Lecture 8: Examples

Today we look at some more examples of affine algebraic varieties.

8.1 Dimension

Definition 1 Let $X$ be an irreducible algebraic set in $\mathbb{A}^n$ (that is, an affine algebraic variety). We define the dimension of $X$ to be the Krull dimension of the coordinate ring $k[X]$ of $X$.

Recall that the Krull dimension of a ring $R$ is the supremum over all integers $n$ such that there exists a strictly increasing chain $J_0 \subset J_1 \subset \ldots \subset J_n$ of prime ideals in $R$.

Some reminders and remarks: recall that the coordinate ring of an affine algebraic set $X$ is defined to be $k[X] = \frac{k[x_1, \ldots, x_n]}{I(X)}$ (1) where $I(X)$ is the radical ideal of all polynomials in $k[x_1, \ldots, x_n]$ vanishing identically on $X$, and $k$ is assumed (at least for now) to be algebraically closed. We have seen that $X$ is irreducible if and only if $I(X)$ is a prime ideal, which is true if and only if the quotient ring $k[X]$ is an integral domain. The quotient field $k(X)$ of $k[X]$, which is an extension field of $k$, is often called the function field of $X$.

This notion of dimension coincides with the topological dimension of $X$ when $X$ is considered with the Zariski topology. This is not difficult to see. Recall that the topological dimension of a topological space $X$ is defined to be the supremum over all integers $n$ such that there exists a strictly decreasing chain of irreducible closed subsets $Y_i$ of $X$:

$$Y_0 \supset \ldots \supset Y_n \quad (2)$$

This corresponds to a strictly increasing chain of prime ideals in $k[x_1, \ldots, x_n]$:

$$I(Y_0) \subset \ldots \subset I(Y_n) \quad (3)$$

Later we will develop another equivalent notion: the dimension of $X$ is equal to the transcendence degree of the function field $k(X)$ over $k$. This is often easier to directly compute than the topological dimension of $X$ or the Krull dimension of $k[X]$.

Recall the inclusion-reversing nature of the correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in $k[x_1, \ldots, x_n]$: If $Y \subset X \subset \mathbb{A}^n$, then $I(X) \supseteq I(Y) \subsetneq k[x_1, \ldots, x_n]$. Thus we can form the quotient $\frac{k[Y]}{I(Y)}$, which we call $\overline{Y}$ for short. $\overline{Y}$ is a subring of the coordinate ring $\frac{k[x_1, \ldots, x_n]}{I(X)}$ of $X$. When discussing the ideal corresponding to an algebraic subset $Y$ of an algebraic set $X$, we may refer either to the ideal $I(Y) \subset k[x_1, \ldots, x_n]$ or the ideal $\overline{Y} \subset k[X]$. The latter viewpoint reflects the intrinsic nature of $X$, apart from its embedding in $\mathbb{A}^n$. 

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This is a manifestation of a basic result in ring theory: Let $R$ be a ring and $J \subseteq R$ an ideal. Then there is a one-to-one correspondence between ideals $I$ in $R$ containing $J$ and ideals $\overline{I}$ in the quotient ring $R/J$.

There are familiar names for low-dimensional algebraic sets: points are 0-dimensional, curves are 1-dimensional, surfaces are 2-dimensional, and 3-folds are 3-dimensional.

8.2 Examples: plane curves

In the last lecture, we briefly examined the plane curve $C = V(f) \subseteq \mathbb{A}^2$, where $f$ is the polynomial $y^2 - x^3 = 0$. This curve is often called the ”cusp.” The coordinate ring of $C$ is given by

$$k[C] = \frac{k[x, y]}{< f >} = \frac{k[x, y]}{< y^2 - x^3 >}$$

where $< f >$ is the principal ideal generated by $f$. That this is in fact the coordinate ring requires justification; by definition $k[C]$ is the quotient of $k[x, y]$ by the radical ideal $I(C)$. Thus, while it is obvious that $< y^2 - x^3 > \subseteq I(C)$, the reverse inclusion must be established. We know by the Nullstellensatz that $\sqrt{< f >} = I(C)$, so we want to show that $< f >$ is radical. It suffices to show that $f$ is irreducible; then $< f >$ is prime and hence radical. This can be accomplished by examining possible factorizations of $f = y^2 - x^3$: suppose $f = gh$ for some polynomials $g, h$ in $k[x, y]$. Since $f$ is quadratic in $y$, then either one of $g, h$ (say $h$) is quadratic in $y$ and $g$ is a polynomial in $x$ alone, or else both $g$ and $h$ are linear in $y$. In the first case:

$$y^2 - x^3 = g(x)(y^2 h_1(x) + h_2(x))$$

which implies

$$g(x)h_1(x) = 1$$
$$g(x)h_2(x) = -x^3$$

The first equation is enough; it implies that $g$ is a unit. In the second case:

$$y^2 - x^3 = (yg_1(x) + g_2(x))(y h_1(x) + h_2(x))$$

we see that $g_1 h_1 = 1$, so without loss of generality,

$$y^2 - x^3 = (y + g_2(x))(y + h_2(x))$$

This yields $g_2 + h_2 = 0$, so $-x^3 = g_2 h_2$ is minus a square, an obvious contradiction. Thus $k[C] = \frac{k[x, y]}{< f >}$ as claimed.

We noted in a previous lecture that there is a bijective morphism $\phi : \mathbb{A}^1 \to C \subseteq \mathbb{A}^2$ which is not an isomorphism. Perhaps the simplest way to see this is to note that the corresponding k-algebra homomorphism $\phi^* : k[C] \to k[\mathbb{A}^1] = k[t]$ is not an isomorphism. We have $\phi^*(\overline{x}) = t^2$ and $\phi^*(\overline{y}) = t^3$, where $\overline{x}$ and $\overline{y}$ are the equivalence
classes of \( x \) and \( y \) in \( k[C] = \frac{k[x,y]}{<y^2 - x^3>} \) (which generate \( k[C] \)). Since any element in the image set \( \phi^*(k[C]) \) is a polynomial in the images of the generators, we see that \( t \), for instance, is not in the image, since \( t \) obviously cannot be written as a polynomial in \( t^2 \) and \( t^3 \). Thus \( \phi^* \) is not a \( k \)-algebra isomorphism, and \( \phi \) is not an isomorphism of affine algebraic varieties.

In fact there is no isomorphism from \( C = V(y^2 - x^3) \) to \( \mathcal{A}^1 \) (that is, the two varieties are not isomorphic). This can also be shown by considering \( k \)-algebra homomorphisms between the coordinate rings.

Now we discuss another plane curve, often called the node. It is also defined by a single polynomial \( f(x,y) = y^2 - x^2(x+1) \). We will use \( C \) again to denote this curve:

\[
C = V(f) = V(y^2 - x^2(x+1))
\]  

(10)

The polynomial \( f = y^2 - x^2(x+1) \) is irreducible, which we can show by considering factorizations, just as in the previous example with the cubic cusp. Thus the ideal \( I(C) \) is just \(<y^2 - x^2(x+1)>\). The map \( \phi : \mathcal{A}^1 \rightarrow C \subset \mathcal{A}^2 \) defined by

\[
\phi(t) = (t^2 - 1, t(t^2 - 1))
\]  

(11)

is a morphism of affine algebraic varieties. To demonstrate this, we first check that every point in the image of \( \phi \) in fact satisfies the defining equation of \( C \); that is, we first show that \( \forall t \in \mathcal{A}^1 \), the point \( (x,y) = (t^2 - 1, t(t^2 - 1)) \) satisfies \( y^2 = x^2(x+1) \). This is a straightforward computation:

\[
x^2(x+1) = (t^2 - 1)^2t^2 = y^2
\]  

(12)

This proves that \( \phi \) maps \( \mathcal{A}^1 \) into \( C \). To show \( \phi \) is onto, we must show that, given \( (x,y) \) satisfying \( y^2 = x^2(x+1) \), there exists \( t \in \mathcal{A}^n \) with \( x = t^2 - 1 \) and \( y = t(t^2 - 1) \). Let \( x \in k \). Recall that \( k \) is algebraically closed; therefore we can take square roots to find (generically) two values of \( t \) such that \( x = t^2 - 1 \), namely \( t = \pm \sqrt{x+1} \). (The usual picture of this curve in the real plane, which shows no points with \( x \) coordinate less than \(-1 \), is misleading because the real number field is not algebraically closed.) Now since \( y^2 = x^2(x+1) \), we have \( y = \pm x\sqrt{x+1} = \pm tx \). Thus, one of the two choices for \( t \) maps into \( (x,y) \) under \( \phi \), so \( \phi \) is onto. We see that the two values of \( t \) corresponding to each \( x \) map to different points (reflected across \( x=0 \)) except for \( t = \pm 1 \), both of which map to \((0,0)\). (There is one value of \( x \), namely \( x = -1 \), for which there is only one value of \( t \).) So \( \phi \) is bijective except for a “double point” at the origin in \( \mathcal{A}^2 \).

8.3 Hypersurfaces

Now we discuss the ”largest” (proper) algebraic sets in \( \mathcal{A}^n \). First, a definition:

**Definition 2** The codimension of an algebraic set \( X \subset \mathcal{A}^n \) is \( n - \text{dim}(X) \)
Codimension is not an intrinsic concept; it depends on the dimension of the space in which \( X \) is embedded. If we instead embed \( X \) in \( \mathbb{A}^m \), \( m \neq n \), then of course the codimension of \( X \) changes. Note that if \( X \subseteq \mathbb{A}^n \) is irreducible, then the codimension of \( X \) is equal to the height of the prime ideal \( I(X) \) in \( k[x_1, \ldots, x_n] \).

**Definition 3** A hypersurface in \( \mathbb{A}^n \) is an algebraic set of codimension one.

Examples include curves in \( \mathbb{A}^2 \) (such as the cusp and the node) and surfaces in \( \mathbb{A}^3 \). One special type of hypersurface is a hyperplane, which is the vanishing set of a linear polynomial in \( k[x_1, \ldots, x_n] \):

\[
H = V(\sum a_i x_i + b)
\]

We state without proof the following theorem:

**Theorem 4** (Krull’s Hauptidealsatz): Any prime ideal \( P \subseteq k[x_1, \ldots, x_n] \) with \( \text{ht}(P) = 1 \) is principal: \( P = \langle f \rangle \), where \( f \) is an irreducible polynomial in \( k[x_1, \ldots, x_n] \).

Since the codimension of \( X \) is equal to the height of the prime ideal \( I(X) \subseteq k[x_1, \ldots, x_n] \), the height of \( I(H) \) is 1 for a hypersurface \( H \). Thus \( I(H) \) is principal by the Hauptidealsatz, so an irreducible hypersurface is the vanishing set of a single polynomial.

**Quadric surfaces** are a rich source of examples of hypersurfaces. A quadric surface is a variety defined by a polynomial \( f \) of second degree:

\[
f = \sum a_{ij} x_i x_j + \sum b_i x_i + c
\]

(We can assume without loss of generality that the \( a_{ij} \) form a symmetric matrix if \( \text{char}(k) \neq 2 \).)

Examples of quadric surfaces include conics in the plane as well as certain familiar surfaces in 3-space such as cones (e.g. \( V(x^2 + y^2 - z^2) \)), ellipsoids (e.g. \( V(x^2 + 2y^2 + 3z^2 - 1) \)), and hyperboloids (e.g. \( V(x^2 + y^2 - z^2 - 1) \)).

We have seen that the radical ideal corresponding to any variety of codimension 1 is principal. It might be tempting to think that every variety of codimension \( m \) has an ideal generated by \( m \) polynomials, but this is false, even for varieties of codimension 2. One example of this is the variety \( C \subseteq \mathbb{A}^3 \) which is the image of \( \mathbb{A}^1 \) under the map \( t \mapsto (t^3, t^4, t^5) \). It is easy to see that every point on \( C \) satisfies the relations \( x^3 - yz = y^2 - xz = x^2 y - z^2 = 0 \). Thus the ideal \( \langle x^3 - yz, y^2 - xz, x^2 y - z^2 \rangle \) is contained in the radical ideal \( I(C) \). The converse is also true, \( I(C) = \langle x^3 - yz = y^2 - xz = x^2 y - z^2 = 0 \rangle \), but this is not so obvious.

**Exercise 5** Show that \( I(C) = \langle x^3 - yz, y^2 - xz, x^2 y - z^2 \rangle \)