6. Zariski Topology

Definition 6.1. A topological space $X$ is Noetherian if it satisfies the descending chain condition for closed sets, that is for closed sets $Y_i$ if $Y_1 \supset Y_2 \supset \cdots$, then for some $R$, we have $Y_r = Y_{r+1}$ for all $r \geq R$.

Example 6.2. (i) $\mathbb{A}^n(k)$ with the Zariski topology is Noetherian. For any descending chain $Y_1 \supset Y_2 \supset \cdots$, we have $Y_i = V(J_i)$, where $J_i = \sqrt{J_i}$ is a radical ideal, and so $J_1 \subset J_2 \subset \cdots \in k[x_1, \ldots, x_n]$. Since $k[x_1, \ldots, x_n]$ is Noetherian, the ascending chain of $J_i$’s satisfies the ascending chain condition, and the result follows.

(ii) Let $X \subset \mathbb{A}^n(k)$. Then $X$ endowed with the subspace topology is Noetherian.

(iii) Let $X = \text{Spec}(A)$ where $A$ is a Noetherian ring. Then the closed sets are $V(I) = \{P:\ P \subset P\}$.

Recall an irreducible subset $Y$ of $X$ is a subset such that if $Y = Y_1 \cup Y_2$ and $Y_1$ and $Y_2$ are closed, then either $Y = Y_1$ or $Y = Y_2$. Also remember that $Y = V(J)$ is irreducible if and only if $\sqrt{J}$ is prime.

Proposition 6.3. In a Noetherian topological space $X$, every nonempty closed subset $Y \subset X$ can be written as a finite union $Y = Y_1 \cup \cdots \cup Y_r$, where each $Y_i$ is irreducible. If $Y_i \nsubseteq Y_j$ for $i \neq j$, then this decomposition is unique. Each $Y_i$ is called an irreducible component of $Y$.

Proof: Let $S$ be the set of nonempty closed subsets $Y \subset X$ that do not have such a decomposition. If $S \neq \emptyset$, the Noetherian hypothesis implies that $S$ has a minimal element, $Y$. Since $Y \in S$, $Y$ admits no finite decomposition into a union of irreducibles. In particular, $Y$ is not irreducible. Thus we may write $Y = Y_1 \cup Y_2$ where $Y_1$, $Y_2$ are closed and $Y_1 \neq Y$ and $Y_2 \neq Y$. Since $Y \in S$ is minimal, neither $Y_1$ nor $Y_2 \in S$. So $Y_1 = Y_1' \cup \cdots \cup Y_s'$ and $Y_2 = Y_2'' \cup \cdots \cup Y_t''$ giving a decomposition for $Y = Y_1' \cup \cdots \cup Y_s' \cup Y_2'' \cup \cdots \cup Y_t''$ and thus $S = \emptyset$.

Now suppose $Y = Y_1' \cup \cdots \cup Y_s'$ is another such representation. Then $Y_1' \subset Y = Y_1 \cup \cdots \cup Y_r$, so $Y_1' = \bigcup(Y_i' \cap Y_i)$. But $Y_i'$ is irreducible, so $Y_1' \subset Y_i$ for some $i$, say $i = 1$. Similarly, $Y_1' \subset Y_j'$ for some $j$. Then $Y_1' \subset Y_j'$, so $j = 1$, and thus $Y_1 = Y_1'$. Now let $Z = (Y - Y_1)$. Then $Z = Y_2 \cup \cdots \cup Y_r$, and also $Z = Y_2' \cup \cdots \cup Y_s'.$ So we proceed by induction on $r$ to obtain uniqueness.

Definition 6.4. An ideal $Q \subset A$ is primary if one of the equivalent conditions holds:

1. Every zero divisor of $A/Q$ is nilpotent.
2. If $xy \in Q$, then $x \in Q$ or $y^n \in Q$ for some $n \geq 1$ (ie $y \in Q$).
If $Q$ is primary, then $\sqrt{Q} = P$ is prime, and we say $Q$ is $P$-primary.

If $Q_1$ and $Q_2$ are $P$-primary, then $Q_1 \cap Q_2$ is $P$-primary. If $M$ is a maximal ideal, then $M'$ is $M$-primary. More generally, if $\sqrt{I} = M$ is maximal, then $I$ is $M$-primary. Observe that it is not true that $P'$ is $P$-primary for any prime ideal $P$.

**Example 6.5.** Let $A = k[x,y,z]/(xy - z^2) = k[\overline{x},\overline{y},\overline{z}]$. Then $P = (\overline{x},\overline{z})$ is prime because $A/P \cong k[\overline{y}]$ is an integral domain. But $P^2$ is not $P$-primary because $\overline{x} \cdot \overline{y} = \overline{z}^2 \in P^2$ yet $\overline{x} \notin P$ and $\overline{y} \notin \sqrt{P^2} = P$.

**Definition 6.6.** An irredundant primary decomposition of an ideal $I \subset A$ is an expression $I = Q_1 \cap \cdots \cap Q_r$ with $Q_i$ primary and

1. the prime $P_i = \sqrt{Q_i}$ are all distinct.
2. No $Q_i$ can be omitted from the decomposition, that is
   \[ \bigcap_{i \neq j} Q_j \not\subset Q_i \text{ for any } i. \]

**Theorem 6.7.** Let $A$ be a commutative Noetherian ring.

1. Every ideal $I \subset A$ has an irredundant primary decomposition: $I = Q_1 \cap \cdots \cap Q_r$.
2. The set of primes $P_1 = \sqrt{Q_1}, \ldots, P_r = \sqrt{Q_r}$ is determined by $I$. These are the associated primes of $I$, denoted $\text{Ass}(I)$.
3. This decomposition of $I$ need not be unique, but the $Q_i$ belonging to the minimal elements of the set $\{P_j\}$ are uniquely determined by $I$. Non-minimal elements are called embedded primes.
4. If $I = \sqrt{I}$, then $I$ has a unique decomposition $\sqrt{I} = P_1 \cap \cdots \cap P_t$ where $P_i$ is prime and $P_i \not\subset P_j$ for $i \neq j$.

**Example 6.8.** This is an example of a non-unique primary decomposition. Let $A = k[x,y]$. Then let $I = (x^2, xy)$. Now, $I$ admits two decompositions $Q_1 \cap Q_2 = (x) \cap (x,y)^2$ and $Q_1 \cap Q_2' = (x) \cap (x^2,y)$. Then $\sqrt{Q_1} = P_1 = (x)$ and $\sqrt{Q_2} = \sqrt{Q_2'} = (x,y) = P_2$. Note that $P_1 \subset P_2$, so $P_2$ is an embedded prime.