Schemes and base-change

**Definition 0.1.** Let $S$ be a scheme. The category of $S$-schemes has objects $f : X \to S$, a morphism of schemes, and morphisms $(f : X \to S) \to (g : Y \to S)$. There is a morphism $h : X \to Y$ such that $f = g \circ h$.

**Theorem 0.2.** Let $S = \text{Spec} R$. Then $\text{Hom}_{\text{Sch}/S}(X, \text{Spec} B) = \text{Hom}_{\text{R-}\text{alg}}(B, \mathcal{O}_x(x))$.

**Definition 0.3.** Let $\alpha : T \to S$ be a morphism of schemes. Then the base-change along $\alpha$ is the covariant functor $(\text{Sch}/S) \to (\text{Sch}/T)$ defined as follows:

$$(X \to S) \mapsto (X \times_S T \to T).$$

We have $\beta_i : X_1 \times_T T \to X_i$, $f_i : X_i \to S$, $g_i : X_i \times_S T \to T$, and $\alpha : T \to S$. Then $f_i \circ \beta_i = \alpha \circ g_i$. We also have $h : X_1 \to X_2$, where $f_1 = f_2 \circ h$. We want to construct $k : X_1 \times_S T \to X_2 \times_S T$. We need $k_1 : X_1 \times S T \to X_2$ and $k_2 : X_1 \times S T \to T$, where $k_1 = h \circ \beta_1$ and $k_2 = g_1$. So $f_2 \circ k_1 = f_2 \circ h \circ \beta_1 = f_1 \circ \beta_1 = \alpha \circ g_1 = \alpha \circ k_2$.

**Definition 0.4.** Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. The fiber of $f$ above $y$, denoted $X_y$, is the base-change $(X \to Y) \mapsto (X \times_Y \text{Spec}(\kappa(y)) \to \text{Spec}(\kappa(y)))$.

**Proposition 0.5.** Let $f : X \to Y$ and $y \in Y$. Then the map $|X_y| \to f^{-1}(y)$ is a homeomorphism.

**Proof.** It will be enough to show that $|U_y| = f^{-1}(y) \cap U$ for all affne open $U \subset X$. We have that $U_y = \text{Spec}(B \otimes_A A_p/pA_p) = \text{Spec}(B_p/pA_p)$, where $A_p/pA_p = \kappa(y)$. Then let $A_p = S^{-1} A = \{ \alpha \in \mathbb{Z} : \text{s is not in p} \}$ and $S_p = A - p$. Then $f : \text{Spec} B \to \text{Spec} A$ if and only if $\phi : A \to B$. We have $B_p$ defined to be $\phi(S_p)^{-1} B$ and $pB_p = \phi(p)B_p$. Now $U_y = \text{Spec}(B_p/pB_p) = \{ \text{primes } Q \subset B_p \text{ s.t. } Q \supseteq pB_p \} = \{ \text{primes } Q' \subset B | Q' \cap \phi(S_p) \text{ and } Q' \supseteq pB_p \cap B, Q' \cap B \}$, where $Q' = Q \cap B$. We have $pB_p \cap B \supseteq \phi(p)B$, which implies that $Q' \supseteq \phi(p)B$, so $p \subseteq \phi^{-1}Q'$. On the other hand, $Q' \cap \phi(S_p) = \emptyset$, which implies that $\phi^{-1}Q' \subseteq p$. Now if $y = [p]$, we have $U_y = \{ [Q] \in \text{Spec} B | \phi^{-1}Q = p \} = f^{-1}([p]) = f^{-1}(y)$ since $f[Q] = [\phi^{-1}Q]$. □

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