Zeta functions, graphs and dynamical systems

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Outline

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1. Zeta functions

A zeta function is a certain kind of counting function. The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$$= \prod_p \frac{1}{1 - p^{-s}}$$

is the Dirichlet series generating function for

$$n \mapsto a_n = \{\text{the number of ideals } I \subset \mathbb{Z} \text{ of norm } n\}$$

where the norm of an ideal is by definition $\#(\mathbb{Z}/I)$.

If $K$ is a finite extension field of $\mathbb{Q}$, then the Dedekind zeta $\zeta_K(s)$ is defined the same way, but with $a_n$ equal to the number of ideals $I \subset O_K$ of norm $n$, where $O_K$ is the ring of integers of $K$. Write the Euler factor as

$$\frac{1}{1 - p^{-s}} = \frac{1}{1 - u} = \exp \left( \sum_{m=1}^{\infty} N_m u^m / m \right).$$

Then $N_m = 1$ is the number of points of the scheme $\text{Spec}(\mathbb{Z})$ in the finite field $\mathbb{F}_{p^n}$. 
To be a zeta function, one expects

1. An Euler product.
3. Functional equation.
5. Riemann hypothesis.

For Riemann’s zeta items 1) - 3) are well-known. Item 5) is the most famous open problem in mathematics. Item 4) is the subject of speculation and conjecture. Current thinking, due to Connes and Deninger, sees strong analogies between the zeta functions occurring in arithmetic geometry and zeta functions coming from dynamical systems. One can give a “trivial proof” of 5) given a good theory of 4) (see Deninger’s talk at the Berlin ICM).
A successful example of this program are the zeta functions of varieties over finite fields. Let \( Y \) be a variety defined over a finite field \( \mathbb{F}_q \). For each integer \( n \geq 1 \) there is a unique extension field of \( \mathbb{F}_q \) of degree \( n \), denoted \( \mathbb{F}_{q^n} \). The set of points of \( Y \) with coordinates in \( \mathbb{F}_{q^n} \) is finite, and its cardinality is denoted by \( \#Y(\mathbb{F}_{q^n}) \). Then the zeta function of \( Y \) is defined as:

\[
Z(Y/\mathbb{F}_q, u) = \exp \left( \sum_{n=1}^{\infty} \frac{\#Y(\mathbb{F}_{q^n})}{n} u^n \right)
\]

\[
Z(Y/\mathbb{F}_q, q^{-s}) = \prod_{y \in |\mathcal{Y}|} \frac{1}{1 - N(y)^{-s}},
\]

where the product is taken over the closed points \( y \) of the scheme \( Y \), and \( N(y) = q^{\deg(y)} \) is the cardinality of the residue field \( \kappa(y) \).

This is known to be a rational function of the variable \( u \).

Also, there is a cohomological formula, due to Grothendieck:

\[
Z(Y/\mathbb{F}_q, u) = \prod_{i=0}^{2 \dim Y} \det \left( 1 - uF : H_c^i \left( Y \otimes \overline{\mathbb{F}}_q, \mathbb{Q}_l \right) \right)^{(-1)^{(i+1)}}
\]

expressing the zeta function in terms of the action of the Frobenius operator on the étale cohomology groups.

The Riemann hypothesis for these was proved by Deligne in 1973.
2. Modular Curves mod $p$

Let $Y$ be the projective curve defined by the equation $y^2 + xy - y = x^3$. We find that there are 18 points on this curve over the field with 13 elements:

$$\infty, \ (0,0), \ (0,1), \ (1,1), \ (1,12), \ (5,3), \ (5,6), \ (7,9), \ (7,11), \ (8,2), \ (8,4), \ (9,7), \ (9,11), \ (10,6), \ (10,11), \ (11,6), \ (11,10), \ (12,1)$$

so

$$Z(Y/\mathbb{F}_{13}, \ u) = \frac{1 + 4u + 13u^2}{(1 - u)(1 - 13u)}$$

In fact this elliptic curve $Y$ is isogenous to the modular curve $X_0(14)$.

For any modular curve $X_0(N)$, associated to a congruence subgroup $\Gamma_0(N) \subset \text{SL}(2, \mathbb{Z})$, the zeta function of the corresponding modular curve at a prime of good reduction has the form

$$Z(X_0(N)/\mathbb{F}_p, \ u) = \frac{P(u)}{(1 - u)(1 - pu)}$$

where $P(u) = \det(1 - T_p u + pR_p u^2)$, with $T_p, R_p$ being the Hecke operators on the space $S_2(\Gamma_0(N))$ of cusp forms of weight 2 for that group. This is the famous Eichler - Shimura theorem.
The essentially unique cusp form of weight 2 for $\Gamma_0(14)$ is
\[
\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)
\]
\[
= q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + q^9
\]
\[
- 2q^{12} - 4q^{13} - q^{14} + q^{16} + 6q^{17} - q^{18} + 2q^{19} + \ldots
\]
where
\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = \exp(2\pi i \tau)
\]

is the Dedekind $\eta$-function.

Thus, for the curve $Y = X_0(14)$, we have
\[
\#Y(\mathbb{F}_p) = p + 1 - c_p
\]
for all primes $p \neq 2, 7$, where $c_p$ is the coefficient of $q^p$ in the above expansion.
3. **Zeta functions of graphs**

A graph $X$ consists of two sets, of vertices $V(X)$ and oriented edges $E(X)$, with a relation “a vertex lies on an edge”. Sometimes we allow weights to be attached to the edges. Given a finite graph $X$ its **adjacency matrix** is defined by $A = (a_{ij})$ where $a_{ij}$ is the weighted sum of edges leaving vertex $i$ and joining vertex $j$.

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$
The zeta function of a graph is defined by

$$Z(X, u) = \prod_{\text{primitive } \gamma} \frac{1}{1 - u^{\deg \gamma}}$$

$$= \exp \left( \sum_{n=1}^{\infty} N_n u^n / n \right)$$

where $N_n$ denotes the number of closed backtrackless cycles of length $n$ on $X$. In the product expansion, $\gamma$ runs through all the nontrivial primitive conjugacy classes in the fundamental group $\pi_1(X)$ ($X$ is assumed connected here).

Recall that the fundamental group of a graph is a free group. If $\gamma$ is any element of $\pi_1$ then its centralizer is an infinite cyclic group, \{\gamma_0^k\}. We say $\gamma$ is primitive if it generates its centralizer, i.e., $\gamma = \gamma_0^{\pm 1}$. It is a rational function:

$$Z(X, u) = \frac{(1 - u^2)^\chi}{\det(1 - Au + Qu^2)}.$$

This theorem is due to Ihara, Hashimoto and Bass. Here $\chi$ is the Euler characteristic of the graph.
Ihara originally introduced these functions as zeta functions of discrete cocompact subgroups $\Gamma$ of the $p$-adic Lie group $\text{GL}(2, \mathbb{Q}_p)$. These were analogues of the zeta functions introduced by Selberg for discrete cocompact subgroups $\Gamma$ of the real Lie group $\text{GL}(2, \mathbb{R})$. It was Serre who pointed out that Ihara’s results could be interpreted in the language of graphs. Such a $\Gamma$ acts on a tree $T$, namely the Bruhat-Tits building associated to $\text{GL}(2, \mathbb{Q}_p)$. This tree plays the same role vis-a-vis $\text{GL}(2, \mathbb{Q}_p)$ as the upper half plane of complex numbers plays for $\text{GL}(2, \mathbb{R})$. The quotient $X = T/\Gamma$ is a finite graph.
Ihara discovered a connection between the zeta functions he introduced and the zeta functions of modular curves mod \( p \). For example, consider the quaternion algebra

\[
D = \mathbb{Q}[1, i, j, k]
\]

where \( i^2 = -1, j^2 = -7, k = ij = -ji \). This has discriminant 7, and so is ramified at 7 and at the archimedian place \( \infty \) of the field \( \mathbb{Q} \). Let

\[
\Gamma = \mathbb{Z} \left[ 1, \frac{1+j}{2}, \frac{i-k}{2}, 2i \right],
\]

which is a nonmaximal order in \( D \), of discriminant 14. Then \((\Gamma \otimes \mathbb{Z}[1/p])^\times\) imbeds as a discrete cocompact subgroup \( \Gamma_p \) of \( \text{GL}(2, \mathbb{Q}_p) \) for every \( p \neq 2, 7 \). For \( p = 13 \) we can take

\[
i \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -7 & 0 \end{pmatrix}
\]

which is valid since \( \sqrt{-1} \in \mathbb{Q}_{13} \).
Ihara’s zeta function for the group $\Gamma_p$ is essentially equal to the zeta function of the modular curve $X_0(14) \mod p$. In the important factor $\det(1 - Au + pu^2)$, the matrix $A$ can be identified with a Brandt matrix $B(p)$ for the quaternion order $\Gamma$. This matrix in turn has eigenvalues $p + 1$ and $c_p$, where $c_p$ is the eigenvalue of the Hecke operator $T_p$ acting on the space of cusp forms of weight 2 for $\Gamma_0(14)$. We have seen that this gives the zeta of the modular curve. Example:

$$B(13) = \begin{pmatrix} 2 & 12 \\ 6 & 8 \end{pmatrix} \sim \begin{pmatrix} 14 & 0 \\ 0 & -4 \end{pmatrix}$$

$B(13)$ can be interpreted as the adjacency matrix of a 14-regular digraph, namely:

![Diagram](image)
Another example: $X_0(37)$. This has genus 2. An equation is 
\[ y^2 = -x^6 - 9x^4 - 11x^2 + 37 \] (Mazur and Swinnerton-Dyer).

The quaternion algebra is

\[ D = \mathbb{Q}[1, i, j, k] \]

\[ i^2 = -2, \ j^2 = -37, \] and the order, maximal this time is,

\[ \Gamma = \mathbb{Z}\left[1, \frac{1 + i + j}{2}, j, \frac{2 + i + k}{4}\right] \]

The essential part of the zeta function of $X_0(37)$ mod $p$ is given by the Brandt matrix $B(p)$, and these can be interpreted as adjacency matrices of (weighted) graphs. For instance:

\[
B(3) = \begin{pmatrix}
2 & 1 & 1 \\
1 & 0 & 3 \\
1 & 3 & 0
\end{pmatrix}
\]

which has eigenvalues 4, 1, -3, so that

\[
Z\left(X_0(37)/\mathbb{F}_3, u\right) = \frac{(1 - u + 3u^2)(1 + 3u + 3u^2)}{(1 - u)(1 - 3u)}
\]
The points of $X_0(37)$ over $\mathbb{F}_3$ are

$$(2, \pm 1), \ (1, \pm 1), \ (0, \pm 1)$$

and the points over $\mathbb{F}_9$ are

$$(2, \pm 1), \ (1, \pm 1), \ (0, \pm 1)$$

$$\left(2\sqrt{-1}, \pm 1\right), \ \left(\sqrt{-1}, \pm 1\right),$$

$\infty_1 \quad \infty_2$

The unique point on the line at infinity $z = 0$ on the projectivization of our equation

$$y^2z^4 = -x^6 - 9x^4z^2 - 11x^2z^4 + 37z^6$$

is singular. Resolving this singularity gives 2 points, but rational over $\mathbb{F}_9 = \mathbb{F}_3(\sqrt{-1})$. 
$X_0(37)$ at the prime 13: Brandt matrix

$$B(13) = \begin{pmatrix} 2 & 6 & 6 \\ 6 & 3 & 5 \\ 6 & 5 & 3 \end{pmatrix}, \quad \text{eigenvalues : 14, } -2, -4$$

Zeta function:

$$Z(X_0(37)/\mathbb{F}_{13}, u) = \frac{(1 + 2u + 13u^2)(1 + 4u + 13u^2)}{(1 - u)(1 - 13u)}$$

Points in $\mathbb{F}_3$

$$(1, \pm 4) \quad (3, \pm 1) \quad (\pm 4, 0) \quad (5, \pm 1) \quad (6, \pm 4)$$

$$(7, \pm 4) \quad (8, \pm 1) \quad (12, \pm 4) \quad (10, \pm 1) \quad \infty_{1,2}$$

There are 202 points in $\mathbb{F}_{13^2}$. 

[Diagram showing a graph with labeled vertices and edges]
4. Zeta functions of dynamical systems

Selberg introduced the zeta functions of discrete cocompact subgroups of $\Gamma \subset \text{SL}(2, \mathbb{R})$ as an application of the trace formula that bears his name. His definition is

$$Z(\Gamma, s) = \prod_{k=0}^{\infty} \prod_{\gamma \text{ prime}} (1 - e^{-(s+k)l(\gamma)})$$

where the product extends over all the primitive closed geodesics $\gamma$ on the Riemann surface $X = H/\Gamma$, and $l(\gamma)$ denotes its length. He established the analytic continuation and functional equation of this as well as a type of Riemann hypothesis. Write this as

$$Z(\Gamma, s) = R(s)R(s + 1)R(s + 2)\ldots$$

where

$$R(s) = R(\Gamma, s) = \prod_{\gamma \text{ prime}} (1 - e^{-s l(\gamma)})$$

is the Ruelle zeta function. It was this function that Ihara took as a template for his zeta function of discrete subgroups of $p$-adic groups.
There have been attempts to extend the idea of Selberg/Ihara zeta functions to other groups besides SL(2). One hope here is to generalize the connection to modular curves so as to say something about the zeta functions of other Shimura varieties (e.g., Siegel modular varieties). In the $p$-adic case, this would amount to studying the action of discrete groups on the Bruhat-Tits building, now a higher dimensional object. No one has seriously attacked this problem. The progress has been in the case of real Lie groups, and here the leading ideas have come from the theory of dynamical systems and foliations.

Selberg’s zeta can be interpreted as the zeta function of a flow. Let $M$ be a compact connected manifold with a smooth flow

$$\Phi : M \times \mathbb{R} \rightarrow M$$

which admits an invariant complementary distribution.
Then:

1. The set of periods of closed orbits of $\Phi$ is a discrete subset of $\mathbb{R}$.

2. The periodic set breaks up into connected components, each of which is compact.

3. For each such component $C'$, there is a Fuller index $\text{ind}(C') \in \mathbb{Q}$.

4. All orbits in $C'$ have the same period, $l(C')$.

Therefore one can define the zeta function of a flow (D. Fried)

$$Z(\Phi, s) = \exp \left( - \sum_{C} \text{ind}(C') e^{-sl(C)} \right)$$

In many cases this can be shown to have good analytic properties and its value at $s = 0$ can be related to Reidemeister torsion.
Definition of the Fuller index in a simple special case: Let $\gamma$ be an isolated closed orbit of period $t$. For $x \in \gamma$ choose a codimension 1 submanifold $S$ transverse to the flow $\Phi$. We define the local return map $r$, which maps a neighborhood $U$ of $x$ in $S$ to $S$ and fixes $x$. Namely, for $y \in U$, $r(y)$ is defined by following the trajectory of the point $y$ under $\Phi$ until it returns to $S$. Assuming $x$ to be an isolated fixed point of $r$ we have its Lefschetz number, $\text{ind}_L(r)$, intersection number of the graph of $r$ with the diagonal of $S$. The orbit $\gamma$ has a multiplicity $m = \text{mult}(\gamma)$ defined as the least positive integer such $t/m$ is a period of $\gamma$. Then the Fuller index is defined as

$$\text{ind}_F(\gamma) = \frac{\text{ind}_L(r)}{\text{mult}(\gamma)}$$
Special case: Let $M$ be the unit sphere bundle $SX$ of a Riemannian manifold $X$. This carries a canonical geodesic flow. If $(x, \vec{v}) \in SX$, let $\alpha$ be the geodesic through $x$ with tangent $\dot{\alpha}(0) = \vec{v}$ then

$$\Phi_t(x, \vec{v}) = (\alpha(t), \dot{\alpha}(t))$$
Let $M = SX$ be the unit sphere bundle to the locally symmetric manifold $\Gamma \backslash G/K$ and $\Phi$ the geodesic flow. Then

1. The connected components of the periodic set are parametrized by the nontrivial conjugacy classes $[\gamma]$ in $\Gamma$.

2. Each connected component $X_\gamma$ is itself a locally symmetric manifold and $\Phi$ restricts to a periodic flow on it.

3. The quotient $\hat{X}_\gamma = X_\gamma / \Phi$ is an orbifold and the Fuller index is

$$\text{ind}(X_\gamma) = \chi(\hat{X}_\gamma) / \mu_\gamma$$

where $\chi$ is the orbifold Euler characteristic and $\mu_\gamma$ is the multiplicity of a generic orbit of $\Phi$ in $X_\gamma$.

In this case the zeta function takes on a familiar form:

$$Z(\Phi, s) = \exp \left( - \sum_{[\gamma] \neq 1} \chi(\hat{X}_\gamma) u^{l_\gamma} / \mu_\gamma \right), \quad u = e^{-s}$$

Selberg’s zeta function for the group $\Gamma$ is equal to the zeta function of the geodesic flow on $SX$, where $X$ is the compact Riemann surface $\Gamma \backslash H$, where $H$ = the upper half-plane.
5. Speculations and hopes

A number of mathematicians: Gangolli and Warner, Moscovici and Stanton, D. Fried, Ruelle, Patterson, Deitmar, ..., have proved various good properties of these zeta functions in special cases. The theory works best in the case of compact locally symmetric manifolds of split rank 1. For higher ranks, the situation is still unclear.

My hope is to be able to compute the zeta functions of Shimura varieties combinatorially via a generalization of the zeta function of a graph. In the 1970’s Langlands and Rapoport gave a conjectural generalization of part of Ihara’s work - that of describing ‘explicitly” the rational points of Shimura varieties in finite fields. For Shimura varieties arising from moduli problems of abelian varieties, Kottwitz (and independently Reimann and Zink) proved unconditionally a formula for the number of these rational points, thereby computing in principle the zeta function of said varieties. These formulas are extremely complicated, and to my knowledge, aside from the classical case of modular curves, they have not been evaluated in any particular example. The idea is to use the higher dimensional version of graphs, Bruhat-Tits buildings, to give a more intuitive and geometric version of these results.