LECTURE 21 - SHEAF THEORY II

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ABSTRACT. This lecture develops the ideas introduced in Lecture 20. In particular, we define a stalk of a sheaf and use this concept along with the concept of a sheaf of continuous sections to show that there is a natural way to associate a sheaf to every presheaf.

1. STALKS

Definition 1.1. Let F be a presheaf on a topological space X and let $x \in X$. The *stalk* F_x of F at x is

$$F_x = \varinjlim F(U),$$

where the *direct limit* is taken over all neighborhoods U of x, via the restriction maps of the presheaf F.

Remark 1.2. Since a definition of a direct limit has not been given, we shall explicate Definition 1.1.

Definition 1.3. Let F be a presheaf on a topolgiacal space X and let $x \in X$. Consider the set of all pairs (s, U), where U is a neighborhood of x and s is a section on U. We define a relation $\sim$ on this set as follows:

$$(s, U) \sim (t, V)$$

if there exists a neighborhood W of x such that $W \subset U \cap V$ and $s_{|W} = t_{|W}$. An equivalence class $[(s, U)]_\sim$ is called a *germ of a section* s at x. The *stalk* F_x of F at x is defined to be the set of all germs of sections of F at x.

Remarks 1.4.

(1) If F is a presheaf of groups (rings), then F_x has a natural structure of a group (ring). Indeed, one can easily check that the addition:

$$[(s, U)]_\sim + [(t, V)]_\sim := [(s + t, U \cap V)]_\sim$$

(and) multiplication:

$$[(s, U)]_\sim \cdot [(t, V)]_\sim := [(s \cdot t, U \cap V)]_\sim$$

are independent of representatives and define a group (ring) structure on F_x.

(2) For every neighborhood U of x, we have a canonical surjection $\tau_U : F(U) \to F_x$ defined by $\tau_U(s) = [(s, U)]_\sim$. If $V \subset U$ are neighborhoods of x and ρ_{VU} is the restriction map, then $\tau_U = \tau_V \circ \rho_{VU}$.

(3) To simplify notation, we will usually use the symbol s_x to denote $[(s, U)]_\sim$.

Examples 1.5.

(1) Let A be an abelian group and let F be a constant presheaf with value A. Then $F_x \cong A$ for every x. Indeed, if U is any neighborhood of x, then $F(U) = A$ and the canonical map $\tau_U : A \to F_x$ is an isomorphism.

(2) Given a topolgical space X and an open set $U \subset X$, let $F(U)$ be the set of all continuous real-valued functions defined on U. Then F is a sheaf and F_x is the set of germs of continuous functions at x.

(3) Similarly, if $X = \mathbb{C}$, and $F(U)$ is the set of holomorphic functions on U, then F is a sheaf and F_x is the set of germs of holomorphic functions at x. Moreover, if f_x denotes a germ of a holomorphic function f at x, then the map

$$f_x \longmapsto \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(z-x)^n$$

establishes a ring isomorphism between F_x and the ring of convergent power series in $z - x$.

2. The sheaf associated to a presheaf

Proposition 2.1. *Let F be a presheaf on a topological space X. There exist a topological space $[F]$ and a continuous projection $\pi : [F] \to X$ which is etale and a natural map $\varphi : F \to F^a$ of presheaves, where F^a is a sheaf of continous sections of π; the map φ is isomorphism if and only if F is a sheaf.*

The sheaf F^a is called *the sheaf associated to the presheaf* F. It has the following universal property:

If C is a sheaf and $\psi : F \to C$ is a morphism of presheaves, then there exists a unique morphism $\psi^* : F^a \to C$ such that $\psi = \psi^* \circ \varphi$.

Proof. A complete proof of this proposition is quite long and tedious. We shall sketch how to construct $[F]$, π, φ and leave details to the reader.

As a set, we define $[F]$ to be a disjoint union of stalks

$$[F] = \coprod_{x \in X} F_x.$$

Next, let $\mathcal{S}$ be a collection consisting of all sets of the form

$$\{\, \{s_x\}_{x \in U} \in \coprod_{x \in X} F_x \mid U \text{ is open in } X \text{ and } s \in F(U) \,\}.$$

We define a topology on $[F]$ to be the topology generated by $\mathcal{S}$. Then $\pi : [F] \to X$ given by $\pi(s_x) = x$ is a continuous projection that is etale. Now, let U be an open subset of X. We define $\varphi(U) : F(U) \to F^a(U)$ by $\varphi(U)(s)(x) = s_x$. This yields a natural map $\varphi : F \to F^a$ with the required property. □

Definition 2.2. Let F, G be sheaves of abelian groups on a topolgical space X, and let $\Phi : F \to G$ be a morphism of sheaves.

(1) We define $\ker \Phi$ to be a subpresheaf of F given by $U \mapsto \ker \Phi(U)$.
(2) The *cokernel* of Φ is defined to be the sheaf associated to the presheaf $U \mapsto \operatorname{Coker} \Phi$.

Remark 2.3. One can show that $\ker \Phi$ is actually a sheaf, whereas the presheaf $U \mapsto \operatorname{Coker} \Phi$ does not need to be a sheaf.

Theorem 2.4. *Let X be a topological space. The category of abelian sheaves on X is an abelian category.*

References

[1] Hartshorne, R., *Algebraic Geometry* , Springer-Verlag., New York, 1977.

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