HUMBERT SURFACES AND HILBERT MODULAR SURFACES

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Abstract. This is a summary of the main properties of the subvarieties of the Siegel modular threefolds that are the loci where the corresponding abelian surface admits nontrivial endomorphisms.

1. Introduction

We want to systematically describe the subvarieties of the moduli spaces of abelian surfaces (with various level structures) where the corresponding abelian variety admits nontrivial endomorphisms. There are countable many such subvarieties. The analysis that follows is due to G. Humbert, who wrote a series of long articles on this around the turn of the century. We follow the exposition of Hirzebruch and van der Geer [5], [4] closely, much of which in turn is taken from Franke's thesis [1].

The key to Humbert's approach is to view the moduli space of abelian surfaces in a different way. As is well - known, there is essentially an isomorphism (ie., an isogeny) between Sp(4) and SO(5), which is visible at the level of Dynkin diagrams. Here we are thinking of the complex Lie groups; over the reals this becomes an isogeny between Sp(4, $\mathbf{R}$) and SO(3, 2). Humbert had the audacity to view the moduli of abelian surfaces from the orthogonal, as opposed to the more orthodox symplectic, perspective. This gives new insights.

Throughout this paper we use the following well-known principle: Let a group G act transitively on the left on a set X. Picking any basepoint $x \in X$ we get a G-set isomorphism

$$G/G_x = X; \quad gG_x \to g \cdot x; \quad G_x = \{g \in G : g \cdot x = x\}$$

The set of orbits $N\backslash X$ of a subgroup$N \subset G$ is thus identified with the set of double cosets $N\backslash G/G_x$. If N is a normal subgroup, this can be identified with the coset space

$$G/NG_x = \bar{G}/\bar{G}_x; \quad \bar{G} = G/N, \ \bar{G}_x = G_x/N \cap G_x.$$

In particular, if $\bar{G}$ is finite, then $\#(N\backslash X) = [\bar{G} : G_x/N \cap G_x]$.

2. The isomorphism $\mathrm{Sp}(4) = \mathrm{SO}(3,\,2)$

Let $V_{\mathbf{Q}} = \mathbf{Q}^4$ with standard basis $e_i,\ i = 1, ..., 4$ and let J be the alternating bilinear form

$$\langle x,\,y\rangle = \langle x,\,y\rangle_J = {}^t x J y, \text{ where } J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

The group of symplectic similitudes is defined as:

$$\mathrm{GSp}(V_{\mathbf{Q}},\,J) = \mathrm{GSp}(J) = \mathrm{GSp}(4) = \{g \in \mathrm{Mat}(4,\,4;\,\mathbf{Q}) \mid {}^t gJg = \eta(g)J\}$$

where $\eta(g) \in \mathbf{Q}^*$, called the multiplier of g. The map $g \to \eta(g)$ is a homomorphism $\eta : \mathrm{GSp}(V_{\mathbf{Q}},\,J) \to \mathbf{Q}^*$ whose kernel is the symplectic group $\mathrm{Sp}(V_{\mathbf{Q}},\,J)$. We can regard $\mathrm{GSp}(J)$ as defining a group scheme over $\mathbf{Z}$, whose points in any ring R, denoted $\mathrm{GSp}(J,\,R)$ or $\mathrm{GSp}(4,\,R)$, is defined by the same formulas as above, but with matrix entries in R, with $\eta(g) \in R^*$ being a unit. The same definitions can be given for any nondegenerate alternating bilinear form, but recall that (over a field) these are all equivalent by a change of coordinates.

Consider the symmetric bilinear form

$$b : \bigwedge^2 V_{\mathbf{Q}} \times \bigwedge^2 V_{\mathbf{Q}} \to \bigwedge^4 V_{\mathbf{Q}} \xrightarrow{\det} \mathbf{Q}$$

where the first arrow is wedge product and the second is the isomorphism defined by $\det(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$. Clearly, the natural action of $\mathrm{GL}(4,\mathbf{Q})$ on $\bigwedge^2 V_{\mathbf{Q}}$ preserves this quadratic form up to scalars, namely the determinant homomorphism. The subgroup $\mathrm{GSp}(J)$ stabilizes the line spanned by $\psi = e_1 \wedge e_3 + e_2 \wedge e_4$ and thus we have a representation on the orthogonal complement relative to b_0:

$$\alpha : \mathrm{GSp}(J) \to \mathrm{GO}_0(\psi^{\perp},\,b_0 \mid \psi^{\perp}) := \mathrm{GO}_0(b_{0,\psi}),$$

where the group $\mathrm{GO}_0(b_{0,\psi})$ is the connected component of the group of orthogonal similitudes. Restricting this to the subgroup $\mathrm{Sp}(J)$ One knows:

Proposition 2.1. *α induces an isomorphism*

$$\alpha : \mathrm{Sp}(J)/\{\pm 1\} \to \mathrm{SO}_0(b_{0,\psi})$$

One checks that the 5-dimensional quadratic form $b_{0,\psi}$ has signature $(3,2)$. For later purposes we want a geometric description of this. Consider, for any ring R, the module $V_R = R^4$ with the standard alternating form J. Viewed this way, V is the affine space $\mathbf{A}^4$ regarded as a scheme over the integers. It is known that the Grassmannian of 2-planes through the origin $\mathbf{Gr}_2(V) = \mathbf{Gr}(2,4)$, or equivalently the Grassmannian of lines in $\mathbf{P}^3$, $\mathbf{Gr}_1(\mathbf{P}(V)) = \mathbf{PGr}(1,3)$ is canonically imbedded

$$\mathbf{Gr}_2(V) = \mathbf{Gr}(2,4) \ \longrightarrow\ \mathbf{P}\left(\bigwedge^2 V\right) = \mathbf{P}^5$$

whose image is a quadric. In coordinates (x_1, x_2, x_3, x_4) on $\mathbf{A}^4$, a plane $h = x \wedge y$ is mapped to the vector of 2 by 2 minors

$$p_{ij}(h) \;=\; \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

The quadric is given by Plücker's relation

$$Q(p) \;=\; p_{12}p_{34} \;-\; p_{13}p_{24} \;+\; p_{14}p_{23} \;=\; 0$$

The alternating form J is given by the formula $p_{13}+p_{24}$, therefore a 2-plane through the origin $\mathbf{A}^4$ is isotropic for this skew-form iff $p_{13}(h)+p_{24}(h)=0$. In other words, the variety of isotropic lines in $\mathbf{P}^3$ is given by these two equations. We can think of this scheme $\mathbf{Gr}_2(V, J)$ as the 3-dimensional quadric in $[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}]$ - space qiven by

$$q(p) \;=\; p_{13}^2 \;+\; p_{12}p_{34} \;+\; p_{14}p_{23} \;=\; 0$$

This variety is the dual symmetric space of the group $G = \mathrm{Sp}(J)$ and is thus isomorphic with G/Q where Q is the parabolic subgroup stabilizing any isotropic 2-plane.

In any ring where 2 is a unit this is easily seen to be equivalent to the form $u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2$, of signature (3, 2).

Now $\mathrm{GSp}(V,\ J)$ operates on V fixing J up to scalar multiple, hence it acts on $\wedge^2 V$ fixing the subvarieties $Q(p) = 0$, $p_{13}+p_{24} = 0$, and hence fixing the variety $q(p) = 0$. We therefore obtain a homomorphism from the symplectic similitudes to the orthogonal similitudes of q, but as $\mathrm{GSp}(J)$ is connected, we get a map

$$\mathrm{GSp}(V, J) \;\longrightarrow\; \mathrm{GO}_0(q)$$

which is an isogeny. The quadratic form q is essentially $b_{0,\psi}$ above. More precisely, consider the wedge product

$$\bigwedge^2 V \times \bigwedge^2 V \;\longrightarrow\; \bigwedge^4 V \;\simeq \mathbf{G}_a$$

where the last isomorphism is gotten by the determinant. Choose any basis $\{e_i\}$ of $V_{\mathbf{Z}} = \mathbf{Z}^4$. Then it is easily seen that the quadratic form associated to this pairing is

$$2Q(a) = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})$$

Consider $\psi = e_1 \wedge e_3 + e_2 \wedge e_4$. Then the orthogonal $U = \psi^{\perp}$ carries the induced form $2(b^2 - ac - de)$ in the basis

$$\{f_1, f_2, f_3, f_4, f_5\} = \{e_1 \wedge e_2, e_1 \wedge e_3 - e_2 \wedge e_4, -e_3 \wedge e_4, -e_1 \wedge e_4, e_2 \wedge e_3\}$$

For our purposes it is better to work with the form on the dual lattice spanned by $\{f_1, (f_2)/2, f_3, f_4, f_5\}$, or rather twice it, given in this basis as

$$\Delta(a, b, c, d, e) \;=\; b^2 - 4ac - 4de$$

This is the quadratic form that appears in the theory of Humbert surfaces.

There is another viewpoint on this isomorphism that will be useful.

Given $\eta = \frac{1}{2}\sum r_{ij} e_i \wedge e_j \in \bigwedge^2 V_{\mathbf{Q}}$ with $r_{ji} = -r_{ij}$, we can associate the skew-symmetric matrix $R_\eta = R = (r_{ij}) \in \mathrm{Skew}(4, \mathbf{Q})$. This assignment sets up a $\mathrm{GL}(4, \mathbf{Q})$-equivariant isomorphism between the alternating 2-tensors and these skew-symmetric matrices, with the action of $g \in \mathrm{GL}(4, \mathbf{Q})$ given on the skew-symmetric matrices as

$$R \to gR^t g.$$

The form ψ above maps to J. We can use this isomorphism to transfer the symmetric bilinear form b to these matrices. Orthogonal projection sets up an isomorphism

$$\mathrm{Skew}(4, \mathbf{Q})/J = J^\perp$$

so that we obtain an isomorphism

$$\mathrm{Sp}(J)/\pm 1 \to \mathrm{SO}_0(\mathrm{Skew}(4, \mathbf{Q})/J,\, b_0)$$

where b_0 is the quadratic form obtained from restriction of b to the subspace $J^\perp$ and transfered to the skew matrices via orthogonal projection.

Let $V^*_{\mathbf{Z}} = \mathrm{Hom}(V_{\mathbf{Z}}, \mathbf{Z})$, and $\varepsilon_i,\ i = 1, ..., 4$ be the dual basis to $e_i,\ i = 1, ..., 4$. We view the elements of $\mathrm{Skew}(4, \mathbf{Q})$ as alternating bilinear forms on $V^*_{\mathbf{Q}}$. Then

$$\eta(u, v) = {}^t u R_\eta v,\ \ \eta \in \bigwedge^2 V_{\mathbf{Q}},\ \ u, v \in V^*_{\mathbf{Q}}$$

where on the right hand side above we write u, v as column vectors relative to the basis ε_i.

3. Symmetric spaces

3.1. **Symplectic viewpoint.** Let $\mathfrak{S}_2$ be the Siegel upper half space of degree 2:

$$\mathfrak{S}_2 \ = \ \{\tau \in \mathrm{M}_2(\mathbf{C}) : {}^t\tau = \tau,\ \mathrm{Im}(\tau) > 0\}.$$

The group $\mathrm{Sp}(4, \mathbf{R})$ acts transitively on $\mathfrak{S}_2$ as

$$\gamma.\tau \ = \ (A\tau + B)(C\tau + D)^{-1},\ \ \gamma \ = \ \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The stabilizer of K of $i1_2$ is a maximal compact subgroup and is the set of symplectic matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix},\ \text{isomorphic to U(2) via } A + iB \in \mathrm{U}(2).$$

Thus we have an isomorphism $\mathrm{Sp}(4, \mathbf{R})/K\mathfrak{S}_2$, which realizes Siegel's space as the symmetric space of $\mathrm{Sp}(4, \mathbf{R})$. We let

$$\mathfrak{D}_2 = \{Z \in \mathrm{M}_2(\mathbf{C}) : {}^t Z = Z,\ 1_2 - Z\bar{Z} > 0\}$$

There is an isomorphism $\Phi : \mathfrak{S}_2 \to \mathfrak{D}_2$ given by $\Phi(\tau) = (\tau - i1_2)(\tau + i1_2)^{-1}$, the Cayley transform.

More intrinsically, $\mathfrak{S}_2$ is the set of 2-planes $F \subset V_{\mathbf{C}} = \mathbf{C}^4$ such that

a) F is J-isotropic, i.e., $\langle x,\, y\rangle_J = 0$ for all $x, y \in F$.
b) $-i\langle x,\, \bar{x}\rangle_J > 0$ for all $x \in F$, where $\bar{x}$ denotes conjugation relative to $V_{\mathbf{R}}$.

$\tau \in \mathfrak{S}_2$ corresponds to

slightly unorthodox convention for period matrices

$$F(\tau) = \mathbf{C}\text{−span of the rows of } \begin{pmatrix}-1_2, \tau\end{pmatrix}$$

Then $(V_{\mathbf{C}}, V_{\mathbf{Z}}, F(\tau), J)$ defines a principally polarized Hodge structure of type $(1,0) + (0,1)$. This is the Hodge structure on $\mathrm{H}^1(A_\tau, \mathbf{C})$ where A_τ is the set of $\mathbf{C}$-points of a principally polarized abelian variety of dimension 2. In fact, $A_\tau = F^*(\tau)\backslash V^*_{\mathbf{C}}/V^*_{\mathbf{Z}}$ for the dual Hodge structure $(V^*_{\mathbf{C}}, V^*_{\mathbf{Z}}, F^*(\tau), J)$ of type $(-1,0) + (0,-1)$. Here $F^*(\tau)$ is the subspace of linear forms that annihilate $F(\tau)$. and Defining

$$\begin{aligned}
\omega_1 &= -e_1 + \tau_{11}e_3 + \tau_{12}e_3\\
\omega_2 &= -e_2 + \tau_{21}e_3 + \tau_{22}e_4\\
\lambda_1 &= \tau_{11}\varepsilon_1 + \tau_{12}\varepsilon_2 + \varepsilon_3\\
\lambda_2 &= \tau_{11}\varepsilon e_1 + \tau_{12}\varepsilon_2 + \varepsilon_4
\end{aligned}$$

we have that ω_1, ω_2, resp. λ_1, λ_2, is a basis of $F(\tau)$, resp. $F^*(\tau)$, with $(\omega_i, \lambda_j) = 0$ for the standard pairing between a vector space and its dual. Thus $F^*(\tau)$ is the column span of $\begin{pmatrix}\tau\\ 1_2\end{pmatrix}$. Then $A_\tau = \mathbf{C}^2/L(\tau)$ where $L(\tau) = \mathbf{Z}^2 + \tau\mathbf{Z}^2$. Let $\mathrm{pr} : V^*_{\mathbf{C}} = \mathbf{C}^4 \to \mathbf{C}^2$ be the projection onto the span of $\varepsilon_1, \varepsilon_2$. This gives an isomorphism $F^*(\tau)\backslash V^*_{\mathbf{C}} \cong \mathbf{C}^2$ with $\mathrm{pr}(V^*_{\mathbf{Z}}) = L(\tau)$, so we see that pr defines an isomorphism

$$F^*(\tau)\backslash V^*_{\mathbf{C}}/V^*_{\mathbf{Z}} \cong A_\tau.$$

Under the identification $V^*_{\mathbf{C}} = \mathrm{H}_1(A_\tau,\, \mathbf{C})$ the ε_i define homology 1-cycles on A_τ which can be viewed as the functionals $\varphi \to \int_\varepsilon \varphi$ where the closed differential forms φ define elements of $V_{\mathbf{C}} = \mathrm{H}^1(A_\tau,\, \mathbf{C})$ by the deRham isomorphism.

Let $T_{\mathbf{Z}} = \bigwedge^2 V_{\mathbf{Z}} = \mathrm{H}^2(A_\tau,\, \mathbf{Z})$. Then τ defines a Hodge structure

$$(T_{\mathbf{C}}, T_{\mathbf{Z}}, F_2(\tau)) \text{ of type } (2,0) + (1,1) + (0,2).$$

$T^{2,0}_{\mathbf{C}}$ is generated by $\omega_1 \wedge \omega_2$, $T^{0,2}_{\mathbf{C}}$ is generated by $\bar{\omega}_1 \wedge \bar{\omega}_2$, and $T^{1,1}_{\mathbf{C}}$ is generated by $\omega_1 \wedge \bar{\omega}_1, \omega_1 \wedge \bar{\omega}_2, \bar{\omega}_1 \wedge \omega_2, \bar{\omega}_2 \wedge \bar{\omega}_2$. Note that a real form $\eta \in \bigwedge^2 V_{\mathbf{R}}$ has type $(1,1)$ if and only if $\eta(u,v) = 0$ for all $u, v \in F^*(\tau)$. Since $F^*(\tau)$ is generated by the rows of $(\tau, 1)$ we see that a real form η is of type $(1,1)$ if and only if

$$(\tau, 1)R_\eta \begin{pmatrix}\tau\\ 1\end{pmatrix} = 0.$$

Since the Néron-Severi group of A_τ is isomorphic with $\mathrm{H}^2(A_\tau, \mathbf{Z}) \cap \mathrm{H}^{1,1}(A_\tau)$ we see

Lemma 3.1.

$$\mathrm{NS}(A_\tau) \cong \{R \in \mathrm{Skew}(4, \mathbf{Z}) : (\tau, 1) R \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0\}$$

We define, for any $R \in \mathrm{Skew}(4, \mathbf{R})$

$$\mathfrak{S}_2 \supset \mathfrak{H}_R := \{\tau \in \mathfrak{S}_2 : (\tau, 1) R \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0\}$$

Clearly $\mathfrak{H}_R = \mathfrak{H}_{tR}$ for any $t \in \mathbf{R}^*$. It can be shown that this set is nonempty if and only if $\Delta(R) > 0$, in which case it is isomorphic with $\mathfrak{S}_1 \times \mathfrak{S}_1$. When the entries of R are integers without common divisor, we call this the Humbert surface associated to R, provided it is nonempty, and we call $\Delta(R)$ the discriminant of the Humert surface. It is a positive integer congruent to 0 or 1 modulo 4.

Dually we have defined, for any $\tau \in \mathfrak{S}_2$,

$$T^{1,1}_{\mathbf{Z}}(\tau) = \{R \in \mathrm{Skew}(4, \mathbf{Z}) : (\tau, 1) R \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0\}.$$

3.2. **Orthgonal viewpoint.** The symmetric space for the group $\mathrm{SO}(3, 2)$ is

$$\mathrm{Pos}_{3,2} = \{Z \in \mathrm{M}_{5,3}(\mathbf{R}) : {}^t M I_{3,2} M > 0\}, \quad I_{3,2} = \begin{pmatrix} 1_3 & 0 \\ 0 & -1_2 \end{pmatrix}$$

More generally, replacing the split form $I_{3,2}$ by any real symmetric B of signature $(3, 2)$, the condition is that the real symmetric 3 by 3 matrix ${}^t MBM$ is positive definite. More intrinsically, fix a 5-dimensional real vector space $U_{\mathbf{R}}$ with a symmetric bilinear form b of signature $(3, 2)$. Then the symmetric space is the open subset of the Grassmannian of 3-planes in $U_{\mathbf{R}}$:

$$\mathrm{Gr}^+_3(U_{\mathbf{R}}) := \{F_{\mathbf{R}} \subset U_{\mathbf{R}} : \dim F_{\mathbf{R}} = 3,\ b \mid F_{\mathbf{R}} > 0\}$$

The isomorphism with $\mathrm{Pos}_{3,2}$ assigns to M the subspace of $U_{\mathbf{R}} = \mathbf{R}^5$ spanned by the rows of M.

To see the isomorphism with $\mathfrak{S}_2$, let $U = \psi^\perp$, and $F_{\mathbf{R}}(\tau) = U_{\mathbf{R}} \cap T^{1,1}_{\mathbf{C}}(\tau)$ One checks direectly or by an appeal to the Hodge index theorem that the restriction of wedge product to $F_{\mathbf{R}}(\tau)$ is positive-definite.

Actually negative

Do the Cayley transform

Proposition 3.2. *The map $\tau \to F_{\mathbf{R}}(\tau) = U_{\mathbf{R}} \cap T^{1,1}_{\mathbf{C}}(\tau)$ sets up an isomorphism*

$$\mathfrak{S}_2 \simeq \mathrm{Gr}^+_3(U_{\mathbf{R}})$$

This map is equivariant for the canonical actions: on the right, of the group $\mathrm{SO}_0(b_{0,\psi})$ on the left of the group $\mathrm{Sp}(J)$ modulo the isomorphism given in proposition 2.1.

The inverse correspondance is

$$F_{\mathbf{R}} \to \tau(F_{\mathbf{R}}) \;=\; \bigcap_{R\in F_{\mathbf{R}}} \mathfrak{H}_R$$

Since the stabilizer of $I_{3,}$ is easliy seen to be $\mathrm{SO}(3)\times\mathrm{SO}(2)$, we see that there is an isomorphism:

$$\mathbf{Gr}^+(3,\ U) \;=\; \mathrm{SO}_0(3,2)/\mathrm{SO}(3)\times\mathrm{SO}(2)$$

where the subscript 0 refers to the connected component.

3.3. **Boundary components.** The rational boundary components are indexed by the nontrivial maximal $\mathbf{Q}$-rational parabolic subgroups of G. In the symplectic realization these are given by J-isotropic lines and 2-plane, $l, h \subset V_{\mathbf{Q}}$. This is defined by setting, for each

$$Z \in \bar{\mathfrak{D}}_2 = \{Z \in \mathrm{Sym}(2,\mathbf{C}) : 1_2 - Z\bar{Z} \geq 0\}$$

the real subspace

$$U(Z) := \mathrm{Ker}\left(v \to v \cdot \begin{pmatrix} i(1+Z) \\ 1-Z \end{pmatrix}\right)$$

The boundary component is rational if and only if the the corresponding space $U(Z)$ is rational, i.e., comes from a subspace of Q^4.

The situation is analogous for the boundary components of the symmetric spaces of quadratic forms. Let b be a symmetric bilinear form over $\mathbf{Q}$ of signature $(3, 2)$. Then the boundary components are indexed by rational b-isotropic subspaces of $\mathbf{Q}^5$. If b is the quadratic form attached to J as in section 2, then we can make a correspondance between these descriptions as follows: If $h \subset V_{\mathbf{Q}} = \mathbf{Q}^4$ is a J-isotropic 2-plane, then the Plücker coordinates $p_{ij}(h)$ satisfy the equation $b(p(h)) = 0$ and hence the line $h^* \subset U_{\mathbf{Q}} = \mathbf{Q}^5$ is isotropic. If $l \subset V_{\mathbf{Q}}$ is an (isotropic) line, then there is a pencil of J-isotropic subspaces $\{h_\alpha\}$ containing l: namely, the J-orthogonal plane $l^\perp$ contains a $\mathbf{P}^1$ of isotropic 2-planes which contain l. The Plučker coordinates of the 2-planes in this pencil $\{p_{ij}(h_\alpha)\}_{\alpha\in\mathbf{P}^1}$ is a straight line in the projective quadric $b = 0$. Viewed in the affine cone over the ambient projective spaces, namely in $U_{\mathbf{Q}}$, this is a b-isotropic 2-plane, l^*.

The quadratic form is called q in section 2

Proposition 3.3. *The assignments $l \to l^*$ and $h \to h^*$ give a bijection between the set of J-isotropic subspaces of $V_{\mathbf{Q}}$ and the set of b-isotropic subspaces of $U_{\mathbf{Q}}$. This correspondance reverses inclusions and is equivariant to the natural actions of $\mathrm{Sp}(V_{\mathbf{Q}}, J)$ on the first set and $\mathrm{SO}_0(U_{\mathbf{Q}}, b)$ on the second set, modulo the isomorphism given in theorem 2.1.*

4. Humbert surfaces

4.1. **Definition.** We apply this to the situation $V = \mathrm{H}_1(A,\ \mathbf{Z})$, for a principally polarized abelian surface. Then $V^* = \mathrm{H}^1(A,\ \mathbf{Z})$, and we choose a basis in which the alternating form given by the polarization $\varphi \in \mathrm{H}_2(A,\ \mathbf{Z})\cap$

$\mathrm{H}^{1,1}(A)$ takes on the canonical form J. The cup product is identified with the ordinary wedge product.

One has $(\varphi \wedge \varphi)[A] = 2$. By the Hodge index theorem, or by more elementary reasoning, one see that the cup - product pairing is negative definite on the 3 - dimensional subspace $U \cap \mathrm{H}^{1,1}(A)$ where U is the orthogonal to φ in $\mathrm{H}_2(A,\ \mathbf{R})$. As van der Geer observes, this provides an alternative description of Siegel's space.

Recall the description of $\mathrm{H}^2(A_\tau,\ \mathbf{R}) \cap \mathrm{H}^{1,1}$of an abelian surface attached to $\tau \in \mathfrak{S}_2$ given at the end of section (1.1.2). It follows easily that $U \cap \mathrm{H}^{1,1}$ described by the equations

$$r_{13} + r_{24} = 0, \quad -(r_{14})\tau_1 + (r_{13} - r_{24})\tau_2 + +r_{23}\tau_3 + r_{12}(\tau_1\tau_3 - \tau_2^2) + r_{34} = 0$$

It is natural to associate, for any $x = (a, b, c, d, e) \in \mathbf{Z}^5$ the subset $\mathfrak{H}_x \subset \mathfrak{S}_2$ given by

$$\{\tau \in \mathfrak{S}_2 \mid a\tau_1 + b\tau_2 + c\tau_3 + d(\tau_2^2 - \tau_1\tau_3) + e = 0\}$$

We call these *Humbert surfaces*. It is clearly no loss of generality to assume that the vector x is primitive (GCD of its entries is 1). Note that $\mathfrak{H}_x = \mathfrak{H}_{-x}$. We also use the notation $F(a, b, c, d, e)$ for this Humbert surface.

For each $\tau \in \mathfrak{S}_2$ we can define $m(\tau) \subset \mathbf{Z}^5$ as

$$\{(a, b, c, d, e) \mid a\tau_1 + b\tau_2 + c\tau_3 + d(\tau_2^2 - \tau_1\tau_3) + e = 0\}$$

The integer $\Delta(a, b, c, d, e) \ = \ b^2 - 4ac - 4de$ is called the *discriminant* of the Humbert surface.

For

$$\begin{aligned} \eta &= d\ dx_1 \wedge dx_2 + b\ (dx_1 \wedge dx_3 - dx_2 \wedge dx_4) \\ &+ a\ dx_1 \wedge dx_4 + c\ dx_2 \wedge dx_3 + e\ dx_3 \wedge dx_4 \end{aligned}$$

in $U \cap \mathrm{H}^{1,1}(A_\tau)$ one finds

$$\eta \wedge \eta \ = \ -2\Delta(-a/2, b, c/2, -d/2, e/2)$$

As the cup - product is negative definite on $U \cap \mathrm{H}^{1,1}(A_\tau)$, we see that $\mathfrak{H}_x$ is nonempty if and only if $\Delta(x) > 0$, and in that case is an imbedded copy of $\mathfrak{S}_1^2$.

Consider $\mathbf{Gr}^+(3,\ U)$ the Grassmannian of 3 - dimensional subspaces of U on which Δ is positive - definite.

We obtain a linear bijection

$$\mathrm{H}^2(A,\ Z) \ = \ \mathbf{Skew}(4,\ \mathbf{Z})$$

by assigning to the form $(1/2)\sum r_{ij} dx_i \wedge dx_j$ the skew matrix

$$r \ = \ \begin{pmatrix} 0 & -r_{12} & -r_{13} & r_{14} \\ r_{12} & 0 & r_{23} & -r_{24} \\ r_{13} & -r_{23} & 0 & -r_{34} \\ -r_{14} & r_{24} & r_{34} & 0 \end{pmatrix}$$

(Note the sign changes from [4, p. 210]). Then the condition that the differential form has type (1, 1) is that

$$r(\tau) \;=\; (\tau,\ 1)\ r\ {}^t(\tau,\ 1) \;=\; 0$$

For any fixed τ this is a set of linear equations in the coefficients of r. The rank of this system is the Picard number of the torus A_τ. Note also that the equations defining the Humbert surfaces in $\mathfrak{S}_2$ are exactly of the form $r(\tau) = 0$ for some $r \in \mathbf{Skew(4,\ Z)}$. Note that $J(\tau) = 0$ for any τ where J is the standard alternating form, so that the equation $r(\tau) = 0$ really only depends on the equivalence class of r in

$$\mathbf{Skew}_0(4,\ \mathbf{Z}) \;=\; \mathbf{Skew}(4,\ \mathbf{Z})/J$$

The invariant $\Delta(r) \;=\; \Delta(a, b, c, d, e)$, with $a = -r_{14},\ b = r_{13} - r_{24} = 2r_{13},\ c = r_{23},\ d = -r_{12},\ e = r_{34}$ is called the *discriminant* of the Humbert surface.

Recall that, for any abelian variety over $\mathbf{C}$ the Néron-Severi group, i.e., the group of divisors modulo algebraic equivalence, or equivalently of algebraic line bundles modulo algebraic equivalence, tensored $\mathbf{Q}$, is isomorphic to

$$\mathrm{NS}_{\mathbf{Q}}(A) = \mathrm{H}^2(A,\ \mathbf{Q}) \cap \mathrm{H}^{1,1}$$

via the first Chern class. If (A, E) is a polarized abelian variety, one also has a canonical identification

$$\lambda_E : \mathrm{End}^{\circ}_{\mathbf{Q}}(A) := \{a \in \mathrm{End}\mathbf{Q}(A) \otimes \mathbf{Q} : a^\circ = a\} \simeq \mathrm{NS}_{\mathbf{Q}}(A)$$

given by

$$a \mapsto E(a-, -) \in \mathrm{H}^2(A,\ \mathbf{Q}) \cap \mathrm{H}^{1,1}.$$

In the above, $a \to a^\circ$ is the Rosati involution, defined by $E(a^\circ x,\ y) = E(x,\ ay)$, and we have identified the alternating form E with a cohomology class via:

$$\mathrm{H}^2(A,\ \mathbf{Q}) = \bigwedge^2 \mathrm{H}^2(A,\ \mathbf{Q}) = \bigwedge^2 V^*_{\mathbf{Q}}.$$

An endomorphism a is defined by a complex matrix $\alpha \in \mathrm{End}_{\mathbf{C}}(V_{\mathbf{R}})$ (recall that $V_{\mathbf{R}} = F\backslash V_{\mathbf{C}}$ has a complex structure), such that $\alpha L \subset L$ where $A = V_{\mathbf{R}}/L$. If $L = \mathbf{Z}^g + \tau Z^g$, this says:

$$\begin{aligned} &\tau\alpha = A\tau + B \\ &\alpha = C\tau + D \qquad \text{where } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}(2g, 2g, \mathbf{Z}) \end{aligned}$$

We can regard E as a skew symmetric matrix, via $E(x,\ y) = {}^t x E y$, then

$$a^\circ = E^t a E^{-1}, \quad \lambda_E(a) = {}^t a E.$$

Multiply the second equation above by τ, subtract from the first; then subtract the resulting equation from its transpose. It gives

$$(\tau,\ 1)M^{\natural}\begin{pmatrix}\tau\\1\end{pmatrix}=0$$

where

$$M^{\natural}=\begin{pmatrix}C-{}^tC & {}^tA+D\\ -A-{}^tD & {}^tB-B\end{pmatrix}={}^tMJ+JM\in\mathrm{Skew}(2g,\mathbf{Z}).$$

Assune $E=J$ is the standard principal polarization, viewed as a skew-symmetric matrix. The assignment $M\to M^{\natural}$ can be viewed as the epimorphism:

$$a\to\lambda_J(a+a^{\circ}):\mathrm{End}_{\mathbf{Q}}(A)\to\mathrm{NS}_{\mathbf{Q}}(A).$$

4.2. **Admissible embeddings.** We are now going to construct explicit embeddings of $\mathfrak{S}_1^2$ into $\mathfrak{S}_2$ which are Humbert surfaces.

Definition 4.1. *A matrix $R\in\mathrm{GL}(2,\,\mathbf{R})$ is called admissible if:*

i. $R_{1,1}R_{2,1},\ R_{1,2}R_{2,2},\ R_{1,1}R_{2,2}+R_{1,2}R_{2,1}$ *are all in* $\mathbf{Z}$ *with GCD equal to 1.*
ii. $R^{-1}WR\in\mathrm{GL}(2,\,\mathbf{Z})$ *where*

$$W=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

Given any $R\in\mathrm{GL}(2,\,\mathbf{R})$ we get an embedding

$$I_R:\mathfrak{S}_1^2\hookrightarrow\mathfrak{S}_2\text{ via }I_R(\tau_1,\tau_2)=R^{-1}\begin{pmatrix}\tau_1&0\\0&\tau_2\end{pmatrix}{}^tR^{-1}.$$

One sees that $\tau\in\mathfrak{S}_2$ is in $\mathfrak{H}_R:=I_R(\mathfrak{S}_1^2)$ if and only if

$$(R\tau{}^tR)_{1,2}=(R_{1,1}R_{2,1})\tau_1+(R_{1,1}R_{2,2}+R_{1,2}R_{2,1})\tau_2+(R_{1,2}R_{2,2})\tau_3=0.$$

Thus $\mathfrak{H}_R=F(R_{1,1}R_{2,1},R_{1,1}R_{2,2}+R_{1,2}R_{2,1},R_{1,2}R_{2,2},0,0)$.

For any $R\in\mathrm{GL}(2,\,\mathbf{R})$ we set

$$S(R):=\begin{pmatrix}R&0\\0&{}^tR^{-1}\end{pmatrix}\in\mathrm{Sp}(4,\,\mathbf{R}).$$

The significance of condition (ii.) in definition 4.1 is that if $R^{-1}WR=H\in\mathrm{GL}(2,\,\mathbf{Z})$ then

$$I_R(\tau_2,\tau_1)=HI_R(\tau_1,\tau_2){}^tH=S(H).I_R(\tau_1,\tau_2)$$

where the right hand side refers to the canonical action of the symplectic group on the Siegel space of degree 2. We let $P(\tau_1,\tau_2)=(\tau_2,\tau_1)$. Also we set $X_R=S(H)$, and note that it is an involution: $X_R^2=1_4$.

Example 4.2. *Here are examples of admissible matrices:*

1. *Every $R\in\mathrm{GL}(2,\,\mathbf{Z})$ is admissible.*

2. *Let D be a positive integer. Define*
$$R_{D^2} = \begin{cases} \begin{pmatrix} 1 & D \\ 1 & -D \end{pmatrix} \text{ if } D \text{ is even.} \\ \begin{pmatrix} 1 & (1+D)/2 \\ 1 & (1-D)/2 \end{pmatrix} \text{ if } D \text{ is odd.} \end{cases}$$
3. *Let $\Delta = f^2 d$ be a positive integer with d square-free, and assume $d > 1$. Let $K = K_d = \mathbf{Q}(\sqrt{d})$, a real quadratic field, and let $x \to x'$ be the nontrivial element of the Galois group. We let $O_f = O_{d,f} = \mathbf{Z} + fO_K$ be the order of conductor f, where O_K is the maximal order of all algebraic integers in K. For any fractional ideal $\mathfrak{a} \subset K$ we let $\mathfrak{a}^\vee$ be the dual relative to the trace form:*
$$\mathfrak{a}^\vee = \{x \in K : \mathrm{Tr}_{K/\mathbf{Q}}(xy) \in \mathbf{Z},\ \text{all } y \in \mathfrak{a}\}$$
and define
$$\mathfrak{a}^{-1} = \{x \in K : x\mathfrak{a} \subset O_K\}$$
Note that $O_K^\vee = \mathfrak{d}^{-1} = \mathfrak{d}_{K/\mathbf{Q}}^{-1}$ is the inverse of the different ideal. We can write $O_f = \mathbf{Z} + fy\mathbf{Z}$ for some $y \in O_K$. In fact we can take $y = \sqrt{d}$ if $d \equiv 2, 3$ mod 4, and $y = (1+\sqrt{d})/2$ if $d \equiv 1$ mod 4. The set
$$R_\Delta = \begin{pmatrix} 1 & fy \\ 1 & -fy' \end{pmatrix}$$

Proposition 4.3. *For any admissible $R \in \mathrm{GL}(2, \mathbf{R})$, $\mathfrak{H}_R := I_R(\mathfrak{S}_1^2) \subset \mathfrak{S}_2$ is a Humbert surface of discriminant $(\det R)^2$. In fact,*
$$I_{R_\Delta}(\mathfrak{S}_1^2) = \begin{cases} F(1, 0, f^2 d, 0, 0) \text{ if } d \equiv 2, 3 \text{ mod } 4. \\ F(1, f, f^2(1-d)/2, 0, 0) \text{ if } d \equiv 1 \text{ mod } 4 \end{cases}$$
$$I_{R_{D^2}}(\mathfrak{S}_1^2) = \begin{cases} F(1, 0, -D^2, 0, 0) \text{ if } D \text{ is even.} \\ F(1, 1, (1-D)/2, 0, 0) \text{ if } D \text{ is odd.} \end{cases}$$
These all have discriminat Δ except for the third case above, which has discriminant 4Δ.

Corollary 4.4. *For every integer $\Delta \geq 1$ there exists a Humbert surface of discriminant Δ. It is a copy of $\mathfrak{S}_1^2$ embedded in $\mathfrak{S}_2$. All Hunbert surfaces of a given discriminant form one orbit for the canonical action of* $\mathrm{Sp}(4, \mathbf{Z})$.

4.3. **Stabilizers.** For every $R \in \mathrm{GL}(2, \mathbf{R})$, define
$$G_R = \{X \in \mathrm{Sp}(4, \mathbf{R}) : X(\mathfrak{H}_R) = \mathfrak{H}_R\}$$
and if R is admissible,
$$\Gamma_R = \{X \in \mathrm{Sp}(4, \mathbf{Z}) : X(\mathfrak{H}_R) = \mathfrak{H}_R\}.$$
Consider the semidirect product $Q = (\mathrm{SL}(2) \times \mathrm{SL}(2)) \ltimes \mathbf{Z}/2$, viewed as a group scheme over $\mathbf{Z}$. We have the quotient $\langle\sigma\rangle = \mathbf{Z}/2$ act on $\mathrm{SL}(2) \times \mathrm{SL}(2)$

by interchange of factors. There is a natural injective homomorphism $j : Q \to \mathrm{Sp}(4)$ given by

$$j\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}$$

$$j(\sigma) = S\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

We will sometimes view $Q \subset \mathrm{Sp}(4)$ via this isomorphism.

Lemma 4.5. *If $R \in \mathrm{GL}(2, \mathbf{R})$ and $X \in \mathrm{Sp}(2, \mathbf{R})$ then*

$$X \in G_R \Leftrightarrow S(R)XS(R)^{-1} \in Q(\mathbf{R}).$$

If R is admissible and $X \in \mathrm{Sp}(2, \mathbf{Z})$ then

$$X \in \Gamma_R \Leftrightarrow S(R)XS(R)^{-1} \in Q(\mathbf{R}).$$

In other words,

$$G_R = S(R)^{-1}Q(\mathbf{R})S(R), \quad \Gamma_R = G_R \cap \mathrm{Sp}(4, \mathbf{Z}).$$

We can write $G_R = \tilde{G}_R \cup \tilde{G}_R X_R$ and $\Gamma_R = \tilde{\Gamma}_R \cup \tilde{\Gamma}_R X_R$, where $\tilde{G}_R$, resp. $\tilde{\Gamma}_R$, is $S(R)^{-1}j(\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R}))S(R)$, resp., $\tilde{G}_R \cap \mathrm{Sp}(4, \mathbf{Z})$, and X_R is the involution defined just before example 4.2.

These stabilizers can be made explicit in the cases of the examples 4.2. We will use the notation Γ_Δ for Γ_{R_Δ}.

Proposition 4.6. 1. $\tilde{\Gamma}_1 \cong \mathrm{SL}(2, \mathbf{Z}) \times \mathrm{SL}(2, \mathbf{Z})$.

2.

$$\tilde{\Gamma}_{D^2} \cong \{(Y_1, Y_2) \in \mathrm{SL}(2, \mathbf{Z}) \times \mathrm{SL}(2, \mathbf{Z}) : Y_1 \equiv MY_2M^{-1} \bmod \varepsilon D\}$$

where $\varepsilon = 1$ if D is odd, 2 if D is even, and

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The index

$$\left[\mathrm{SL}(2, \mathbf{Z}) \times \mathrm{SL}(2, \mathbf{Z}) : \tilde{\Gamma}_{D^2}\right] = \varepsilon^3 D^3 \prod_{p|D} \left(1 - \frac{1}{p^2}\right).$$

3. *If $\Delta = f^2 d$ with $d > 1$ square-free, then*

$$\tilde{\Gamma}_\Delta \cong \mathrm{SL}(O_f, O_f^\vee)$$

where

$$\mathrm{SL}(O_f,\, O_f^\vee) = \{x \in \mathrm{SL}(2,\, K) : x(O_f \oplus O_f^\vee) \subset O_f \oplus O_f^\vee\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in O_f, c \in O_f^\vee, b \in (O_f^\vee)^{-1} \right\}$$

5. Tits building with scaffolding

Recall that the group $\mathrm{Sp}(J)$ acts on the space of tuples (a, b, c, d, e) in the 5-dimensional representation giving the isogeny $\mathrm{Sp}(J) \to \mathrm{SO}_0(\Delta)$. The following theorem is due to Humbert. See [1, Thm. 3.1.9].

Theorem 5.1. *Fix any $D \in \mathbf{Z}$. Then $\mathrm{Sp}(V_{\mathbf{Z}}, J) = \mathrm{Sp}(4, \mathbf{Z})$ acts transitively on the set $\Delta_D(\mathbf{Z})$ of primitive $(a, b, c, d, e) \in \mathbf{Z}^5$ with $\Delta(a, b, c, d, e) = D$. The same conclusion holds with the ring $\mathbf{Z}$ replaced by $\mathbf{Z}_p$ for any prime p.*

The proof for the p-adic case is essentially the same as the proof for the integers; only step 2 i'n his proof needs an easy modification. In fact, Franke proved that, up to $\mathrm{Sp}(4, \mathbf{Z})$-equivalence, every element of $\Delta_D(\mathbf{Z})$ is equivalent to

$$(-D/4, 0, 1, 0, 0), \text{ if } D \equiv 0 \bmod 4.$$
$$((1-D)/4, 1, 1, 0, 0), \text{ if } D \equiv 1 \bmod 4.$$

Now let $\Gamma(N) \subset \mathrm{Sp}(4, \mathbf{Z})$ be the principal congruence subgroup of level N. Let $\Delta_D(\mathbf{Z}/N)$ be the set of primitive vectors $(a, b, c, d, e) \in (\mathbf{Z}/N)^5$ with $\Delta(a, b, c, d, e) = D \bmod N$.

Proposition 5.2. *Suppose that N and D are relatively prime. Then the reduction map $\Gamma(N)\backslash\Delta_D(\mathbf{Z}) \to \Delta_D(\mathbf{Z}/N)$ is bijective.*

If N is even we should replace Δ by the Plücker form.

Proof. Let $x = (a, b, c, d, e) \in \Delta_D(\mathbf{Z}/N)$. Write $N = \prod p_i^{k_i}$, the prime factorization. Fix any one prime $p = p_i$ and let $k = k_i$.

Then the algebraic variety $\Delta_D : \Delta(z) = D$ in affine 5-space is smooth over $\mathbf{Z}$. Then x defines a point, also denoted x in $\Delta_D(\mathbf{Z}/p^k)$. By Hensel's lemma, we can find a lift of this point $\tilde{x} \in \Delta_D(\mathbf{Z}_p)$ under the natural projection $\Delta_D(\mathbf{Z}_p) \to \Delta_D(\mathbf{Z}/p^k)$. This vector is primitive, since its reduction mod p^k is. By Humbert's theorem 5.1, we can find $\gamma \in \mathrm{Sp}(4, \mathbf{Z}_p)$ with $\gamma \cdot x_0 = \tilde{x}$ where $x_0 \in \Delta_D(\mathbf{Z})$ is chosen as in the remarks following 5.1. Doing this for each prime factor of N gives a collection of liftings $\tilde{x}_i \in \Delta_D(\mathbf{Z}_{p_i})$ and $\gamma_i \in \mathrm{Sp}(4, \mathbf{Z}_{p_i})$ such that $\gamma_i \cdot x_0 = \tilde{x}_i$. By the strong approximation theorem [13], we can find $\gamma \in \mathrm{Sp}(4, \mathbf{Z})$ such that $\gamma \equiv \gamma_i \bmod p_i^{k_i}$ for all i. Then $\gamma \cdot v_0 \in \Delta_D(\mathbf{Z})$ maps to x under reduction mod N. This proves the surjectivity of the map $\Gamma(N)\backslash\Delta_D(\mathbf{Z}) \to \Delta_D(\mathbf{Z}/N)$.

False in char = 2; replace by Plücker form.

To prove injectivity, we must verify the following: From the surjectivity just proved, it follows that $\mathrm{Sp}(4, \mathbf{Z}/N)$ acts transitively on the primitive vectors in $\Delta_D(\mathbf{Z}/N)$. Let $x_0 \in \Delta_D(\mathbf{Z})$ be chosen as in the remarks following 5.1. Let $\mathrm{Stab}_N(x_0) \subset \mathrm{Sp}(4, \mathbf{Z}/N)$ and let $\mathrm{Stab}(x_0) \subset \mathrm{Sp}(4, \mathbf{Z})$ be the respective stabilizing subgoups. We must show that reduction mod N defines an epimorphism $\mathrm{Stab}(x_0) \to \mathrm{Stab}_N(x_0)$. This again follows from the strong approximation theorem because $\mathrm{Stab}(x_0)$ is a simply-connected semisimple algebraic group without compact factor. □

That is, each connected component of the stabilizer is.

6. Special cycles

The theory of Humbert surfaces is a particular case of a more general theory of special cycles on locally symmetric spaces. The origin of this is the discovery, by Hirzebruch and Zagier, of special curves on Hilbert modular surfaces and the connection of these with modular forms, [7]: the intersection numbers of these special curves appear as Fourier coefficients of modular forms. This was vastly generalized by two groups working in parallel: Kudla and Millson, [14], [16], [17], [18], [19], [22], [23]; and Tong and Wang, [27], [28], [29], [30], [31]. The symmetric spaces in question are those associated to one of three classes of groups: $O(p,q)$, $U(p,q)$ and $Sp(p,q)$. The relevant case for us is $O(3,2)$; $O(2,1)$ corresponds to classical modular curves, quotients of the complex upper half plane. This was treated by Shintani, [26]. The case $O(2,2)$ gives the Hilbert modular surfaces. Recently there have been extensions of this to arithmetic cycles and height pairings. For an overview, see [15].

We will follow the notations of the papers of Kudla-Millson, specialized to the case at hand. Let $V_{\mathbf{Q}}$ be a 5-dimensional vector space over $\mathbf{Q}$ with a quadratic form $(\ ,\ )$ of signature $(3,2)$. Let $L \subset V_{\mathbf{Q}}$ be a $\mathbf{Z}$-lattice on which $(\ ,\ )$ is $\mathbf{Z}$-valued. Let G be the algebraic group over $\mathbf{Q}$ which is the group of isometries of the form $(\ ,\ )$, so that $G(\mathbf{R}) = \mathrm{O}(3,2)$. Let G^0 be the connected component of the identity (Zariski topology). Note that $G^0(\mathbf{R})) = \mathrm{SO}(3,2)$ has two topological components: The component of the identity is denoted $\mathrm{SO}_0(3,2)$. We identify the symmetric space

$$D = \mathrm{SO}_0(3,2)/(\mathrm{SO}(3) \times \mathrm{SO}(2))$$

with the subspace of the Grassmannian $\mathrm{Gr}_2(V_{\mathbf{R}})$ of 2 planes Z such that $Z \mid (\ ,\ )$ is negative definite. This is isomorphic with the Siegel space of genus 2. Let $U \subset V_{\mathbf{Q}}$ be a subspace such that $(\ ,\ ) \mid U_{\mathbf{R}}$ is nondegenerate. Then we have a decomposition $V_{\mathbf{R}} = U_{\mathbf{R}} \oplus U_{\mathbf{R}}^{\perp}$. We define

$$D_U = \{Z \in D : Z = Z \cap U_{\mathbf{R}} + Z \cap U_{\mathbf{R}}^{\perp}\} \subset D$$

and let G_U be the stabilizer of U in G and G_U^0 the connected component of the identity. Let $\Gamma \subset G(\mathbf{Q})$ be a (torsion-free) congruence subgroup which preserves the lattice L. Put $\Gamma_U = \Gamma \cap G_U$ and $\Gamma_U^0 = \Gamma \cap G_U^0$. Define $C_U = \Gamma_U^0 \backslash D_U$. The natural map $\pi : C_U \to \Gamma\backslash D := M$ is proper, and thus the pair (C_U, π) is a locally finite singular cycle in M. This will orientable if Γ is a subgroup of $\mathrm{SO}_0(3,2)$, which will always be the case in our examples, since the Γ we work with will be the images of corresponding subgroups of $\mathrm{Sp}(4,\mathbf{R})$, which is the spin covering of $\mathrm{SO}_0(3,2)$.

Given subspace U as above we define an involution $\tau = \tau_U$ via

$$\tau = \begin{cases} +1 \text{ on } U \\ -1 \text{ on } U^{\perp} \end{cases}.$$

Then G_U is the centralizer of τ and $D_U = \{Z : \tau Z = Z\}$. If $(\ , \ ) \mid U_{\mathbf{R}}$ has signature (r, s) then $G_U(\mathbf{R}) \cong \mathrm{O}(r, s) \times \mathrm{O}(p - r, q - s)$ and D_U has codinension $ps + qr - 2rs$ in D. Most important for us is the case where the plane U is positive definite. In these cases

$$D_U = \{Z \in D : Z \subset U_{\mathbf{R}}^{\perp}\}$$

and clearly $\dim U \leq 3$. In these cases, D_U is an embedded copy of $\mathfrak{S}_1 \times \mathfrak{S}_1$, resp. $\mathfrak{S}_1$, resp. a point, in $D \cong \mathfrak{S}_2$, corresponding to $\dim U = 1$, resp. 2, resp. 3. The cycles C_U are analytic cycles in case U is positive definite; in general, they are totally geodesic (singular) submanifolds of D. From now on, we only consider positive definite 2-planes.

We now define certain linear combinations of the cycles C_U. Let n be an integer in the range 1 to 3. Let $X = \{x_1, ..., x_n\} \subset V_{\mathbf{R}}^n$; such an X is called a (n-)frame. Let (X, X) be the symmetric matrix given by $(X, X)_{ij} = (x_i, x_j$. We consider the orbit $\mathcal{O} = G(\mathbf{R})X$ in $V_{\mathbf{R}}^n$. We call the orbit *nonsingular* if $\operatorname{rank}(X, X) = n$ and *nondegenerate* if $\operatorname{rank}(X, X) = \dim_{\mathbf{R}} \operatorname{span} X$. The zero orbit is nondegenerate but singular. It is known that the orbit $\mathcal{O}$ is closed if and only if it is nondegenerate. If $\mathcal{O}$ is a closed orbit, then by a theorem of Borel, $\mathcal{O} \cap L^n$ consists of a finite number of Γ-orbits. Letting $Y_1, ..., Y_l$ be a set of representatives of these orbits and $U_j = \operatorname{span} Y_j$ we define

$$C_{\mathcal{O}} = \sum_{j=1}^{l} C_{U_j}.$$

Given any symmetric $n \times n$ matrix β, define

$$\mathcal{Q}_\beta = \{X \in V_{\mathbf{R}}^n : (X, X) = \beta\}.$$

If $X \in \mathcal{Q}_\beta$ then $\mathcal{O} = G(\mathbf{R})X \subset \mathcal{Q}_\beta$. In case β is positive definite, $G(\mathbf{R})$ acts transitively on $\mathcal{Q}_\beta$, thus $\mathcal{O} = \mathcal{Q}_\beta$ and we may write C_β for $C_{\mathcal{O}}$. If β is positive semidefinite of rank $t < n$, $\mathcal{Q}_\beta$ contains a unique closed orbit defined by

$$\mathcal{Q}_\beta^c = \{X \in \mathcal{Q}_\beta : \dim \operatorname{span} X = t\}$$

and it is known that $G(\mathbf{R})$ acts transitively on $\mathcal{Q}_\beta^c$. If β is positive semidefinite, we let C_β denote $C_{\mathcal{O}}$ for this unique closed orbit.

As Kudla-Millson observe, these cycles are often zero for trivial reasons. By their conventions on orientations the frames $(x_1, ..., x_n)$ and $(-x_1, x_2, ...x_n)$ could both occur and give cancelling contributions. Therefore they fix a $h \in L^n$ and an integer $N \geq 1$ such that $\gamma \in \Gamma$ implies $\gamma \equiv 1 \bmod N$, and replace $\mathcal{Q}_\beta \cap L^n$ in the above definition with $\mathcal{Q}_\beta \cap (h + NL^n)$. Taking Γ-orbit representatives in this set then defines cycles $C_{\beta,h,N}$ as above. Note that Kudla-Millson simplify this notaion to C_β in all their susbsequent work.

7. Theta correspondence

Let G be $\mathrm{O}(p,q)$ (the theory also works for $\mathrm{U}(p,q)$ and $\mathrm{Sp}(p,q)$). Let V be the real vector space of dimension $m = p+q$ on which G is acting in the obvious representation. We let $K \subset G(\mathbf{R})$ be a maximal compact and $D = G/K$ the symmetric space. For a torsion free arithmetic subgroup $\Gamma \subset G$ we let $M_\Gamma = \Gamma\backslash D$. Let $L \subset V$ be a $\mathbf{Z}$ lattice which is stable under Γ. We also assume that the quadratic form $(\ ,\ )$ on V of signature (p,q) is integer valued on L. Let $1 \le n \le p$ be an integer. We fix a $h \in L^n$ and a positive integer N such that $\gamma \in \Gamma$ implies $\gamma \equiv 1 \bmod N$.

The theta correspondence is a linear map

$$\Lambda : S_{m/2}(\Gamma') \to \mathbf{H}^{nq}(M_\Gamma)$$

where the left side above is the space of holomorphic cusp forms of weight $m/2$ on a congruence subgroup $\Gamma' \subset \tilde{G}' = \mathrm{Mp}(2n,\mathbf{R})$ of the metaplectic two-fold cover of $G' = \mathrm{Sp}(2n,\mathbf{R})$, and the right hand side is the space of closed harmonic nq forms on M_Γ. The construction is an application of the theory of dual reductive pairs: (G, G') as above forms just such a pair. The operator $\mathcal{L}$ is an integral transform:

$$\Lambda(f) = \int_{\Gamma'\backslash\mathfrak{S}_n} \theta_\varphi(\tau, z)\overline{f(\tau)}d\tau, \quad z \in D,\ \tau \in \mathfrak{S}_n.$$

The kernel is a C^∞ function on $\mathfrak{S}_n$ with values in $\mathbf{H}^{nq}(M_\Gamma)$ defined as

$$\theta_\varphi(\tau, z) = \det(v)^{-m/4}\theta_\varphi(g'_\tau, z)$$

where $\theta_\varphi(g, z)$ is a C^∞ function on $\mathrm{Mp}(n,\mathbf{R})$ with values in $\mathbf{H}^{nq}(M_\Gamma)$. The element $g'_\tau \in \mathrm{Mp}(n,\mathbf{R})$ is defined by

$$g'_\tau = \begin{pmatrix} \sqrt{v} & \sqrt{v}^{-1}u \\ 0 & \sqrt{v}^{-1} \end{pmatrix} \text{ so that } g'_\tau(i\,1_n) = \tau = u + iv.$$

The kernel $\theta_\varphi(g', z)$ is obtained as

$$\theta_\varphi(g', z) = \sum_{\substack{x \in L^n \\ x \equiv h \bmod N}} \omega(g')\varphi(z, x)$$

where ω is the Weil representation. $\varphi(z,x) \in (\mathbf{H}^{nq}(D) \otimes \mathcal{S}(V^n))^G$ is a canonically defined element where $\mathcal{S}(V^n)$ is the Schwartz space of C^∞ functions all of whose derivatives decrease rapidly to 0 at infinity. Recall that the space $\mathcal{S}(V^n)$ carries an obvious representation of G and the Weil (or oscillator) representation of $\tilde{G}'$ and that these actions commute. One has:

i. $\theta_\varphi(\gamma' g', z) = \theta_\varphi(g', z)$ for all $\gamma' \in \Gamma'$.
ii. $\theta_\varphi(g'k', z) = [\sqrt{\det(k')}]^m \theta_\varphi(g', z)$ for all $k' \in K'$ where $K' \subset G'$ is the maximal compact stabilizing $i1_n \in \mathfrak{S}_n$.

The above properties imply that $\theta_\varphi(\gamma' g', z)$ defines a section of the line bundle $\mathcal{L}_m$ on $\Gamma'\backslash\mathfrak{S}_n$ given by the character $[\sqrt{\det(k')}]^m$ of K'. This is the line bundle whose sections are the holomorphic Siegel modular forms of weight $m/2$.

We are now going to state one of the key results of Kudla-Millson. First we will give this under the assumption that Γ is cocompact. This is equivalent to the condition that the quadratic form $(\ ,\ )$ restricted to $V_{\mathbf{Q}} = L \otimes \mathbf{Q}$ is anisotropic (does not represent 0 nontrivially). Let $\mathbf{H}_{\text{cycle}}$ be the subspace of $\mathbf{H}^{nq}$ generated by the duals of the cycles C_β where β is positive-definite. Let $\mathbf{H}_\theta$ be the image of $S_{m/2}(\Gamma')$ under Λ. Observe that these spaces depend on the choice of h and N.

Theorem 7.1. *If $n < m/4$ then $\mathbf{H}_\theta = \mathbf{H}_{\text{cycle}}$.*

Proof. We will sketch the proof. Let

$$[\eta,\ \omega] = \int_{M_\Gamma} \eta \wedge \omega : \mathbf{H}^{(p-n)q} \times \mathbf{H}^{nq} \to \mathbf{H}^{pq} = \mathbf{C}$$

be the natural nonsingular pairing, and define $\theta_\varphi(\eta)$ for a η harmonic $(p-n)q$ form by $\theta_\varphi(\eta) = [\eta,\ \theta_\varphi]$. This a function of $\tau \in \mathfrak{S}_n$ and it is periodic with respect to the lattice $L' = \Gamma' \cap N$ where $N \subset \mathrm{Mp}(n, \mathbf{R})$ is the group of matrices of the form

$$\begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix}, \quad S \text{ is symmetric.}$$

Let $(L')^*$ be the dual lattice relative to the form $\mathrm{Tr}(XY)$. Thus $\theta_\varphi(\eta)$ has a Fourier expansion indexed by elements $\beta \in (L')^*$, which are symmetric n by n matrices with rational entries. Each Fourier coefficient $a_\beta(\theta_\varphi(\eta))$ is a function of v where $\tau = u + iv$. The key fact is

Theorem 7.2.

$$a_\beta(\theta_\varphi(\eta))(v) = \mathrm{e}^{-2\pi \mathrm{Tr}(\beta v)} \int_{C_\beta} \eta$$

□

To prove the theorem it is equivalent to show that $\mathbf{H}_\theta^\perp = \mathbf{H}_{\text{cycle}}^\perp$.

$\mathbf{H}_{\text{cycle}}^\perp \subset \mathbf{H}_\theta^\perp$: Let η be orthogonal to all the dual classes of the cycles C_β. Thus $\int_{C_\beta} \eta = 0$ for all these cycles and hence $a_\beta(\theta_\varphi(\eta)) = 0$ for all positive definite β. Then if $f \in S_{m/2}(\Gamma')$ we have

$$[(\theta_\varphi(f), \eta] = (f, \theta_\varphi(\eta)) = 0$$

where the left-hand side is the Petersson inner product, and it vanishes because the Fourier coefficients of $\theta_\varphi(\eta)$ are disjoint from those of f. Thus $\eta \in \mathbf{H}_\theta^\perp$. This does not require the hypothesis that $n < m/4$.

$\mathbf{H}_\theta^\perp \subset \mathbf{H}_{\text{cycle}}^\perp$ Assume $\theta_\varphi(\eta)$ is orthogonal to all the $f \in S_{m/2}(\Gamma')$. Define the Poincaré series

$$p_\beta(\tau) = c \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma'} \frac{\gamma^* \mathrm{e}^{2\pi i \mathrm{Tr}(\beta\tau)}}{j(\gamma, \tau)^m}$$

where

$$j(\gamma,\tau) = (C\tau + D), \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and $\Gamma'_\infty = \Gamma' \cap N$. This series is absolutely convergent, provided $n < m/4$ and is a holomorphic cusp form then. One also introduces a kind of Eisenstein series

$$p_\beta(\tau, s) = c(s) \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma'} \frac{\gamma^* \mathrm{e}^{2\pi i \mathrm{Tr}(\beta\tau)}}{j(\gamma,\tau)^m} v(\gamma\tau)^s$$

with $c(s)$ chosen so that $(p_\beta(\tau, s), f(\tau)) = a_\beta(f)$ for any $f \in S_{m/2}(\Gamma')$. Again under the restriction $n < m/4$ this is a holomorphic cusp form for any s in a vertical strip that includes 0, and clearly $p_\beta(\tau, 0) = p_\beta(\tau)$. Under our assumption on η we have

$$(p_\beta(\tau), \theta_\varphi(\eta)) = (p_\beta(\tau, s), \theta_\varphi(\eta))|_{s=0} = 0.$$

A direct computation shows that

$$(p_\beta(\tau, s), \theta_\varphi(\eta)) = c(s) \left(\int_{C_\beta} \eta \right) \int_{\mathcal{P}_n} \mathrm{e}^{-4\pi \mathrm{Tr}(\beta v)} (\det v)^{m/2+s-n+1/2} \frac{dv}{(\det v)^{n+1/2}}$$

when $\mathrm{Re}(s)$ is sufficiently large. Here $\mathcal{P}_n$ is the space of positive definite symmetric n by n matrices. Siegel evaluated the last integral and showed that it is convergent and nonzero for $\mathrm{Re}(s) > n - m/2$, a region that includes 0 by our assumptions. By the principle of analytic continuation, we may evaluate the above integral at $s = 0$ and it will coincide with $(p_\beta(\tau, s), \theta_\varphi(\eta))|_{s=0}$. Thus,

$$cc' \int_{C_\beta} \eta = (p_\beta(\tau), \theta_\varphi(\eta)) = 0$$

and therefore $\eta \in \mathbf{H}^\perp_{\mathrm{cycle}}$.

8. Humbert surfaces with square discriminants

Consider a Humbert surface of discriminant D. This is $\mathrm{Sp}(4, \mathbf{Z})$-equivalent to

$$\mathfrak{H}_D = \left\{ \begin{pmatrix} \tau_1 & 1/D \\ 1/D & \tau_3 \end{pmatrix} \right\}$$

Let

$$\mathfrak{H}_0 = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \right\}$$

and not that this is a Humbert surface of discriminant 1. Observe that

$$\mathfrak{H}_0 = \mathrm{Fix}\begin{pmatrix} -1 & 0 & & \\ 0 & 1 & & \\ & & -1 & 0 \\ & & 0 & 1 \end{pmatrix} = \mathrm{Fix}(j_0)$$

and conversely that

$$\begin{aligned} Q_0 &:= \{g \in \mathrm{Sp}(4, \mathbf{Q}) \mid g(\mathfrak{H}_0) = \mathfrak{H}_0\} \\ &= \{g \in \mathrm{Sp}(4, \mathbf{Q}) \mid g j_0 = \pm j_0 g\} \\ &= \text{Centralizer of } j_0 \end{aligned}$$

i.e., that $g j_0 g^{-1} = j_0$ in $\mathrm{PSp}(4, \mathbf{Q})$. We have a split exact sequence

$$1 \longrightarrow \mathrm{SL}(2, \mathbf{Q}) \times \mathrm{SL}(2, \mathbf{Q}) \longrightarrow Q_0 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 1$$

where $\mathbf{Z}/2\mathbf{Z}$ is the group generated by

$$\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

which is an element of order 4 in $\mathrm{Sp}(4, \mathbf{Q})$ but is of order 2 in $\mathrm{PSp}(4, \mathbf{Q})$.

Next observe that $t_D(\mathfrak{H}_0) = \mathfrak{H}_D$ where

$$t_D = \begin{pmatrix} 1 & 0 & 0 & 1/D \\ 0 & 1 & 1/D & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}(2, \mathbf{Q}).$$

hence $\mathfrak{H}_D = \mathrm{Fix}(j_D)$ where $j_D = t_D\, j_0\, t_D^{-1}$. Direct computation shows that

$$j_D = \begin{pmatrix} -1 & 0 & 0 & 2/D \\ 0 & 1 & -2/D & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and then that

$$\begin{aligned} Q_D &:= \{g \in \mathrm{Sp}(4, \mathbf{Q}) \mid g(\mathfrak{H}_D) = \mathfrak{H}_D\} \\ &= \{g \in \mathrm{Sp}(4, \mathbf{Q}) \mid g j_D = \pm j_D g\} \\ &= \text{Centralizer of } j_D \end{aligned}$$

and then $Q_D = t_D Q_0 t_D^{-1}$.

The relation $g j_D = j_D g$ forces g to be of the form

$$g = \begin{pmatrix} a & n/D & c & (p-a)/D \\ i/D & f & (k-f)/D & h \\ i & 0 & k & -i/D \\ 0 & n & -n/D & p \end{pmatrix}$$

and the condition that g be symplectic forces the conditions

$$ak - ic = fp - nh$$
$$(ak - ic) - in/D^2 = (fp - nh) - in/D^2 = 1.$$

We can rule out the case $g j_D = -j_D g$ except for $N = 1$ or $N = 2$, D odd; these cases will be treated later. So Q_D consists of the matrices of the above shape satisfying the symplectic conditions above. Then $Q_{D,N} := Q_D \cap \Gamma(N)$ consists of these matrices such that all of a, c, f, h, i, k, n, p are integers satisfying the congruences

$$(*)\begin{cases} a \equiv p \equiv 1\ (N) & a \equiv p\ (DN) \\ f \equiv k \equiv 1\ (N) & f \equiv k\ (DN) \\ \quad c \equiv 0\ (N) & i \equiv 0\ (DN) \\ \quad h \equiv 0\ (N) & n \equiv 0\ (DN) \end{cases}$$

We now determine the action of $Q_{D,N}$ on $\mathfrak{H}_D$. By conjugating, this is the action of $t_D^{-1} Q_{D,N} t_D$ on $t_D^{-1} \mathfrak{H}_D t_D = \mathfrak{H}_0$. If g is the element above, we find

$$t_D^{-1}\, g\, t_D = \begin{pmatrix} a & 0 & c + n/D & 0 \\ 0 & f & 0 & h + i/D^2 \\ i & 0 & k & 0 \\ 0 & n & 0 & p \end{pmatrix}$$

with the conditions given by $(*)$ above. In particular, n/D^2, $i/D^2 \in (1/D)\mathbf{Z}$. Clearly, the above matrix acts on $\mathfrak{H}_0 = \mathfrak{S}_1 \times \mathfrak{S}_1$ in the obvious product action, with the two-by-two block of $(g_{i,j})$ with both i and j odd (resp. even) acting on the first (resp. second) factor.

Let us now consider the action of

$$\left\{ \begin{pmatrix} a & c + n/D^2 \\ i & k \end{pmatrix} \middle| \begin{array}{lll} a \equiv 1\ (N) & c \equiv 0\ (N) & \\ k \equiv 1\ (N) & i \equiv 0\ (DN) & n \equiv 0\ (DN) \end{array} \right\}$$

on $\mathfrak{S}_1$. We may conjugate this matrix by

$$\begin{pmatrix} 0 & -1/\sqrt{D} \\ 1/\sqrt{D} & 0 \end{pmatrix} \in \mathrm{SL}(2,\, \mathbf{R})$$

to get

$$\begin{pmatrix} k & -i/D \\ -Dc - n/D & a \end{pmatrix} \in \mathrm{SL}(2,\, \mathbf{Z});$$

observe that this matrix has integer entries and has determinant one. The same conjugation applied to the other block of $t_D\, g\, t_D$, namely

$$\begin{pmatrix} f & h + i/D^2 \\ n & p \end{pmatrix}$$

leads to

$$\begin{pmatrix} p & -n/D \\ -Dh - i/D & f \end{pmatrix} \in \mathrm{SL}(2,\, \mathbf{Z}).$$

Define $\tilde{i} = i/D$ and $\tilde{n} = n/D$ and observe that these integers are both divisible by N. By the congruences already shown, both of these conjugated matrices are in the principal congruence subgroup of level N: $\Gamma_1(N) \subset \mathrm{SL}(2, \mathbf{Z})$. Our stabilizer $Q_{D,N}$ is thus isomorphic with a subgroup

$$\begin{pmatrix} k & -\tilde{i} \\ -Dc - \tilde{n} & a \end{pmatrix} \times \begin{pmatrix} p & -\tilde{n} \\ -Dh - \tilde{i} & f \end{pmatrix} \subset \Gamma_1(N) \times \Gamma_1(N).$$

The congruences $c, h \equiv 0\ (N)$ show that modulo ND these matrices are congruent to

$$\begin{pmatrix} k & -\tilde{i} \\ -\tilde{n} & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p & -\tilde{n} \\ -\tilde{i} & f \end{pmatrix}$$

Since $a \equiv p,\ f \equiv k\ (DN)$, we see that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k & -\tilde{i} \\ -\tilde{n} & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} p & -\tilde{n} \\ -\tilde{i} & f \end{pmatrix} \mod DN$$

Clearly this conjugate of $Q_{D,N}$ contains $\Gamma_1(DN) \times \Gamma_1(DN)$ as a normal subgroup.

We consider the cases excluded above:

Lemma 8.1. *With the above notations, there is no* $g \in \Gamma(N) \subset \mathrm{Sp}(4, \mathbf{Z})$ *with* $gj_D = -j_D g$ *unless* $N = 1$ *or* $N = 2$ *and* D *is even.*

Proof. the equation $gj_D = -j_D g$ leads to

$$g = \begin{pmatrix} m/D & b & (o-b)/D & d \\ e & j/D & g & (l-e)/D \\ 0 & j & -j/D & l \\ m & 0 & o & -m/D \end{pmatrix}$$

For this to be in $\mathrm{Sp}(4, \mathbf{Z})$ yields

$$m \equiv 0\ (D), \quad j \equiv 0\ (D), \quad o \equiv b\ (D), \quad l \equiv e\ (D).$$

For $N > 2$ this matrix cannot be in $\Gamma(N)$ as it is impossible to have both $m/D \equiv 1\ (N)$ and $-m/D \equiv 1\ (N)$.

If $N = 2$, then we must have $m \equiv 1\ (D)$, $m/D \equiv 1\ (2)$ and $m \equiv 1\ (2)$, which is impossible for D odd.

For D even this is possible and we have the involution

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -D & 1 & 0 \\ D & 0 & 0 & -1 \end{pmatrix} \in \Gamma(2).$$

This is conjugate to

$$\begin{pmatrix} 0 & 0 & 0 & 1/D \\ 0 & 0 & -1/D & 0 \\ 0 & -D & 0 & 0 \\ D & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1/D & 0 \\ 0 & 0 & 0 & 1/D \\ -D & 0 & 0 & 0 \\ 0 & -D & 0 & 0 \end{pmatrix}.$$

The second factor is the automorphism

$$\begin{pmatrix} 0 & 1/D \\ -D & 0 \end{pmatrix} \text{ of } \mathfrak{S}_1 \text{ acting ``diagonally'' on } \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}$$

while the second factor acts by

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_1 \end{pmatrix} \text{ i.e., interchanging the factors.}$$

□

8.1. **When N and D are relatively prime.** If N and D are relatively prime and greater or equal to 3, then there is an isomorphism

$$\Gamma_1(N)/\Gamma_1(DN) \cong \Gamma_1(\mathbf{Z}/D) = \mathrm{SL}(2, \mathbf{Z}/D).$$

Consequently, $G_{D,N} := Q_{D,N}/(\Gamma_1(DN) \times \Gamma_1(DN))$ may be identified with a subgroup of $\Gamma_1(D) \times \Gamma_1(D)$. We have shown:

Lemma 8.2. *Let N and D be positive integers. Let $\Gamma(N) \subset \mathrm{Sp}(4, \mathbf{Z})$ be the principal congruence subgroup of level N. The stabilizer $Q_D \subset \mathrm{Sp}(4, \mathbf{Z})$ of the standard Humbert surface $\mathfrak{H}_D$ of discriminant D^2 is the centralizer of the involution j_D in $\mathrm{Sp}(4, \mathbf{Z})/\pm 1$. If $N = 1$ and $(N, D) = (2, \mathrm{odd})$ are excluded, Q_D is the centralizer of j_D in $\mathrm{Sp}(4, \mathbf{Z})$. In this case $Q_{D,N} = Q_D \cap \Gamma(N)$ is isomorphic to the subgroup of $\Gamma_1(N) \times \Gamma_1(N)$ consisting of (A, B) such that $wAw^{-1} \equiv B$ mod DN, where*

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If N and D are relatively prime and greater or equal to 3,

$$G_{D,N} = Q_{D,N}/(\Gamma_1(DN) \times \Gamma_1(DN))$$

may be identified with the subgroup of $\Gamma_1(D) \times \Gamma_1(D)$ consisting of (A, B) such that $wAw^{-1} = B$, so that $G_{D,N}$ isomorphic with $\Gamma_1(D)$.

Hence we have a tower of covers:

$$\begin{tikzcd}
 & \Gamma(DN) \times \Gamma(DN)\backslash\mathfrak{S}_1 \times \mathfrak{S}_1 = B_{DN} \times B_{DN} \arrow[dl, "\pi_1"'] \arrow[dd, "\pi"] \\
Q_{D,N}\backslash\mathfrak{H}_D \arrow[dr, "\pi_2"'] & \\
 & \Gamma(N) \times \Gamma(N)\backslash\mathfrak{S}_1 \times \mathfrak{S}_1 = B_N \times B_N
\end{tikzcd}$$

Here, π is Galois with group $\Gamma_1(\mathbf{Z}/D) \times \Gamma_1(\mathbf{Z}/D)$, of order $(D(D^2-1)^2)^2$; π_1 is Galois with group $Q_{D,N}$, of order $D(D^2-1)^2$; π_2 is not Galois. Thus over a generic point $(\tau_1, \tau_2) \in B_N \times B_N$, π_2 has $D(D^2-1)^2$ preimages, and

π has $(D(D^2-1)^2)^2$ preimages. Note that all these are nonsingular complex surfaces, as $B_{DN} \times B_{DN}$ is nonsingular, and $\Gamma_1(\mathbf{Z}/D) \times \Gamma_1(\mathbf{Z}/D)$, and hence also $Q_{D,N}$ act freely. Now let us consider the compactifications $B^*_{ND} \times B^*_{ND}$. The action of $\Gamma_1(\mathbf{Z}/D) \times \Gamma_1(\mathbf{Z}/D)$ extends to this, so we have a diagram:

$$\begin{tikzcd}
 & B^*_{DN} \times B^*_{DN} \arrow[dl, "\pi_1"'] \arrow[dd, "\pi"] \\
(Q_{D,N}\backslash\mathfrak{H}_D)^* \arrow[dr, "\pi_2"'] & \\
 & B^*_N \times B^*_N
\end{tikzcd}$$

There will be branching in this cover, which we now describe.

Let the situation over a point $(\tau, \infty) \in B^*_N \times B^*_N$ where τ is generic. Note that the stabilizer of ∞ is the subgroup

$$\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\} \text{ of } \Gamma_1(D), \text{ isomorphic to } \mathbf{Z}/D\mathbf{Z},$$

while the stabilizer of τ is trivial. The inverse images of this point in $B^*_{ND} \times B^*_{ND}$correspond to the cosets of this subgroup in $\Gamma_1(D) \times \Gamma_1(D)$, so the degree is $(D(D^2-1))^2/D = D(D^2-1)^2$ over such a point.

Lemma 8.3. *$G_{N,D}$ acts freely on these $D(D^2-1)^2$ points.*

Since $G_{N,D}$ has order $D(D^2-1)$ we see that there are D^2-1 points of $(Q_{D,N}\backslash\mathfrak{H}_D)^*$ lying over this point. Furthermore, since $B^*_{DN} \times B^*_{DN}$ is nonsinglar near any of these points, and $G_{N,D}$ acts on them, the quotient is nonsingular around any of these points. Thus we see the situation as:

$$\begin{tikzcd}
 & \pi_1^{-1}(y) = \{D(D^2-1) \text{ points}\} \in B^*_{DN} \times B^*_{DN} \arrow[dl, "\pi_1"'] \arrow[dd, "\pi"] \\
\pi_2^{-1}(x) = \{D^2-1 \text{ points } y\} \in (Q_{D,N}\backslash\mathfrak{H}_D)^* \arrow[dr, "\pi_2"'] & \\
 & x = (\tau, \infty) \in B^*_N \times B^*_N
\end{tikzcd}$$

Proof. The stabilizer of (τ, ∞) is

$$K_1 = \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) \right\}$$

and $G_{N,D}$ acts by

$$(g, \, wgw^{-1}) \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) = \left(g, \, wgw^{-1} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right)$$

and examining the first factor we see that this action is obviously free on the cosets of K_1. □

Lemma 8.4. *The intersection of the corank 1 boundary compnents with the union of the Humbert surfaces of discriminant D^2 consists of the multisection of the $N^2(D^2-1)$ points of order D or ND. Recall that the corank boundary components are all isomorphic to the elliptic modular surface over B_N^*, so that the points of order k for any k refer to the points of that order on the universal elliptic curve over B_N.*

Proof. There are $N^2(D^2-1)$ points in this multisection that are in the fiber over any point in B_N, and we have just seen that there are D^2-1 points in $Q_{D,N}\backslash\mathfrak{H}_D$ over any point in the base, and furthermore each open corank 1 boundary component is intersected by N^2 Humbert surfaces of discriminant D^2. But the $N^2(D^2-1)$ points:

$$\begin{pmatrix} \tau_1 & (-c'\tau_3 - e')/D - c''\tau_3 - e'' \\ (-c'\tau_3 - e')/D - c''\tau_3 - e'' & \tau_3 \end{pmatrix}$$

$$c' = 0, ..., D-1; \ e' = 0, ..., D-1, \text{ not both } 0.$$

$$c' = 0, ..., N-1; \ e' = 0, ..., N-1,$$

which are representatives of the points in this multisection all lie on Humbert surfaces of discriminant D^2. Implicitly in this argument we have used the next proposition 8.5. □

Proposition 8.5. *The closures of the Humbert surfaces inside* $(\Gamma(N)\backslash\mathfrak{S}_2)^*$ *coincide with the smooth compact models* $(Q_{N,D}\backslash\mathfrak{H}_D)^*$.

Now let us consider the cover over a point of the form

$$(\text{cusp, cusp}) \in B_N^* \times B_N^*.$$

Note that there are $(D^2-1)^2$ points in $B_{ND}^* \times B_{ND}^*$ lying over any such point, i.e., $\#\pi^{-1}(\text{cusp, cusp}) = (D^2-1)^2$. To see this it is enough to consider the special case (∞, ∞) since the action of the group $\Gamma(1) \times \Gamma(1)$ acts on the diagram given by π and permutes the (cusp, cusp) points transitively. But the stabilizer of (∞, ∞) is

$$\left\{ \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) \right\} \subset \Gamma_1(\mathbf{Z}/D) \times \Gamma_1(\mathbf{Z}/D)$$

which is of order D^2 and it acts freely on the points over (∞, ∞), so the inverse image of this point in $B^*_{ND} \times B^*_{ND}$ correspond to the cosets, and the degree is $(D(D^2-1))^2/D^2 = (D^2-1)^2$ over such a point. Recall that $G_{N,D}$ has order $D(D^2-1)$.

Proposition 8.6. *The action of $G_{N,D}$ on these $(D^2-1)^2$ points is as follows:*

1. *There are $D-1$ free orbits of G_D. These give $D-1$ nonsingular points in $(Q_{N,D}\backslash\mathfrak{H}_D)^*$.*
2. *There are $D-1$ orbits on which $G_{N,D}$ acts with isotropy group at a point a cyclic subgroup of order D. These give $D-1$ singular points in $(Q_{N,D}\backslash\mathfrak{H}_D)^*$. Locally, a neighborhood of any of these points looks like a neighborhood of $(0,0)$ in the quotient of $\mathbf{C}^2$ under $(z_1, z_2) \to (\zeta z_1, \zeta z_2)$ where $\zeta = \exp(2\pi i/D)$.*

Proof. It is clear that we may conjugate within $G_{N,D}$ to achieve that the first cusp is ∞.

We will need a convenient way to index the cosets. Note that $\Gamma_1(\mathbf{Z}/D)$ acts on $\mathbf{Z}/D\mathbf{Z} \times \mathbf{Z}/D\mathbf{Z}$ as a group of matrices, transitively on the nonzero points (warning: we are assuming D is a prime here; in general it will act transitively on the primitive vectors), and the stabilizer of any one of them is a cyclic subgroup of order D. More precisely,

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ fixes } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the cosets of $\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\}$ are therefore indexed by all the transforms of the vector $(1,0)$, in other words by all the nonzero vectors (a,b) in $(\mathbf{Z}/D)^2$.

Let (cusp, cusp) $= (\infty, r)$, where

$$\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r = \begin{pmatrix} \gamma_0 \\ \delta_0 \end{pmatrix},$$

The stabilizer of this is

$$K_2 = \left\{ \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad V \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} V^{-1} \right) \right\} \subset \Gamma_1(\mathbf{Z}/D) \times \Gamma_1(\mathbf{Z}/D)$$

where

$$V \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \delta_0 \end{pmatrix}.$$

The fiber of π over (∞, r) is indexed by pairs $(\xi, \eta) \in (\mathbf{Z}/D)^2 - \{(0,0)\}$, and we must determine the orbits of $G_{N,D}$ on these. $G_{N,D}$ acts via

$$(g, wgw^{-1}) \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, V \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} V^{-1} \right) = \left(g \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, wgw^{-1} V \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} V^{-1} \right)$$

Suppose that a coset of K_2 is indexed by a pair of vectors (α, β), (γ, δ), and let $(h_1, h_2$ be such that

$$h_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad h_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

We have $(g,\, wgw^{-1})(h_1, h_2) = (gh_1, wgw^{-1}h_2)$ and

$$gh_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = g \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad wgw^{-1}h_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = wgw^{-1} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

Suppose that $(\alpha, \beta) = (1, 0)$. For this to be fixed we must have

$$g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad wgw^{-1} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

The first one implies that g is of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and the second equality holds if and only if $\gamma = 0$. So:

$$\text{the cosets of } \left\{ V \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} V^{-1} \right\} \text{ indexed by } \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ D-1 \end{pmatrix}$$

or equivalently

$$\text{the cosets of } \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right\} \text{ indexed by } V \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ldots, V \begin{pmatrix} 0 \\ D-1 \end{pmatrix}$$

have isotropy which is cyclic of order D, and the remaining cosets have trivial isotropy. Clearly these latter give $D-1$ nonsingular points in the quotient $(Q_{N,D} \backslash \mathfrak{H}_D)^*$. As for the other points with nontrivial isotropy, we observe that the action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $(\Gamma_1(DN)\mathfrak{S}_1)^*$ near a cusp is given by $z \to \zeta z$ where $\zeta = \exp(2\pi i/D)$. On the other hand, the derivative of wgw^{-1} is the same as the derivative of g, so that in an analytic corrdinate wgw^{-1} also acts by $z \to \zeta z$; thus the action in suitable coordinates (z_1, z_2) is given by $(z_1, z_2) \to (\zeta z_1, \zeta z_2)$ as claimed. □

Corollary 8.7. *There are* $(D^2-1)^2$ *points in* $\pi^{-1}(\text{cusp}, \text{cusp})$. *These are distributed as follows:* $\pi_2^{-1}(\text{cusp}, \text{cusp})$ *has* $2(D-1)$ *points;* $D-1$ *(resp.* $D-1$*) of these are singular (resp. nonsingular) points of* $(Q_{N,D} \backslash \mathfrak{H}_D)^*$. *For each singular point* $x \in \pi_2^{-1}(\text{cusp}, \text{cusp})$, $\pi_1^{-1}(x)$ *has* D^2-1 *points. For each nonsingular point* $x \in \pi_2^{-1}(\text{cusp}, \text{cusp})$, $\pi_1^{-1}(x)$ *has* $D(D^2-1)$ *points.*

8.2. **When $N = D$.** The anlysis at the beginning of section 8 goes through unchanged and leads to the following result: The group $Q_{D,D}$ sits in a tower

$$\Gamma_1(D^2) \times \Gamma_1(D^2) \subset Q_{D,D} \subset \Gamma_1(D) \times \Gamma_1(D)$$

so that $G_{D,D} = Q_{D,D}/(\Gamma_1(D^2) \times \Gamma_1(D^2))$ can be identified with a subgroup of

$$(\Gamma_1(D)/\Gamma_1(D^2)) \times (\Gamma_1(D)/\Gamma_1(D^2) \cong (\mathbf{Z}/D)^3 \times (\mathbf{Z}/D)^3$$

In fact,

$$G_{D,D} = \left\{(g_1, g_2) \mid w g_1 w^{-1} = g_2\right\} \text{ where } w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

hence $G_{D,D} \cong (\mathbf{Z}/D)^3$. Thus we get a diagram of nonsingular complex algebraic surfaces

$$\begin{tikzcd} & B_{D^2} \times B_{D^2} \arrow[dl, "\pi_1"'] \arrow[dd, "\pi"] \\ Q_{D,D}\backslash\mathfrak{H}_D \arrow[dr, "\pi_2"'] & \\ & B_D \times B_D \end{tikzcd}$$

where π is Galois with group isomorphic with $(\mathbf{Z}/D)^3 \times (\mathbf{Z}/D)^3$, and both π_1 and π_2 are Galois with group isomorphic with $(\mathbf{Z}/D)^3$. These are unramified coverings. As before, this diagram with its group actions extends to the compactifications. There will be branching in these coverings occuring at the points at infinity that we have added. We will analyze the branching.

First consider the situation over a point $(\tau, \infty) \in B_D^* \times B_D^*$ with τ generic. The stabilizer of ∞ is the subgroup

$$\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\} \text{ with } x \in D(\mathbf{Z}/D^2\mathbf{Z}) \cong \mathbf{Z}/D\mathbf{Z}$$

and the stabilizer of τ is trivial, so the stabilizer of (τ, ∞) is cyclic of order D. Since the set $\pi^{-1}(\tau, \infty)$ is in bijective correspondance with the cosets of this stabilizer in the Galois group of the cover π, we see that there are $D^6/D = D^5$ points in this fiber.

Lemma 8.8. *$G_{D,D}$ acts freely on these D^5 points.*

Since $G_{D,D}$ has order D^3 we see that there are D^2 points of $(Q_{D,D}\backslash\mathfrak{H}_D)^*$ lying over this point. Furthermore, since $B_{D^2}^* \times B_{D^2}^*$ is nonsinglar near any of these points, and $G_{D,D}$ acts on them, the quotient is nonsingular around any of these points. Thus we see the situation as:

$$\begin{tikzcd}
 & \pi_1^{-1}(y) = \{D^3) \text{ points}\} \in B^*_{D^2} \times B^*_{D^2} \arrow[dl, "\pi_1"'] \arrow[dd, "\pi"] \\
\pi_2^{-1}(x) = \{D^2 \text{ points } y\} \in (Q_{D,D} \backslash \mathfrak{H}_D)^* \arrow[dr, "\pi_2"'] & \\
 & x = (\tau, \infty) \in B^*_D \times B^*_D
\end{tikzcd}$$

Proof. Exactly as in the proof of lemma 8.3. $\square$

Now let us consider the cover over a point of the form

$$(\text{cusp, cusp}) \in B^*_D \times B^*_D.$$

We may assume the first cusp is ∞ and let second be r, so

$$\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r = \begin{pmatrix} \gamma_0 \\ \delta_0 \end{pmatrix}.$$

The stabilizer for this is

$$K_2 = \left\{ \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ V \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} V^{-1} \right) \right\} \subset \Gamma_1(\mathbf{Z}/D) \times \Gamma_1(\mathbf{Z}/D)$$

where

$$V \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \delta_0 \end{pmatrix}.$$

and $G_{D,D}$ acts by

$$(g, \, wgw^{-1}) \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ V \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} V^{-1} \right) = \left(g, \, wgw^{-1} V \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} V^{-1} \right)$$

We wish to examine the action of $G_{D,D}$ on the cosets of K_2 in

$$(\Gamma_1(D)/\Gamma_1(D^2)) \times (\Gamma_1(D)/\Gamma_1(D^2)).$$

But this latter group is abelian. Hence we see that

- an element of $G_{D,D}$ acts trivially if that element is in K_2.
- an element of $G_{D,D}$ acts freely if that element is not in K_2.

Thus we must simply investigate $G_{D,D} \cap K_2$. Let $(g, wgw^{-1}) \in G_{D,D}$. Clearly this element is in K_2 if and only if

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ for some } x \text{ and } wgw^{-1} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \text{ for some } y.$$

But

$$wgw^{-1} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \text{ and } wgw^{-1} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \text{ holds if and only if}$$

wgw^{-1} is the stabilizer of the cusp r. By direct calculation,

$$\text{if } r = \begin{pmatrix}\gamma_0\\ \delta_0\end{pmatrix} \text{ for } \delta_0 = \pm 1, ..., \pm(D-1)/2, \text{ then } \begin{pmatrix}1 & 0\\ x & 1\end{pmatrix} \text{ stabilizes } r.$$

if r is any other cusp the only element of the form $\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$ stabilizing r is $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, as $\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\begin{pmatrix}\gamma_0\\ \delta_0\end{pmatrix} = \begin{pmatrix}\gamma_0\\ x\gamma_0 + \delta_0\end{pmatrix}$.

Note that B_D^* has $(D^2-1)/2$ cusps. We have shown:

Lemma 8.9. *Over $(D-1)/2$ cusps where $G_{D,D}$ has isotropy $\mathbf{Z}/D$ we have the situation:*

$$\begin{tikzcd}
 & \pi_1^{-1}(y) = \{D^2 \text{ points}\} \in B^*_{D^2} \times B^*_{D^2} \arrow[dl, "\pi_1"'] \arrow[dd, "\pi"] \\
\pi_2^{-1}(x) = \{D^2 \text{ points } y\} \in (Q_{D,D}\backslash \mathfrak{H}_D)^* \arrow[dr, "\pi_2"'] & \\
 & x = (\infty, r) \in B_D^* \times B_D^*
\end{tikzcd}$$

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