Arithmetic properties of Picard-Fuchs differential equations

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1. Gauss-Manin connections

2. Families of elliptic curves

3. Dwork’s theory

4. Modular forms

5. Examples

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Arithmetic properties of Picard-Fuchs differential equations
Given a family $X_s$ of algebraic varieties depending on a parameter $s$ and an integer $i$, there is a differential equation that expresses the variation in the cohomology $H^i(X_s, \mathbb{C})$.

More formally let $f : X \to S$ be a proper and smooth morphism, with $S$ smooth. Then there is canonical integrable connection

$$\nabla : H^i_{DR}(X/S) := R^i f_* \Omega^\bullet_{X/S} \to H^i_{DR}(X/S) \otimes O_S \Omega^1_S$$

called the Gauss-Manin connection. Note that

$$H^i_{DR}(X/S)^{an} = R^i f_* \mathbb{C} \otimes_{\mathbb{C}} O_S^{an}, \quad (R^i f_* \mathbb{C})_S = H^i(X_s, \mathbb{C}).$$

It has regular singular points at infinity (relative to any smooth compactification of $S$; theorem of Griffiths, Deligne, Katz).

regular singular = of the Fuchsian class.
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It has regular singular points at infinity (relative to any smooth compactification of $S$; theorem of Griffiths, Deligne, Katz).

regular singular = of the Fuchsian class.
To get a differential equation for (multivalued) **functions** rather than **cohomology classes** we can consider the **periods**

\[ p(\delta, \omega, s) := \int_{\delta}^{s} \omega, \quad \omega \in H_{DR}^i(X/S), \quad \delta \in R^i f_* \mathbb{C} \]

Then \( \nabla(\omega) = 0 \Rightarrow \nabla(p(\delta, \omega, s)) = 0 \) for any \( \delta \). In other words, the Gauss-Manin connection is equivalent to a first-order linear system

\[ \frac{dy}{dt} = My \]

for a matrix \( M \) of rational functions on \( S \) (assumed of dimension 1 with local coordinate \( t \) for simplicity). We call these **Picard-Fuchs equations**.

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Elliptic curves

Example

Let $E_t$ be the family of projective plane curves

$$y^2 = x^3 - g_2(t)x - g_3(t).$$

For all $t \in S := \{g_2(t)^3 - 27g_3(t)^2 \neq 0\} \subset \mathbb{P}^1$ these are elliptic curves.

Let

$$\omega_1 = \frac{dx}{y}, \quad \omega_2 = \frac{x dx}{y}.$$  

These form a basis of $H^1_{DR}(E/S)$.

The DE is

$$\frac{d}{dt} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$
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Example

Where

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\begin{align*}
A_1 &= \frac{-1}{12\Delta} \frac{d\Delta}{dt}, \\
A_2 &= \frac{3G}{2\Delta}, \\
A_3 &= \frac{-g_2 G}{8\Delta}, \\
A_4 &= \frac{1}{12\Delta} \frac{d\Delta}{dt}
\end{align*}
\]

\[
\Delta = g_2^3 - 27g_3^2, \quad G = 3g_3 \frac{dg_2}{dt} - 2g_2 \frac{dg_3}{dt}
\]

We can write this first order system as a second order equation in the shape

\[
A(t) \frac{d^2 y}{dt^2} + B(t) \frac{dy}{dt} + C(t)y = 0
\]

for rational functions \( A(t), B(t), C(t) \).
Example

Where

\[ A_1 = -\frac{1}{12\Delta} \frac{d\Delta}{dt}, \quad A_2 = \frac{3G}{2\Delta} \]
\[ A_3 = -\frac{g_2 G}{8\Delta}, \quad A_4 = \frac{1}{12\Delta} \frac{d\Delta}{dt} \]

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\[ A(t) \frac{d^2 y}{dt^2} + B(t) \frac{dy}{dt} + C(t)y = 0 \]

for rational functions \( A(t), B(t), C(t) \).
Consider the $\Gamma(2)$ modular family: Legendre’s family. This is $E_t : y^2 = x(x - 1)(x - t)$, which gives the DE

$$t(t - 1)\frac{d^2 y}{dt^2} + (2t - 1)\frac{dy}{dt} + \frac{1}{4}y = 0.$$ 

This is hypergeometric with parameters $1/2, 1/2, 1$. A period is a solution

$$2\pi F \left( \frac{1}{2}, \frac{1}{2}, 1; t \right) = 2 \int_0^1 \frac{dx}{\sqrt{x(1 - x)(1 - xt)}}$$

$$F \left( \frac{1}{2}, \frac{1}{2}, 1; t \right) = 1 + \frac{t}{4} + \frac{9t^2}{64} + \frac{25t^3}{256} + \frac{1225t^4}{16384} + \frac{3969t^5}{65536} + \ldots$$
Dwork’s theory

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Now regard the series

\[ F(1/2, 1/2, 1; t) = \sum_{j=0}^{\infty} \left( -\frac{1}{2} \right)^j t^j \]

as a function of a \( p \)-adic variable \( t \). (\( p \) odd). Let \( U(t) = F(1/2, 1/2, 1; t)/F(1/2, 1/2, 1; t^p) \).

Then Dwork showed:

1. The series \( U(t) \) can be analytically continued to a function defined in \( p \)-adic disk of radius \( \geq 1 \).

2. Let \( \lambda_0 \in \mathbb{P}^1(\mathbb{F}_q) - \{0, 1, \infty\} \), \( q = p^s \). Suppose that \( \text{Hasse}(\lambda_0) \neq 0 \). Let \( t_0 \in \mathcal{W}(\mathbb{F}_q) \) be the Teichmüller lifting. Let \( 1 - a(\lambda_0)X + qX^2 \) be the numerator of the zeta function of the elliptic curve \( y^2 = x(x-1)(x-\lambda_0) \) defined over \( \mathbb{F}_q \). Then:

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1 - a(\lambda_0)X + qX^2 = (1 - \alpha(t_0)X)(1 - (q/\alpha(t_0))X)
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where

\[ \alpha(t) = U(t)U(t^p)U(t^{p^2})...U(t^{p^{s-1}}) \]

In other words, the unit root of the zeta function of the elliptic curve is given by a \( p \)-adic power series that comes from the solution to the Picard-Fuchs differential equation of the family of elliptic curves.

Conceptually, the family \( f : X \to S \) defines an \( F \)-crystal on the rigid analytic space \( S(\mathbb{C}_p) \) — supersingular disks. Roughly speaking, \( F \)-crystal = DE (connection) with a Frobenius structure.
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Here are some key points in Dwork’s proof.

1. The truncated hypergeometric series

\[ H(\lambda) = (-1)^{(p-1)/2} \sum_{j=0}^{(p-1)/2} \left( \frac{-1}{2} \right)^2 \lambda_j^0 \]

is the Hasse invariant = Frobenius mod \( p \) of the elliptic curve
\[ y^2 = x(x - 1)(x - \lambda_0). \]
This was discovered by Igusa and generalized by Manin.
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Next

2. Let \( u \) be a parameter at the origin of the elliptic curve \( E : y^2 = x(x - 1)(x - t) \), regarded as a scheme over \( \mathbb{Z}[t, 1/2t(t - 1)] \). Then expanding the differential

\[
\omega = \frac{dx}{\sqrt{x(x - 1)(x - t)}} = \sum_{n \geq 1} q_n(t)u^{n-1}du
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is the logarithm of the formal group \( \hat{E} \).
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Then

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\int \omega = \sum_{n \geq 1} n^{-1} q_n(t)u^n
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is the logarithm of the formal group \( \hat{E} \).
Therefore

2 cont. We have Cartier-Honda congruences: This was first observed by Lazard and Tate, and this gives a connection to the zeta function of the elliptic curves in the family. For simplicity assume \( \lambda \in \mathbb{Z} \). Then

\[
q_{mp^a}(\lambda) - a(\lambda)q_{mp^{a-1}}(\lambda) + pq_{mp^{a-2}}(\lambda) \equiv 0 \mod p^a
\]

\( a(\lambda) = \text{trace of Frob}_p \) on the Tate module of
\( y^2 = x(x - 1)(x - \lambda) \). For the parameter \( u = 1/\sqrt{x} \) we have

\[
q_{2n+1}(t) = (-1)^n \sum_{i=0}^{n} \binom{-1/2}{n-i} \binom{-1/2}{i} t^i
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Then: 2 cont. If \( q_p(\lambda) = \text{Hasse}(\lambda) \not\equiv 0 \mod p \), then these congruences show that

\[
\lim_{a \to \infty} \frac{q_{mp^a+1}(\lambda)}{q_{mp^a}(\lambda^p)}
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converges \( p \)-adically to the unit root of the zeta function of the elliptic curve \( y^2 = x(x - 1)(x - \lambda) \mod p \).  

3. Finally we have Dwork’s congruence:

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\frac{q_{mp^a+1}(\lambda)}{q_{mp^a}(\lambda^p)} \equiv \frac{F(1/2, 1/2, 1; \lambda)}{F(1/2, 1/2, 1; \lambda^p)} \mod p^a
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Dwork set up a general theory of zeta functions of families of hypersurfaces. This theory related the zeta function to the period matrices in the family, but this theory was only valid for regular values of the parameter $t$, that is, values where $X_t$ is nonsingular.

The Legendre family of elliptic curves is different in this respect: the expression for the unit root of the zeta function of the elliptic curves in terms of the series $F(1/2, 1/2, 1; t)$ which is the holomorphic solution to the Picard-Fuchs differential equation at a singular point $t = 0$. This holomorphic solution is the period of the differential over the vanishing cycle at $t = 0$. 
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Here are the key points: Give a family of hypersurfaces $X_\lambda$ for $\lambda \in \mathbb{C}_p$ there are finite-dimensional $\mathbb{C}_p$-vector spaces (cohomology spaces) $W(\lambda)$, such that

1. For all $\lambda_0$ with $|\lambda_0| \leq 1$ there is a map
   $$\alpha(\lambda_0) : W(\lambda_0) \to W(\lambda_0^p).$$

2. If $|\lambda_0 - \lambda_1| < 1$ there are maps
   $$C(\lambda_0, \lambda_1) : W(\lambda_0) \to W(\lambda_1)$$
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2. If $|\lambda_0 - \lambda_1| < 1$ there are maps

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3. The diagram commutes:

\[
\begin{align*}
\mathcal{W}(\lambda_0) & \xrightarrow{\alpha(\lambda_0)} \mathcal{W}(\lambda_0^p) \\
C(\lambda_0, \lambda_1) \downarrow & \quad \quad \downarrow C(\lambda_0^p, \lambda_1^p) \\
\mathcal{W}(\lambda_1) & \xrightarrow{\alpha(\lambda_0)} \mathcal{W}(\lambda_1^p)
\end{align*}
\]
4. If $\lambda_0^p = \lambda_0$ then the trace of the endomorphism

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of $W(\lambda_0)$ is essentially the number of points in

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Modular forms

The following theorem was essentially known in the 19th century:

**Theorem.** (Zagier) Let \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \) be a subgroup of finite index. Let \( f(z) \) be a (meromorphic) modular form of weight \( k \) for \( \Gamma \) \((z \in \mathbb{H})\). Let \( t(z) \) be a modular function (=meromorphic modular form of weight \( 0 \)) for \( \Gamma \).

The (many-valued) function \( F(t) \) defined by \( F(t(z)) = f(z) \) satisfies a differential equation of order \( k + 1 \) with algebraic coefficients.

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That is, there is an equation

$$a_0(t) \frac{d^{k+1}F}{dt^{k+1}} + \ldots + a_k(t) \frac{dF}{dt} + a_{k+1}(t)F = 0$$

with the $a_i(t)$ in the function field $\mathbb{C}(\Gamma) = \mathbb{C}(X_\Gamma)$ (hence algebraic functions of the $t$).
Examples

\[ t(z) = \frac{\eta(6z)^8 \eta(z)^4}{\eta(2z)^8 \eta(3z)^4} \]

is a generator of the function field \( \mathbb{C}(\Gamma_1(6)) \).

\[ f(z) = \frac{\eta(2z)^6 \eta(3z)}{\eta(z)^3 \eta(6z)^2} \]

has weight 1.
The function $F(t)$ defined by $F(t(z)) = f(z)$ satisfies a 2nd order DE:

$$t(t-1)(9t-1)\frac{d^2F}{dt^2} + (27t^2 - 20t + 1)\frac{dF}{dt} + 3(3t - 1)F = 0$$

This is the PF equation for the universal family of elliptic curves with a point of order 6:

$$(x + y + z)(xy + yz + zx) = \frac{1}{t}xyz$$
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The expansion for the solution has all integer coefficients:

\[ F(t) = 1 + 3t + 15t^2 + 93t^3 + 639t^4 + 4653t^5 + 35169t^6 + \ldots \]

Let

\[ F(t^2) = \sum_{m \geq 0} c_{2m+1} t^{2m} \]

Then we have ASDCH congruences:

\[ c_{mp^{r+1}} - \alpha_p c_{mp^r} + p^2 c_{mp^r-1} \equiv 0 \mod p^{r+1} \]
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$$\pi : Y \to U, \quad j : U \to \mathbb{P}^1 = \text{t-line},$$

where $Y \subset X$ is the corresponding open subset of the surface

$$(x + y + z)(xy + yz + zx) = \frac{1}{t^2} xyz.$$  

This example is due to Beukers and Stienstra.
The key is that \( \int F(t^2)dt = \sum_{n \geq 1} n^{-1}c_n t^n \) is the logarithm of the formal Brauer group \( Br_{X_p} \) of the elliptic K3 surface \( X_p = X \mod p \).

The zeta function has the shape:

\[
Z(X_p/\mathbb{F}_p, T) = \frac{1}{(1 - T)(1 - p^2 T)P(T)}
\]

where the polynomial of degree 22 is

\[
P(T) = \det(1 - T \text{Frob}_p | H^2_{\text{cris}}(X_p/\mathbb{Z}_p)).
\]

In the deRham-Witt decomposition

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H^2_{\text{cris}}(X_p/\mathbb{Z}_p)) = H^2(X_p, W\mathcal{O}) \oplus H^1(X_p, W\Omega^1) \oplus H^0(X_p, W\Omega^2)
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If $p$ is not supersingular, then $H^2(X_p, W\mathcal{O})$ is free over $\mathbb{Z}_p$ of rank $= \text{the height of } Br_{X_p} = 1$. Moreover, for this example, the Neron-Severi group has rank 20, which means that the interesting part of the zeta function is determined by the action of Frobenius on the transcendental part, $H^2(X_p, W\mathcal{O}) \oplus H^0(X_p, W\mathcal{O}^2)$, of rank 2, and this in turn is determined by the formal Brauer group. In the ASDCH congruences,

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y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2
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The solution to the PF equation is

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F(t) = 1 - t + 6t^2 - 25t^3 + 125t^4 - 642t^5 + 3423t^6 - \ldots
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The coefficients satisfy congruences

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Symmetric square of the $\Gamma_0(8)$ family of elliptic curves. Also worked out in the 2010 REU. Here the DE is of third order:

$$t^2(16t^2 + 1)F''' + 3t(16t^2 + 1)(48t^2 + 1)F'' + (4864t^4 + 256t^2 + 1)F' + 64t(32t^2 + 1)F = 0,$$

related to a family of K3-surfaces. Solution:

$$F(t) = 1 - 8t^2 + 88t^4 - 1088t^6 + 14296t^8 + ...$$

(Experimentally) these satisfy congruences

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Jerome William Hoffman 

Arithmetic properties of Picard-Fuchs differential equations
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The **formal Brauer group**, namely the functor $H^2(X, \mathbb{G}_m^\wedge)$, is a special case of **Artin-Mazur** formal groups, defined by $H^i(X, \mathbb{G}_m^\wedge)$. **Stienstra** has generalized these methods to **AM** formal groups in various situations (complete intersections, cyclic branched coverings).
Here is a

**Problem.** Katz has shown that one can define a **unit root part of an $F$-crystal** for a family $X \to S$. Moreover this unit root crystal can be reconstructed (locally) from the expansion coefficients of $H^0(X, \Omega^i_{X/S})$. Hence the unit root part of the zeta functions of the fibers $X_s$ can be determined by this data.

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This work continues in the 2011 REU directed with my colleague Chris Bremer, Students: Cody Gunton, Zane Li, Jason Steinberg, Avi Steiner, Alex Walker,
Thanks to

KingFai Lai and Winnie Li!