PERIODICITY FOR THE $\mathbb{Z}/p^r$-HOMOLOGY OF CYCLIC COVERS OF KNOTS AND $\mathbb{Z}$-HOMOLOGY CIRCLES

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ABSTRACT. We consider the sequence of finite branched cyclic covers of a homology sphere with branching locus a codimension-two homology sphere and show that the first homology with coefficients in $\mathbb{Z}/p^r$ of this sequence of manifolds is periodic. We also establish that the $\mathbb{Z}/p^r$-homology of the finite (unbranched) cyclic covers of any integral homology circle is periodic in every dimension.

§0 INTRODUCTION

In [D] Dellomo proves a lemma about the mod-p^r homology of the branched cyclic covers of S^3 associated to a knot which says that for all multiples mn of some integer n the mn-fold branched cyclic covers have first homology with coefficients in $\mathbb{Z}/p^r$ isomorphic to $\bigoplus_{i=1}^{d_p} \mathbb{Z}/p^r$, where d_p is the degree of the Alexander polynomial of the knot with coefficients reduced modulo p. Although it is not mentioned in that paper, it follows from his proof of the lemma that the sequence of first homology groups with coefficients in $\mathbb{Z}/p^r$ is in fact periodic (as Abelian groups), that is, letting M_k denote the k-fold branched cyclic cover, $H_1(M_{k+n};\mathbb{Z}/p^r) \cong H_1(M_k;\mathbb{Z}/p^r)$ for every k (n is as above). Hillman gave a simplified proof of this lemma [H], which added the feature that for every m, $H_1(M_{mn};\mathbb{Z}/p^r)$ and $H_1(M_n;\mathbb{Z}/p^r)$ are isomorphic as modules over the ring of integral Laurent polynomials, $\mathbb{Z}[t,t^{-1}]$, however, overall periodicity does not seem to be immediate from this proof. In §1 of this paper we offer a yet simpler proof of the more general result that the 1-dimensional mod-p^r homology of the branched cyclic covers of any homology sphere with branching locus a codimension-two homology sphere is periodic as a sequence of $\mathbb{Z}[t,t^{-1}]$-modules and, moreover, that the isomorphisms are given canonically. Neither the proofs of [D] or [H] apply to non-classical knots. Interestingly, this result implies that the t-actions on any pair $H_1(M_k;\mathbb{Z}/p^r)$ and $H_1(M_{k'};\mathbb{Z}/p^r)$ in the same periodicity class (thus $k-k'$ is divisible by the period n) induced by a generator of the group of covering translations of M_k and $M_{k'}$ respectively, must be identical. We have recently learned that subsequent to our proof of this result (Theorem 1 below), this same theorem was obtained independently by Dan Silver and Susan Williams using techniques developed in their paper [SW].

In §2 we change our object of study to any finite complex having the integral homology of S^1 and show that the mod-p^r homology groups of the finite (unbranched) cyclic covers of such a homology circle are periodic in every dimension. In §3 we return to the branched cyclic covers of homology spheres in the case $r = 1$ to take a more explicit look afforded by having field coefficients.

Note that since the homology functor is "linear" in the coefficient argument in the sense that $H_*(X_k; G_1 \oplus G_2) \cong H_*(X_k; G_1) \oplus H_*(X_k; G_2)$, all the results mentioned above and in this paper apply to homology with coefficients in any finite Abelian group. Throughout this paper we will work with a fixed prime power p^r, $r \geq 1$.

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§1 Branched Cyclic Covers of Homology Spheres

Let S be an $(m+2)$-dimensional homology sphere ($m \geq 1$) and K be a codimension-two homology sphere in S. Denote the exterior of K in S by X (thus X is the closure in S of the complement of a tubular neighborhood of K). It is easy to see using Alexander duality that X is a homology circle, one consequence of which is that the abelianization map provides a surjection $\pi_1(X) \twoheadrightarrow H_1(X) \cong \mathbb{Z}$, and thence onto the cyclic group $\mathbb{Z}/k$ of any order. Thus, X is endowed with an infinite cyclic covering space X_∞, as well as finite cyclic covers X_k, for all positive k. Note that every X_k arises as a quotient space of X_∞. Let Λ denote the integral Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. The homology groups of all the cyclic covers, $H_*(X_k)$, $1 \leq k \leq \infty$, become modules over Λ by giving t the action induced by a fixed generator of the group of covering translations of X_∞.

Now, the k-fold branched cyclic covers of S branched along K can be constructed from the k-fold cyclic cover X_k of X as follows. A tubular neighborhood of K in S is $\nu(K) \cong K \times D^2$ and we have $\partial(X) = \partial(\nu(K)) \cong K \times S^1$. Also, $\partial(X_k) \cong K \times S^1$ and we may take the covering map $X_k \to X$ restricted to the boundary to be given by $(z_1, z_2) \mapsto (z_1, z_2^k) : K \times S^1 \to K \times S^1$ (complex multiplication in the second coordinate). Then the k-fold branched covering M_k of S is obtained by attaching the boundary of $N = K \times D^2$ to the boundary of X_k and extending the covering map to $X_k \bigcup_{K \times S^1} N \to X \bigcup_{K \times S^1} \nu(K)$ by mapping $N \to \nu(K)$ via $(z_1, z_2) \mapsto (z_1, z_2^k/|z_2^k|) : K \times D^2 \to K \times D^2$. Note that if μ is a meridian of $\partial(X_k)$, then attaching N to X_k results in μ bounding a disk, thus killing the homology class $[\mu] \in H_1(M_k)$. Therefore, the inclusions $i : S^1 \hookrightarrow * \times S^1 \subset K \times S^1 \cong \partial(X_k) \subset X_k$ and $j : X_k \hookrightarrow M_k$ induce a natural exact sequence of Λ-modules

$$0 \to H_1(S^1) \xrightarrow{i_*} H_1(X_k) \xrightarrow{j_*} H_1(M_k) \to 0 \tag{1}$$

(the Λ-module structure on $H_1(S^1)$, induced by the infinite cyclic cover $\mathbb{R} \to S^1$, is trivial).

Theorem 1. *There exists an integer n such that for every k, $H_1(M_{k+n}; \mathbb{Z}/p^r)$ and $H_1(M_k; \mathbb{Z}/p^r)$ are naturally isomorphic as Λ-modules.*

Proof. We first relate the homology of the branched cyclic covers to that of the infinite cyclic cover via the following short exact sequence of chain complexes adapted from Milnor [M] (see also [G]):

$$0 \to C_*(X_\infty; \mathbb{Z}/p^r) \xrightarrow{t^k - 1} C_*(X_\infty; \mathbb{Z}/p^r) \xrightarrow{\pi} C_*(X_k; \mathbb{Z}/p^r) \to 0. \tag{2}$$

Combining the resulting long exact sequence in homology with the sequence (1) we get the diagram of Λ-modules

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H_1(S^1;\mathbb{Z}/p^r) & & & & \\
 & & & & \downarrow{\scriptstyle i_*} & & & & \\
\to H_1(X_\infty;\mathbb{Z}/p^r) & \xrightarrow{t^k-1} & H_1(X_\infty;\mathbb{Z}/p^r) & \xrightarrow{\pi_*} & H_1(X_k;\mathbb{Z}/p^r) & \xrightarrow{\partial_*} & H_0(X_\infty;\mathbb{Z}/p^r) \to & & \\
 & & & & \downarrow{\scriptstyle j_*} & & \wr\wr & & \\
 & & & & H_1(M_k;\mathbb{Z}/p^r) & & \mathbb{Z}/p^r & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

where, by looking at the maps on the chain level, we can see that the composition $\partial_* i_*$ is an isomorphism, whence the composition $j_*\pi_*$ is an epimorphism with kernel the image of $t^k - 1$. Thus, we have a natural isomorphism

$$\overline{j_*\pi_*} : \frac{H_1(X_\infty;\mathbb{Z}/p^r)}{(t^k-1)H_1(X_\infty;\mathbb{Z}/p^r)} \xrightarrow{\cong} H_1(M_k;\mathbb{Z}/p^r), \tag{3}$$

and we proceed to establish the result by showing that the Λ-module homomorphism of $H_1(X_\infty;\mathbb{Z}/p^r)$ given by multiplication by $t^k - 1$ is a periodic function of k.

To this end, consider the long exact homology sequence

$$\to H_1(X_\infty;\mathbb{Z}/p) \to H_1(X_\infty;\mathbb{Z}/p^i) \to H_1(X_\infty;\mathbb{Z}/p^{i-1}) \to H_0(X_\infty;\mathbb{Z}/p) \to \tag{4}$$

induced by the short exact sequence of coefficient groups $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^i \to \mathbb{Z}/p^{i-1} \to 0$. Now, $H_1(X_\infty;\mathbb{Z}/p)$ is finite since we know from Milnor [M] that the homology of the infinite cyclic cover of a homology circle with coefficients in a field is finitely generated as a vector space, and so it follows by induction that $H_1(X_\infty;\mathbb{Z}/p^i)$ is finite for all i. Hence, also the group of automorphisms $\mathrm{Aut}(H_1(X_\infty;\mathbb{Z}/p^i))$ is finite, so, in particular, the automorphism given by multiplication by t has finite order for all i; let n be the order of $t \in \mathrm{Aut}(H_1(X_\infty;\mathbb{Z}/p^r))$. Then $t^n = 1$ implying that for any k, $t^{k+n} - 1 = t^k - 1$ as maps of $H_1(X_\infty;\mathbb{Z}/p^r)$. The result then follows from the isomorphism (3). □

Remark. It follows that the (fundamental) period of $H_1(M_k;\mathbb{Z}/p^r)$ is precisely n (the order of t in $\mathrm{Aut}(H_1(X_\infty;\mathbb{Z}/p^r))$) and this is also the first k such that $H_1(M_k;\mathbb{Z}/p^r) \cong H_1(X_\infty;\mathbb{Z}/p^r)$. Moreover, at every multiple mn of the order of t, $H_1(M_{mn};\mathbb{Z}/p^r) \cong H_1(X_\infty;\mathbb{Z}/p^r)$. Now, if K is a knotted S^1 in S^3 we have $H_1(X_\infty)$ is $\mathbb{Z}$-torsion free [C] so that $H_1(X_\infty;\mathbb{Z}/p^r) \cong \bigoplus_{i=1}^{d} \mathbb{Z}/p^r$ where by Lemma 1 (see the end of this section) d is the dimension of $H_1(X_\infty;\mathbb{Z}/p)$ over $\mathbb{Z}/p$. But also $H_1(X_\infty;\mathbb{Z}/p) \cong H_1(X_\infty)/pH_1(X_\infty)$ (see §3) so d is in fact d_p, the degree of the Alexander polynomial of K with coefficients reduced modulo p. Combining all this, we have in the classical case that $H_1(M_{mn};\mathbb{Z}/p^r) \cong \bigoplus_{i=1}^{d_p} \mathbb{Z}/p^r$ for every m; this is the lemma of Dellomo mentioned in the introduction.

Consider now the order of t as a function of r (for a fixed prime p). Let t_i denote t as an automorphism of $H_1(X_\infty;\mathbb{Z}/p^i)$ and n_i denote the corresponding order. Then we have the following regularity exhibited.

Theorem 2. *With the above notation, either* $n_i = n_{i-1}$ *or* $n_i = pn_{i-1}$ $(i \geq 2)$.

Proof. Assume by induction on i that $n_{i-1} = p^{s_i} n_1$ for some $0 \leq s_i \leq i-1$. Of course, this hypothesis does hold for $i = 2$. By the naturality of the Bockstein sequence (4), we have a commutative diagram

$$\begin{array}{ccccc} \to H_1(X_\infty;\mathbb{Z}/p) & \xrightarrow{\phi_*} & H_1(X_\infty;\mathbb{Z}/p^i) & \xrightarrow{\psi_*} & H_1(X_\infty;\mathbb{Z}/p^{i-1})\to \\ {\scriptstyle t_1}\downarrow & & {\scriptstyle t_i}\downarrow & & {\scriptstyle t_{i-1}}\downarrow \\ \to H_1(X_\infty;\mathbb{Z}/p) & \xrightarrow{\phi_*} & H_1(X_\infty;\mathbb{Z}/p^i) & \xrightarrow{\psi_*} & H_1(X_\infty;\mathbb{Z}/p^{i-1})\to \end{array}$$

Let $a \in H_1(X_\infty;\mathbb{Z}/p^i)$ be arbitrary. We want to see that $t_i^{pn_{i-1}}a = a$. $\psi_*(t_i^{n_{i-1}}a) = t_{i-1}^{n_{i-1}}\psi_*(a) = 1\psi_*(a) = \psi_*(a)$ so $t_i^{n_{i-1}}a$ and a differ by something from $H_1(X_\infty;\mathbb{Z}/p)$, say $t_i^{n_{i-1}}a = a + \phi_*(a')$. Using this and the fact that $t_i^{n_{i-1}}\phi_*(a') = \phi_*(t_1^{n_{i-1}}a') = \phi_*(a')$ we have

$$\begin{aligned} t_i^{pn_{i-1}}a = t_i^{(p-1)n_{i-1}}t_i^{n_{i-1}}a &= t_i^{(p-1)n_{i-1}}(a+\phi_*(a')) \\ &= t_i^{(p-2)n_{i-1}}(a+2\phi_*(a')) \\ &\cdots \\ &= a + p\phi_*(a') \\ &= a. \end{aligned}$$

Therefore $n_i | pn_{i-1}$. But we also have $H_1(X_\infty;\mathbb{Z}/p^{i-1}) \cong H_1(X_\infty;\mathbb{Z}/p^i)\otimes\mathbb{Z}/p^{i-1}$ (Lemma 1 below) with $t_{i-1} = t_i \otimes \mathrm{id}$, so certainly $n_{i-1}|n_i$. □

Lemma 1. *For any space* Y, $H_q(Y;\mathbb{Z}/p^r)\otimes\mathbb{Z}/p^{r-1} \cong H_q(Y;\mathbb{Z}/p^{r-1})$.

Proof. This follows from the universal-coefficient theorem since one has for any Abelian group A, $A\otimes\mathbb{Z}/p^r\otimes\mathbb{Z}/p^{r-1} \cong A\otimes\mathbb{Z}/p^{r-1}$ and $\mathrm{Tor}(A,\mathbb{Z}/p^r)\otimes\mathbb{Z}/p^{r-1} \cong \mathrm{Tor}(A,\mathbb{Z}/p^{r-1})$. The second fact is obvious for finitely generated A, and follows for arbitrary A since direct limits commute with the torsion and tensor products. □

If we now make the observation that $H_1(X_\infty;\mathbb{Z}/p) \cong \mathrm{GL}(d,\mathbb{Z}/p)$ for some d, we can produce a list of possible periods for $H_1(X_\infty;\mathbb{Z}/p^r)$ by consulting, for example, [N] which gives formulas for the orders of all elements of $\mathrm{GL}(d,\mathbb{Z}/p)$. (Note that $\mathrm{GL}(d,\mathbb{Z}/p)$ has order $\prod_{i=1}^d p^{i-1}(p^i-1)$.) We will make a hands-on investigation of the periods of $H_1(M_k;\mathbb{Z}/p)$ in §3 of this paper.

§2 Finite Cyclic Covers of Homology Circles

Now let X be any finite complex having the homology of a circle, $H_*(X) \cong H_*(S^1)$.

Theorem 3. *The mod-*p^r *homology of the finite cyclic covers of* X *is periodic, i.e., for every* $q \geq 0$ *there exists an* n_q *such that* $H_q(X_{k+n_q};\mathbb{Z}/p^r) \cong H_q(X_k;\mathbb{Z}/p^r)$ *for every* $k \geq 1$.

Proof. The zeroth homology satisfies this trivially. Fix an arbitrary dimension $q \geq 1$ and a cover of order k. From the long exact sequence in homology

$$\longrightarrow H_q(X_\infty;\mathbb{Z}/p^r) \xrightarrow{t^k-1} H_q(X_\infty;\mathbb{Z}/p^r) \longrightarrow H_q(X_k;\mathbb{Z}/p^r) \longrightarrow$$
$$\longrightarrow H_{q-1}(X_\infty;\mathbb{Z}/p^r) \xrightarrow{t^k-1} H_{q-1}(X_\infty;\mathbb{Z}/p^r) \longrightarrow$$

induced by (2) we extract the short exact sequence

$$0 \to \operatorname{coker}_q(t^k-1) \longrightarrow H_q(X_k;\mathbb{Z}/p^r) \longrightarrow \ker_{q-1}(t^k-1) \to 0, \tag{5}$$

where $(\mathrm{co})\ker_i(t^k-1)$ is the (co)kernel of multiplication by t^k-1 on $H_i(X_\infty;\mathbb{Z}/p^r)$, $i = q, q-1$.

As X is a homology circle, we again have that the automorphism t has finite order in $\operatorname{Aut}(H_i(X_\infty;\mathbb{Z}/p^r))$ for every i; let n be the least common multiple of the orders of $t \in \operatorname{Aut}(H_{q-1}(X_\infty;\mathbb{Z}/p^r))$ and $t \in \operatorname{Aut}(H_q(X_\infty;\mathbb{Z}/p^r))$. Then both $\operatorname{coker}_q(t^{k+n}-1) = \operatorname{coker}_q(t^k-1)$ and $\ker_{q-1}(t^{k+n}-1) = \ker_{q-1}(t^k-1)$ so from (5) we can conclude that $\operatorname{order}(H_q(X_{k+n};\mathbb{Z}/p^r)) = \operatorname{order}(H_q(X_k;\mathbb{Z}/p^r))$ and, in the case $r=1$, $H_q(X_{k+n};\mathbb{Z}/p) \cong H_q(X_k;\mathbb{Z}/p)$.

Proceeding by induction on r, assume $H_q(X_{k+n};\mathbb{Z}/p^{r-1}) \cong H_q(X_k;\mathbb{Z}/p^{r-1})$. Lemma 1 and the induction hypothesis give $H_q(X_{k+n};\mathbb{Z}/p^r) \otimes \mathbb{Z}/p^{r-1} \cong H_q(X_k;\mathbb{Z}/p^r) \otimes \mathbb{Z}/p^{r-1}$, and since we already have that the orders of $H_q(X_{k+n};\mathbb{Z}/p^r)$ and $H_q(X_k;\mathbb{Z}/p^r)$ are the same, Lemma 2 (below) applies to give $H_q(X_{k+n};\mathbb{Z}/p^r) \cong H_q(X_k;\mathbb{Z}/p^r)$. □

Lemma 2. *Let A and B be two finite Abelian p-groups of the same order such that every element of A and B has order at most p^r. If $A \otimes \mathbb{Z}/p^{r-1} \cong B \otimes \mathbb{Z}/p^{r-1}$, then $A \cong B$.*

Proof. Write $A \cong \bigoplus_{i=1}^r (\mathbb{Z}/p^i)^{m_i}$ and $B \cong \bigoplus_{i=1}^r (\mathbb{Z}/p^i)^{m'_i}$. Then

$$A \otimes \mathbb{Z}/p^{r-1} \cong (\mathbb{Z}/p)^{m_1} \oplus \cdots \oplus (\mathbb{Z}/p^{r-2})^{m_{r-2}} \oplus (\mathbb{Z}/p^{r-1})^{m_{r-1}+m_r} \quad \text{and}$$
$$B \otimes \mathbb{Z}/p^{r-1} \cong (\mathbb{Z}/p)^{m'_1} \oplus \cdots \oplus (\mathbb{Z}/p^{r-2})^{m'_{r-2}} \oplus (\mathbb{Z}/p^{r-1})^{m'_{r-1}+m'_r},$$

so under the hypothesis that $A \otimes \mathbb{Z}/p^{r-1} \cong B \otimes \mathbb{Z}/p^{r-1}$ we must have $m_i = m'_i$ for $1 \leq i \leq r-2$ and $m_{r-1}+m_r = m'_{r-1}+m'_r$. Additionally, since $\prod_{i=1}^r (p^i)^{m_i} = \operatorname{order}(A) = \operatorname{order}(B) = \prod_{i=1}^r (p^i)^{m'_i}$, we have $(r-1)m_{r-1} + rm_r = (r-1)m'_{r-1} + rm'_r$, forcing $m_{r-1} = m'_{r-1}$ and $m_r = m'_r$. Therefore $A \cong B$. □

One easily obtains:

Corollary. *The mod-p^r homology of the sequence of branched cyclic covers of a homology sphere branched along a codimension-two homology sphere is periodic in every dimension.*

§3 A Note on Periods in the Case $r = 1$

Now return to the situation where K is a codimension-two homology sphere in a homology sphere S. Then Theorem 1 applies and we have the first homology with coefficients in $\mathbb{Z}/p^r$ of the sequence of branched cyclic covers of S branched along K is periodic. Presently we want to investigate what the period of $H_1(M_k;\mathbb{Z}/p)$ is. Using the isomorphism (3) we study the modules $H_1(M_k;\mathbb{Z}/p)$ by examining the effect of multiplication by t^k-1 on $H_1(X_\infty;\mathbb{Z}/p)$ in a manner similar to Sumner's analysis of $H_1(M_k;\mathbb{C})$ [S]. This produces an explicit formula for the period in this case.

The short exact sequence of coefficient groups $0\to\mathbb{Z}\xrightarrow{p}\mathbb{Z}\to\mathbb{Z}/p\to 0$ gives the long exact sequence in homology,

$$\to H_1(X_\infty)\xrightarrow{p}H_1(X_\infty)\to H_1(X_\infty;\mathbb{Z}/p)\to H_0(X_\infty)\xrightarrow{p}H_0(X_\infty)\to,$$

and since multiplication by p on $H_0(X_\infty)\cong\mathbb{Z}$ is injective, we have $H_1(X_\infty;\mathbb{Z}/p)\cong H_1(X_\infty)/pH_1(X_\infty)$. Thus, we may consider $H_1(X_\infty;\mathbb{Z}/p)$ as a module over $\Lambda_p=\Lambda/p\Lambda$, and since $\Lambda_p\cong\mathbb{Z}/p[t,t^{-1}]$ is a principal ideal domain, $H_1(X_\infty;\mathbb{Z}/p)$ decomposes into a direct sum of cyclic Λ_p-modules, $H_1(X_\infty;\mathbb{Z}/p)\cong\bigoplus_{i=1}^{l}\Lambda_p/(\lambda_i)$ with $(\lambda_1)\subseteq(\lambda_2)\subseteq\cdots\subseteq(\lambda_l)$. We have, moreover, from [M] that in this situation all the λ_i's are nonzero. Then since the representatives $\lambda_1,\ldots,\lambda_l$ are defined only up to multiplication by units in Λ_p, we take each λ_i to be the unique monic polynomial (thus having no negative powers of t) with nonzero constant term in the ideal (λ_i), and we call this preferred generator the i-th invariant factor of the Λ_p-module $H_1(X_\infty;\mathbb{Z}/p)$. Then

$$H_1(M_k;\mathbb{Z}/p)\cong H_1(X_\infty;\mathbb{Z}/p)/(t^k-1)H_1(X_\infty;\mathbb{Z}/p)\cong\bigoplus_{i=1}^{l}\Lambda_p/(\lambda_i,t^k-1).$$

Over an algebraic closure, $\overline{\mathbb{Z}/p}$, the first invariant factor λ_1 splits as, say, $\lambda_1=\prod_{j=1}^{m}(t-\alpha_j)^{e_j}$. For each $j=1,\ldots,m$ let μ_j be the order of α_j (thus we may think of α_j as being a primitive μ_j-th root of unity in $\overline{\mathbb{Z}/p}$), and let h be the least common multiple of $\mu_1,\ldots,\mu_m$. Then, writing $k=up^v$ with $p\nmid u$ we have (in Λ_p) $t^k-1=(t^u-1)^{p^v}$ and so

$$(\lambda_1,t^{up^v}-1)=\Big(\prod_{\{j:\mu_j|u\}}(t-\alpha_j)^{\min(e_j,p^v)}\Big).$$

Thus when v is such that $p^v\geq e_j$ for every $1\leq j\leq m$, then $(\lambda_1,t^{hp^v}-1)=(\lambda_1)$ and similarly for the other λ_i (due to the divisibility conditions), and we have $H_1(M_{hp^v};\mathbb{Z}/p)\cong\bigoplus_{i=1}^{l}\Lambda_p/(\lambda_i)$. Taking v minimal we then have the first occurrence of $H_1(M_k;\mathbb{Z}/p)\cong H_1(X_\infty;\mathbb{Z}/p)$ and therefore hp^v is the period of $H_1(M_k;\mathbb{Z}/p)$. Furthermore, by Theorem 2, the period of $H_1(M_k;\mathbb{Z}/p^r)$ is hp^{v+s} for some s, $0\leq s<r$.

Example. Let K be the knotted S^1 in S^3 enumerated 6_1 in the Alexander-Briggs table. K has $H_1(X_\infty)\cong\Lambda/(2t^2-5t+2)$ with Alexander polynomial $\Delta(t)=2t^2-5t+2$ (see [G]). Notice that on reducing coefficients modulo 3 the polynomial $2t^2-5t+2\mapsto(t+1)^2$ which has a single root of multiplicity two which is the primitive second root of unity. Thus $H_1(X_\infty;\mathbb{Z}/3)\cong\Lambda_3/(t+1)^2$ and we have

$$H_1(M_k;\mathbb{Z}/3)\cong\begin{cases}\Lambda_3/(t+1) & \text{if } k\in(2),\ k\notin(6)\\ \Lambda_3/(t+1)^2 & \text{if } k\in(6)\\ 0 & \text{otherwise}\end{cases}$$

which clearly has the period $6 = 2 \cdot 3^1$.

Now let K' be the knot 9_{46}. K' has the same Alexander polynomial as K although $H_1(X'_\infty) \cong \Lambda/(1-2t) \oplus \Lambda/(2-t)$. Then $H_1(X'_\infty; \mathbb{Z}/3) \cong \Lambda_3/(t+1) \oplus \Lambda_3/(t+1)$ so

$$H_1(M'_k; \mathbb{Z}/3) \cong \begin{cases} \Lambda_3/(t+1) \oplus \Lambda_3/(t+1) & \text{if } k \in (2) \\ 0 & \text{otherwise} \end{cases}$$

which has period 2. (Notice for this knot the annihilator of $H_1(X'_\infty; \mathbb{Z}/3)$ is $(t+1) \subset \Lambda_3$ which is not the reduction modulo 3 of the annihilator $(2t^2 - 5t + 2) \subset \Lambda$ of $H_1(X'_\infty)$.)

Modulo 2 the infinite cyclic covers of both knots have trivial homology. Modulo any other prime $p \neq 2, 3$, $H_1(X_\infty; \mathbb{Z}/p) \cong \Lambda_p/(2t^2 - 5t + 2) \cong \Lambda_p/(1-2t) \oplus \Lambda_p/(2-t)$ since the polynomials $1 - 2t$ and $2 - t$ reduce to nonassociate polynomials in Λ_p. Thus, if $p \neq 2, 3$, we have

$$H_1(M_k; \mathbb{Z}/p) \cong H_1(M'_k; \mathbb{Z}/p) \cong \begin{cases} \Lambda_p/(1-2t) \oplus \Lambda_p/(2-t) & \text{if } k \in (\text{order}_p(2)) \\ 0 & \text{otherwise} \end{cases}$$

where $\text{order}_p(2)$ is the order of 2 modulo p.

For the knot K' we have further that for any odd integer d,

$$H_1(M'_k; \mathbb{Z}/d) \cong \Lambda_d/(1-2t, 2^k - 1) \oplus \Lambda_d/(2-t, 2^k - 1),$$

hence the period of $H_1(M'_k; \mathbb{Z}/d)$ is the first occurrence of k such that $2^k \equiv 1 \mod d$. Thus, if p is any prime with $2^{p-1} \not\equiv 1 \mod p^2$, then since the period of $H_1(M'_k; \mathbb{Z}/p)$ divides $p-1$, the period of $H_1(M'_k; \mathbb{Z}/p^2)$ must be p times the period of $H_1(M'_k; \mathbb{Z}/p)$, while if $2^{p-1} \equiv 1 \mod p^2$, i.e., p is a so-called Wieferich prime, then the period of $H_1(M'_k; \mathbb{Z}/p^2)$ must be the same as the period of $H_1(M'_k; \mathbb{Z}/p)$, thereby illustrating both of the possibilities in Theorem 2. Note that 1093 and 3511 are known to be the only Wieferich primes less than 3×10^9 [IR]. We remark that $\text{order}_{1093}(2) = 364$ and $\text{order}_{3511}(2) = 1755$.

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