Lecture 11: Proof of the NSZ

11.1 Theorem (NSZ): Let \( k \) be an algebraically closed field. (\( k = \bar{k} \)) Let \( J \subseteq k[x_1, \ldots, x_n] \) be an ideal. If \( J \neq k[x_1, \ldots, x_n] \), then \( V(J) \neq \emptyset \).

Rather than prove this directly, we shall instead prove an equivalent theorem.

11.2 Theorem: Let \( K \) be a field and \( K \subseteq L \) an extension field. such that \( L \) is a \( K \)-algebra of finite type. Then, \( L \) is an algebraic extension of fields.

\( L \) is obtained from \( K \) by adjoining finitely many algebraic elements. i.e., \( L = K[\xi_1, \ldots, \xi_n] \), with each \( \xi_i \) algebraic over \( K \).

11.3 Proposition: Theorem 11.1 is equivalent to Theorem 11.2.

Proof: (\( \Leftarrow \)) Let \( J \subseteq M \subseteq k[x_1, \ldots, x_n] \), \( M \) a maximal ideal. Let \( L = k[x_1, \ldots, x_n]/M = k[\xi_1, \ldots, \xi_n] \). \( M \) is maximal, so \( L \) is a field and it is finitely generated as an algebra over \( k \). So, \( L/K \) is algebraic. But, \( k = \bar{k} \), thus \( k = L \), as \( k \) has no strictly larger algebraic extensions. This means that \( \xi_i \in k \), \( 1 \leq i \leq n \). So, there exists \( x_i \in \xi_i \) (mod \( M \)). So \( (\xi_1, \ldots, \xi_n) \in k^n \) is a zero of \( M \), and thus of \( J \). \( (x_i - \xi_i) \in M \) implies \( < x_1 - \xi_1, \ldots, x_n - \xi_n > \subseteq M \). But, that ideal is maximal, so it equals all of \( M \). So, \( (\xi_1, \ldots, \xi_n) \in V(J) \), hence \( V(J) \neq \emptyset \).

(\( \Rightarrow \)) Let \( L = k[x_1, \ldots, x_n]/M \), \( M \) a maximal ideal. \( M \subseteq k[x_1, \ldots, x_n] \subseteq \bar{k}[x_1, \ldots, x_n] \). Theorem 11.1 implies that there exists \( (\xi_1, \ldots, \xi_n) \in \bar{k}^n \) which is a zero for \( M \). Using this zero, we can construct a ring homomorphism \( \phi : k[x_1, \ldots, x_n] \rightarrow \bar{k} \)

\[
\begin{align*}
x_i &\mapsto \xi_i
\end{align*}
\]

with \( \ker(\phi) = M \); \( M \subseteq \ker(\phi) \), since everything in \( M \) goes to zero. But, \( M \) is maximal, so it must be all of \( \ker(\phi) \). Since every element of \( \bar{k} \) is algebraic of \( k \), every element of \( L \subseteq \bar{k} \) is algebraic over \( k \).
Now that we have established the equivalency of the two theorems, we will work on proving Theorem 11.2.

11.4 Lemma: Let $S = k(z_1, \ldots, z_t)$ be a rational function field in $t > 0$ variables. Then $S$ is not finitely generated (over $k$) as a $k$-algebra. (Proof later)

Proof of 11.2 (assuming 11.4): We can write $L = k[x_1, \ldots, x_n]/M$, $M$ maximal ideal. Will use proof by contradiction; Assume $L$ is not algebraic over $k$. So $L$ is a mixed algebraic and transcendental extension of $k$ and can be written as a purely algebraic extension $B$ of a purely transcendental extension $S$ of $k$. So, we have $A = k \subseteq S \subseteq B = L$, with $B/S$ algebraic and $S/k$ transcendental.

11.5 Lemma: Let $A \subseteq S \subseteq B$ be rings, with $A$ Noetherian. Let $B = A[\xi_1, \ldots, \xi_n]$ be finitely generated as an $A$-algebra. Assume $B$ is finite as an $S$-module. Then, $S$ is finitely generated as an $A$-algebra.

In the proof of 11.2, we have $B$ as a finitely generated $S$-module, and $B$ is finitely generated as a $k$-algebra. But, by Lemma 11.4, $S$ is not finitely generated as a $k$-algebra. $\rightarrow$ So, $L$ is algebraic over $k$. QED

All that remains is to prove Lemma 11.5.

Proof of Lemma 11.5: We can write $B = S\xi_1 + \ldots + S\xi_n$. Also, we can assume the set \{${\xi_1, \ldots, \xi_n}$\} to contain $A_1, \ldots, A_n$. We get products

$$w_iw_j = \sum_{k=1}^{m} a_{ij}^k w_k, \quad a_{ij}^k \in S$$

Look at $S' = A[a_{ij}^k] \subseteq B$; Claim $B = S'\xi_1 + \ldots + S'\xi_n$. $B = A[\xi_1, \ldots, \xi_n] \ni \xi_1^{a_1}, \ldots, \xi_n^{a_n} \in S'\xi_1 + \ldots + S'\xi_n$, as every $\xi_i^{a_i}$ can be rewritten in terms of $w_i$'s and $a_{ij}^k$'s. By the Hilbert Basis Theorem, $S' = A[a_{ij}^k]$ is a Noetherian ring. $B$ is a finitely generated $S'$-module, so $B$ is Noetherian. $S' \subseteq S \subseteq B$ implies $S$ is finitely generated as an $S'$-module. Thus, it is finitely generated as an $A$-algebra.

$k(z) \neq k[A_1, \ldots, A_n]$ for any finite set of polynomial generators $A_i$. There exists infinitely-many irreducible polynomials $p_1(z), \ldots, p_k(z), \ldots$ by a proof similar to Euclid’s infinitely-many primes proof.