LECTURE 26

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1. Lecture 26 Schemes I

Definition 1.1. Let $X = \text{Spec}(R)$. For each distinguished open $X_f$, we define

$$\mathcal{O}_X(X_f) = R_f.$$ 

Recall that for any $f \in R$,

$$X_f = \{x = [p] \in \text{Spec}(R) | f(x) \neq 0 \iff f \notin p\}.$$ 

Proposition 1.2. Let $\mathcal{B} = \{X_f : f \in R\}$ be a basis for topology. (Prop. 25.5 (i)) Then,

$$X_f \rightarrow R_f$$

is a $\mathcal{B}$ pre-sheaf.

Proof: Let $X_g \subset X_f$. We need to define restriction maps

$$R_g \leftarrow R_f,$$

so that

$$X_n \subset X_g \subset X_f$$

where $R_n \leftarrow R_g \leftarrow R_f \rightarrow R_n$. Last time,

$$\sqrt{I} = \bigcap_{p \supseteq I} p.$$ 

Note:

$$V(I) \subseteq V(J) \iff \sqrt{I} \supseteq \sqrt{J}.$$ 

Then,

$$X_g = X - V(g) \subseteq X_f = X - V(f)$$

$$\iff V(g) \supseteq V(f) \iff \sqrt{f} \subseteq \sqrt{g} \Rightarrow g \in \sqrt{f}.$$ 

So,

$$g^n = af; \ n\geq 1 \ & \ a \in R.$$ 

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This shows: \[ R_g = R_{fg}, \]
since there are maps \[ R_g \rightarrow R_{fg}, \]
where the map \[ R_g \rightarrow R_{fg} \]
is given by
\[
\frac{r}{g^\nu} \mapsto \frac{rg^\nu}{(fg)^\nu}
\]
and the map \[ R_{fg} \rightarrow R_g \]
where
\[
\frac{r}{(fg)^\nu} = \frac{rg^{n\nu}f^{n\nu}}{(fg)^{n\nu}(fg)^\nu} = \frac{rf^{n\nu}(af)^\nu}{f^{(n+1)\nu}g^{(n+1)\nu}} = \frac{ra^\nu}{g^{(n+1)\nu}} \in R_g.
\]
So
\[ R_f \xrightarrow{rest} R_g \]
is
\[ R_f \rightarrow R_{fg} = R_g \]
where
\[
\frac{r}{f^\nu} \mapsto \frac{rg^\nu}{(fg)^\nu} = \frac{?}{g^\nu}.
\]
Now let us prove our note above
\[ V(I) \subseteq V(J) \iff \sqrt{I} \supseteq \sqrt{J}. \]
Let us first show that
\[ V(I) \subseteq V(J) \iff \sqrt{I} \supseteq \sqrt{J}. \]
Suppose
\[ \sqrt{I} \supseteq \sqrt{J}, \]
then if \( p \) is prime we have \( [p] \in V(J) \), so
\[ p \supset I \Rightarrow p \supset \sqrt{I} \Rightarrow p \supset \sqrt{J} \supseteq J \Rightarrow [p] \in V(J). \]
We now want to show
\[ V(I) \subseteq V(J) \Rightarrow \sqrt{I} \supseteq \sqrt{J}. \]
Suppose
\[ V(I) \subseteq V(J), \]
by definition we have the following,

\[ V(I) = \{ [p] | p \supseteq I \} \]

and

\[ V(J) = \{ [q] | q \supseteq J \}. \]

So let \( f \in \sqrt{J} \), be arbitrary, we want to show

\[ f \in \sqrt{I}. \]

Since \( f \in \sqrt{J} \), it follows that

\[ f \in q, \ \forall [q] \in V(J) \iff q \supseteq J. \]

Notice that

\[ \{ q \supseteq J \} \subseteq \{ p \supseteq I \} \]

which implies \( f \in p, \forall p \supseteq I \implies f \in \sqrt{I} \), as desired.

\[ \square \]

**Theorem 1.3.** The \( \mathcal{B} \) pre-sheaf \( \mathcal{O}_X \) is a \( \mathcal{B} \)-sheaf and hence it extends uniquely to a sheaf on \( X \) [Prob. 23.3 (1)] called the **structure sheaf**.

Explicitly: for all \( f \in \mathcal{R} \), suppose \( X_f \) is covered by \( X_{fa} \subset X_f \).

1. If \( g, h \in R_f \), become equal in \( R_{fa} \) for all \( a \), then \( g = h \).
2. If for each \( a \), there exits \( g \in R_f \), such that \( g \rightarrow g_a \) under \( R_f \rightarrow R_{fa} \).

**Proof:** WLOG suppose \( f = 1 \). In

\[ X_f = \text{Spec } R_f \]

replace

\[ R \rightarrow R_f \]

so that

\[ X = \text{Spec } R \]

and

\[ X = \bigcup_{fa \in R} X_{fa}. \]

\[ \square \]

**Lemma 1.4.**

\( X_{fa} \) covers \( X \) \iff \( \langle fa \rangle = R \).

Moreover, \( X \) is quasicompact (every open cover has a finite subcover).
Proof of lemma:

$X_{fa}$ covers $X \iff$ no prime $p \subset R$ contains all $f_a$. Also,

$$[p] \in X_{fa} \iff f_a \not\in p \iff \langle f_a \rangle = R.$$ 

If $X = \bigcup U_a$, then we can refine to a cover by distinguished open sets. So

$$X = \bigcup_a X_{fa}$$
$$R = \langle f_a \rangle$$
$$\sum_{a=1}^n g_a f_a = 1$$

So,

$$R = \langle f_1, \ldots, f_n \rangle$$

implies

$$X = \bigcup_{i=1}^n X_{f_i}.$$ 

Now back to the proof of the theorem, for part (1) we have $g = h$ in $R_{fa}$. This implies,

$$\frac{g}{1} = \frac{h}{1} \quad \text{in} \quad \frac{\langle f_a \rangle}{f_a^N} \iff f_a^N(g - h) = 0 \quad \text{in} \quad R.$$ 

We can then extract a finite subcover of $X_{fa}$, we can do so by the Lemma. Let $a = 1, \ldots, n$ so $f_a^N(g - h) = 0$ and $\langle f_a \rangle = R$. We now want to show that

$$\langle f_a^N \rangle = R,$$

since

$$\langle f_1^N, \ldots, f_n^N \rangle \supseteq \langle f_1, \ldots, f_n \rangle^M,$$

for some big power $M$, where $s_1 + s_2 + \cdots + s_n = M$ and $f_1^{s_1} \cdots f_n^{s_n}$. 

Note that $f_a^N(g - h) = 0$ implies that $g - h$ is annihilated by

$$\langle f_1^N, \ldots, f_n^N \rangle = R,$$

so

$$g - h = 0.$$ 

Let us now prove part (2) of this theorem. Suppose

$$g_a \in R_{fa},$$

then

$$g_a = \frac{h_a}{f_a^N} = \frac{h_a f}{f_a^{N(s+a)}}.$$
We can just consider $f_1, \ldots, f_n$, so that
\[ g_a = \frac{h_a}{f_a^{N_a}}, \text{ where } a = 1, \ldots, n. \]

Now,
\[ g_a = g_b \text{ in } R_{f_a g_a} \]
if and only if
\[ \exists N' \text{ such that } f_b^{N'} h_a = (f_a f_b)^{N'} g_a = (f_a f_b)^{N'} g_b = f^{N'} h_b. \]

Therefore, there exits an $N$ such that
\[ f_b^N h_a = (f_a f_b)^N g_a = (f_a f_b)^N g_b = f^N h_b. \]
So,
\[ f_b^N = \sum_{a=1}^{n} f_b^N h_a e_a = h_b \sum_{a=1}^{n} e_a f_a^N = h_b. \]

This is since,
\[ \langle f_a^N \rangle = R \]
implies
\[ \sum_{a=1}^{n} e_a f_a^N = 1, \quad e_a \in R. \]

Thus,
\[ f_b^N g = h_b, \]
so
\[ g = \frac{h_b}{f_b^N} \text{ in } R_{f_b}, \]
which in turn implies
\[ g = g_b. \]

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