Consider an open set in the Zariski topology 

\[ U_i = \mathbb{P}^n \setminus V(x_i) = \{ [x_0, \ldots, x_n] \in \mathbb{P}^n | x_i \neq 0 \} . \]

Note that a conventional picture for a projective space (without a common zero for all variables) is on the left, and the algebraic set \( V(x_i) \cong \mathbb{P}^{n-1} \). Consider a canonical isomorphism \( \phi_i : U_i \rightarrow \mathbb{A}^n \), by mapping \( \begin{bmatrix} x_0, x_1, \ldots, x_n \end{bmatrix} \rightarrow (x_0/x_i, x_1/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i) \). We’ll see today that \( \phi_i \) is a homeomorphism for Zariski topology.

Given a projective algebraic set \( X \subseteq \mathbb{P}^n \), we can ”view it” in the \( i \)th coordinate chart \( \phi_i(U_i \cap X) = \mathbb{A}^n \).

**Example 0.1.** \( X = V(y^2 - xz) \subseteq \mathbb{P}^2 \).

Now we can consider it in the charts

\( U_0 = \mathbb{P}^2 \setminus V(x) = \{ [x, y, z] \in \mathbb{P}^2 | x \neq 0 \} \),
\( U_1 = \mathbb{P}^2 \setminus V(y) = \{ [x, y, z] \in \mathbb{P}^3 | y \neq 0 \} \), and
\( U_3 = \mathbb{P}^2 \setminus V(z) = \{ [x, y, z] \in \mathbb{P}^2 | z \neq 0 \} \).

In \( U_0 \) we have \( \phi_0(X \cap U_0) = V(y^2 - z) \subset \mathbb{A}^2 \).
(Remark: when you have a homogeneous polynomial, it is enough to take \( x = 1 \) for the image of it under \( \phi_0 \).)

In \( U_1 \) we have \( \phi_1(X \cap U_1) = V((1 - zx) \subset \mathbb{A}^2 \), and in \( U_2 \) we have \( \phi_2(X \cap U_2) = V(y^2 - x) \subset \mathbb{A}^2 \). (See the corresponding pictures below.)
Example 0.2. Let $X = V(y^3 - x^2 z) \subset \mathbb{P}^2$.

In $U_0$ we have $\phi_0(X \cap U_0) = V(y^3 - z) \subset \mathbb{A}^2$; in $U_1$ we have $\phi_1(X \cap U_1) = V(x^2 z - 1)$, and in $U_2$ we have $\phi_2(X \cap U_2) = V(y^3 - x^2)$.

Let $X$ be a projective algebraic set in $\mathbb{P}^n$.

Lemma 0.3. If $X = V(F_1, F_2, \ldots, F_s) \subset \mathbb{P}^n$, and $\phi_i(X \cap U_i) = V(f_1, f_2, \ldots, f_s) \subset \mathbb{A}^n$, where $f_j(x_0, \ldots, x_i, \ldots, x_n) = F_j(x_0, \ldots, x_i = 1, \ldots, x_n)$, then

$$X = (X \cap U_i) \cup (X \cap V(x_i)),$$

Remark: The first set in the union above is an affine piece in the $i$th chart, the second is a piece at infinity, in a hyperplane at $x_i = 0$, which homeomorphic to $\mathbb{P}^{n-1}$.

Let’s fix a chart for $i = 0$ and consider an inverse of $\phi_0$. We can see that $\phi_0^{-1} : \mathbb{A}^n \sim U_0 \hookrightarrow \mathbb{P}^n$.

Definition 0.4. (1) Let $X \subseteq \mathbb{A}^n$ be affine algebraic set. The closure of $\phi_0^{-1}(X)$ in $\mathbb{P}^n$ is called the projective closure of $X$, (relative to a chart) and denoted by $\overline{X}$.

$$X_\infty = \overline{X} \cap H_\infty, H_\infty = V(x_0) \equiv \mathbb{P}^{n-1} \subset \mathbb{P}^n.$$  

(2) Let $J$ be an ideal of $k[x_1, \ldots, x_n]$. Define $J^h = \langle f^h | f \in J \rangle$, where

$$f^h(x_0, x_1, \ldots, x_n) := x_0^{\deg f} f\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

(3) $J_\infty = \langle g_\infty | g \in J \rangle$, where $g_\infty(x_1, \ldots, x_n) = g(x_0 = 0, x_1, \ldots, x_n)$.

(4) If $X \subseteq \mathbb{P}^n$ is a projective algebraic set, denote $X^a = \phi_0(X \cap U_0) \subseteq \mathbb{A}^n$ - an affine part of $X$.

(5) If $J$ is a homogeneous ideal of $k[x_0, x_1, \ldots, x_n]$, define an ideal $J^a$ of $k[x_1, \ldots, x_n]$ as $J^a = \langle F^a | F \in J, F$ is homogeneous $\rangle$, where $F^a = F(x_0 = 1, x_1, \ldots, x_n)$.

Lemma 0.5. If $F \in k[x_0, x_1, \ldots, x_n]$ is a homogeneous polynomial, $x_0 \nmid F$, then $F = (F^a)^h$.

Consider an example when the lemma does not hold: $F = x^2 y^3 + x^3 y z + x^2 z^3$, $F^a = y^3 + y z + z^3$, $(F^a)^h = y^3 + x y z + z^3$.

Proposition 0.6. (1) If $J$ is an ideal of $k[x_1, \ldots, x_n]$, and $X = V(J) \subseteq \mathbb{A}^n$, then $I(\overline{X}) = I(X)^h, V(J^h) = \overline{X}, I(X_\infty) = \sqrt{I(\overline{X})_\infty}, V(J_\infty) = X_\infty$. 


(2) If \( J \) is a homogeneous ideal of \( k[x_0, x_1, \ldots, x_n] \), \( X = V(J) \subseteq \mathbb{P}^n \), then \( I(X^a) = I(X)^a \), \( V(J^a) = X^a \).

Warning: If \( J = \langle f_1, \ldots, f_s \rangle \subseteq k[x_1, \ldots, x_n] \) (can be not homogeneous), it is not true in general that \( J^h = \langle f_1^h, \ldots, f_s^h \rangle \).

Example 0.7. Let \( X = \{(t, t^2, t^3) \mid t \in k\} \subseteq \mathbb{A}^3 \). Note that \( X = V(y - x^2, z - xy) \).

Consider variables \([w, x, y, z]\) in \( \mathbb{P}^3 \). Then

\[
f_1^h = wy - x^2, \quad f_2^h = zw - xy, \quad V(f_1^h, f_2^h) = X \cup \{\text{line } w = x = 0\},
\]

and if we denote \( J = \langle y - x^2, z - xy \rangle \), then

\[
J^h = \langle wy - x^2, zw - xy, y^2 - xz \rangle \quad \text{and} \quad X = V(wy - x^2, zw - xy, y^2 - xz).
\]