Algebraic curves of $GL_2$-type

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1. Galois representations of GL$_2$-type

2. Explicit moduli for genus two curves

3. Humbert surfaces

4. Examples
For any number field $K$ let $G_K = \text{Gal}(\overline{Q}/K)$. Let

$$\rho : G_Q \to \text{GL}(V_\ell) \cong \text{GL}_{2d}(\mathbb{Q}_\ell)$$

be a representation. Mostly we will be concerned with $d = 2$.

Assume that

1. There is a number field $E/\mathbb{Q}$ of degree $d$ and a homomorphism $E \to \text{End}(V_\ell)$.

2. There is a finite extension $K/\mathbb{Q}$ such that $\rho(G_K)$ commutes with the image of $E$.

Then we have a factorization:

$$\rho_K := \rho \mid G_K \to \text{GL}_{E \otimes \mathbb{Q}_\ell}(V_\ell) \cong \text{GL}_2(E \otimes \mathbb{Q}_\ell) \subset \text{GL}(V_\ell) \cong \text{GL}_{2d}(\mathbb{Q}_\ell).$$

We say that $\rho_K$ has an $E$-linear structure. or $\rho$ has an $\text{GL}_2$-structure over $K$. 
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We say that $\rho_K$ has an $E$-linear structure. or $\rho$ has an $\text{GL}_2$-structure over $K$. 
Example: Abelian varieties of \( \text{GL}_2 \)-type. For an abelian variety \( A/k \) let \( V_\ell(A) \) be its Tate module: a \( 2 \text{dim } A \)-dimensional representation of \( G_k \).

**Theorem.** (Ribet + Serre’s conjecture)

1. Let \( A/\mathbb{Q} \) be a \( \mathbb{Q} \)-simple abelian variety. Suppose that \( E = \mathbb{Q} \otimes \text{End}_\mathbb{Q}(A) \) is a number field of degree \( \text{dim } A \). Then the Tate module \( V_\ell(A) \) defines a representation of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) with values in \( \text{GL}_2(E \otimes \mathbb{Q}_\ell) \). Moreover \( A \) is isogenous to a \( \mathbb{Q} \)-simple factor of \( J_1(N) \) for some \( N \geq 1 \).

2. Let \( C/\overline{\mathbb{Q}} \) be an elliptic curve. Then \( C \) is a quotient of \( J_1(N)_{\overline{\mathbb{Q}}} \) for some \( N \geq 1 \) if and only if \( C \) is a \( \mathbb{Q} \)-curve, i.e., \( C \) is isogenous to each of its conjugates \( \sigma C, \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \).

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**Problem:** Explicitly construct $\mathbb{Q}$-curves.
Theorem. (Shimura) Let \( f = \sum a_n q^n \) be a normalized cuspidal Hecke eigenform of weight 2 on \( \Gamma_1(N) \) for some \( N \geq 1 \). Then there is an abelian variety \( A_f \) defined over \( \mathbb{Q} \) with an action of the field \( E = \mathbb{Q}(..., a_n, ...) \). This is a quotient of \( J_1(N) \). We have \( \dim A_f = [E : \mathbb{Q}] \), \( E = \text{End}_\mathbb{Q}(A_f) \otimes \mathbb{Q} \), and thus \( V_\ell(A_f) \) is of \( \text{GL}_2 \)-type over \( \mathbb{Q} \).

Example. \( N = 29 \). There exist Hecke eigenforms \( f, \overline{f} \in S_2(29, (*/29)) \) with field of coefficients \( E = \mathbb{Q}(\sqrt{-5}) \). Shimura’s \( A_f \) is two-dimensional, and in fact it is isogenous to \( C \times \sigma C \) for an elliptic \( \mathbb{Q} \)-curve \( C/\mathbb{Q} (\sqrt{5}) \). Also we have an isogeny \( A_f \cong \text{Jac}(X) \) where \( X \) is the genus 2 curve

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Note that for a given $\rho$, there can be more than one pair $E, K$ such that $\rho_K$ has an $E$-linear structure.

Examples are given by quaternion structures. Here $d = 2$. That is, assume:

1. there is a quaternion algebra $B/\mathbb{Q}$ and a homomorphism $B \rightarrow \text{End}(V_\ell)$ and
2. a finite extension $K/\mathbb{Q}$ such that the image of $B$ commutes with $\rho_K$.

Then $\rho_K$ has an $E$-linear structure for every quadratic subfield $E \subset B$. 

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Suppose $\rho$ has an $E$-linear structure defined over $K$ (so $[E : \mathbb{Q}] = 2$).

Let $\lambda, \overline{\lambda}$ be the places of $E$ lying over the prime number $\ell$ (possibly $\lambda = \overline{\lambda}$). Then we get a decomposition $E \otimes \mathbb{Q}_\ell = \oplus_{\lambda, \overline{\lambda}} E_\lambda$ hence

$$\rho_K \otimes E = \rho_{K, \lambda} \oplus \overline{\rho}_{K, \lambda}, \quad L(\rho_K, s) = L(\rho_{K, \lambda}, s)L(\overline{\rho}_{K, \lambda}, s).$$
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Special case: Suppose that:

1. $K/\mathbb{Q}$ is also a quadratic extension.
2. The generator $s \in \text{Gal}(K/\mathbb{Q})$ induces an isomorphism $s : \rho_{K, \lambda} \cong \rho_{K, \lambda}$.

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Then

$$V_\ell = H^1(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell) = H^1(Jac(X) \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$$

gives a four dimensional representation of $G_{\mathbb{Q}}$. These are of $GL_2$-type if the Jacobian $Jac(X)$ has extra endomorphisms.

For any abelian variety $A$ defined over a field $k$ we let

$End^0_k(A) := \text{End}_k(A) \otimes \mathbb{Q}$

be the semisimple $\mathbb{Q}$-algebra of endomorphisms defined over $k$; $End^0(A) = End^0_k(A)$.
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Then if $A$ is a two dimensional absolutely simple abelian variety defined over a number field $k$ then $\text{End}^0(A)$ is one of the following:

1. $\mathbb{Q}$;
2. a real quadratic field $E/\mathbb{Q}$;
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Items 2 (RM) and 3 (QM) lead to representations of $\text{GL}_2$-type.

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The familiar invariants of an elliptic curve, e.g., $j$, $g_2$, $g_3$ arise as invariants and covariants of the action of $\text{PGL}(2)$ on binary quartic forms $f(x, y) = a_0 x^4 + a_1 x^3 y + a_2 x^2 y^2 + a_3 x y^3 + a_4 y^4$. These were worked out in the 19th century.

Reason: every genus 1 curve can be expressed as a double cover of $\mathbb{P}^1$ with four branch points. Then $f(x, y) = 0$ give the coordinates of the branch points. This leads to coordinate systems on the moduli spaces of elliptic curves: these are the modular curves. Riemann already knew that the moduli space $\mathcal{M}_g$ of genus $g \geq 2$ curves had dimension $3g - 3$, but explicit models of these as algebraic varieties are not easy to find.
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$$y^2 = f(x) \quad \text{deg } f = 6, \text{ with distinct roots.}$$

As for elliptic curves, the moduli for genus 2 curves then can be expressed via the in- and covariants of the action of $\text{PGL}(2)$ on binary sextic forms. The expression of the moduli of genus 2 curves via projective invariants of binary sextics was done by Clebsch, and in more modern times by Igusa and Mestre.

Note that $\dim \mathcal{M}_2 = 3$, so there are three “J-invariants” $j_1, j_2, j_3$. Recall also that the map $X \mapsto \text{Jac}(X) : \mathcal{M}_2 \to \mathcal{A}_2$ to the moduli space of principally polarized abelian varieties of dimension 2 is a birational correspondence.
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Analytic moduli.
Let
\[ \mathcal{H}_2 = \{ \tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \text{M}_2(\mathbb{C}) \mid \text{Im}(\tau) > 0 \} \]
the Siegel space of genus 2.
As an analytic space, \( \mathcal{A}_2^{an} = \Gamma \backslash \mathcal{H}_2 \) where
\[ \Gamma = \{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_4(\mathbb{Z}) \} \]
acting via \( \tau \mapsto (A\tau + B)(C\tau + D)^{-1} \).
We get coverings by taking subgroups of finite index in \( \Gamma \) (congruence subgroups!)
Analytic moduli.

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\[ \mathcal{H}_2 = \{ \tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \text{M}_2(\mathbb{C}) \mid \text{Im}(\tau) > 0 \} \]

the Siegel space of genus 2.

As an analytic space, \( \mathcal{A}_2^{an} = \Gamma \backslash \mathcal{H}_2 \) where

\[ \Gamma = \{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_4(\mathbb{Z}) \} \]

acting via \( \tau \mapsto (A\tau + B)(C\tau + D)^{-1} \).

We get coverings by taking subgroups of finite index in \( \Gamma \) (congruence subgroups!)
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Algebraic moduli. A natural set of coordinates on the covering of $M_2$ given by level 2 structure is gotten by taking the cross ratios of the roots $e_i$, $i = 1, ..., 6$ of

$$y^2 = f(x) = (x - e_1) ... (x - e_6).$$

level 2 structure = an ordering of the 6 roots $e_1, ..., e_6$:
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We are interested in the subvariety $H_\Delta \subset \mathcal{M}_2$ of those genus 2 curves $X$ where $\text{Jac}(X)$ has endomorphisms by an order in a real quadratic field $\mathbb{Q}(\sqrt{\Delta})$. We have $\dim H_\Delta = 2$ and these are called Humbert surfaces.

1. When the integer $\Delta$ is a square, $H_\Delta$ is a product of modular curves; $\text{Jac}(X)$ factors into 2 elliptic curves for $X \in H_\Delta$.

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Explicit equations for Humbert surfaces were written down for \( \Delta = 5, 8 \) by G. Humbert. Modern treatment given by P. Bending, Hashimoto, Hirzebruch, Murabayashii, R. M. Wilson, Sakai, Shephard-Barron, R. Taylor, van der Geer.

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The analytic equations of Humbert surfaces are quite simple: Each point of $H_\Delta$ is $Sp_4(\mathbb{Z})$-equivalent to a point $\tau \in \mathfrak{H}_2$ which satisfies

$$a\tau_1 + b\tau_2 + \tau_3 = 0, \ a, b, \in \mathbb{Z}, \ b^2 - 4a = \Delta, \ b = 0 \ or \ 1.$$  

We want equations in the algebraic moduli. Humbert’s construction is based on Poncelet’s theorem. Given two projective plane conics $C$ and $D$, if an $n$-gon can be inscribed in $C$ in such a way that each edge of the polygon is tangent to $D$ (i.e., the $n$-gon is circumscribed about $D$) then, given any point $P \in C$, there is an $n$-gon inscribed in $C$ and circumscribed about $D$ which passes through $P$. 
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Algebraic curves of GL$_2$-type
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Proof (Cayley): Consider the dual projective plane: \((P^2)^* = \) the variety of lines in \(P^2\).
Let \(D^* \subset (P^2)^*\) be the variety of lines tangent to \(D\). This is also a conic.
Define
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E(C, D) := \{(P, \ell) \in C \times D^* \mid P \in \ell\},
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the incidence correspondence.
Then \(E(C, D)\) is an elliptic curve, and a Poncelet \(n\)-gon corresponds to a point of order \(n\) on \(E(C, D)\).
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Proof (Cayley): Consider the dual projective plane: 

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Humbert, Mestre: Given a Poncelet $n$-gon, one constructs a hyperelliptic curve $X$ together with an action of $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ on $\text{Jac}(X)$. $X$ is a double cover of $C$ branched over the vertices of the $n$-gon. When $n = 5, 8$ this curve $X$ has genus 2, with endomorphisms by $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{2})$ respectively. $\Delta = 8: -2\tau_1 + 2\tau_3 = 0$. This corresponds to a Poncelet 4-gon.
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$\Delta = 8$: $-2\tau_1 + 2\tau_3 = 0$. This corresponds to a Poncelet 4-gon.
Quadrilateral $\alpha\beta\gamma\delta$ is inscribed on conic $C$, tangent to conic $D$.

Genus 2 curve $X$ is double cover of $C$ branched over $p, q, \alpha, \beta, \gamma, \delta$.

**Humbert 8 = Poncelet 4**
This configuration leads to the explicit equation for $H_8 = 0$ in terms of the roots $e_i$ of the sextic $f(x)$ where the genus 2 curve is represented by $y^2 = f(x)$.

$$H_8(e_1, \ldots, e_8) =$$

$$(e_3 - e_1)(e_3 - e_2)(e_3 - e_4)(e_4 - e_2)^2(e_3 - e_5)(e_6 - e_1)(e_6 - e_2)(e_6 - e_4)(e_6 - e_5)(e_1 - e_5)^2$$

$$+ (e_1 - e_2)(e_1 - e_3)(e_1 - e_4)(e_2 - e_4)^2(e_3 - e_5)^2(e_6 - e_2)(e_6 - e_3)(e_6 - e_4)(e_6 - e_5)(e_1 - e_5)$$

$$+ (e_3 - e_1)^2(e_4 - e_1)(e_4 - e_2)(e_4 - e_3)(e_2 - e_5)^2(e_4 - e_5)(e_6 - e_1)(e_6 - e_2)(e_6 - e_3)(e_6 - e_5)$$

$$+ (e_2 - e_1)(e_1 - e_3)^2(e_2 - e_3)(e_2 - e_4)(e_2 - e_5)(e_4 - e_5)^2(e_6 - e_1)(e_6 - e_3)(e_6 - e_4)(e_6 - e_5)$$

$$+ 16(e_1 - e_2)(e_2 - e_3)(e_1 - e_4)(e_3 - e_4)(e_2 - e_5)(e_3 - e_5)$$

$$(e_4 - e_5)(e_1 - e_6)(e_2 - e_6)(e_3 - e_6)(e_4 - e_6)(e_1 - e_5)$$
\[ \Delta = 5: -\tau_1 + \tau_2 + \tau_3 = 0. \] This corresponds to a Poncelet 5-gon.
Pentagon $\alpha \beta \gamma \delta \varepsilon$
inscribes conic $C$
circumscribes conic $D$

Genus 2 curve $X$ is the
double cover of $C$ branched
above $\alpha, \beta, \gamma, \delta, \varepsilon$ and
a point $q$ in $C$ intersect $D$.

The correspondence

$P \rightarrow P' + P''$

lifts to a correspondence

$\phi$ of $X$ with $\phi^2 + \phi - 1 = 0$
in $\text{Jac}(X)$. 
This configuration leads to the explicit equation for $H_5 = 0$ in terms of the roots $e_i$ of the sextic $f(x)$ where the genus 2 curve is represented by $y^2 = f(x)$.

$$H_5(e_1, ..., e_6) =$$

$$(e_2 - e_3)^2 (e_5 - e_1) (e_5 - e_2) (e_5 - e_3) (e_5 - e_4) (e_1 - e_6) (e_2 - e_6) (e_3 - e_6) (e_4 - e_6) (e_1 - e_4)^2$$

+$$(e_1 - e_2) (e_1 - e_3) (e_3 - e_4)^2 (e_1 - e_5) (e_2 - e_5)^2 (e_2 - e_6) (e_3 - e_6) (e_4 - e_6) (e_5 - e_6) (e_1 - e_4)$$

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Let $B/\mathbb{Q}$ be an indefinite quaternion algebra. Two-dimensional principally polarized abelian varieties $A$ with $B \subset \text{End}(A)$ are parametrized by the points of the Shimura curve $S_B$ associated to $B$. There is a universal family $X \to S_B$ of genus two curves which have Jacobians that have QM by $B$.

Problem. Find explicit equations for these universal families, for small values of the discriminant of $B$.

So far the only known examples are for discriminants 6, 10.
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The idea of Hashimoto and Murabayashi is to consider the intersection of Humbert surfaces.
If a simple abelian surface \( A \) has \( \text{End}^0(A) \supset \mathbb{Q}(\sqrt{\Delta_1}), \mathbb{Q}(\sqrt{\Delta_2}) \) for two different real quadratic fields, then \( \text{End}^0(A) \) is a quaternion algebra. Thus

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$$H_{\Delta_1} \cap H_{\Delta_2} = \text{union of Shimura curves}.$$
Example 1 (Brumer/Hashimoto) Let

\[ f(X; a, b, c) = X^6 - (4 + 2b + 3c)X^5 + (2 + 2b + b^2 - ac)X^4 \]
\[ - (6 + 4a + 6b - 2b^2 + 5c + 2ac)X^3 \]
\[ + (1 + b^2 - ac)X^2 + (2 - 2b)X + (c + 1) \]

Let \( C(a, b, c) : Y^2 = f(X; a, b, c) \), assume \( f(X, a, b, c) \) has 6 distinct roots. Then \( C(a, b, c) \) is a genus two curve with RM by \( \mathbb{Q}(\sqrt{5}) \). These endomorphisms are defined over \( \mathbb{Q}(a, b, c) \). Hence if \( a, b, c, \in \mathbb{Q} \), the curve is modular.
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Example 2 (P. Bending) Let $K \subset \mathbb{C}$ be a field. Let $A, P \neq 0, Q, D \neq 0$ in $K$. Define

$$B = \frac{Q(PA - Q) + 4P^2 + 1}{P^2}, \quad C = \frac{4(PA - Q)}{P}.$$ 

Define $\alpha_i$ by $\prod_{i=0}^{2}(X - \alpha_i) = X^3 + AX^2 + BX + C$. Then the genus 2 curve

$$Y^2 = D \prod_{i=0}^{2}(X^2 - \alpha_iX + P\alpha_i^2 + Q\alpha_i + 4P)$$

is defined over $K$ and has RM by $\mathbb{Q}(\sqrt{2})$, also defined over $K$. Hence if $K = \mathbb{Q}$, these curves are modular.
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$$Y^2 = D[(N-1)X^6 - 6NX^5 + 3(N+1)X^4 - 8N^2X^3 + 3(N-1)X^2 + 6N + N+1]$$

Then $\text{End}_K^0(\text{Jac}(C))$ contains the division quaternion algebra $B_6$ of discriminant 6, namely $\left(\frac{2,-3}{\mathbb{Q}}\right)$. The action of $\mathbb{Q}(\sqrt{2}) \subset B_6$ is defined over $K$ and the action of $\mathbb{Q}(\sqrt{-3}) \subset B_6$ is defined over $K(\sqrt{-3})$. Hence if $N, D \in \mathbb{Q}$ these are modular.
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$$Y^2 = D[(N-1)X^6 - 6NX^5 + 3(N+1)X^4 - 8N^2X^3 + 3(N-1)X^2 + 6N + N+1]$$

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Example 4 (QM by $B_{10}$) Let

$$C : y^2 = x^6 - 16x^5 + 40x^4 + 140x^3 + 80x^2 - 64x + 64.$$  

Then $\text{Jac}(C)$ has multiplications by the quaternion division ring of discriminant 10. We have

$$\det(X - \rho(\text{Frob}_p)) = g_p(X)\overline{g_p(X)}$$

for $g_p(X) \in \mathbb{Q}(\sqrt{5})[X]$. 

Jerome William Hoffman  
Louisiana State University

Algebraic curves of $\text{GL}_2$-type
<table>
<thead>
<tr>
<th>$p$</th>
<th>$g_p(X)$</th>
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<tr>
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<tr>
<td>11</td>
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<td>29</td>
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<tr>
<td>43</td>
<td>$X^2 + 43$</td>
</tr>
<tr>
<td>47</td>
<td>$X^2 + 5\sqrt{5}X + 47$</td>
</tr>
</tbody>
</table>
Example 5 (RM by $\mathbb{Q}(\sqrt{2})$) Let

$$C : y^2 = 21x^5 + 50x^4 + 5x^3 - 20x^2 + 4x.$$  

Then $\text{Jac}(C)$ has multiplications by $\mathbb{Q}(\sqrt{2})$, but the endomorphisms are defined over $\mathbb{Q}(\sqrt{2})$. The $\ell$-adic representation here is not obviously automorphic. The conductor is $2^23^27^2$. This is a special case of a 1-parameter family of Hashimoto/Sakai.
<table>
<thead>
<tr>
<th>$p$</th>
<th>$\det(X - \rho(\text{Frob}_p))$</th>
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<tbody>
<tr>
<td>5</td>
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<td>$(X^2 + 4\sqrt{2}X + 11)(X^2 - 4\sqrt{2}X + 11)$</td>
</tr>
<tr>
<td>13</td>
<td>$X^4 + 6X^2 + 13^2$</td>
</tr>
<tr>
<td>17</td>
<td>$(X^2 + (4\sqrt{2} - 2)X + 17)(X^2 + (-4\sqrt{2} - 2)X + 17)$</td>
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<td>19</td>
<td>$X^4 - 10X^2 + 19^2$</td>
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<td>$(X^2 + 4\sqrt{2}X + 23)(X^2 - 4\sqrt{2}X + 23)$</td>
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<td>29</td>
<td>$(X^2 + 6\sqrt{2}X + 29)(X^2 - 6\sqrt{2}X + 29)$</td>
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<tr>
<td>31</td>
<td>$(X^2 + 31)^2$</td>
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<tr>
<td>37</td>
<td>$(X^2 + 10X + 37)(X^2 - 10X + 37)$</td>
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<tr>
<td>41</td>
<td>$(X^2 + (4\sqrt{2} - 2)X + 41)(X^2 + (-4\sqrt{2} - 2)X + 41)$</td>
</tr>
<tr>
<td>43</td>
<td>$X^4 + 3X^2 + 43^2$</td>
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<tr>
<td>47</td>
<td>$(X^2 - 8X + 47)^2$</td>
</tr>
</tbody>
</table>
Example 6 (RM by $\mathbb{Q}(\sqrt{2})$) Let

$$C : y^2 = 9x^5 - 210x^4 + 165x^3 + 16740x^2 - 74844x$$

Then $\text{Jac}(C)$ has multiplications by $\mathbb{Q}(\sqrt{2})$, but the endomorphisms are defined over $\mathbb{Q}(\sqrt{-2})$. The $\ell$-adic representation here is not obviously automorphic. The conductor is $2^33^77^111$. This is special case of a 1-parameter family of Hashimoto/Sakai.
<table>
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<tr>
<th>$p$</th>
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<tr>
<td>29</td>
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<td>41</td>
<td>$(X^2 + (4\sqrt{2} + 6)X + 41)(X^2 + (-4\sqrt{2} + 6)X + 41)$</td>
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<tr>
<td>43</td>
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<td>47</td>
<td>$(X^2 + 8X + 47)(X^2 - 8X + 47)$</td>
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</tbody>
</table>
Thanks to Ramin Takloo-Bighash and Winnie Li!