PERIODICITY FOR THE $\mathbb{Z}/p^r$-HOMOLOGY OF CYCLIC COVERS OF KNOTS AND $\mathbb{Z}$-HOMOLOGY CIRCLES

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ABSTRACT. We consider the sequence of finite branched cyclic covers of a homology sphere with branching locus a codimension-two homology sphere and show that the first homology with coefficients in $\mathbb{Z}/p^r$ of this sequence of manifolds is periodic. We also establish that the $\mathbb{Z}/p^r$-homology of the finite (unbranched) cyclic covers of any integral homology circle is periodic in every dimension.

§0 Introduction

In [D] Dellomo proves a lemma about the mod-$p^r$ homology of the branched cyclic covers of $S^3$ associated to a knot which says that for all multiples $mn$ of some integer $n$ the $mn$-fold branched cyclic covers have first homology with coefficients in $\mathbb{Z}/p^r$ isomorphic to $\bigoplus_{i=1}^{d_p} \mathbb{Z}/p^r$, where $d_p$ is the degree of the Alexander polynomial of the knot with coefficients reduced modulo $p$. Although it is not mentioned in that paper, it follows from his proof of the lemma that the sequence of first homology groups with coefficients in $\mathbb{Z}/p^r$ is in fact periodic (as Abelian groups), that is, letting $M_k$ denote the $k$-fold branched cyclic cover, $H_1(M_{kn}; \mathbb{Z}/p^r) \cong H_1(M_k; \mathbb{Z}/p^r)$ for every $k$ ($n$ is as above). Hillman gave a simplified proof of this lemma [H], which added the feature that for every $m$, $H_1(M_{mn}; \mathbb{Z}/p^r)$ and $H_1(M_n; \mathbb{Z}/p^r)$ are isomorphic as modules over the ring of integral Laurent polynomials, $\mathbb{Z}[t, t^{-1}]$, however, overall periodicity does not seem to be immediate from this proof. In §1 of this paper we offer a yet simpler proof of the more general result that the 1-dimensional mod-$p^r$ homology of the branched cyclic covers of any homology sphere with branching locus a codimension-two homology sphere is periodic as a sequence of $\mathbb{Z}[t, t^{-1}]$-modules and, moreover, that the isomorphisms are given canonically. Neither the proofs of [D] or [H] apply to non-classical knots. Interestingly, this result implies that the $t$-actions on any pair $H_1(M_k; \mathbb{Z}/p^r)$ and $H_1(M_{k'}; \mathbb{Z}/p^r)$ in the same periodicity class (thus $k-k'$ is divisible by the period $n$) induced by a generator of the group of covering translations of $M_k$ and $M_{k'}$ respectively, must be identical. We have recently learned that subsequent to our proof of this result (Theorem 1 below), this same theorem was obtained independently by Dan Silver and Susan Williams using techniques developed in their paper [SW].

In §2 we change our object of study to any finite complex having the integral homology of $S^1$ and show that the mod-$p^r$ homology groups of the finite (unbranched) cyclic covers of such a homology circle are periodic in every dimension. In §3 we return to the branched
cyclic covers of homology spheres in the case $r = 1$ to take a more explicit look afforded by having field coefficients.

Note that since the homology functor is “linear” in the coefficient argument in the sense that $H_*(X_k; G_1 \oplus G_2) \cong H_*(X_k; G_1) \oplus H_*(X_k; G_2)$, all the results mentioned above and in this paper apply to homology with coefficients in any finite Abelian group. Throughout this paper we will work with a fixed prime power $p^n$, $r \geq 1$.

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§1 Branched Cyclic Covers of Homology Spheres

Let $S$ be an $(m+2)$-dimensional homology sphere $(m \geq 1)$ and $K$ be a codimension-two homology sphere in $S$. Denote the exterior of $K$ in $S$ by $X$ (thus $X$ is the closure in $S$ of the complement of a tubular neighborhood of $K$). It is easy to see using Alexander duality that $X$ is a homology circle, one consequence of which is that the abelianization map provides a surjection $\pi_1(X) \to H_1(X) \cong \mathbb{Z}$, and thence onto the cyclic group $\mathbb{Z}/k$ of any order. Thus, $X$ is endowed with an infinite cyclic covering space $X_\infty$, as well as finite cyclic covers $X_k$, for all positive $k$. Note that every $X_k$ arises as a quotient space of $X_\infty$. Let $\Lambda$ denote the integral Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. The homology groups of all the cyclic covers, $H_*(X_k)$, $1 \leq k \leq \infty$, become modules over $\Lambda$ by giving $t$ the action induced by a fixed generator of the group of covering translations of $X_\infty$.

Now, the $k$-fold branched cyclic covers of $S$ branched along $K$ can be constructed from the $k$-fold cyclic cover $X_k$ of $X$ as follows. A tubular neighborhood of $K$ in $S$ is $\nu(K) = K \times D^2$ and we have $\partial(X) = \partial(\nu(K)) = K \times S^1$. Also, $\partial(X_k) = K \times S^1$ and we may take the covering map $X_k \to X$ restricted to the boundary to be given by $(z_1, z_2) \mapsto (z_1, z_2^k) : K \times S^1 \to K \times S^1$ (complex multiplication in the second coordinate). Then the $k$-fold branched covering $M_k$ of $S$ is obtained by attaching the boundary of $N = K \times D^2$ to the boundary of $X_k$ and extending the covering map to $X_k \cup_{K \times S^1} N \to X \cup_{K \times S^1} \nu(K)$ by mapping $N \to \nu(K)$ via $(z_1, z_2) \mapsto (z_1, z_2^k / |z_2^k|) : K \times D^2 \to K \times D^2$. Note that if $\mu$ is a meridian of $\partial(X_k)$, then attaching $N$ to $X_k$ results in $\mu$ bounding a disk, thus killing the homology class $[\mu] \in H_1(M_k)$. Therefore, the inclusions $i : S^1 \hookrightarrow K \times S^1 \subset K \times S^1 \cong \partial(X_k) \subset X_k$ and $j : X_k \hookrightarrow M_k$ induce a natural exact sequence of $\Lambda$-modules

\begin{equation}
0 \to H_1(S^1) \xrightarrow{i_*} H_1(X_k) \xrightarrow{j_*} H_1(M_k) \to 0
\end{equation}

(the $\Lambda$-module structure on $H_1(S^1)$, induced by the infinite cyclic cover $\mathbb{R} \to S^1$, is trivial).

Theorem 1. There exists an integer $n$ such that for every $k$, $H_1(M_k; \mathbb{Z}/p^n)$ and $H_1(M_k; \mathbb{Z}/p^n)$ are naturally isomorphic as $\Lambda$-modules.

Proof. We first relate the homology of the branched cyclic covers to that of the infinite cyclic cover via the following short exact sequence of chain complexes adapted from Milnor [M] (see also [G]):

\begin{equation}
0 \to C_*(X_\infty; \mathbb{Z}/p^n) \xrightarrow{t^k - 1} C_*(X_\infty; \mathbb{Z}/p^n) \xrightarrow{r} C_*(X_k; \mathbb{Z}/p^n) \to 0.
\end{equation}
Combining the resulting long exact sequence in homology with the sequence (1) we get the diagram of $\Lambda$-modules

$$
\begin{array}{cccccccc}
\text{0} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \text{0} \\
& & & & & & & & \\
\downarrow & & & & & & & & \\
H_1(S^1; \mathbb{Z}/p^r) & \xrightarrow{i_*} & H_1(X_\infty; \mathbb{Z}/p^r) & \xrightarrow{\pi_*} & H_1(X_k; \mathbb{Z}/p^r) & \xrightarrow{\partial_*} & H_0(X_\infty; \mathbb{Z}/p^r) & \xrightarrow{0} & H_1(M_k; \mathbb{Z}/p^r) \\
& & \downarrow & & & & & & \downarrow & \\
& & i_* & & & & & & \pi_* \\
& & \downarrow & & & & & & \downarrow & \\
& & \downarrow & & & & & & \downarrow & \\
& & \text{Z}/p^r & & & & & & \text{Z}/p^r \\
\end{array}
$$

where, by looking at the maps on the chain level, we can see that the composition $\partial_* i_*$ is an isomorphism, whence the composition $j_* \pi_*$ is an epimorphism with kernel the image of $t^k - 1$. Thus, we have a natural isomorphism

$$
(3) \quad \frac{H_1(X_\infty; \mathbb{Z}/p^r)}{(t^k - 1)H_1(X_\infty; \mathbb{Z}/p^r)} \cong H_1(M_k; \mathbb{Z}/p^r),
$$

and we proceed to establish the result by showing that the $\Lambda$-module homomorphism of $H_1(X_\infty; \mathbb{Z}/p^r)$ given by multiplication by $t^k - 1$ is a periodic function of $k$.

To this end, consider the long exact homology sequence

$$
(4) \quad \to H_1(X_\infty; \mathbb{Z}/p) \to H_1(X_\infty; \mathbb{Z}/p^i) \to H_1(X_\infty; \mathbb{Z}/p^{i-1}) \to H_0(X_\infty; \mathbb{Z}/p) \to
$$

induced by the short exact sequence of coefficient groups $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^i \to \mathbb{Z}/p^{i-1} \to 0$. Now, $H_1(X_\infty; \mathbb{Z}/p)$ is finite since we know from Milnor [M] that the homology of the infinite cyclic cover of a homology circle with coefficients in a field is finitely generated as a vector space, and so it follows by induction that $H_1(X_\infty; \mathbb{Z}/p^i)$ is finite for all $i$. Hence, also the group of automorphisms $\text{Aut}(H_1(X_\infty; \mathbb{Z}/p^i))$ is finite, so, in particular, the automorphism given by multiplication by $t$ has finite order for all $i$; let $n$ be the order of $t \in \text{Aut}(H_1(X_\infty; \mathbb{Z}/p^r))$. Then $t^n = 1$ implying that for any $k$, $t^{k+n} - 1 = t^k - 1$ as maps of $H_1(X_\infty; \mathbb{Z}/p^r)$. The result then follows from the isomorphism (3). □

**Remark.** It follows that the (fundamental) period of $H_1(M_k; \mathbb{Z}/p^r)$ is precisely $n$ (the order of $t$ in $\text{Aut}(H_1(X_\infty; \mathbb{Z}/p^r))$) and this is also the first $k$ such that $H_1(M_k; \mathbb{Z}/p^r) \cong H_1(X_\infty; \mathbb{Z}/p^r)$. Moreover, at every multiple $mn$ of the order of $t$, $H_1(M_{mn}; \mathbb{Z}/p^r) \cong H_1(X_\infty; \mathbb{Z}/p^r)$. Now, if $K$ is a knotted $S^1$ in $S^3$ we have $H_1(X_\infty)$ is $\mathbb{Z}$-torsion free [C] so that $H_1(X_\infty; \mathbb{Z}/p^r) \cong \bigoplus_{i=1}^d \mathbb{Z}/p^r$ where by Lemma 1 (see the end of this section) $d$ is the dimension of $H_1(X_\infty; \mathbb{Z}/p)$ over $\mathbb{Z}/p$. But also $H_1(X_\infty; \mathbb{Z}/p) \cong H_1(X_\infty)/pH_1(X_\infty)$ (see §3) so $d$ is in fact $d_p$, the degree of the Alexander polynomial of $K$ with coefficients reduced modulo $p$. Combining all this, we have in the classical case that $H_1(M_{mn}; \mathbb{Z}/p^r) \cong \bigoplus_{i=1}^{d_p} \mathbb{Z}/p^r$ for every $m$; this is the lemma of Dellomo mentioned in the introduction.

Consider now the order of $t$ as a function of $r$ (for a fixed prime $p$). Let $t_i$ denote $t$ as an automorphism of $H_1(X_\infty; \mathbb{Z}/p^i)$ and $n_i$ denote the corresponding order. Then we have the following regularity exhibited.
Theorem 2. With the above notation, either \( n_i = n_{i-1} \) or \( n_i = pn_{i-1} \) (\( i \geq 2 \)).

Proof. Assume by induction on \( i \) that \( n_{i-1} = p^{s_i}n_1 \) for some \( 0 \leq s_i \leq i - 1 \). Of course, this hypothesis does hold for \( i = 2 \). By the naturality of the Bockstein sequence (4), we have a commutative diagram

\[
\begin{array}{c}
\rightarrow H_1(X_\infty; \mathbb{Z}/p) \xrightarrow{\phi_*} H_1(X_\infty; \mathbb{Z}/p^i) \xrightarrow{\psi_*} H_1(X_\infty; \mathbb{Z}/p^{i-1}) \rightarrow \\
\downarrow t_i \quad \downarrow t_i \quad \downarrow t_{i-1} \\
\rightarrow H_1(X_\infty; \mathbb{Z}/p) \xrightarrow{\phi_*} H_1(X_\infty; \mathbb{Z}/p^i) \xrightarrow{\psi_*} H_1(X_\infty; \mathbb{Z}/p^{i-1}) \rightarrow
\end{array}
\]

Let \( a \in H_1(X_\infty; \mathbb{Z}/p^i) \) be arbitrary. We want to see that \( t_i^{n_{i-1}}a = a \). \( \psi_*(t_i^{n_{i-1}}a) = t_i^{n_{i-1}}\psi_*(a) = \psi_*(a) \psi_*(a) \) so \( t_i^{n_{i-1}}a \) and \( a \) differ by something from \( H_1(X_\infty; \mathbb{Z}/p^i) \), say \( t_i^{n_{i-1}}a = a + \phi_*(a') \). Using this and the fact that \( t_i^{n_{i-1}}\phi_*(a') = \phi_*(t_i^{n_{i-1}}a') = \phi_*(a') \) we have

\[
t_i^{p^{n_{i-1}}}a = t_i^{(p-1)n_{i-1}}t_i^{n_{i-1}}a = t_i^{(p-1)n_{i-1}}(a + \phi_*(a')) = t_i^{(p-2)n_{i-1}}(a + 2\phi_*(a')) = \ldots = a + p\phi_*(a') = a.
\]

Therefore \( n_i \mid pn_{i-1} \). But we also have \( H_1(X_\infty; \mathbb{Z}/p^{i-1}) \cong H_1(X_\infty; \mathbb{Z}/p^i) \otimes \mathbb{Z}/p^{i-1} \) (Lemma 1 below) with \( t_{i-1} = t_i \otimes \text{id} \), so certainly \( n_{i-1} \mid n_i \). \( \square \)

Lemma 1. For any space \( Y \), \( H_q(Y; \mathbb{Z}/p^r) \otimes \mathbb{Z}/p^{r-1} \cong H_q(Y; \mathbb{Z}/p^{r-1}) \).

Proof. This follows from the universal-coefficient theorem since one has for any Abelian group \( A \), \( A \otimes \mathbb{Z}/p^r \otimes \mathbb{Z}/p^{r-1} \cong A \otimes \mathbb{Z}/p^{r-1} \) and \( \text{Tor}(A, \mathbb{Z}/p^r) \otimes \mathbb{Z}/p^{r-1} \cong \text{Tor}(A, \mathbb{Z}/p^{r-1}) \). The second fact is obvious for finitely generated \( A \), and follows for arbitrary \( A \) since direct limits commute with the torsion and tensor products. \( \square \)

If we now make the observation that \( H_1(X_\infty; \mathbb{Z}/p) \cong \text{GL}(d, \mathbb{Z}/p) \) for some \( d \), we can produce a list of possible periods for \( H_1(X_\infty; \mathbb{Z}/p^i) \) by consulting, for example, [N] which gives formulas for the orders of all elements of \( \text{GL}(d, \mathbb{Z}/p) \). (Note that \( \text{GL}(d, \mathbb{Z}/p) \) has order \( \prod_{i=1}^{d} p^{i-1}(p^i - 1) \).) We will make a hands-on investigation of the periods of \( H_1(M_k; \mathbb{Z}/p) \) in \( \S 3 \) of this paper.

\section*{§2 Finite Cyclic Covers of Homology Circles}

Now let \( X \) be any finite complex having the homology of a circle, \( H_*(X) \cong H_*(S^1) \).

Theorem 3. The \( \text{mod}-p^r \) homology of the finite cyclic covers of \( X \) is periodic, i.e., for every \( q \geq 0 \) there exists an \( n_q \) such that \( H_q(X_{k+n_q}; \mathbb{Z}/p^r) \cong H_q(X_k; \mathbb{Z}/p^r) \) for every \( k \geq 1 \).
Proof. The zeroth homology satisfies this trivially. Fix an arbitrary dimension \( q \geq 1 \) and a cover of order \( k \). From the long exact sequence in homology

\[
\rightarrow H_q(X_\infty; \mathbb{Z}/p^r) \xrightarrow{t^k-1} H_q(X_\infty; \mathbb{Z}/p^r) \rightarrow H_q(X_k; \mathbb{Z}/p^r) \rightarrow H_{q-1}(X_\infty; \mathbb{Z}/p^r) \xrightarrow{t^k-1} H_{q-1}(X_\infty; \mathbb{Z}/p^r) \rightarrow
\]

induced by (2) we extract the short exact sequence

\[
0 \rightarrow \text{coker}_q(t^k - 1) \rightarrow H_q(X_k; \mathbb{Z}/p^r) \rightarrow \text{ker}_{q-1}(t^k - 1) \rightarrow 0,
\]

where \((\text{co})\ker_i(t^k - 1)\) is the (co)kernel of multiplication by \( t^k - 1 \) on \( H_i(X_\infty; \mathbb{Z}/p^r) \), \( i = q, q - 1 \).

As \( X \) is a homology circle, we again have that the automorphism \( t \) has finite order in \( \text{Aut}(H_i(X_\infty; \mathbb{Z}/p^r)) \) for every \( i \); let \( n \) be the least common multiple of the orders of \( t \in \text{Aut}(H_{q-1}(X_\infty; \mathbb{Z}/p^r)) \) and \( t \in \text{Aut}(H_q(X_\infty; \mathbb{Z}/p^r)) \). Then both \( \text{coker}_q(t^{k+n} - 1) = \text{coker}_q(t^k - 1) \) and \( \text{ker}_{q-1}(t^{k+n} - 1) = \text{ker}_{q-1}(t^k - 1) \) so from (5) we can conclude that \( \text{order}(H_q(X_{k+n}; \mathbb{Z}/p^r)) = \text{order}(H_q(X_k; \mathbb{Z}/p^r)) \) and, in the case \( r = 1 \), \( H_q(X_{k+n}; \mathbb{Z}/p) \cong H_q(X_k; \mathbb{Z}/p) \).

Proceeding by induction on \( r \), assume \( H_q(X_{k+n}; \mathbb{Z}/p^{r-1}) \cong H_q(X_k; \mathbb{Z}/p^{r-1}) \). Lemma 1 and the induction hypothesis give \( H_q(X_{k+n}; \mathbb{Z}/p^r) \otimes \mathbb{Z}/p^{r-1} \cong H_q(X_k; \mathbb{Z}/p^r) \otimes \mathbb{Z}/p^{r-1} \), and since we already have that the orders of \( H_q(X_{k+n}; \mathbb{Z}/p^r) \) and \( H_q(X_k; \mathbb{Z}/p^r) \) are the same, Lemma 2 (below) applies to give \( H_q(X_{k+n}; \mathbb{Z}/p^r) \cong H_q(X_k; \mathbb{Z}/p^r) \). \( \square \)

Lemma 2. Let \( A \) and \( B \) be two finite Abelian \( p \)-groups of the same order such that every element of \( A \) and \( B \) has order at most \( p^r \). If \( A \otimes \mathbb{Z}/p^{r-1} \cong B \otimes \mathbb{Z}/p^{r-1} \), then \( A \cong B \).

Proof. Write \( A \cong \bigoplus_{i=1}^r (\mathbb{Z}/p^i)^{m_i} \) and \( B \cong \bigoplus_{i=1}^r (\mathbb{Z}/p^i)^{m'_i} \). Then

\[
A \otimes \mathbb{Z}/p^{r-1} \cong (\mathbb{Z}/p)^{m_1} \oplus \cdots \oplus (\mathbb{Z}/p^{r-2})^{m_{r-2}} \oplus (\mathbb{Z}/p^{r-1})^{m_{r-1}+m_r} \quad \text{and}
\]

\[
B \otimes \mathbb{Z}/p^{r-1} \cong (\mathbb{Z}/p)^{m'_1} \oplus \cdots \oplus (\mathbb{Z}/p^{r-2})^{m'_{r-2}} \oplus (\mathbb{Z}/p^{r-1})^{m'_{r-1}+m'_r},
\]

so under the hypothesis that \( A \otimes \mathbb{Z}/p^{r-1} \cong B \otimes \mathbb{Z}/p^{r-1} \) we must have \( m_i = m'_i \) for \( 1 \leq i \leq r - 2 \) and \( m_{r-1} + m_r = m'_{r-1} + m'_r \). Additionally, since \( \prod_{i=1}^r (p^i)^{m_i} = \text{order}(A) = \text{order}(B) = \prod_{i=1}^r (p^i)^{m'_i} \), we have \( (r-1)m_{r-1} + rm_r = (r-1)m'_{r-1} + rm'_r \), forcing \( m_{r-1} = m'_{r-1} \) and \( m_r = m'_r \). Therefore \( A \cong B \). \( \square \)

One easily obtains:

Corollary. The mod-\( p^r \) homology of the sequence of branched cyclic covers of a homology sphere branched along a codimension-two homology sphere is periodic in every dimension.

\section{§3 A Note on Periods in the Case \( r = 1 \)}

Now return to the situation where \( K \) is a codimension-two homology sphere in a homology sphere \( S \). Then Theorem 1 applies and we have the first homology with coefficients in \( \mathbb{Z}/p^r \) of the sequence of branched cyclic covers of \( S \) branched along \( K \) is periodic. Presently
we want to investigate what the period of \( H_1(M_k;\mathbb{Z}/p) \) is. Using the isomorphism (3) we study the modules \( H_1(M_k;\mathbb{Z}/p) \) by examining the effect of multiplication by \( t^k - 1 \) on \( H_1(X_{\infty};\mathbb{Z}/p) \) in a manner similar to Sunner’s analysis of \( H_1(M_k;\mathbb{C}) \) [S]. This produces an explicit formula for the period in this case.

The short exact sequence of coefficient groups \( 0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0 \) gives the long exact sequence in homology,

\[
\rightarrow H_1(X_{\infty}) \xrightarrow{p} H_1(X_{\infty}) \rightarrow H_1(X_{\infty};\mathbb{Z}/p) \rightarrow H_0(X_{\infty}) \xrightarrow{p} H_0(X_{\infty}) \rightarrow,\]

and since multiplication by \( p \) on \( H_0(X_{\infty}) \cong \mathbb{Z} \) is injective, we have \( H_1(X_{\infty};\mathbb{Z}/p) \cong H_1(X_{\infty})/pH_1(X_{\infty}) \). Thus, we may consider \( H_1(X_{\infty};\mathbb{Z}/p) \) as a module over \( \Lambda_p = \mathbb{Z}/p\Lambda \), and since \( \Lambda_p \cong \mathbb{Z}/p[t, t^{-1}] \) is a principal ideal domain, \( H_1(X_{\infty};\mathbb{Z}/p) \) decomposes into a direct sum of cyclic \( \Lambda_p \)-modules, \( H_1(X_{\infty};\mathbb{Z}/p) \cong \bigoplus_{i=1}^l \Lambda_p/(\lambda_i) \) with \( (\lambda_1) \subset (\lambda_2) \subset \cdots \subset (\lambda_l) \). We have, moreover, from [M] that in this situation all the \( \lambda_i \)'s are nonzero. Then since the representatives \( \lambda_1, \ldots, \lambda_l \) are defined only up to multiplication by units in \( \Lambda_p \), we take each \( \lambda_i \) to be the unique monic polynomial (thus having no negative powers of \( t \)) with nonzero constant term in the ideal \( (\lambda_i) \), and we call this preferred generator the \( i \)-th invariant factor of the \( \Lambda_p \)-module \( H_1(X_{\infty};\mathbb{Z}/p) \). Then

\[
H_1(M_k;\mathbb{Z}/p) \cong H_1(X_{\infty};\mathbb{Z}/p)/(t^k - 1)H_1(X_{\infty};\mathbb{Z}/p) \cong \bigoplus_{i=1}^l \Lambda_p/(\lambda_i, t^k - 1).
\]

Over an algebraic closure, \( \mathbb{Z}/p \), the first invariant factor \( \lambda_1 \) splits as, say, \( \lambda_1 = \prod_{j=1}^m (t - \alpha_j)^{e_j} \). For each \( j = 1, \ldots, m \) let \( \mu_j \) be the order of \( \alpha_j \) (thus we may think of \( \alpha_j \) as being a primitive \( \mu_j \)-th root of unity in \( \mathbb{Z}/p \)), and let \( h \) be the least common multiple of \( \mu_1, \ldots, \mu_m \). Then, writing \( k = up^v \) with \( p \nmid u \) we have \( (\Lambda_p) t^k - 1 = (t^u - 1)^{p^v} \) and so

\[
(\lambda_1, t^{up^v} - 1) = \left( \prod_{\{j: \mu_j \mid u\}} (t - \alpha_j)^{\min(e_j, p^v)} \right).
\]

Thus when \( v \) is such that \( p^v \geq e_j \) for every \( 1 \leq j \leq m \), then \( (\lambda_1, t^{hp^v} - 1) = (\lambda_1) \) and similarly for the other \( \lambda_i \) (due to the divisibility conditions), and we have \( H_1(M_{hp^v};\mathbb{Z}/p) \cong \bigoplus_{i=1}^l \Lambda_p/(\lambda_i) \). Taking \( v \) minimal we then have the first occurrence of \( H_1(M_k;\mathbb{Z}/p) \cong H_1(X_{\infty};\mathbb{Z}/p) \) and therefore \( hp^v \) is the period of \( H_1(M_k;\mathbb{Z}/p) \). Furthermore, by Theorem 2, the period of \( H_1(M_k;\mathbb{Z}/p) \) is \( hp^{v+s} \) for some \( s, 0 \leq s < r \).

Example. Let \( K \) be the knotted \( S^1 \) in \( S^3 \) enumerated 6 in the Alexander-Briggs table. \( K \) has \( H_1(X_{\infty}) \cong \Lambda/(2t^2 - 5t + 2) \) with Alexander polynomial \( \Delta(t) = 2t^2 - 5t + 2 \) (see [G]). Notice that on reducing coefficients modulo 3 the polynomial \( 2t^2 - 5t + 2 \rightarrow (t + 1)^2 \) which has a single root of multiplicity two which is the primitive second root of unity. Thus \( H_1(X_{\infty};\mathbb{Z}/3) \cong \Lambda_3/(t + 1)^2 \) and we have

\[
H_1(M_k;\mathbb{Z}/3) \cong \begin{cases} 
\Lambda_3/(t + 1) & \text{if } k \in (2), k \notin (6) \\
\Lambda_3/(t + 1)^2 & \text{if } k \in (6) \\
0 & \text{otherwise}
\end{cases}
\]
which clearly has the period $6 = 2 \cdot 3^1$.

Now let $K'$ be the knot $9_{46}$. $K'$ has the same Alexander polynomial as $K$ although $H_1(X_{\infty}^e) \cong \Lambda/(1 - 2t) \oplus \Lambda/(2 - t)$. Then $H_1(X_{\infty}^e; \mathbb{Z}/3) \cong \Lambda_3/(t + 1) \oplus \Lambda_3/(t + 1)$ so

$$H_1(M_k^e; \mathbb{Z}/3) \cong \begin{cases} \Lambda_3/(t + 1) \oplus \Lambda_3/(t + 1) & \text{if } k \in (2) \\ 0 & \text{otherwise} \end{cases}$$

which has period 2. (Notice for this knot the annihilator of $H_1(X_{\infty}^e; \mathbb{Z}/3)$ is $(t + 1) \subset \Lambda_3$ which is not the reduction modulo 3 of the annihilator $(2t^2 - 5t + 2) \subset \Lambda$ of $H_1(X_{\infty}^e)$.)

Modulo 2 the infinite cyclic covers of both knots have trivial homology. Modulo any other prime $p \neq 2, 3$, $H_1(X_{\infty}^e; \mathbb{Z}/p) \cong \Lambda_p/(2t^2 - 5t + 2) \cong \Lambda_p/(1 - 2t) \oplus \Lambda_p/(2 - t)$ since the polynomials $1 - 2t$ and $2 - t$ reduce to nonassociate polynomials in $\Lambda_p$. Thus, if $p \neq 2, 3$, we have

$$H_1(M_k^e; \mathbb{Z}/p) \cong H_1(M_k; \mathbb{Z}/p) \cong \begin{cases} \Lambda_p/(1 - 2t) \oplus \Lambda_p/(2 - t) & \text{if } k \in (\text{order}_p(2)) \\ 0 & \text{otherwise} \end{cases}$$

where $\text{order}_p(2)$ is the order of 2 modulo $p$.

For the knot $K'$ we have further that for any odd integer $d$,

$$H_1(M_k^e; \mathbb{Z}/d) \cong \Lambda_d/(1 - 2t, 2^k - 1) \oplus \Lambda_d/(2 - t, 2^k - 1),$$

hence the period of $H_1(M_k^e; \mathbb{Z}/d)$ is the first occurrence of $k$ such that $2^k \equiv 1 \mod d$. Thus, if $p$ is any prime with $2^p - 1 \equiv 1 \mod p^2$, then since the period of $H_1(M_k^e; \mathbb{Z}/p)$ divides $p - 1$, the period of $H_1(M_k^e; \mathbb{Z}/p^2)$ must be $p$ times the period of $H_1(M_k^e; \mathbb{Z}/p)$, while if $2^p - 1 \equiv 1 \mod p^3$, i.e., $p$ is a so-called Wieferich prime, then the period of $H_1(M_k^e; \mathbb{Z}/p^2)$ must be the same as the period of $H_1(M_k^e; \mathbb{Z}/p)$, thereby illustrating both of the possibilities in Theorem 2. Note that 1093 and 3511 are known to be the only Wieferich primes less than $3 \times 10^9$ [IR]. We remark that $\text{order}_{1093}(2) = 364$ and $\text{order}_{3511}(2) = 1755$.

REFERENCES


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