Arithmetic properties of Picard-Fuchs differential equations

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Gauss-Manin connections

Given a family X_s of algebraic varieties depending on a parameter s and an integer i, there is a differential equation that expresses the variation in the cohomology $H^i(X_s, \mathbb{C})$.

More formally let $f : X \to S$ be a proper and smooth morphism, with S smooth. Then there is canonical integrable connection

$$\nabla : H^i_{DR}(X/S) := \mathbf{R}^i f_* \Omega^\bullet_{X/S} \longrightarrow H^i_{DR}(X/S) \otimes_{\mathcal{O}_S} \Omega^1_S$$

called the Gauss-Manin connection. Note that

$$H^i_{DR}(X/S)^{an} = R^i f_* \mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_{S^{an}}, \quad (R^i f_* \mathbb{C})_s = H^i(X_s, \mathbb{C}).$$

It has regular singular points at infinity (relative to any smooth compactification of S; theorem of Griffiths, Deligne, Katz).
regular singular = of the Fuchsian class.

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To get a differential equation for (multivalued) functions rather than cohomology classes we can consider the periods

$$p(\delta,\omega,s) := \int_{\delta_s} \omega_s, \quad \omega \in H^i_{DR}(X/S),\ \delta \in R^i f_* \mathbb{C}$$

Then $\nabla(\omega) = 0 \Rightarrow \nabla(p(\delta,\omega,s)) = 0$ for any δ. In other words, the Gauss-Manin connection is equivalent to a first-order linear system

$$\frac{dy}{dt} = My$$

for a matrix M of rational functions on S (assumed of dimension 1 with local corrdinate t for simplicity).
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Elliptic curves

Example

Let E_t be the family of projective plane curves

$$y^2 = x^3 - g_2(t)x - g_3(t).$$

For all $t \in S := \{g_2(t)^3 - 27g_3(t)^2 \neq 0\} \subset \mathbf{P}^1$ these are elliptic curves.

Let

$$\omega_1 = \frac{dx}{y}, \quad \omega_2 = \frac{xdx}{y}.$$

These form a basis of $H^1_{DR}(E/S)$.

The DE is

$$\frac{d}{dt}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

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Example

Where

$$A_1 = \frac{-1}{12\Delta}\frac{d\Delta}{dt}, \qquad A_2 = \frac{3G}{2\Delta}$$
$$A_3 = \frac{-g_2 G}{8\Delta}, \quad A_4 = \frac{1}{12\Delta}\frac{d\Delta}{dt}$$

$$\Delta = g_2^3 - 27g_3^2, \quad G = 3g_3\frac{dg_2}{dt} - 2g_2\frac{dg_3}{dt}$$

We can write this first order system as a second order equation in the shape

$$A(t)\frac{d^2y}{dt^2} + B(t)\frac{dy}{dt} + C(t)y = 0$$

for rational functions $A(t), B(t), C(t)$.

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Dwork's theory

Consider the $\Gamma(2)$ modular family: Legendre's family. This is $E_t : y^2 = x(x-1)(x-t)$, which gives the DE

$$t(t-1)\frac{d^2y}{dt^2} + (2t-1)\frac{dy}{dt} + \frac{1}{4}y = 0.$$

This is hypergeometric with parameters $1/2, 1/2, 1$. A period is a solution

$$2\pi F\left(\frac{1}{2}, \frac{1}{2}, 1; t\right) = 2\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-xt)}}$$

$$F(1/2, 1/2, 1; t) = 1 + \frac{t}{4} + \frac{9t^2}{64} + \frac{25t^3}{256} + \frac{1225t^4}{16384} + \frac{3969t^5}{65536} + \ldots$$

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Now regard the series

$$F(1/2,1/2,1;t) = \sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{j}^2 t^j$$

as a function of a p-adic variable t. (p odd). Let

$$U(t) = F(1/2,1/2,1;t)/F(1/2,1/2,1;t^p).$$

Then Dwork showed:

1. The series $U(t)$ can be analytically continued to a function defined in p-adic disk of radius ≥ 1.
2. Let $\lambda_0 \in \mathbf{P}^1(\mathbb{F}_q) - \{0,1,\infty\}$, $q = p^s$. Suppose that $\mathrm{Hasse}(\lambda_0) \neq 0$. Let $t_0 \in W(\mathbb{F}_q)$ be the Teichmuller lifting.Let $1 - a(\lambda_0)X + qX^2$ be the numerator of the zeta function of the elliptic curve $y^2 = x(x-1)(x-\lambda_0)$ defined over $\mathbb{F}_q$. Then:

$$1 - a(\lambda_0)X + qX^2 = (1-\alpha(t_0)X)(1-(q/\alpha(t_0))X)$$

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where

$$\alpha(t) = U(t)U(t^p)U(t^{p^2})...U(t^{p^{s-1}})$$

In other words, the unit root of the zeta function of the elliptic curve is given by a p-adic power series that comes from the solution to the Picard-Fuchs differential equation of the family of elliptic curves.

Conceptually, the family $f : X \to S$ defines an F-crystal on the rigid analytic space $S(\mathbb{C}_p) - \text{supersingular disks.}$ Roughly speaking, F-crystal = DE (connection) with a Frobenius structure.

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Here are some key points in Dwork's proof.

1. The truncated hypergeometric series

$$H(\lambda) = (-1)^{(p-1)/2} \sum_{j=0}^{(p-1)/2} \binom{-\frac{1}{2}}{j}^2 \lambda_0^j$$

is the Hasse invariant = Frobenius mod p of the elliptic curve $y^2 = x(x-1)(x-\lambda_0)$. This was discovered by Igusa and generalized by Manin.

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Next

2. Let u be a parameter at the origin of the elliptic curve $E : y^2 = x(x-1)(x-t)$, regarded as a scheme over $\mathbb{Z}[t, 1/2t(t-1)]$. Then expanding the differential

$$\omega = \frac{dx}{\sqrt{x(x-1)(x-t)}} = \sum_{n \geq 1} q_n(t) u^{n-1} du$$

Then

$$\int \omega = \sum_{n \geq 1} n^{-1} q_n(t) u^n$$

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Therefore
2 cont. We have Cartier-Honda congruences: This was first observed by Lazard and Tate, and this gives a connection to the zeta function of the elliptic curves in the family. For simplicity assume $\lambda \in \mathbb{Z}$. Then

$$q_{mp^a}(\lambda) - a(\lambda) q_{mp^{a-1}}(\lambda) + p q_{mp^{a-2}}(\lambda) \equiv 0 \bmod p^a$$

$a(\lambda) =$ trace of Frob_p on the Tate module of $y^2 = x(x-1)(x-\lambda)$. For the parameter $u = 1/\sqrt{x}$ we have

$$q_{2n+1}(t) = (-1)^n \sum_{i=0}^{n} \binom{-1/2}{n-i} \binom{-1/2}{i} t^i$$

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Then: **2 cont.** If $q_p(\lambda) = \mathrm{Hasse}(\lambda) \neq 0$ mod p, then these congruences show that

$$\lim_{a\to\infty} \frac{q_{mp^{a+1}}(\lambda)}{q_{mp^a}(\lambda^p)}$$

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3. Finally we have Dwork's congruence:

$$\frac{q_{mp^{a+1}}(\lambda)}{q_{mp^a}(\lambda^p)} \equiv \frac{F(1/2,1/2,1;\lambda)}{F(1/2,1/2,1;\lambda^p)} \mod p^a$$

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Dwork set up a general theory of zeta functions of families of hypersurfaces. This theory related the zeta function to the period matrices in the family, but this theory was only valid for regular values of the parameter t, that is, values where X_t is nonsingular. The Legendre family of elliptic curves is different in this respect: the expression for the unit root of the zeta function of the elliptic curves in terms of the series $F(1/2, 1/2, 1; t)$ which is the holomorphic solution to the Picard-Fuchs differential equation at a singular point $t = 0$. This holomorphic solution is the period of the differential over the vanishing cycle at $t = 0$.

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Here are the key points: Give a family of hypersurfaces X_λ for $\lambda \in \mathbb{C}_p$ there are finite-dimensional $\mathbb{C}_p$-vector spaces (cohomology spaces) $W(\lambda)$, such that

1. For all λ_0 with $|\lambda_0| \leq 1$ there is a map
$$\alpha(\lambda_0) : W(\lambda_0) \to W(\lambda_0^p).$$
2. If $|\lambda_0 - \lambda_1| < 1$ there are maps
$$C(\lambda_0, \lambda_1) : W(\lambda_0) \to W(\lambda_1)$$

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3. The diagram commutes:

$$\begin{tikzcd} W(\lambda_0) \arrow[r, "\alpha(\lambda_0)"] \arrow[d, "C(\lambda_0{,}\lambda_1)"'] & W(\lambda_0^p) \arrow[d, "C(\lambda_0^p{,}\lambda_1^p)"] \\ W(\lambda_1) \arrow[r, "\alpha(\lambda_0)"] & W(\lambda_1^p) \end{tikzcd}$$

4. If $\lambda_0^{p^s} = \lambda_0$ then the trace of the endomorphism

$$\alpha(\lambda^{p^{s-1}})\alpha(\lambda^{p^{s-2}})\ldots\alpha(\lambda_0)$$

of $W(\lambda_0)$ is essentially the number of points in

$$X_{\lambda_0}(\mathbf{F}_{p^s}).$$

5. The matrix $C(\lambda) = C(0, \lambda)$ satisfies a differential equation

$$\frac{dC(\lambda)}{d\lambda} = C(\lambda)B(\lambda)$$

for a matrix of rational functions (Picard-Fuchs DE).

4. If $\lambda_0^{p^s} = \lambda_0$ then the trace of the endomorphism

$$\alpha(\lambda^{p^{s-1}})\alpha(\lambda^{p^{s-2}})...\alpha(\lambda_0)$$

of $W(\lambda_0)$ is essentially the number of points in

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Modular forms

The following theorem was essentially known in the 19th century:

Theorem.(Zagier) Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index. Let $f(z)$ be a (meromorphic) modular form of weight k for Γ ($z \in \mathfrak{H}$). Let $t(z)$ be a modular function (=meromorphic modular form of weight 0) for Γ.

The (many-valued) function $F(t)$ defined by $F(t(z)) = f(z)$ satisfies a differential equation of order $k+1$ with algebraic coefficients.

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That is, there is an equation

$$a_0(t)\frac{d^{k+1}F}{dt^{k+1}} + \ldots + a_k(t)\frac{dF}{dt} + a_{k+1}(t)F = 0$$

with the $a_i(t)$ in the function field $\mathbb{C}(\Gamma) = \mathbb{C}(X_\Gamma)$ (hence algebraic functions of the t).

Examples

$\Gamma_1(6)$.

$$t(z) = \frac{\eta(6z)^8\eta(z)^4}{\eta(2z)^8\eta(3z)^4}$$

is a generator of the function field $\mathbb{C}(\Gamma_1(6))$.

$$f(z) = \frac{\eta(2z)^6\eta(3z)}{\eta(z)^3\eta(6z)^2}$$

has weight 1.

The function $F(t)$ defined by $F(t(z)) = f(z)$ satisfies a 2nd order DE:

$$t(t-1)(9t-1)\frac{d^2F}{dt^2} + (27t^2 - 20t + 1)\frac{dF}{dt} + 3(3t-1)F = 0$$

This is the PF equation for the universal family of elliptic curves with a point of order 6:

$$(x+y+z)(xy+yz+zx) = \frac{1}{t}xyz$$

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The expansion for the solution has all integer coefficients:

$$F(t) = 1 + 3t + 15t^2 + 93t^3 + 639t^4 + 4653t^5 + 35169t^6 + \ldots$$

Let

$$F(t^2) = \sum_{m \geq 0} c_{2m+1} t^{2m}$$

Then we have ASDCH congruences:

$$c_{mp^{r+1}} - \alpha_p c_{mp^r} + p^2 c_{mp^{r-1}} \equiv 0 \bmod p^{r+1}$$

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Where $\alpha_p = \mathrm{Trace}(\mathrm{Frob}_p \mid H^1(\mathbf{P}^1, j_* R^1\pi_*\mathbb{Q}_l))$

$$\pi : Y \to U, \quad j : U \to \mathbf{P}^1 = t - \text{ line},$$

where $Y \subset X$ is the corresponding open subset of the surface

$$(x+y+z)(xy+yz+zx) = \frac{1}{t^2}xyz.$$

This example is due to Beukers and Stienstra.

The key is that $\int F(t^2)dt = \sum_{n\geq 1} n^{-1}c_n t^n$ is the logarithm of the formal Brauer group Br_{X_p} of the elliptic K3 surface $X_p = X \bmod p$.
The zeta function has the shape:

$$Z(X_p/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-p^2T)P(T)}$$

where the polynomial of degree 22 is

$$P(T) = \det(1 - T\mathrm{Frob}_p \mid H^2_{\mathrm{cris}}(X_p/\mathbb{Z}_p)).$$

In the deRham-Witt decomposition

$$H^2_{\mathrm{cris}}(X_p/\mathbb{Z}_p)) = H^2(X_p, W\mathcal{O}) \oplus H^1(X_p, W\Omega^1) \oplus H^0(X_p, W\Omega^2)$$

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If p is not supersingular, then $H^2(X_p, W\mathcal{O})$ is free over $\mathbb{Z}_p$ of rank $=$ the height of $Br_{X_p} = 1$. Moreover, for this example, the Neron-Severi group has rank 20, which means that the interesting part of the zeta function is determined by the action of Frobenius on the transcendental part, $H^2(X_p, W\mathcal{O}) \oplus H^0(X_p, W\Omega^2)$, of rank 2, and this in turn is determined by the formal Brauer group. In the ASDCH congruences,

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$\Gamma_1(7)$. This was worked out by my REU students in 2010 (Suzanne Carter, Shaunak Das, Steffen Docken). The family of elliptic curves is

$$y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2$$

The solution to the PF equation is

$$F(t) = 1 - t + 6t^2 - 25t^3 + 125t^4 - 642t^5 + 3423t^6 - \dots$$

The coefficients satisfy congruences

$$c_{mp^{r+1}} - \alpha_p c_{mp^r} + p^2 c_{mp^{r-1}} \equiv 0 \mod p^{r+1}$$

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p	α_p
5	0
11	-6
13	0
17	0
19	0
23	18
29	-54
31	0
37	-38
41	0
43	58
47	0
53	-6

Symmetric square of the $\Gamma_0(8)$ family of elliptic curves. Also worked out in the 2010 REU. Here the DE is of third order:

$$t^2(16t^2+1)F''' + 3t(16t^2+1)(48t^2+1)F'' + (4864t^4+256t^2+1)F' + 64t(32t^2+1)F = 0,$$

related to a family of K3-surfaces. Solution:

$$F(t) = 1 - 8t^2 + 88t^4 - 1088t^6 + 14296t^8 + \ldots$$

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p	α_p
5	-2
7	24
11	-44
13	22
17	50
19	44
23	-56
29	128
31	-160
37	-162
41	-198
43	52
47	528
53	-242

The formal Brauer group, namely the functor $H^2(X, \mathbf{G}_m^\wedge)$, is a special case of Artin-Mazur formal groups, defined by $H^i(X, \mathbf{G}_m^\wedge)$. Stienstra has generalized these methods to AM formal groups in various situations (complete intersections, cyclic branched coverings).

Here is a

Problem. Katz has shown that one can define a unit root part of an F-crystal for a family $X \to S$. Moreover this unit root crystal can be reconstructed (locally) from the expansion coefficients of $H^0(X, \Omega^i_{X/S})$. Hence the unit root part of the zeta functions of the fibers X_s can be determined by this data.

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Can one (or: under what conditions can one) reconstruct the unit root F-crystals by expansions at singular points of solutions to Picard-Fuchs differential equations?

This work continues in the 2011 REU directed with my colleague Chris Bremer, Students: Cody Gunton, Zane Li, Jason Steinberg, Avi Steiner, Alex Walker,

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