm{d}q^2\big) \\
&\quad + \sum_{i=1}^{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} (2\chi^{i+1} - 1) \zeta'(\xi) (\, \mathrm{d}p^i + \mathrm{d}q^i) \\
&\leq 4 \int_{\mathbb{R}} \int_{\mathbb{T}^d} \Phi'(\xi) \delta_{\rho^1} \delta_{\rho^2} \nabla \rho^1 \cdot \nabla \rho^2 \zeta(\rho^1) - 2 \sum_{i=1}^{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \zeta(\rho^1) \delta_{\rho^1} \delta_{\rho^2} \Phi'(\xi) |\nabla \rho^i|^2 \\
&\quad + \sum_{i=1}^{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} (2\chi^{i+1} - 1) \zeta'(\xi) (\, \mathrm{d}p^i + \mathrm{d}q^i) \\
&\leq \sum_{i=1}^{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} (2\chi^{i+1} - 1) |\zeta'(\xi)| (\, \mathrm{d}p^i + \mathrm{d}q^i).
\end{aligned}$$

The final term vanishes as $\zeta \to 1$ using the vanishing of the defect measures at infinity. We conclude that $\partial_t \int_{\mathbb{T}^d} |\chi^1 - \chi^2|^2 \leq 0$ and, therefore,

$$\int_{\mathbb{T}^d} |\rho^1(x,t) - \rho^2(x,t)| = \int_{\mathbb{R}} \int_{\mathbb{T}^d} |\chi^1 - \chi^2|^2 \leq \int_{\mathbb{T}^d} |\rho_0^1 - \rho_0^2|.$$

Uniqueness of kinetic solutions [Perthame; 1998]

Let $A\colon \mathbb{R}^d \to \mathbb{R}^d$ be locally Lipschitz continuous and let $\rho_0 \in L^1(\mathbb{R}^d)$. Then there exists a unique kinetic solution of the equation

$$\partial_t \rho + \nabla \cdot A(\rho) = 0 \ \text{ in } \ \mathbb{R}^d \times (0, \infty) \ \text{ with } \ \rho(\cdot, 0) = \rho_0.$$

Furthermore, if ρ^1 and ρ^2 are two solutions with initial data ρ_0^1 and ρ_0^2,

$$\sup_{t \in [0,\infty)} \left\| \rho^1 - \rho^2 \right\|_{L^1(\mathbb{R}^d)} \leq \left\| \rho_0^1 - \rho_0^2 \right\|_{L^1(\mathbb{R}^d)}.$$

Viscous Burger's equation: in one-dimension,

$$\partial_t \rho + \partial_x^2 \rho + \partial_x \left(\frac{1}{2} \rho^2 \right) = 0.$$

The entropy formulation is, for every convex S,

$$\partial_t S(\rho) + \partial_x^2 S(\rho) + \rho \partial_x S(\rho) \leq -S''(\rho) |\partial_x \rho|^2,$$

and the kinetic formulation is, for some nonnegative measure q,

$$\partial_t \chi + \partial_x^2 \chi + \xi \partial_x \chi = \partial_\xi q + \partial_\xi \left(\delta_\rho |\partial_x \rho|^2 \right).$$

IV. The skeleton equation

The rate function: for $\rho \in L^1(\mathbb{T}^d \times [0,T])$,

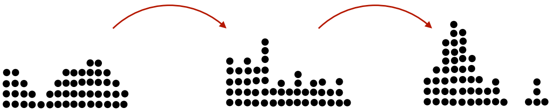
$$I(\rho) = \frac{1}{2} \inf \left\{ \|g\|^2_{L^2} : \partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \right\}.$$

The skeleton equation: for controls $g \in L^2(\mathbb{T}^d \times [0,T])^d$,

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \;\text{ in }\; \mathbb{T}^d \times (0,T) \;\text{ with }\; \rho(\cdot,0) = \rho_0.$$

The porous media case: for nonnegative data and $\Phi(\xi) = \xi^m$,

$$\partial_t \rho = \Delta\rho^m - \nabla \cdot (\rho^{\frac{m}{2}} g) \;\text{ in }\; \mathbb{T}^d \times (0,T) \;\text{ with }\; \rho(\cdot,0) = \rho_0.$$



The skeleton equation: in the porous media case $\partial_t \rho = \Delta \rho^m - \nabla \cdot \left(\rho^{\frac{m}{2}} g\right)$.

Zooming in: consider the rescaling $\tilde{\rho}(x,t) = \lambda \rho(\eta x, \tau t)$ which solves

$$\partial_t \tilde{\rho} = \left(\frac{\tau}{\eta^2 \lambda^{m-1}}\right) \Delta \left(\tilde{\rho}^m\right) - \nabla \cdot \left(\tilde{\rho}^{\frac{m}{2}} \tilde{g}\right),$$

for $\tilde{g}$ defined by

$$\tilde{g}(x,t) = \left(\frac{\tau}{\eta \lambda^{\frac{m}{2}-1}}\right) g(\eta x, \tau t).$$

We preserve the diffusion by fixing $\frac{\tau}{\eta^2 \lambda^{m-1}} = 1$ and for $r \in [1, \infty)$ we preserve the L^r-norm of the initial data by fixing $\lambda = \eta^{\frac{d}{r}}$. Then,

$$\|\tilde{g}\|_{L^p([0,T];L^q(\mathbb{R}^d;\mathbb{R}^d))} = \eta^{1-\frac{d}{p}+\frac{2}{q}+\frac{d}{r}\left(\frac{m}{2}-\frac{m}{q}+\frac{1}{q}\right)} \|g\|_{L^p([0,T];L^q(\mathbb{R}^d;\mathbb{R}^d))}.$$

To ensure that this norm does not diverge as $\eta \to 0$, we require that

$$1 + \frac{d}{r}\left(\frac{m}{2} + \frac{1}{q}\right) \geq \frac{2}{q} + \frac{d}{p} + \frac{dm}{rq}.$$

If $p = q = 2$, we conclude that ${}^{d}/_{2r} \geq {}^{d}/_{2}$ and therefore that $r = 1$.

The skeleton equation: we have

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g).$$

The a priori estimate: test the equation with $\psi(\rho)$ to find, for $\Psi' = \psi$,

$$\partial_t \int \Psi(\rho) + \int \Phi'(\rho)\psi'(\rho)|\nabla\rho|^2 = \int \Phi^{\frac{1}{2}}(\rho)\psi'(\rho) g\cdot\nabla\rho.$$

It follows from Hölder's and Young's inequality that, for every $\varepsilon \in (0,1)$,

$$\partial_t \int \Psi(\rho) + \int \Phi'(\rho)\psi'(\rho)|\nabla\rho|^2 \leq \frac{\varepsilon}{2}\int \Phi(\rho)\psi'(\rho)^2|\nabla\rho|^2 + \frac{1}{2\varepsilon}\int |g|^2.$$

To close the estimate, we require that $\Phi'(\xi)\psi'(\xi) \leq \psi'(\xi)^2\Phi(\xi)$, or that

$$\frac{\Phi'}{\Phi} \leq \psi' \text{ and, hence, we take } \psi(\xi) = \log(\Phi(\xi)).$$

Using the equality $\nabla\Phi^{\frac{1}{2}}(\rho) = \frac{\Phi'(\rho)}{2\Phi^{\frac{1}{2}}(\rho)}\nabla\rho$ we have

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^d} \Psi(\rho) + \int_0^T\int_{\mathbb{T}^d} |\nabla\Phi^{\frac{1}{2}}(\rho)|^2 \lesssim \int_{\mathbb{T}^d}\Psi(\rho_0) + \int_0^T\int_{\mathbb{T}^d}|g|^2.$$

If $\Phi(\xi) = \xi^m$ then $\Psi(\xi) = m(\xi\log(\xi) - \xi)$ is the (negative) physical entropy.

The skeleton equation: we have

$$\partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g) \ \text{ in } \ \mathbb{T}^d\times(0,T) \ \text{ with } \ \rho(\cdot,0)=\rho_0,$$

and we have the a priori estimate

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^d}\Psi_\Phi(\rho) + \int_0^T\int_{\mathbb{T}^d}|\nabla\Phi^{\frac{1}{2}}(\rho)|^2 \lesssim \int_{\mathbb{T}^d}\Psi_\Phi(\rho_0) + \int_0^T\int_{\mathbb{T}^d}|g|^2,$$

for $\Psi'_\Phi(\xi)=\log(\Phi(\xi))$ and for ρ_0 in the entropy space

$$\mathrm{Ent}_\Phi(\mathbb{T}^d) = \left\{\rho\in L^1(\mathbb{T}^d)\colon \rho\geq 0 \ \text{ and } \ \int_{\mathbb{T}^d}\Psi_\Phi(\rho_0)<\infty\right\}.$$

The kinetic form: for the kinetic function χ, for a nonnegative measure q,

$$\begin{aligned}\partial_t\chi &= \Phi'(\xi)\Delta\chi - \frac{\Phi'(\xi)}{2\Phi^{\frac{1}{2}}(\xi)}\nabla\chi\cdot g + \Phi^{\frac{1}{2}}(\xi)(\nabla\cdot g)\partial_\xi\chi + \partial_\xi\left(\delta_\rho\Phi'(\xi)|\nabla\rho|^2\right) + \partial_\xi q \\ &= \Phi'(\xi)\Delta\chi - \partial_\xi\left(\Phi^{\frac{1}{2}}(\xi)\nabla\chi\cdot g\right) + \nabla\cdot\left(\Phi^{\frac{1}{2}}(\xi)g\partial_\xi\chi\right) \\ &\quad + \partial_\xi\left(\delta_\rho\frac{4\Phi(\xi)}{\Phi'(\xi)}|\nabla\Phi^{\frac{1}{2}}(\rho)|^2\right) + \partial_\xi q.\end{aligned}$$

The skeleton equation: we have

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g) \text{ in } \mathbb{T}^d \times (0,T) \text{ with } \rho(\cdot,0)=\rho_0,$$

and the conservative kinetic form

$$\partial_t\chi = \Phi'(\xi)\Delta\chi - \partial_\xi\left(\Phi^{\frac{1}{2}}(\xi)\nabla\chi\cdot g\right) + \nabla\cdot(\Phi^{\frac{1}{2}}(\xi)g\partial_\xi\chi) + \partial_\xi\left(\delta_\rho\frac{4\Phi(\xi)}{\Phi'(\xi)}|\nabla\Phi^{\frac{1}{2}}(\rho)|^2\right) + \partial_\xi q.$$

For every $\phi\in \mathrm{C}_c^\infty(\mathbb{T}^d\times(0,\infty))$, for $\chi_t=\chi(x,\xi,t)$, using $\nabla\chi=\delta_\rho\nabla\rho$, the equality

$$\int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\Phi'(\xi)\chi\Delta\phi = -2\int_0^t\int_{\mathbb{T}^d}\Phi^{\frac{1}{2}}(\rho)\nabla\Phi^{\frac{1}{2}}(\rho)\cdot(\nabla\phi)(x,\rho),$$

and $\partial_\xi\chi=\delta_0-\delta_\rho$ we have that

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{T}^d}\phi\chi_t = {} & \int_{\mathbb{R}}\int_{\mathbb{T}^d}\phi\chi_0 - 2\int_0^t\int_{\mathbb{T}^d}\Phi^{\frac{1}{2}}(\rho)\nabla\Phi^{\frac{1}{2}}(\rho)\cdot(\nabla\phi)(x,\rho)\\ & + \int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\frac{2\Phi(\rho)}{\Phi'(\rho)}\nabla\Phi^{\frac{1}{2}}(\rho)\cdot g + \int_0^t\int_{\mathbb{T}^d}\Phi^{\frac{1}{2}}(\rho)g\cdot(\nabla\phi)(x,\rho)\\ & - \int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\frac{4\Phi(\rho)}{\Phi'(\rho)}|\nabla\Phi^{\frac{1}{2}}(\rho)|^2 - \int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\partial_\xi\phi\,\mathrm{d}q.\end{aligned}$$

If $\Phi(\xi)=\xi^m$ then $\frac{\Phi(\xi)}{\Phi'(\xi)}=m^{-1}\xi$ and the products are not integrable.

The skeleton equation: the kinetic formulation, for $\Phi(\xi)=\xi^M$,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{T}^d}\phi\chi_t = \int_{\mathbb{R}}\int_{\mathbb{T}^d}\phi\chi_0 - 2\int_0^t\int_{\mathbb{T}^d}\rho^{\frac{m}{2}}\nabla\rho^{\frac{m}{2}}\cdot(\nabla\phi)(x,\rho) \\ + \frac{2}{m}\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\rho\nabla\rho^{\frac{m}{2}}\cdot g + \int_0^t\int_{\mathbb{T}^d}\rho^{\frac{m}{2}}g\cdot(\nabla\phi)(x,\rho) \\ - \frac{4}{m}\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\rho|\nabla\rho^{\frac{m}{2}}|^2 - \int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\partial_\xi\phi\,\mathrm{d}q.\end{aligned}$$

The cutoff: if ζ_M is a smooth cutoff of $[M^{-1},M]$ on $[(2M)^{-1},M+1]$,

$$\begin{aligned}|\frac{4}{m}\int_0^t\int_{\mathbb{T}^d}\zeta_M'(\rho)\rho|\nabla\rho^{\frac{m}{2}}|^2| \lesssim \int_0^t\int_{\mathbb{T}^d}\mathbf{1}_{\{(2M)^{-1}<\rho<M^{-1}\}}|\nabla\rho^{\frac{m}{2}}|^2 \\ + (M+1)\int_0^t\int_{\mathbb{T}^d}\mathbf{1}_{\{M<\rho<M+1\}}|\nabla\rho^{\frac{m}{2}}|^2.\end{aligned}$$

The decay of the parabolic defect measure: by dominated convergence

$$\lim_{M\to\infty}\int_0^t\int_{\mathbb{T}^d}\mathbf{1}_{\{(2M)^{-1}<\rho<M^{-1}\}}|\nabla\rho^{\frac{m}{2}}|^2=0,$$

and, essentially by the fact that $\sum_n a_n<\infty$ implies $\liminf_n(na_n)=0$,

$$\liminf_{M\to\infty}(M+1)\int_0^t\int_{\mathbb{T}^d}\mathbf{1}_{\{M<\rho<M+1\}}|\nabla\rho^{\frac{m}{2}}|^2=0.$$

Renormalized kinetic solutions [F., Gess; 2022]

A function $\rho \in \mathrm{C}([0,T];L^1(\mathbb{T}^d))$ is a stochastic kinetic solution of the equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g) \text{ in } \mathbb{T}^d\times(0,T) \text{ with } \rho(\cdot,0)=\rho_0,$$

for $\rho_0 \in \mathrm{Ent}_\Phi(\mathbb{T}^d)$ if $\Phi^{\frac{1}{2}}(\rho)\in L^2([0,T];H^1(\mathbb{T}^d))$ and, for every $\phi\in \mathrm{C}_c^\infty(\mathbb{T}^d\times(0,\infty))$,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{T}^d}\phi\chi_t = \int_{\mathbb{R}}\int_{\mathbb{T}^d}\phi\chi_0 &- 2\int_0^t\int_{\mathbb{T}^d}\Phi^{\frac{1}{2}}(\rho)\nabla\Phi^{\frac{1}{2}}(\rho)\cdot(\nabla\phi)(x,\rho)\\ &+\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\frac{2\Phi(\rho)}{\Phi'(\rho)}\nabla\Phi^{\frac{1}{2}}(\rho)\cdot g + \int_0^t\int_{\mathbb{T}^d}\Phi^{\frac{1}{2}}(\rho)g\cdot(\nabla\phi)(x,\rho)\\ &-\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\frac{4\Phi(\rho)}{\Phi'(\rho)}|\nabla\Phi^{\frac{1}{2}}(\rho)|^2.\end{aligned}$$

Uniqueness and existence of kinetic solutions [F., Gess; 2022]

Assume that $\Phi(0)=0$, that $\Phi'>0$ on $(0,\infty)$, that Φ' is locally $1/2$-Hölder continuous on $(0,\infty)$, and $\sup_{0<\xi\leq M}|\frac{\Phi(\xi)}{\Phi'(\xi)}|\leq cM$. Then renormalized kinetic solutions are unique. Existence under general assumptions including $\Phi(\xi)=\xi^m$ for $m\in(0,\infty)$.

IV. The skeleton equation

A weak solution: a *weak solution* of the equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g) \ \text{ in } \ \mathbb{T}^d\times(0,T) \ \text{ with } \ \rho(\cdot,0)=\rho_0,$$

for $\rho_0 \in \mathrm{Ent}_\Phi(\mathbb{T}^d)$ is a function $\rho\in \mathrm{C}([0,T];L^1(\mathbb{T}^d))$ that satisfies

$$\Phi^{\frac{1}{2}}(\rho)\in L^2([0,T];H^1(\mathbb{T}^d)),$$

and, for every $\phi\in \mathrm{C}^\infty(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d}\rho_t\phi = \int_{\mathbb{T}^d}\rho_0\phi - 2\int_0^t\int_{\mathbb{T}^d}\Phi^{\frac{1}{2}}(\rho)\nabla\Phi^{\frac{1}{2}}(\rho)\cdot\nabla\phi + \int_0^t\int_{\mathbb{T}^d}\Phi^{\frac{1}{2}}(\rho)g\cdot\nabla\phi.$$

Deriving the kinetic form: for $\partial_\xi\Psi(x,\xi)=\psi(x,\xi)$, for $\rho^\varepsilon=(\rho*\kappa^\varepsilon)$,

$$\begin{aligned}\partial_t\int\Psi(x,\rho^\varepsilon) &= \int\psi(x,\rho^\varepsilon)\partial_t\rho^\varepsilon\\ &= -2\int(\nabla\psi)(x,\rho^\varepsilon)\cdot(\Phi^{\frac{1}{2}}(\rho)\nabla\Phi^{\frac{1}{2}}(\rho))^\varepsilon - \int(\nabla\psi)(x,\rho^\varepsilon)\cdot(\Phi^{\frac{1}{2}}(\rho)g)^\varepsilon\\ &\quad - 2\int_{\mathbb{T}^d}(\partial_\xi\psi)(x,\rho^\varepsilon)\nabla\rho^\varepsilon\cdot(\Phi^{\frac{1}{2}}(\rho)\nabla\Phi^{\frac{1}{2}}(\rho))^\varepsilon - \int(\partial_\xi\psi)(x,\rho^\varepsilon)\nabla\rho^\varepsilon\cdot(\Phi^{\frac{1}{2}}(\rho)g)^\varepsilon.\end{aligned}$$

Here $\nabla\phi^\varepsilon$ is not defined and $(\Phi^{\frac{1}{2}}(\rho)\nabla\Phi^{\frac{1}{2}}(\rho))^\varepsilon$ converges essentially in L^1.

IV. The skeleton equation

Equivalence of weak and renormalized kinetic solutions [F., Gess; 2022]

Under general assumptions including $\Phi(\xi) = \xi^m$ for every $m \in [1, \infty)$, a function $\rho \in \mathrm{C}([0,T]; L^1(\mathbb{T}^d))$ that satisfies $\Phi^{\frac{1}{2}}(\rho) \in L^2([0,T]; H^1(\mathbb{T}^d))$ is a renormalized kinetic solution of

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ \rho(\cdot, 0) = \rho_0,$$

for $\rho_0 \in \mathrm{Ent}_\Phi(\mathbb{T}^d)$, if and only if ρ is a weak solution.

- equivalence of renormalized and weak solutions [Ambrosio; 2004], [DiPerna, Lions; 1989]
- strong continuity with respect to weak convergence of the control

Weak-strong continuity [F., Gess; 2022]

If ρ_n are solutions of the skeleton equation with controls $g_n \rightharpoonup g$ and initial data $\rho_0^n \rightharpoonup \rho_0$ then $\rho_n \to \rho$ for ρ the solution of the skeleton equation with control g and initial data ρ_0.

IV. References

L. Ambrosio
Transport equation and Cauchy problem for BV vector fields.
Invent. Math. 158(2): 227-260, 2004.

P Bénilan and J. Carrillo and P. Wittbold
Renormalized entropy solutions of scalar conservation laws.
Annali della Scuola Normale Superiore di Pisa - Classe di Scienze 4(29), 313-327, 2000.

R.J. DiPerna and P.-L. Lions
Ordinary differential equations, transport theory and Sobolev spaces.
Invent. Math. 98(3): 511-547, 1989.

L.C. Evans
Partial Differential Equations, 2nd edn. Graduate Studies in Mathematics.
American Mathematical Society, Providence, RI vol. 19, p. 749, 2010.

B. Fehrman and B. Gess
Non-equilibrium large deviations and parabolic-hyperbolic PDE with irregular drift.
arXiv:1910.11860, 2022.

S.N. Kružkov
First Order Quasilinear Equations in Several Independent Variables.
Mathematics of the USSR-Sbornik 10, 217-243, 1970.

B. Perthame
Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure.
J. Math. Pures Appl. 77(10), 1055-1064, 1998.