1. PROJECTIVE V-I AND THE AFFINE CONE

**Key Point:** If \( F(x_0, ..., x_n) \in k[x_0, ..., x_n] \) where \( k \) is a field, then \( a = [a_0, ..., a_n] = [\lambda a_0, ..., \lambda a_n] \in \mathbb{P}^n(k) \) where \( \lambda \in k^* \). Notice that \( F(a) = F(a_0, ..., a_n) \neq F(\lambda a_0, ..., \lambda a_n) \) so \( F \) is not well defined.

Assume \( F \) is homogeneous of degree \( d \).

Example: \( 3xyz - z^3 + x^2y \) is homogeneous whereas \( 2 + x^3y^2 \) is not.

Now, \( F(\lambda a_0, ..., \lambda a_n) = \lambda^{\text{deg} F} F(a_0, ..., a_n) \), so \( F(a) = 0 \) is a well-defined condition for \( a \in \mathbb{P}^n(k) \).

**Definition 1.1.** There are mappings

\[
\begin{align*}
\left\{ \text{homogeneous ideals } J \subseteq k[x_0, ..., x_n] \right\} & \xrightarrow{\text{V}} \left\{ \text{subsets } X \subseteq \mathbb{P}^n(k) \right\}, \\
\end{align*}
\]

where

\[
V(J) = \{ a \in \mathbb{P}^n(k) : F(a) = 0, \forall \text{ homogeneous } F \in J \}
\]

and

\[
I(X) = \langle \text{homogeneous } F \in k[x_0, ..., x_n] : F|_X = 0 \rangle.
\]

A *projective* algebraic set is contained in a subset \( X \subseteq \mathbb{P}^n(k) \) of the form \( X = V(J) \) for some homogeneous ideal \( J \).

**Recall:** An ideal \( J \subseteq k[x_0, ..., x_n] \) is homogeneous \( \iff \) (TFAE)

1. \( J \) is generated by homogeneous polynomials.
2. If \( F \in J \) then every homogeneous part of \( F \) is in \( J \), where \( F = F_d + F_{d+1} + ... + F_e; F_i \in J, J = \langle F_1, ..., F_s \rangle \), and \( V(J) = \{ a \in \mathbb{P}^n(k) : F_j(a) = 0, \ j = 1, ..., s \} \).

**Theorem 1.2.** Suppose the following are homogeneous ideals.

1. If \( J_1 \subseteq J_2 \Rightarrow V(J_1) \supseteq V(J_2) \)

*Date: October 3, 2005.*
(2) $X_1 \subseteq X_2 \Rightarrow I(X_1) \supseteq I(X_2)$
(3) $\bigcup_{j=1}^{s} V(I_j) = V\left(\bigcap_{j=1}^{s} I_j\right)$
(4) $\cap_{\lambda \in \Lambda} V(I_\lambda) = V\left(\sum_{\lambda \in \Lambda} I_\lambda\right)$
(5) If $J \supseteq \langle x_0, ..., x_n \rangle \Rightarrow V(J) = \phi$
(6) $V(0) = \mathbb{P}^n(k)$

**Corollary:** The sets $\{V(J) : J$ is a homogeneous ideal$\}$ form the closed sets for a topology on $\mathbb{P}^n(k)$, the Zariski Topology.

**Theorem 1.3. Projective NSZ** Let $k = \bar{k}$, be an algebraically closed field, we have the following:

1. $V(J) = \phi \iff J \supseteq \langle x_0, ..., x_n \rangle$, an irregular ideal.
2. If $V(J) \neq \phi$, then $IV(J) = \sqrt{J}$.

\[
\begin{align*}
\text{homogeneous radical ideals } J \left( = \sqrt{J} \right) & \xrightarrow{V} \text{projective algebraic sets } X \subseteq \mathbb{P}^n(k) \\
\bigcup \text{homogeneous prime ideals } P & \xleftarrow{\text{irreducible projective algebraic}} \text{= projective varieties} \\
\bigcup \text{homogeneous maximal ideals } M & \xleftarrow{\text{points in } \mathbb{P}^n(\mathbb{R})}
\end{align*}
\]

**Idea for proof of Theorem 1.3:**
We define the affine cone $CX$. Recall, $\mathbb{P}^n = (\mathbb{A}^{n+1} - \{0\})/\sim$ where $\lambda \in k^*$. Notice, $0 + F_j(\lambda a) = \lambda^{deg} F(a)$ and $X = V(F_1, F_2, ..., F_s)$.

\[
\begin{align*}
\mathbb{A}^{n+1} - \{0\} & \xrightarrow{\pi} \mathbb{P}^n \\
V^n(F_1, ..., F_2) & \subseteq \mathbb{A}^{n+1}
\end{align*}
\]

$CX := V^a(F_1, ..., F_s) = \bigcup \text{lines through } \bar{0} = \text{affine cone over } X.$

When is $V(J)$ empty? Now by the Affine NSZ Theorem, we have that $J = k[x_0, ..., x_n]$ or $J = \langle x_0, ..., x_n \rangle$.

For part two of the theorem, we have that if $V(J) \neq \phi$, ...
then

\[ I(V(J)) = I(V^a(J)), \]

so if \( F \) homogeneous, then

\[ F|_{V(J)} \equiv 0 \iff F|_{CV(J)} \equiv 0 \iff F \in I(V^a(J)) \]

And now by the affine NSZ it follows that this is true \( \iff F \in V(J) \).

\[ \square \]

**Definition 1.4.** A *graded* ring \( A \) is a ring with a decomposition

\[ A = A_0 \oplus A_1 \oplus ... \]

such that \( A_nA_m \subseteq A_n + A_m \), where \( A_i \) are abelian subgroups.

**Example**

\[ A = k[x_0, \ldots, x_n] \]

and \( A_d = (\text{homogeneous polynomials of degree } d) \). Similarly a graded module \( M \) over a graded ring \( A \) is an \( A \)-module with a decomposition into \( M = \bigoplus_{d \in \mathbb{Z}} M_d \), where \( M_d \) are abelian subgroups of \( M \), furthermore

\[ A_d \cdot M_e \subseteq M_{d+e}. \]

**Example** Let \( M = J \), where \( J \) is a homogeneous ideal, then

\[ J = J_0 \oplus J_1 \oplus J_2 \oplus ... \]

where \( J_d = J \cap k[x_0, \ldots, x_n]_d \). If \( J \) is homogeneous, then

\[ A = k[x_0, \ldots, x_n]/J \]

is graded. And

\[ k[x_0, \ldots, x_n]/J \cong \bigoplus_{d \geq 0} k[x_0, \ldots, x_n]_d/J_d. \]